IEOR 6711, HMWK 4, Professor Sigman

1. Consider an irreducible Markov chain with transition matrix $P = (P_{i,j})$ and state space $\mathcal{S}$. Prove that if you can find both a probability distribution π on $\mathcal{S}$ and a Markov transition matrix $Q = (Q_{i,j})$ such that

$$\pi_i P_{i,j} = \pi_j Q_{j,i},$$

then the chain is positive recurrent with stationary distribution π, and $Q = P(r)$, the transition matrix for the time-reversed chain.

SOLUTION: Assuming the hypothesis, $\sum_i Q_{j,i} = 1$, so by summing up both sides over i we get, for each j,

$$\sum_i \pi_i P_{i,j} = \pi_j,$$

which is $\pi = \pi P$. Thus the chain is positive recurrent with stationary distribution π. Moreover, $Q_{j,i} = \pi_i P_{i,j}/\pi_j = P_{i,j}(r)$.

2. Let X be any non-negative rv.

   (a) Derive

$$E(XI\{X > x\}) = xP(X > x) + \int_x^\infty P(X > y)dy.$$

   SOLUTION: $Y = XI\{X > x\}$ is a non-negative rv, so we can obtain its expected value by integrating the tail:

$$\begin{aligned} E(Y) &= \int_0^\infty P(Y > y)dy \\ &= \int_0^x P(X > x)dy + \int_x^\infty P(X > y)dy \\ &= xP(X > x) + \int_x^\infty P(X > y)dy. \end{aligned}$$

   (b) Use (directly) the dominated convergence theorem to prove that that if $E(X) < \infty$, then $E(XI\{X > x\}) \to 0$ as $x \to \infty$.
   SOLUTION: wp1, $\lim_{x\to\infty} XI\{X > x\} = 0$, and $\sup_{x\geq 0} XI\{X > x\} \leq X$ and $E(X) < \infty$; the result follows.

   (c) Conclude that $xP(|X| > x) \to 0$ for any non-negative rv with $E(|X|) < \infty$.

SOLUTION: From the above,

$$E(|X|I\{|X| > x\}) = xP(|X| > x) + \int_x^\infty P(|X| > y)dy,$$

and $E(|X|I\{|X| > x\}) \to 0$; the result follows.

3. Consider two non-negative rvs, X and Y. We say that X is *stochastically smaller* than Y, denoted by

$$X \le_{st} Y,$$

if $P(X > x) \le P(Y > x),\ x \ge 0$.

(a) Show that if $X \le Y$ wp1, then $X \le_{st} Y$.
SOLUTION: Immediate: If $X \le Y$ wp1, then $I\{X > x\} \le I\{Y > x\},\ x \ge 0$, wp1. Taking expected values yields the result.

(b) Give a counterexample of two rvs X and Y on the same probability space such that $X \le_{st} Y$ but it does not hold that $X \le Y$ wp1.
SOLUTION: Let U have a uniform distribution on $(0, 1)$. Define $X = U$ and $Y = 1 - U$. Then both X and Y are uniform distribution on $(0, 1)$, so $P(X > x) = P(Y > x),\ x \ge 0$; in particular $P(X > x) \le P(Y > x),\ x \ge 0$. But if for example $U = 0.80$, then $X = 0.80$ while $Y = 0.20$.

(c) Given a rv U having a uniform distribution on $(0, 1)$, show (by using U somehow) that if $X \le_{st} Y$, then there exists rvs X' and Y' on the same probability space such that X' is distributed as X, and Y' is distributed as Y and $X' \le Y'$ wp1.
SOLUTION: Let $F(x) = P(X \le x)$ and $G(x) = P(Y \le x)$ be the cdfs, and let $F^{-1}(u)$, $G^{-1}(u)$ denote the generalized inverses;

$$F^{-1}(u) = \min\{x : F(x) \ge u\},\ u \in [0, 1].$$

$$G^{-1}(u) = \min\{x : G(x) \ge u\},\ u \in [0, 1].$$

$X \le_{st} Y$ is equivalent to $F(x) \ge G(x),\ x \ge 0$ which thus implies (looking at the definition of the generalized inverses) that $F^{-1}(u) \le G^{-1}(u),\ u \in [0, 1]$.
Using the inverse transform method with the one U given, we set $X' = F^{-1}(U)$ and $Y' = G^{-1}(U)$ yielding the result.

4. You plan to retire at deterministic time R years in the future, and you wish to buy a summer home right before you retire (e.g., as close to retirement as possible). Suppose that over time you receive offers to buy a summer home at times $\{t_n : n \ge 1\}$ that form a Poisson process at rate μ such that $R > \mu^{-1}$. Your objective is to accept the last offer before time R, but such a "last" time strategy is not a stopping time; if you were to choose $t_n < R$ (say), then if you waited a little bit more, maybe $t_{n+1} < R$ also. Thus you decide upon the following strategy: fix a time $0 < t < R$, and accept the first offer (if any) received after time t but before time R. If no such offer occurs, then you fail at your objective (as you do if more than one offer occurs after t but before R); otherwise you succeed. What is the optimal value of t to use (e.g., what value of t maximizes the probability that you succeed)?

SOLUTION: You succeed if and only if there is exactly one Poisson event during $(t, R]$. That happens with probability $P(N(R) - N(t) = 1)$ which by stationary increments is the same as $P(N(R-t) = 1) = \mu(R-t)e^{-\mu(R-t)}$. Changing variables $x = \mu(R - t)$, we must maximixe the function $f(x) = xe^{-x}$ on the interval $(0, R\mu)$ via calculus: $\frac{d(xe^{-x})}{dx} = -xe^{-x} + e^{-x} := 0$.

Hence, $x = 1$ and $f(1) = e^{-1}$.
So the probability is maximized when $\mu(R - t) = 1$; $t = R - 1/\mu$. (That our solution is a maximum as opposed to minimum is easily seen since since $f(x)$ is increasing on $[0, 1]$ then decreasing on $(1, R\mu]$.)

5. A straight road of length L miles connects two points A and B. There is a gas station located at A, and then additional stations are located along the road according to a Poisson process with rate λ (per unit mile). At a fixed location y miles along the road $(0 < y < L)$, John's car runs out of gas. Compute the expected distance from John's car to the nearest gas station.

   SOLUTION: Let D_1 denote the distance to the nearest station to the left of y, and let D_2 denote the the distance to the nearest station to the right of y, where we set $D_2 = \infty$ if there are no stations to the right of y. Let $D = \min\{D_1, D_2\}$. We need to compute $E(D)$. We will do so via $E(D) = \int_0^\infty P(D > x)dx$.

   Let X_1 and X_2 denote iid exponentials at rate λ,

   Then (in distribution) $D_2 = X_2$ if $X_2 \leq L - y$; $D_2 = \infty$ if $X_2 > L - y$.

   $D_1 = \min\{X_1, y\}$. Thus

$$P(D_1 > x) = \begin{cases} e^{-\lambda x} & \text{if } 0 \leq x < y \ , \\ 0 & \text{if } x \geq y. \end{cases}$$

$$P(D_2 > x) = \begin{cases} e^{-\lambda x} & \text{if } 0 \leq x < L - y \ , \\ P(D_2 = \infty) = e^{-\lambda(L-y)} & \text{if } x \geq L - y. \end{cases}$$

   Thus $P(D > x) = P(D_1 > x, D_2 > x) = P(D_1 > x)P(D_2 > x)$ yielding

$$P(D > x) = \begin{cases} e^{-2\lambda x} & \text{if } 0 \leq x < \min\{y, L - y\} \ , \\ e^{-\lambda(L-y)}e^{-\lambda x} & \text{if } L - y \leq x < y, \\ 0 & \text{if } x \geq y. \end{cases}$$

   We need to consider the two cases: $\min\{y, L - y\} = y$ and $\min\{y, L - y\} = L - y$; or equivalently $y \leq L/2$ and $y > L/2$.

   The first case yields

$$P(D > x) \begin{cases} e^{-2\lambda x} & \text{if } 0 \leq x < y \ , \\ 0 & \text{if } x \geq y, \end{cases}$$

   in which case

$$E(D) = \int_0^y e^{-2\lambda x}dx = (1/2\lambda)(1 - e^{-2\lambda y}).$$

   The second case yields

$$P(D > x) = \begin{cases} e^{-2\lambda x} & \text{if } 0 \le x < L - y \ , \\ e^{-\lambda(L-y)}e^{-\lambda x} & \text{if } L - y \le x < y, \\ 0 & \text{if } x \ge y, \end{cases}$$

in which case

$$\begin{aligned} E(D) &= \int_0^{L-y} e^{-2\lambda x}dx + e^{-\lambda(L-y)} \int_{L-y}^{y} e^{-\lambda x}dx \\ &= (1/2\lambda)(1 - e^{-2\lambda(L-y)}) + (1/\lambda)e^{-\lambda(L-y)}(e^{-\lambda(L-y)} - e^{-\lambda y}) \\ &= (1/2\lambda)(1 - e^{-2\lambda(L-y)}) + (1/\lambda)(e^{-2\lambda(L-y)} - e^{-\lambda L}) \\ &= (1/2\lambda)(1 + e^{-2\lambda(L-y)}) - (1/\lambda)e^{-\lambda L}. \end{aligned}$$

6. Cars pass a certain street location according to a Poisson process at rate λ. Mary is standing at that location and will wait to cross the street until she sees that no cars will pass in the next T time units. Let W denote Mary's waiting time.

   (a) Compute $P(W = 0)$.

   (b) Compute $E(W)$.

   SOLUTION:
   $P(W = 0) = P(X_1 > T) = e^{-\lambda T}$ and

   $$E(W) = e^{\lambda T}(1/\lambda) - T - (1/\lambda)$$

   derived as follows:

   We can use the method of conditioning on the length of the first interarrival time X_1. Observe that if $X_1 > T$, then $W = 0$, whereas if $X_1 \le T$, then (in distribution) $W = (X_1|X_1 \le T) + W$ since if $X_1 \le T$, then we start waiting again at time X_1 as if things started all over. Thus $E(W) = E(W|X_1 > T)P(X_1 > T) + E(W|X_1 \le T)P(X_1 \le T) = E(W|X_1 \le T)P(X_1 \le T)$
   $= (E(X_1|X_1 \le T) + E(W))P(X_1 \le T)$. Thus we can solve for $E(W)$: $E(W) = E(X_1|X_1 \le T)(1-p)/p$, where $p = e^{-\lambda T} = P(X_1 > T)$. We can re-write this as $E(W) = E(X_1; X_1 \le T)/p$, and can compute $E(X_1; X_1 \le T) = \int_0^T x\lambda e^{-\lambda x} = (1/\lambda)(1-p) - Tp$.

7. Consider a Poisson process ψ at rate λ with counting process $\{N(t)\}$. Let Y be a rv that is independent of ψ with mean $E(Y)$ second moment $E(Y^2)$ and variance $\sigma^2 = Var(Y)$. Compute $E(N(Y))$ and $Cov(Y, N(Y)$ and $Var(N(Y))$.

   SOLUTION: Conditioning on $Y = y$ yields $E(N(Y) \mid Y = y) = \lambda y$ and so $E(N(Y) \mid Y) = \lambda Y$. Thus $E(N(Y)) = E[E(N(Y) \mid Y)] = \lambda E(Y)$. $Cov(Y, N(Y) = E(YN(Y)) - E(Y)E(N(Y))$, so we need to compute $E(YN(Y))$. Once again by conditioning on $Y = y$ we get $E(YN(Y) \mid Y = y) = \lambda y^2$ hence $E(YN(Y) \mid Y) =$

λY^2. So $E(YN(Y)) = E[E(YN(Y) \mid Y)] = \lambda E(Y^2)$. To compute $Var(N(Y))$, we can use the conditional variance formula:

$$Var(N(Y)) = E(Var(N(Y) \mid Y)) + Var(E(N(Y) \mid Y)),$$

where here $Var(N(Y) \mid Y = y) = Var(N(y)) = \lambda y$, so $Var(N(Y) \mid Y) = \lambda Y$. Finally $Var(N(Y)) = E(\lambda Y) + Var(\lambda Y) = \lambda E(Y) + \lambda^2 Var(Y)$.

8. Let $X \sim exp(\lambda_1)$ and $Y \sim exp(\lambda_2)$ be independent, representing the first arrival time from Express and Local trains respectively. $\lambda_1 = 4,\ \lambda_2 = 7$.

   (a) $P(X < Y) = \frac{\lambda_1}{\lambda_1+\lambda_2} = \frac{4}{11}$.

   (b) Let $Z = \min\{X, Y\}$. $E(Z) = \frac{1}{\lambda} = \frac{1}{11}$. Here $\lambda = \lambda_1 + \lambda_2 = 11$, and $Z \sim exp(\lambda)$.

   (c) $E(Z|Z = X) = E(Z) = \frac{1}{11}$. The point here is that conditioning on which of the two r.v.s. is the minimum does not change the fact that the minimum has an exponential distribution with paramater $\lambda = \lambda_1 + \lambda_2 = 11$.

   (d) Let $N(t) = N_1(t) + N_2(t)$ denote the counting process for the superposition of both types (forms a Poisson process at rate $\lambda = 11$). 12 minutes is 1/5 of an hour; $P(N(1/5) = 0) = e^{-\frac{\lambda}{5}} = e^{-\frac{11}{5}}$. One can also do this directly: $P(N(1/5) = 0) = P(N_1(1/5) = 0, N_2(1/5) = 0) = P(N_1(1/5) = 0)P(N_2(1/5) = 0)$ by the assumed independence of the two Poisson processes we got $e^{-\frac{4}{5}}e^{-\frac{7}{5}} = e^{-\frac{11}{5}}$.

   (e) $t_3 \sim gamma(3, \lambda)$ (*Erlang* with 3 phases, each at rate λ) because it is the third arrival time from a Poisson process at rate $\lambda = 11$. It has representation $t_3 = X_1 + X_2 + X_3$, where $X_i \sim exp(11)$. Thus $E(t_3) = 3E(X) = 3/11$ and from independence

$$\begin{aligned} Var(t_3) &= Var(X_1) + Var(X_2) + Var(X_3) \\ &= 3Var(X_1) \\ &= 3 \times \frac{1}{\lambda^2} \\ &= 3 \times \frac{1}{121}. \end{aligned}$$

   (f) Let $U \sim Uniform(0, 0.5)$ be a rv independent of $X \sim exp(7)$. We want $P(X > U) = (0.5)^{-1} \int_0^{0.5} P(X > u)du = (0.5)^{-1} \int_0^{0.5} (e^{-7u})du = \frac{2}{7}(1 - e^{-3.5})$.

9. Two ATM machines work in parallel (and have one common queue/line for both). You arrive and find both in use but no one waiting in line. (So you are the only one in line now and will begin whenever a machine becomes free.) Suppose each user of an ATM spends an iid exponential amount of time (called a *service time*) at rate λ using the machine before departing.

(a) What is the expected length of time until you depart?

SOLUTION: Let W denote this *sojourn* time. Let S_1 and S_2 denote the remaining service times of the two in service. By the memoryless property, they are iid exponentials at rate λ. You will enter service after an amount of time $D = \min\{S_1, S_2\} \sim exp(2\lambda)$ (your *delay in queue*), and then will spend S_3 (your service time) in service. So $W = D+S_3$ and $E(W) = E(D)+E(S_3) = 1/(2\lambda) + 1/\lambda = 3/(2\lambda)$.

(b) What is the expected length of time until both users (the two you found) have departed?

SOLUTION: Denote this time by T; $T = \max\{S_1, S_2\} = S_1+S_2-\min\{S_1, S_2\} = S_1 + S_2 - D$. Thus $E(T) = 2E(S) - E(D) = 2/\lambda - 1/(2\lambda)$.

(c) Repeat in the case when ATM machine 1 has iid exponential service times at rate λ_1 and ATM machine 2 has iid exponential service times at rate λ_2.

SOLUTION: We now need to know which of the two were first to depart. Let $S_i \sim exp(\lambda_i),\ i = 1, 2$.

Let $p = P(S_1 < S_2) = \lambda_1/(\lambda_1 + \lambda_2)$ and let $q = 1 - p$.

Given that $S_1 < S_2$, or given $S_2 < S_1$, the rv D has the same distribution, namely $D = \min\{S_1, S_2\}$ (recall that for $Z = \min\{X_1, X_2\}$ of two independent exponentials, the conditional distribution of Z given that $Z = X_1$ (or similarly if $Z = X_2$) is still the same as Z itself.) Now let $S_3(i)$ denote an independent exponential service at rate $\lambda_i,\ i = 1, 2$. Then

$$W = D + S_3(1)|\{S_1 < S_2\} + S_3(2)|\{S_2 < S_1\}.$$

$E(W) = E(D) + p(1/\lambda_1) + q(1/\lambda_2)$
$= 3/(\lambda_1 + \lambda_2)$.

As for T: once again, $T = \max\{S_1, S_2\} = S_1 + S_2 - \min\{S_1, S_2\}$; $E(T) = 1/\lambda_1 - 1/\lambda_2 - 1/(\lambda_1 + \lambda_2)$.

10. Consider an infinite sequence of independent Poisson processes at rates $\lambda_i,\ i \geq 1$, where it is assumed that

$$\sum_{i=1}^{\infty} \lambda_i < \infty.$$

Let ψ denote the superposition of all of them together. Is ψ a Poisson process (prove that it is or explain why it is not)?

SOLUTION: Yes; a Poisson process at rate $\lambda = \sum_{i=1}^{\infty} \lambda_i$.

First note that the superposition counting process $N(t) = \sum_{i=1}^{\infty} N_i(t)$ is a well defined finite for each t integer valued process:

$$\begin{aligned} E(N(t)) &= E\Big\{\sum_{i=1}^{\infty} N_i(t)\Big\} \\ &= \sum_{i=1}^{\infty} E(N_i(t)) \\ &= \sum_{i=1}^{\infty} \lambda_i t \\ &= \lambda t < \infty, \end{aligned}$$

where we have used non-negativity of all the rvs to justify the interchange of expected value and sum. (This implies in particular that $P(N(t) < \infty) = 1$ for each t.) The resulting point process is also simple because the probability that any number (finite or countable infinite) of continuous rvs are equal is zero.

It is next easily verified that ψ has both stationary and independent increments (because each of the individual Poisson processes does), so it must be a Poisson process. The rate of a Poisson process is always given by $\lambda = E(N(1))$.

11. Consider a Poisson process at rate λ, define $t_0 = 0$, and for fixed $t > 0$, let $B(t)$ denote the amount of time since the last arrival before time t: $t_{N(t)} \leq t < t_{N(t)+1}$, $B(t) = t - t_{N(t)}$. Find $P(B(t) \leq x),\ x \geq 0$. What happens as $t \to \infty$?

    SOLUTION: Let $X \sim exp(\lambda)$. Then by the memoryless property (think in reverse time) $B(t)$ can be expressed (in distribution) as $B(t) = \min\{X, t\}$ because the last arrival can occur at most t time units ago (where the origin is). $P(B(t) \leq x) = 1 - e^{-\lambda x},\ x < t,\ P(B(t) \leq x) = 1,\ x \geq t$. Note how as $t \to \infty$ we do obtain the $exp(\lambda)$ distribution.

    On the other hand, $A(t) = t_{N(t)+1} - t$, the time until the next arrival, always has the $exp(\lambda)$ distribution since there is no upper bound; no limit is required to get there.

12. Consider a point process ψ defined as follows: Flip a fair coin. If it lands heads, then ψ is a Poisson process at rate 1, otherwise it is a Poisson process at rate 2.

    (a) Is ψ a Poisson process (prove that it is or explain why it is not)?

    (b) Does ψ have stationary increments?

    (c) Does ψ have independent increments?

    (d) Is ψ a renewal process?

    SOLUTION:

    ψ is not Poisson. Consider the first arrival time t_1; its distribution is $P(t_1 > t) = (1/2)e^{-t} + (1/2)e^{-2t}$ a mixture of two distinct exponential distributions which is not exponential. We can write that $t_1 \sim (0.5)exp(1) + (0.5)exp(2)$.

    $\{N(t)\}$ does, however, have stationary increments: The distribution of $N(t+h) - N(t)$ only depends on h; it is a mixture of two distinct Poisson distributions, $\sim (0.5)Poisson(h) + (0.5)Poisson(2h)$.

    $\{N(t)\}$ does not have independent increments: Knowing $N(t)$ for t large would give you a fine estimate of the value of λ, either 1 or 2, via the elementary renewal theorem, $N(t)/t \approx \lambda$, which thus would tell you which way the coin landed and hence which distribution, $Poisson(h)$ or $Poisson(2h)$, the future increment $N(t+h) - N(t)$ is from; $N(t+h) - N(t)$ is not independent of $N(t)$.

    ψ a not renewal process for similar reasons as to why it does not have independent increments. Whereas the interarrival times are all identically distributed as t_1, $P(X_i > t) = P(t_1 > t) = (1/2)e^{-t} + (1/2)e^{-2t},\ i \geq 1$, they are not independent. Knowing (say) the first n values (n large) $X_1, \ldots, X_n$ gives us information to estimate λ, and hence to predict X_{n+1}.