IEOR 6711, HMWK 4, Professor Sigman

1. Consider an irreducible Markov chain with transition matrix $P = (P_{i,j})$ and state space $\mathcal{S}$. Prove that if you can find both a probability distribution π on $\mathcal{S}$ and a Markov transition matrix $Q = (Q_{i,j})$ such that
$$\pi_i P_{i,j} = \pi_j Q_{j,i},$$
then the chain is positive recurrent with stationary distribution π, and $Q = P(r)$, the transition matrix for the time-reversed chain.

2. Let X be any non-negative rv.
   (a) Derive
   $$E(XI\{X > x\}) = xP(X > x) + \int_x^\infty P(X > y)dy.$$
   (b) Use (directly) the dominated convergence theorem to prove that that if $E(X) < \infty$, then $E(XI\{X > x\}) \to 0$ as $x \to \infty$.
   (c) Conclude that $xP(|X| > x) \to 0$ for any non-negative rv with $E(|X|) < \infty$.

3. Consider two non-negative rvs, X and Y. We say that X is *stochastically smaller* than Y, denoted by
$$X \leq_{st} Y,$$
if $P(X > x) \leq P(Y > x),\ x \geq 0$.
   (a) Show that if $X \leq Y$ wp1, then $X \leq_{st} Y$.
   (b) Give a counterexample of two rvs X and Y on the same probability space such that $X \leq_{st} Y$ but it does not hold that $X \leq Y$ wp1.
   (c) Given a rv U having a uniform distribution on $(0, 1)$, show (by using U somehow) that if $X \leq_{st} Y$, then there exists rvs X' and Y' on the same probability space such that X' is distributed as X, and Y' is distributed as Y and $X' \leq Y'$ wp1.

4. You plan to retire at deterministic time R years in the future, and you wish to buy a summer home right before you retire (e.g., as close to retirement as possible). Suppose that over time you receive offers to buy a summer home at times $\{t_n : n \geq 1\}$ that form a Poisson process at rate μ such that $R > \mu^{-1}$. Your objective is to accept the last offer before time R, but such a "last" time strategy is not a stopping time; if you were to choose $t_n < R$ (say), then if you waited a little bit more, maybe $t_{n+1} < R$ also. Thus you decide upon the following strategy: fix a time $0 < t < R$, and accept the first offer (if any) received after time t but before time R. If no such offer occurs, then you fail at your objective (as you do if more than one offer occurs after t but before R); otherwise you succeed. What is the optimal value of t to use (e.g., what value of t maximizes the probability that you succeed)?

5. A straight road of length L miles connects two points A and B. There is a gas station located at A, and then additional stations are located along the road according to a Poisson process with rate λ (per unit mile). At a fixed location y miles along the road $(0 < y < L)$, John's car runs out of gas. Compute the expected distance from John's car to the nearest gas station.

6. Cars pass a certain street location according to a Poisson process at rate λ. Mary is standing at that location and will wait to cross the street until she sees that no cars will pass in the next T time units. Let W denote Mary's waiting time.
   (a) Compute $P(W = 0)$.
   (b) Compute $E(W)$.

7. Consider a Poisson process ψ at rate λ with counting process $\{N(t)\}$. Let Y be a rv that is independent of ψ with mean $E(Y)$ and variance $Var(Y)$. Compute $E(N(Y))$ and $Cov(Y, N(Y)$ and $Var(N(Y))$.

8. Downtown Express trains arrive to West 96th Street station according to a Poisson process at rate 4 per hour, and independent of this, Downtown Local trains arrive according to a Poisson process at rate 7 per hour. Suppose you arrive at the station (to go Downtown) and decide to take the first train that arrives.
   (a) What is the probability that you take an Express train?
   (b) What is the expected amount of time you wait for a train to arrive?
   (c) Given that you took an Express train (it arrived first), what was the expected amount of time you waited for it to arrive?
   (d) What is the probability that no trains (of any type) arrive during the next 12 minutes?
   (e) Let t_3 denote the time at which the third train arrives (regardless of type). What is the variance, $Var(t_3)$?
   (f) Suppose that Express trains do not arrive according to a Poisson process, but instead arrive according to a renewal point process with iid interarrival times uniformly distributed over the time interval $(0, 0.5)$ (hours). (Local trains are Poisson as before.) Further suppose that when you arrived at the station, an Express train had just departed so you missed it. What is the probability that you take an Express train?

9. Two ATM machines work in parallel (and have one common queue/line for both). You arrive and find both in use but no one waiting in line. (So you are the only one in line now and will begin whenever a machine becomes free.) Suppose each user of an ATM spends an iid exponential amount of time (called a *service time*) at rate λ using the machine before departing.
   (a) What is the expected length of time until you depart?
   (b) What is the expected length of time until both users (the two you found) have departed?
   (c) Repeat in the case when ATM machine 1 has iid exponential service times at rate λ_1 and ATM machine 2 has iid exponential service times at rate λ_2.

10. Consider an infinite sequence of independent Poisson processes at rates $\lambda_i,\ i \geq 1$, where it is assumed that
$$\sum_{i=1}^{\infty} \lambda_i < \infty.$$
Let ψ denote the superposition of all of them together. Is ψ a Poisson process (prove that it is or explain why it is not)?

11. Consider a Poisson process at rate λ, define $t_0 = 0$, and for fixed $t > 0$, let $B(t)$ denote the amount of time since the last arrival before time t: $t_{N(t)} \leq t < t_{N(t)+1}$, $B(t) = t - t_{N(t)}$. Find $P(B(t) \leq x),\ x \geq 0$.

12. Consider a point process ψ defined as follows: Flip a fair coin. If it lands heads, then ψ is a Poisson process at rate 1, otherwise it is a Poisson process at rate 2.

    (a) Is ψ a Poisson process (prove that it is or explain why it is not)?

    (b) Does ψ have stationary increments?

    (c) Does ψ have independent increments?

    (d) Is ψ a renewal process?