IEOR 6711, HMWK 5, Professor Sigman

1. Consider a FIFO $M/M/1$ queue (arrival rate $0 < \lambda < \infty$, service rate $0 < \mu < \infty$; $\rho = \lambda/\mu$). Let $X(t)$ denote the number of customers in the system at time t, a birth and death process.

   (a) Suppose that $\rho > 1$. Let $\{X_n : n \geq 0\}$ denote the embedded discrete-time Markov chain, and suppose that $X_0 = 1$. What is the probability that this chain will go off to ∞ without ever going to state 0 first?

   (b) *Continuation:* Use (a) to prove that $\{X(t)\}$ is transient when $\rho > 1$.

   (c) Suppose now that $\rho = 1$. Prove that $\{X(t)\}$ is null recurrent.

2. Consider a stable FIFO $M/M/1$ queue, $0 < \rho < 1$. Let $X(t)$ denote the number in system at time t, and let $P_n = (1-\rho)\rho^n,\ n \geq 0$ denote the stationary distribution. Show (direct calculation) that if $X(0) \sim (P_n)$ (e.g., the chain is started off with its stationary distribution, hence is a stationary process), then the time until the first departure (after time $t = 0$), t_1^d, has an exponential distribution at rate λ.

3. Consider the M/M/1 queue (arrival rate λ service time rate μ) with the following twist: Each customer independently will get impatient after an amount of time that is exponentially distributed at rate γ while waiting in line (queue) and leave before ever entering service, and without ever returning. A customer who does enter service completes service (e.g., customers are only impatient while waiting in the line, not when in service.)

   (a) You arrive finding exactly one customer in the system (hence they are in service) and you join the queue to wait. What is the probability that you will get served?

   (b) You arrive finding exactly two (2) customers in the system (one in service, one in line)) and you join the end of the queue to wait. What is the probability that you will get served?

   (c) Model as a Birth and Death process, give the birth and death rates, $\lambda_n, \mu_n,\ n \geq 0$.

   (d) Set up the Birth and Death balance equations for the limiting probabilities P_n (but do not try to solve.)

   (e) Compute the ratio P_{n+1}/P_n and prove using the "ratio test" from calculus, that the limiting probabilities exist for all values of $\lambda > 0, \mu > 0, \gamma > 0$. Thus this chain is always positive recurrent (e.g., a condition such as $\rho < 1$ is not needed); explain intuitively why this should be so.

4. Cars arrive to a parking lot according to a Poisson process at rate λ. Each car, when parking, independently remains parked for an amount of time that is iid with an exponential distribution at rate μ. The parking lot only has c spots, and any car who arrives finding all c spots taken immediately goes nearby to a huge (infinite capacity) lot, and parks there instead (same iid exponential parking times). What is the long-run average number of parked cars in the infinite capacity lot?

5. For the M/G/∞ queue, with $L(t)$ denoting number in system at time t with $L(0) = 0$, let $S_r(t)$ denote the remaining service time at time t of the one customer in service, *conditional on the event* $\{L(t) = 1\}$.

   Compute $\lim_{t\to\infty} P(S_r(t) > x | L(t) = 1)$.

6. Suppose that $\{X(t)\}$ is an irreducible (non-explosive) CTMC with transition rates matrix $Q = (q_{i,j})$. Independently suppose that $\{t_n : n \geq 1\}$ is a Poisson process at rate λ.

   (a) Let $Z_n = X(t_n)$, $n \geq 1$, $Z_0 = X(0)$. Argue that $\{Z_n : n \geq 0\}$ is a discrete-time Markov chain with transition matrix $\tilde{P} = (\tilde{P}_{i,j})$ given by

   $$\tilde{P} = [I - (Q/\lambda)]^{-1}.$$

   (b) Prove that $\{X(t)\}$ is positive recurrent if and only if $\{Z_n\}$ is positive recurrent in which case they have the same stationary distribution: $\pi = \vec{P}$.

7. Just as in discrete time, if we consider a positive recurrent CTMC in stationarity that has been started since the infinite past, $\{X^*(t) : -\infty < t < \infty\}$, then the (stationary) time-reversal $X^{(r)}(t) = X^*(-t) : t \geq 0$ is itself a CTMC. It has the same holding time rates $\{a_i\}$ and the same stationary distribution $\vec{P} = (P_j)$ as the original forward time CTMC. The only thing that can differ are the transition rate matrices Q and $Q^{(r)}$.

   A positive recurrent CTMC is called *time-reversible* if the time-reversed process $\{X^{(r)} : t \geq 0\}$ has the same distribution as the forward-time process $\{X^*(t) : t \geq 0\}$. This is equivalent to saying that $Q = Q^{(r)}$, which can be stated in terms of the forward-time chain:

   > *the long-run rate that the chain moves from i to j equals the long-run rate that the chain moves from from j to i, for any two states $i, j \in \mathcal{S}$.*

   Thus a positive recurrent CTMC is time-reversible if for all pairs of states $i, j \in \mathcal{S}$,

   $$a_i P_i P_{i,j} = a_j P_j P_{j,i}. \tag{1}$$

   Summing up both sides of (1) over i yields the balance equations, and hence

   Proposition 0.1 *If, for an irreducible CTMC, you find a probability distribution (P_j) satisfying (1), then the CTMC is positive recurrent with stationary distribution (P_j), and the chain is time-reversible*

   Since a birth and death process can only make transitions of magnitude ± 1 we know that Equation (1) becomes "the long-run rate that the chain moves from i to $i+1$ equals the long-run rate that the chain moves from from $i+1$ to i, for all states $i \in \mathcal{S}$"; the birth and death balance equations. We conclude:

   > *Every positive recurrent birth and death (B&D) process is time-reversible.*

   We are now ready for some exercises based on time-reversibility:

(a) *M/M/1 queue:* $X(t)$ = the number of customers in a FIFO M/M/1 queue is a B& D process, so we conclude that when $\rho < 1$, it is time-reversible. Assume that it is started at time $t = 0$ with its stationary distribution. Let $\psi = \{t_n : n \geq 1\}$ denote the Poisson arrival times starting from time $t = 0$: $0 < t_1 < t_2 < \cdots$, and let $\psi^{(d)} = \{t_n^d : n \geq 1\}$ denote the point process of departure times after time $t = 0$: $0 < t_1^d < t_2^d < \cdots$. We know from Exercise 2 above that t_1^d has an exponential distribution at rate λ, but here you will deduce more.

Argue (from time-reversibility) that $\psi^{(d)}$ must have the same distribution as ψ, and hence must itself be a Poisson process at rate λ: *the departure process from a stationary M/M/1 queue is itself a Poisson process.*

(b) Consider an $M/M/\infty$ queue (arrival rate λ service rate μ) and suppose that departures from it immediately attend another facility: a FIFO single-server queue with its own iid exponential service times at rate μ_2. Assuming that $\lambda < \mu_2$, find the long-run average number of customers in the second facility.

8. Consider a CTMC $\{X(t)\}$ (with $P_{i,i} = 0,\ i \in \mathcal{S}$) with embedded chain transition matrix $P = (P_{i,j})$ and holding time rates $\{a_i\}$. Assume that $a = \sup\{a_i : i \in \mathcal{S}\} < \infty$. Consider an alternative CTMC $\{\overline{X}(t)\}$for which all holding time rates are fixed at the constant rate a; $\overline{a}_i = a$, and has embedded transition probabilities given by

$$\overline{P}_{i,j} = \begin{cases} \frac{a_i}{a} P_{i,j} & \text{if } j \neq i, \\ 1 - \frac{a_i}{a} & \text{if } j = i. \end{cases}$$

(So $\overline{P}_{i,i} > 0$ is possible.)

(a) Let $\overline{N}(t)$ denote the number of transitions by time t for $\{\overline{X}(t)\}$. Explain why $\{\overline{N}(t) : t \geq 0\}$ forms a Poisson process at rate a.

(b) Show that the balance equations are the same for the two chains.

(c) Explain why $\{X(t)\}$ and $\{\overline{X}(t)\}$ have the same distribution as stochastic processes; $P_{i,j}(t) = \overline{P}_{i,j}(t),\ t \geq 0$ for all pairs i, j.