IEOR 6711, Solutions to HMWK 6, Professor Sigman

1. For a (random) point process $\psi = \{t_n : n \geq 1\}$, explain why, for any given fixed $t > 0$, the random time $N(t)$ is not a stopping time with respect to the $\{t_n\}$, but $N(t) + 1$ is a stopping time.

   SOLUTION: For example, $\{N(t) = 1\} = \{t_1 \leq t,\ t_2 > t\}$ depends on both t_1 and t_2, not just t_1; $N(t)$ is not a stopping time.

   Meanwhile $\{N(t) + 1 = 1\} = \{N(t) = 0\} = \{t_1 > t\}$, and more generally, $\{N(t) + 1 = n\} = \{N(t) = n - 1\} = \{t_1 \leq t, \ldots, t_{n-1} \leq t, t_n > t\}$; $N(t) + 1$ is a stopping time.

   Equivalently, $N(t) + 1$ is a stopping time with respect to the interarrival times $X_n = t_n - t_{n-1}$, since $t_n = X_1 + \cdots X_n$. $\{N(t) + 1 = 1\} = \{N(t) = 0\} = \{X_1 > t\}$, and more generally, $\{N(t) + 1 = n\} = \{N(t) = n - 1\} = \{X_1 \leq t, \ldots, X_1 + \cdots X_{n-1} \leq t, X_1 + \cdots + X_n > t\}$.

2. Consider a FIFO M/M/1 queue (arrival rate λ, service rate μ) for modeling a printer (say), but with *disasters* (arriving as an independent Poisson process at rate γ) that are so serious that ALL jobs are removed and lost: Whenever a disaster occurs, all jobs (if any) are removed. Let $X(t)$ denote the number of jobs in system at time t. Argue that it forms a CTMC (not a birth and death process though).

   (a) **SOLUTION:** $\lambda P_0 = \mu P_1 + \gamma(1 - P_0)$ and $(\lambda + \mu + \gamma)P_n = \lambda P_{n-1} + \mu P_{n+1},\ n \geq 1$.

   (b) **SOLUTION:** Let us "guess" that the solution to the P_n is geometric: $P_n = (1 - \alpha)\alpha^n,\ n \geq 0$ for some $0 < \alpha < 1$.
   Plugging into the first equation yields $\lambda(1 - \alpha) = \mu(1 - \alpha)\alpha + \gamma\alpha$ which simplifies to the quadratic equation $f(\alpha) \overset{\text{def}}{=} \mu\alpha^2 - (\lambda + \mu + \gamma)\alpha + \lambda = 0$. Similarly, plugging into any other equation yields exactly the same quadratic. Thus the solution is given by the appropriate solution to the quadratic (e.g., the solution falling in $(0, 1)$).

   $$\frac{\lambda + \mu + \gamma}{2\mu} \pm \frac{\sqrt{(\lambda + \mu + \gamma)^2 - 4\lambda\mu}}{2\mu}.$$

   We need to prove that one of these solutions always falls in $(0, 1)$ when $\gamma > 0$, $\lambda > 0$ and $\mu > 0$. Observer that $f(\alpha)$ is a continuous function such that $f(0) = \lambda > 0$ and $f(1) = -\gamma < 0$. So by continuity it must hit 0 in between; there must exist an $\alpha^* \in (0, 1)$, such that $f(\alpha^*) = 0$. Moreover, $f(\alpha) \to +\infty$ as $\alpha \to \infty$ thus, since $f(1) < 0$ it must hit 0 yet again (by continuity) for some $\alpha > 1$; we thus conclude that the smaller of the two roots given above is the required α^*:

   $$\alpha^* = \frac{\lambda + \mu + \gamma}{2\mu} - \frac{\sqrt{(\lambda + \mu + \gamma)^2 - 4\lambda\mu}}{2\mu}.$$

   (c) On average, how many jobs are removed by a jam?
   SOLUTION: Jams see time averages via PASTA, so on average they remove $l = \sum_n nP_n = \alpha/(1 - \alpha)$.

(d) What is the proportion of jobs that get printed?
SOLUTION: $\lambda_r = \gamma l = \gamma\alpha/(1-\alpha)$ is the rate at which jobs are removed, so $(\lambda - \lambda_r)/\lambda$ is the proportion that are printed.

3. Consider a renewal process $\{t_n\}$ with iid interarrival times X_n, with $0 < E(X) = 1/\mu < \infty$. $t_n = X_1 + \cdots + X_n$; $N(t) = \max\{n : t_n \le t\}$. Let $A(t) = t_{N(t)+1} - t$ denote the forward recurrence time, the time until the next renewal strictly after time t. Prove that $A(t)/t \to 0$ as $t \to \infty$. Also prove that $E(A(t))/t \to 0$.

SOLUTION:

$A(t)/t = -1 + t_{N(t)+1}/t$, so it suffices to prove that $t_{N(t)+1}/t \to 1$.

$$t_{N(t)+1} = \sum_{n=1}^{N(t)+1} X_n = \frac{N(t)+1}{t}\frac{1}{N(t)+1}\sum_{n=1}^{N(t)+1} X_n,$$

using the elementary renewal theorem and the SLLN then yields $t_{N(t)+1}/t \to \lambda E(X) = 1$, w.p.1. Thus $A(t)/t \to 0$, w.p.1.

$E(A(t))/t = -1 + E(t_{N(t)+1})/t$, so it suffices to prove that $E(t_{N(t)+1})/t \to 1$. By Wald's equation, $E(t_{N(t)+1}) = \lambda^{-1}E(N(t)+1)$. But by the elementary renewal theorem, $E(N(t)+1)/t \to \lambda$, thus completing the proof.

4. (25 points) A light has $b \ge 2$ bulbs, and the lifetimes $L_1, L_2, \ldots$ of new bulbs are iid with (continuous) distribution $F(x) = P(L \le x)$. For fixed $T > 0$, a *group* replacement policy operates as follows: All bulbs (together as a group) are replaced with new ones at (deterministic) times $T, 2T, 3T, \ldots$. (They are replaced wether they are dead or not.) The cost of each new bulb is C_N, and there is a fixed additional work cost of C_R each time the group is replaced. Moreover, there is an inconvenience cost of C_I per bulb that dies.

(a) (10 points) What is the long-run cost per unit time of using this policy?
SOLUTION: Renewal reward theorem. $X = T$ is deterministic and

$$R = C_R + bC_N + C_I\sum_{j=1}^{b} I\{L_j \le T\}.$$

$E(R) = C_R + bC_N + bC_IF(T)$. $E(R)/E(X) = [C_R + b(C_N + C_IF(T))]/T$.

(b) (10 points) Assume that F has density $f(x) = 2x,\ x \in (0,1)$ (months) and that $T < 1$. Also assume that $b = 10$, $C_R = 0.5$, $C_N = 1$, $C_I = 12$. Find the value of $T < 1$ that minimizes cost.
SOLUTION: $F(x) = \int_0^x 2y\,dy = x^2,\ x \in (0,1)$, so $F(T) = T^2$ and thus $g(T) = E(R)/E(X) = (10.5)/T + 120T$. Solve $g'(T) = -(10.5)/T^2 + 120 = 0$; $T = \sqrt{10.5/120} = 0.725$.

(c) (5 points) Suppose that instead of being a fixed inconvenience cost per bulb, C_I is the cost per bulb per unit time that a dead bulb remains in the light. So if a bulb dies s units of time before the next group replacement, the cost will be sC_I. What is the long-run cost per unit time of using this policy (e.g., re-answer (a))?

SOLUTION:
In this case, the j^{th} bulb in a batch will yield a cost of $C_I(T-L_j)^+$ instead of $C_I I\{L_i \le T\}$ because $(T-L_j)^+$ is the amount of time the bulb remains dead in the light before replacement. Thus

$$R = C_R + bC_N + C_I \sum_{j=1}^{b} (T-L_j)^+,$$

and $E(R) = C_R + bC_N + bE(T-L)^+$. Letting $f(y) = F'(y)$ denote the density,

$$E(T-L)^+ = \int_0^T (T-y)f(y)dy = TF(T) - \int_0^T yf(y)dy.$$

5. Suppose that $\psi = \{t_n : n \ge 1\}$ is a renewal process. Suppose we partition it into two types: independently each point t_n is of type I or type II with probability p and $q = 1-p$ respectively. Let $\psi_i,\ i = 1, 2$ denote the two resulting renewal processes, of type I and II arrivals. Are each of ψ_1 and ψ_2 renewal processes? Are they independent?

   SOLUTION: (i) Yes, partitioning a renewal process results in a renewal process, but with a new (geometric sum of the old), interarrival time distribution: Let $X_n = t_n - t_{n-1}$ denote the interrrival times of ψ. Independently, let $\{K_l : l \ge 1\}$ denote iid geometric rvs, $P(K = n) = (1-p)^{n-1}p,\ n \ge 1$, the number of trials (points) until the first "success". Then the points of ψ_1 are given by $t_{K_1}, t_{K_1+K_2}, \cdots$ and so ψ_1 has iid interarrival times distributed as

   $$T(1) = \sum_{n=1}^{K_1} X_n.$$

   The new rate is $\lambda_1 = \{E(T(1))\}^{-1} = \{E(K_1)E(X)\}^{-1} = p\lambda$. ψ_2 is handled in the same way with $\lambda_2 = q\lambda$.

   (ii) No, they are not independent in general. (Of course the famous case when they are independent is the Poisson process case in which case $T(1) \sim \exp(p\lambda)$.) The easiest counter example is when $P(X = 1) = 1$, the interarrival times are deterministic of length 1. For then $N_1(n) + N_2(n) = N(n) = n,\ n \ge 1$, and so if $N_1(n) = m$, then we know that $N_2(n) = n - m$; $N_1(n)$ and $N_2(n)$ are negatively correlated.

6. For a renewal process with cycle length distribution $F(x) = P(X \le x)$ and mean $E(X) = 1/\lambda$, let $B(t) = t - t_{N(t)}$ denote the age (backwards recurrence time), and let $S(t) = t_{N(t)+1} - t_{N(t)}$ denote the spread. Let $Y(t) = B(t)/S(t)$. Show that

   $$\lim_{t\to\infty} \frac{1}{t} \int_0^t I\{Y(s) \le y\}ds = y,\ y \in (0,1);$$

   $Y(t)$ has a $Unif(0,1)$ limiting distribution.

   SOLUTION:

Renewal reward theorem with reward rate $Y(t)$ at time t. During the first cycle $X = X_1$, we have $B(s) = s$ and $S(s) = X,\ s \in [0, X)$. Thus For $y \in (0, 1)$, $R = \int_0^X I\{s/X \le y\}ds = \int_0^X I\{s \le yX\}ds = \int_0^{yX} 1ds = yX$. Thus $E(R) = yE(X)$ and $E(R)/E(X) = y$.

Now suppose (X_b, X_e) has the joint limiting distribution of age and excess, $P(X_b > y, X_e > x) = \overline{F}_e(x + y)$. ($X_s = X_b + X_e$ thus has the limiting distribution of spread.)

Prove directly that X_b/X_s has the $Unif(0, 1)$ distribution.

SOLUTION: Let us assume for simplicity that F has a density $f(x)$, so that we can retrieve the joint density of (X_b, X_e) via

$$g(y, x) = \frac{\partial}{\partial y}\frac{\partial}{\partial y}\overline{F}_e(x + y) = \lambda f(x + y).$$

It suffices to show that the Laplace transform of X_b/X_s has the $Unif(0, 1)$ Laplace transform, $\tilde{U}(s) = \int_0^1 e^{-sx}dx = s^{-1}(1 - e^{-s}),\ s \ge 0$. To this end:

$$\begin{aligned}
E(e^{-s(X_b/X_s)}) &= \int_0^\infty \int_0^\infty e^{-s(y/(x+y))}\lambda f(x + y)dydx && (1)\\
&= \lambda \int_0^\infty \int_x^\infty e^{-s((u-x)/u)} f(u)dudx \text{ (U=x+y, dU=dy)} && (2)\\
&= \lambda \int_0^\infty f(u)\left(\int_0^u e^{-s((u-x)/u)}dx\right)du \text{ (Fubini's theorem)} && (3)\\
&= \lambda \int_0^\infty f(u)\left(u\tilde{U}(s)\right)du && (4)\\
&= \tilde{U}(s)\lambda \int_0^\infty uf(u)du && (5)\\
&= \tilde{U}(s)\lambda\lambda^{-1} && (6)\\
&= \tilde{U}(s). && (7)
\end{aligned}$$

7. Let $X \sim F$, and let $X_e \sim F_e$ (the equilibrium distribution of F). Let

$$\mathcal{L}_X(s) = E(e^{-sX}),\ s \ge 0,$$

the Laplace transform of X. Let

$$\mathcal{L}_e(s) = E(e^{-sX_e}),\ s \ge 0,$$

the Laplace transform of X_e.

(a) Show that

$$\mathcal{L}_e(s) = \frac{\lambda}{s}(1 - \mathcal{L}_X(s)).$$

SOLUTION:
Using the density $f_e(x) = \lambda\overline{F}(x)$,

$$\begin{aligned}
\mathcal{L}_e(s) &= \int_0^\infty e^{-sx}\lambda\overline{F}(x)dx && (8)\\
&= \lambda\int_0^\infty E(e^{-sx}I\{X > x\})dx && (9)\\
&= \lambda E\left(\int_0^\infty e^{-sx}I\{X > x\}\right)dx && (10)\\
&= \lambda E(\int_0^X e^{-sx})dx && (11)\\
&= \frac{\lambda}{s}E(1 - e^{-sX}) && (12)\\
&= \frac{\lambda}{s}(1 - E(e^{-sX})) && (13)\\
&= \frac{\lambda}{s}(1 - \mathcal{L}_X(s)). && (14)
\end{aligned}$$

We also could use the renewal reward theorem by recognizing that wp1:

$$\mathcal{L}_e(s) = \lim_{t\to\infty}\frac{1}{t}\int_0^t e^{-sB(u)}du.$$

Thus we can right away use $R = \int_0^X e^{-su}du$, and get the result very fast! This is the same lesson we learn from Exercise 6: Time average approach is must easier and faster!

(b) Use (a) to show that for any integer $n \geq 1$,

$$E(X_e^n) = \frac{E(X^{n+1})}{(n+1)E(X)}.$$

SOLUTION:
Use the basic fact that for any non-negative rv X, moments are retrieved via differentiation at 0:

$$(-1)^n\mathcal{L}_X^{(n)}(0) = E(X^n).$$

(c) Use the renewal reward theorem to show that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t A^n(s)ds = \frac{E(X^{n+1})}{(n+1)E(X)};$$

explain why this is the same answer as in (b).

SOLUTION:

Renewal reward theorem with $R = \int_0^X A^n(s)ds = \int_0^X (X-s)^n ds = \int_0^X s^n ds = X^{n+1}/(n+1)$.

The answers are the same because wp1,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t A^n(s)ds = E(X_e^n).$$

More generally, for any function $g = g(x)$, wp1,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t g(A(s))ds = E(g(X_e)).$$