# IEOR 6711, HMWK 6, Professor Sigman

1. For a (random) point process $\psi = \{t_n : n \geq 1\}$, explain why, for any given fixed $t > 0$, the random time $N(t)$ is not a stopping time with respect to the $\{t_n\}$, but $N(t) + 1$ is a stopping time.

2. Consider a FIFO M/M/1 queue (arrival rate $\lambda$, service rate $\mu$) for modeling a printer (say), but with *disasters* (arriving as an independent Poisson process at rate $\gamma$) that are so serious that ALL jobs are removed and lost: Whenever a disaster occurs, all jobs (if any) are removed. Let $X(t)$ denote the number of jobs in system at time $t$. Argue that it forms a CTMC (not a birth and death process though).

   (a) Set up the balance equations (rate out of $n$ equals rate into $n$) for finding the limiting probabilities $P_n$. (Don't try to solve yet.)

   (b) Let us "guess" that the solution to the $P_n$ is geometric: $P_n = (1-\alpha)\alpha^n,\ n \geq 0$, for some $0 < \alpha < 1$. Plug in this guess into the balance equations and solve for $\alpha$.

   (c) On average, how many jobs are removed by a disaster?

   (d) What is the proportion of jobs that get printed?

3. Consider a renewal process $\{t_n\}$ with iid interarrival times $X_n$, with $0 < E(X) = 1/\mu < \infty$. $t_n = X_1 + \cdots + X_n$; $N(t) = \max\{n : t_n \leq t\}$. Let $A(t) = t_{N(t)+1} - t$ denote the forward recurrence time, the time until the next renewal strictly after time $t$. Prove that $A(t)/t \to 0$ as $t \to \infty$. Also prove that $E(A(t))/t \to 0$.

4. A light has $b \geq 2$ bulbs, and the lifetimes $L_1, L_2, \ldots$ of new bulbs are iid with (continuous) distribution $F(x) = P(L \leq x)$. For fixed $T > 0$, a *group* replacement policy operates as follows: All bulbs (together as a group) are replaced with new ones at (deterministic) times $T, 2T, 3T, \ldots$. (They are replaced wether they are dead or not.) The cost of each new bulb is $C_N$, and there is a fixed additional work cost of $C_R$ each time the group is replaced. Moreover, there is an inconvenience cost of $C_I$ per bulb that dies.

   (a) What is the long-run cost per unit time of using this policy?

   (b) Assume that $F$ has density $f(x) = 2x,\ x \in (0, 1)$ (months) and that $T < 1$. Also assume that $b = 10$, $C_R = 0.5$, $C_N = 1$, $C_I = 12$. Find the value of $T < 1$ that minimizes cost.

   (c) Suppose that instead of being a fixed inconvenience cost per bulb, $C_I$ is the cost per bulb per unit time that a dead bulb remains in the light. So if a bulb dies $s$ units of time before the next group replacement, the cost will be $sC_I$. What is the long-run cost per unit time of using this policy (e.g., re-answer (a))?

5. Suppose that $\psi = \{t_n : n \geq 1\}$ is a renewal process. Suppose we partition it into two types: independently each point $t_n$ is of type I or type II with probability $p$ and $q = 1 - p$ respectively. Let $\psi_i,\ i = 1, 2$ denote the two resulting renewal processes, of type I and II arrivals. Are each of $\psi_1$ and $\psi_2$ renewal processes? Are they independent?

6. For a renewal process with cycle length distribution $F(x) = P(X \leq x)$ and mean $E(X) = 1/\lambda$, let $B(t) = t - t_{N(t)}$ denote the age (backwards recurrence time), and let $S(t) = t_{N(t)+1} - t_{N(t)}$ denote the spread. Let $Y(t) = B(t)/S(t)$. Show that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t I\{Y(s) \leq y\}ds = y, \ \ y \in (0,1);$$

$Y(t)$ has a $Unif(0,1)$ limiting distribution.

Now suppose $(X_b, X_e)$ has the joint limiting distribution of age and excess, $P(X_b > y, X_e > x) = \overline{F}_e(x+y)$. ($X_s = X_b + X_e$ thus has the limiting distribution of spread.)

Prove directly that $X_b/X_s$ has the $Unif(0,1)$ distribution.

7. Let $X \sim F$, and let $X_e \sim F_e$ (the equilibrium distribution of $F$). Let

$$\mathcal{L}_X(s) = E(e^{-sX}), \ \ s \geq 0,$$

the Laplace transform of $X$. Let

$$\mathcal{L}_e(s) = E(e^{-sX_e}), \ \ s \geq 0,$$

the Laplace transform of $X_e$.

(a) Show that

$$\mathcal{L}_e(s) = \frac{\lambda}{s}(1 - \mathcal{L}_X(s)).$$

(b) Use (a) to show that for any integer $n \geq 1$,

$$E(X_e^n) = \frac{E(X^{n+1})}{(n+1)E(X)}.$$

(c) Use the renewal reward theorem to show that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t A^n(s)ds = \frac{E(X^{n+1})}{(n+1)E(X)};$$

explain why this is the same answer as in (b).