IEOR 6711, Solutions to HMWK 7, Professor Sigman

1. Give an example showing that if $\{X_n\}$ (or $\{X(t)\}$) is a Markov chain it is NOT true in general that $\{f(X_n)\}$ (or $\{f(X(t))\}$) still satisfies the Markov property. ($f = f(x)$ is a real-valued function.)

   SOLUTION: In general, picking a function f that is not invertable yields a counterexample (but there are exceptions). For example, let $\{X(t)\}$ denote the CTMC for the M/M/1 queue. Let $f(x) = (x-1)^+$. Then $f(X(t)) = Q(t)$, the number in queue (line) at time t. Given $Q(t) = 0$, we do not know if $X(t) = 0$ or $X(t) = 1$; the past would give us that information; $\{Q(t)\}$ is not a Markov chain.

2. Consider a FIFO M/M/1 queue with $\rho < 1$. Consider the stochastic process $\mathbf{L} = \{L(t) : t \geq 0\}$, the number of customers in the system. Explain which of the following do or do not define regeneration times (and why) for $\mathbf{L}$. If they do, explain if they make $\mathbf{L}$ positive or null recurrent.

   (a) The consecutive times at which an arrival finds the system empty.
   SOLUTION:
   Yes. Right after the arrival, $X(t)$ jumps up to 1 ($X(t_n+) = 1$) and the future is independent of the past by the (strong) Markov property. The rate of this renewal process is the rate at which transitions from 0 to 1 occur, $\lambda P_0 = \lambda(1-\rho) > 0$; hence inter-regeneration times have mean $E(X) = \{\lambda(1-\rho)\}^{-1} < \infty$; positive recurrence.

   (b) The consecutive times at which a departure leaves behind an empty system.
   SOLUTION: Yes. Right after the departure, $X(t)$ jumps down to 0 ($X(t_n^d+) = 0$) and the future is independent of the past by the (strong) Markov property. The rate of this renewal process is the rate at which transitions from 1 to 0 occur, same as in (a) ($\lambda(1-\rho)$); positive recurrence.

   (c) The consecutive times at which an arrival finds ≤ 2 people in the system.
   SOLUTION: No. We would need to know which of the three states, $0, 1, 2$, $X(t)$ is in at such an arrival time. The past would tell us that additional information.

   (d) The consecutive times at which a departure leaves behind exactly 10 people in the system.
   SOLUTION: Yes. same reasoning as in (b). Right after the departure, $X(t)$ jumps down from 11 to 10 ($X(t_n^d+) = 10$) and the future is independent of the past by the (strong) Markov property. The rate of this renewal process is the rate at which transitions from 11 to 10 occur, $\mu P_{11} = \lambda\rho^{10}(1-\rho) > 0$; hence inter-regeneration times have mean $E(X) = \{\lambda\rho^{10}(1-\rho)\}^{-1} < \infty$; positive recurrence.

   (e) The superposition of (d) above with the consecutive times at which an arrival finds 9 people in the system.
   SOLUTION: Yes. These times form the consecutive visits of the chain to state $i = 10$. $\rho < 1$ implies positive recurrent of the chain hence $E(\tau_{i,i}) < \infty$ for any i; the regeneration points are positive recurrent.

3. *Continuation:* Let $Q(t)$ = the number of customers in the queue (line) at time t. Is $\{Q(t) : t \geq 0\}$ a regenerative process?

SOLUTION: Yes. $Q(t) = f(X(t))$ for $f(x) = (x-1)^+$; functions of regenerative processes are regenerative with the same regeneration times. (Compare with Exercise 1.)

4. Buses arrive to a certain bus stop according to a point process, $\psi = \{t_n : n \geq 1\}$, which is *cyclic regenerative* defined as follows:

   $T_n \stackrel{\text{def}}{=} t_n - t_{n-1}, t_0 = 0$ and these interarrival times they are broken up into iid blocks of length $c \geq 1$ (fixed). The random vectors $(T_1, \ldots, T_c)$, $(T_{c+1}, \ldots, T_{2c}) \ldots$ are i.i.d. with common joint cdf $H(x_1, \ldots, x_c) \stackrel{\text{def}}{=} P(T_1 \leq x_1, \ldots, T_c \leq x_c)$, and marginal distributions $G_i(x) \stackrel{\text{def}}{=} P(T_i \leq x)$, $1 \leq i \leq c$. Observe that a renewal process is the special case when $c = 1$. When $c \geq 2$, observe that within a block, the c interarrival times are allowed to be dependent. The point process thus regenerates in continuous time at times $\tau_k \stackrel{\text{def}}{=} t_{kc}, k \geq 1$, and hence, every c^{th} point from ψ is a regeneration point. We have i.i.d. cycle lengths $L_1 = T_1 + \cdots + T_c$, $L_2 = T_{c+1} + \ldots + T_{2c}$, and so on. We let $F(x) \stackrel{\text{def}}{=} P(L_1 \leq x)$, denote this cycle length distribution. We assume a non-delayed set-up.

   Let $A(t)$ denote the forward recurrence time for ψ; $A(t)$ represents how long you must wait for a bus if you arrive at the stop at time t. Then $\{A(t) : t \geq 0\}$ is a regenerative process with embedded renewal process $\{\tau_k : k \geq 1\}$. We assume that $EL_1 < \infty$, so that $A(t)$ is positive recurrent. Let $1/\mu_i \stackrel{\text{def}}{=} E(T_i), 1 \leq i \leq c$.

   (a) Let $\{N(t)\}$ denote the counting process for ψ. Find the long run arrival rate of buses (w.p.1):

   $$\lambda = \lim_{t \to \infty} \frac{N(t)}{t}.$$

   SOLUTION:
   We use renewal reward (say) to conclude that $\lambda = c/E(L_1)$; the reward over a cycle is deterministic $R = c$. $E(L_1) = E(T_1 + \cdots + T_c) = \sum_{i=1}^{c} 1/\mu_i$.

   $$\lambda = c\left\{\sum_{i=1}^{c} 1/\mu_i\right\}^{-1}.$$

   (b) Draw the graph of $A(t)$ over a cycle, and then find the (w.p.1.) time average of $A(t)$:

   $$\lim_{t \to \infty} \frac{1}{t} \int_0^t A(s)ds.$$

   Does it also hold that

   $$\lim_{t \to \infty} \frac{1}{t} \int_0^t E(A(s))ds$$

   exists and is the same as the sample path limit?
   SOLUTION:

The graph over L_1 consists of c triangles, the i^{th} one having area $T_i^2/2$. Thus we can use renewal reward with reward rate at time t given by $A(t)$ so that the reward R_1 is given by

$$R_1 = \sum_{i=1}^{c} T_i^2/2,$$

and if we now assume that all the T_i second moments are finite, $E(T_i^2) < \infty,\ i = 1, \ldots, c$, then
$E(R_1) < \infty$ and

$$E(R)/E(L) = \frac{\sum_{i=1}^{c} \frac{E(T_i^2)}{2}}{\sum_{i=1}^{c} 1/\mu_i} < \infty.$$

As in the renewal reward theorem (or Theorem 2.1 from Lecture Notes on regenerative processes) taking expected values is indeed allowed so as to get the same limit because here $E(|R|) = E(R) < \infty$ (assuming finite second moments of the T_i as we did above). (e.g., uniform integrability always holds.)

(c) If you arrive "randomly" in the infinite future to the bus stop, we wish to find the distribution of how long you must wait for a bus. Compute this (tail) as the limit (w.p.1)

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t I(A(s) > x)ds,\ x \geq 0.$$

Reduce your expression to be in terms of the c equilibrium distributions G_{ie} of the G_i, and basic system parameters.

SOLUTION:

Similar to (b), now

$$R_1 = \sum_{i=1}^{c} (T_i - x)^+,$$

yielding

$$E(R)/E(L) = \frac{\sum_{i=1}^{c} E(T_i - x)^+}{\sum_{i=1}^{c} 1/\mu_i}.$$

Recalling that $\overline{G}_{ie}(x) = \mu_i E(T_i - x)^+$, we can re-write the answer as

$$\frac{\sum_{i=1}^{c} \frac{1}{\mu_i}\overline{G}_{ie}(x)}{\sum_{i=1}^{c} 1/\mu_i} = \sum_{i=1}^{c} p_i \overline{G}_{ie}(x),$$

where

$$p_i = \frac{\frac{1}{\mu_i}}{\sum_{i=1}^{c} 1/\mu_i},$$

denotes (via renewal reward), the long-run proportion of time a bus interarrival time of type T_i is in progress. Thus the answer is quite intuitive: If you randomly way out in the future arrive at the bus stop, then with probability p_i you will be in the middle of a T_i and hence face a remaining wait distributed as G_{ie}.

(d) Assume that $c = 2$ and that T_1 and T_2 are independent; thus $H(x, y) = G_1(x)G_2(y)$. Let $B(t)$ denote the *backward* recurrence time for ψ; e.g., the length of time since the last bus came (it also is regenerative with the same embedded renewal process). What is $E(B(t)|T_1 > t)$? What is $E(B(t)|T_1 = s < t, T_2 > t - s)$?. Now use this to find an expression for $E(B(t)|L_1 > t)$ (by considering the fact that $\{L_1 > t\} = \{T_1 + T_2 > t\} = \{T_1 > t\} \cup \{T_1 < t, T_1 + T_2 > t\}$). Finally, use all this to obtain a renewal equation (and its solution) for $E(B(t))$ by conditioning on L_1.

SOLUTION: $E(B(t)|T_1 > t) = t$ and $E(B(t)|T_1 = s < t, T_2 > t - s\} = t - s$. Thus

$E(B(t); T_1 > t) = t\overline{G}_1(t)$ and $E(B(t); T_1 = s < t, T_2 > t - s) = (t - s)\overline{G}_2(t - s)$ and integrating the second over s yields

$$E(B(t); T_1 < t < T_1 + T_2) = \int_0^t (t - s)\overline{G}_2(t - s)dG_1(s).$$

Using

$$E(B(t); L_1 > t) = E(B(t); T_1 > t) + E(B(t); T_1 < t < T_1 + T_2)$$

then yields

$$E(B(t); L_1 > t) = t\overline{G}_1(t) + \int_0^t (t - s)\overline{G}_2(t - s)dG_1(s). \tag{1}$$

(Divide (1) by $P(L_1 > t)$ to get $E(B(t)|L_1 > t)$.)

Meanwhile, $E(B(t)|L_1 = s \le t) = E(B(t - s))$ by our standard renewal arguments because L_1 is a regeneration point, and thus

$$E(B(t); L_1 \le t) = \int_0^t E(B(t - s))dF(s),$$

where

$$F(t) = P(L_1 \le t) = P(T_1 + T_2 \le t) = \int_0^t G_2(t - s)dG_1(s) = \int_0^t G_1(t - s)dG_2(s).$$

(e.g., $F = G_1 * G_2$.) Finally, using $E(B(t)) = E(B(t); L_1 > t) + E(B(t); L_1 \le t)$ yields the renewal equation

$$E(B(t)) = t\overline{G}_1(t) + \int_0^t (t - s)\overline{G}_2(t - s)dG_1(s) + \int_0^t E(B(t - s))dF(s). \tag{2}$$

Here $Q(t) = t\overline{G}_1(t) + \int_0^t (t - s)\overline{G}_2(t - s)dG_1(s)$, and $H(t) = E(B(t))$ and the solution is $H(t) = Q(t) + (Q * m)(t)$ where $m(t) = \sum_{j=1}^{\infty} F^{(*j)}(t)$.

(e) Assume F is non-lattice. Without any new calculations, what must be the value of the following limit and why:

$$\lim_{t\to\infty} P(A(t) > x).$$

SOLUTION:
The same as the time average answer given in (c) above. As with any positive recurrent regenerative process, the non-lattice condition only ensures that $A(t)$ also converges weakly to its limiting distribution (as opposed to only in a time-average sense).

5. A stochastic process $\{X(t)\}$ that possesses the *Markov property* (Given $X(t)$ (the present state at time t), the future $X(t+h)$ is independent of the past $\{X(s) : 0 \leq s < t\}$) is called a Markov process. We have so far only studied the discrete state space case. Now we will consider the case when the state space is more general: $\mathcal{S} = \mathbb{R}$.

(a) Suppose that $\{\Delta_n : n \geq 0\}$ are iid rvs with common distribution $F(x) = P(\Delta \leq x)$. Define

$$D_{n+1} = (D_n + \Delta_n)^+, \ n \geq 0.$$

Argue that $\{D_n\}$ forms a discrete-time Markov process.
Find the transition probabilities $P(D_{n+1} \leq y \mid D_n = x), \ x \geq 0, \ y \geq 0.$
SOLUTION:
This is a recursion of the form $X_{n+1} = f(X_n, U_n)$ where the $U_n = \Delta_{n-1}, \ n \geq 1$ are iid, hence from general results on Markov chains it is Markovian. Here $f(x, u) = (x+u)^+$. (See for Example Proposition1.1 of Section 1.3 in Lecture Notes "Introduction to discrete-time Markov chains".)
$P(D_{n+1} \leq y \mid D_n = x) = P(x + \Delta)^+ \leq y) = P((x + \Delta) \leq y) =$
$P(\Delta \leq y - x) = F(y - x).$

(b) For a renewal process $\psi = \{t_n\}$, Argue that both $\{A(t)\}$ and $\{B(t)\}$ are Markov processes. Find the transition probabilities $P(A(t+h) \leq y \mid A(t) = x), \ h \geq 0, \ x \geq 0, \ y \geq 0.$
SOLUTION:
Conditional on the value of $A(t) = x$, the future after time t only depends on (in addition to the now known value of $A(t) = x$) the remaining iid interarrival times $X_{N(t)+1}, X_{N(t)+2}, \ldots$ which are iid copies of X and independent of the past; the Markov property is satisfied for $\{A(t)\}$..

For $\{B(t)\}$: Conditional on the value of $B(t) = y$, the future depends only on the value of $A(t)$ and then the future remaining iid interarrival times $X_{N(t)+1}, X_{N(t)+2}, \ldots$ which are iid copies of X independent of the past. So we need to show that given $B(t) = y$, $A(t)$ only depends at most on y and is otherwise independent of the past. To this end: $P(A(t) \leq x | B(t) = y) = P(X \leq x + y | X \geq y) = (F(x + y) - F(x))/\overline{F}(y), \ x \geq 0$; this conditional distribution only depends on y. The Markov property is satisfied.

To compute $P(A(t + h) \leq y | A(t) = x)$, we consider the two cases $h \leq x$ or $h > x$. If $h \leq x$, we have $A(t + h) = x - h$; if $h > x$, a new renewal happens at time $t + x$, therefore,

$$P(A(t+h) \le y | A(t) = x) = \begin{cases} 1_{\{x-h \le y\}}, & if \quad h \le x, \\ P(A(h-x) \le y), & if \quad h > x. \end{cases}$$

It remains to determine $P(A(t) \le x)$ for $t \ge 0, x \ge 0$. The best we can do is derive a renewal equation so as to express the answer in terms of $m(t) = E(N(t))$. For example, using the same method used to derive Equation (8) on page 4 of the Lecture Notes on renewal theory II, we condition on the first renewal $X_1 = t_1$:

$$\begin{aligned} P(A(t) \le x) &= P(A(t) \le x; X_1 > t) + P(A(t) \le x; X_1 \le t) \\ &= P(X_1 - t \le x, X_1 > t) + \int_0^t P(A(t-s) \le x) dF(s) \\ &= F(x+t) - F(t) + \int_0^t P(A(t-s) \le x) dF(s), \end{aligned}$$

which is a renewal equation. The solution is then of the form

$$P(A(t) \le x) = Q(t) + \int_0^t Q(t-s) dm(t),$$

where $Q(t) \equiv F(x+t) - F(t)$ and $m(t) \equiv \sum_{j=1}^{\infty} F^{*j}(t)$.