IEOR 6711, HMWK 7, Professor Sigman

1. Give an example showing that if $\{X_n\}$ (or $\{X(t)\}$) is a Markov chain it is NOT true in general that $\{f(X_n)\}$ (or $\{f(X(t))\}$) still satisfies the Markov property. ($f = f(x)$ is a real-valued function.)

2. Consider a FIFO M/M/1 queue with $\rho < 1$. Consider the stochastic process $\mathbf{L} = \{L(t) : t \geq 0\}$, the number of customers in the system. Explain which of the following do or do not define regeneration times (and why) for $\mathbf{L}$. If they do, explain if they make $\mathbf{L}$ positive or null recurrent.

   (a) The consecutive times at which an arrival finds the system empty.

   (b) The consecutive times at which a departure leaves behind an empty system.

   (c) The consecutive times at which an arrival finds ≤ 2 people in the system.

   (d) The consecutive times at which a departure leaves behind exactly 10 people in the system.

   (e) The superposition of (d) above with the consecutive times at which an arrival finds 9 people in the system.

3. *Continuation:* Let $Q(t)$ = the number of customers in the queue (line) at time t. Is $\{Q(t) : t \geq 0\}$ a regenerative process?

4. Buses arrive to a certain bus stop according to a point process, $\psi = \{t_n : n \geq 1\}$, which is *cyclic regenerative* defined as follows:

   $T_n \stackrel{\text{def}}{=} t_n - t_{n-1}, t_0 = 0$ and these interarrival times they are broken up into iid blocks of length $c \geq 1$ (fixed). The random vectors $(T_1, \ldots, T_c)$, $(T_{c+1}, \ldots, T_{2c}) \ldots$ are i.i.d. with common joint cdf $H(x_1, \ldots, x_c) \stackrel{\text{def}}{=} P(T_1 \leq x_1, \ldots, T_c \leq x_c)$, and marginal distributions $G_i(x) \stackrel{\text{def}}{=} P(T_i \leq x)$, $1 \leq i \leq c$. Observe that a renewal process is the special case when $c = 1$. When $c \geq 2$, observe that within a block, the c interarrival times are allowed to be dependent. The point process thus regenerates in continuous time at times $\tau_k \stackrel{\text{def}}{=} t_{kc}, k \geq 1$, and hence, every c^{th} point from ψ is a regeneration point. We have i.i.d. cycle lengths $L_1 = T_1 + \cdots + T_c$, $L_2 = T_{c+1} + \ldots + T_{2c}$, and so on. We let $F(x) \stackrel{\text{def}}{=} P(L_1 \leq x)$, denote this cycle length distribution. We assume a non-delayed set-up.

   Let $A(t)$ denote the forward recurrence time for ψ; $A(t)$ represents how long you must wait for a bus if you arrive at the stop at time t. Then $\{A(t) : t \geq 0\}$ is a regenerative process with embedded renewal process $\{\tau_k : k \geq 1\}$. We assume that $EL_1 < \infty$, so that $A(t)$ is positive recurrent. Let $1/\mu_i \stackrel{\text{def}}{=} E(T_i), 1 \leq i \leq c$.

   (a) Let $\{N(t)\}$ denote the counting process for ψ. Find the long run arrival rate of buses (w.p.1):

   $$\lambda = \lim_{t \to \infty} \frac{N(t)}{t}.$$

   (b) Draw the graph of $A(t)$ over a cycle, and then find the (w.p.1.) time average of $A(t)$:

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t A(s)ds.$$

Does it also hold that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t E(A(s))ds$$

exists and is the same as the sample path limit?

(c) If you arrive "randomly" in the infinite future to the bus stop, we wish to find the distribution of how long you must wait for a bus. Compute this (tail) as the limit (w.p.1)

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t I(A(s) > x)ds, \ x \geq 0.$$

Reduce your expression to be in terms of the c equilibrium distributions G_{ie} of the G_i, and basic system parameters.

(d) Assume that $c = 2$ and that T_1 and T_2 are independent; thus $H(x, y) = G_1(x)G_2(y)$. Let $B(t)$ denote the *backward* recurrence time for ψ; e.g., the length of time since the last bus came (it also is regenerative with the same embedded renewal process). What is $E(B(t)|T_1 > t)$? What is $E(B(t)|T_1 = s < t, T_2 > t - s\}$?. Now use this to find an expression for $E(B(t)|L_1 > t)$ (by considering the fact that $\{L_1 > t\} = \{T_1 + T_2 > t\} = \{T_1 > t\} \cup \{T_1 < t, T_1 + T_2 > t\}$). Finally, use all this to obtain a renewal equation (and its solution) for $E(B(t))$ by conditioning on L_1.

(e) Assume F is non-lattice. Without any new calculations, what must be the value of the following limit and why:

$$\lim_{t\to\infty} P(A(t) > x).$$

5. A stochastic process $\{X(t)\}$ that possesses the *Markov property* (Given $X(t)$ (the present state at time t), the future $X(t+h)$ is independent of the past $\{X(s) : 0 \leq s < t\}$) is called a Markov process. We have so far only studied the discrete state space case. Now we will consider the case when the state space is more general: $\mathcal{S} = \mathbb{R}$.

(a) Suppose that $\{\Delta_n : n \geq 0\}$ are iid rvs with common distribution $F(x) = P(\Delta \leq x)$. Define

$$D_{n+1} = (D_n + \Delta_n)^+, \ n \geq 0.$$

Argue that $\{D_n\}$ forms a discrete-time Markov process.
Find the transition probabilities $P(D_{n+1} \leq y \mid D_n = x), \ x \geq 0, \ y \geq 0$.

(b) For a renewal process $\psi = \{t_n\}$, Argue that both $\{A(t)\}$ and $\{B(t)\}$ are Markov processes. Find the transition probabilities $P(A(t+h) \leq y \mid A(t) = x), \ h \geq 0, \ x \geq 0, \ y \geq 0$.