IEOR 6711, HMWK 8, Professor Sigman

Throughout, we consider the binomial lattice model, $S_n = S_0 \times Y_1 \cdots \times Y_n,\ n \geq 0$, with given parameters $0 < d < 1 + r < u$, and $0 < p < 1$.

1. Consider the case $u = 1.1,\ d = 0.9,\ p = 0.5$; $P(Y = 1.1) = 0.5$ and $P(Y = 0.9) = 0.5$; thus $E(Y) = 1$. (Assume that $S_0 > 0$.) Prove (using straightforward basic probability theory) that $S_n \to 0$, wp1, even though $E(S_n) = S_0,\ n \geq 1$. (Hint: Take logarithms.)

   SOLUTION:

   $S_n \to 0$, wp1 if and only if $\ln(S_n) \to -\infty$, wp1. But $\ln(S_n) = \ln(S_0) + \ln(Y_1) + \cdots + \ln(Y_n),\ n \geq 0$, a random walk with iid increments distributed as $\ln(Y)$. Thus we need to prove that this random walk has negative drift, that is, that $E(\ln(Y)) < 0$. This is easily shown directly: $E(\ln(Y)) = (1/2)(\ln(u) + \ln(d)) = (1/2)\ln(ud) = (1/2)\ln(0.99) < 0$.

   Meanwhile, $E(Y) = 1$ and so $E(S_n) = S_0,\ n \geq 1$.

2. *Continuation:* Change u to 1.11 leaving all else the same. Prove that now, $S_n \to 0$, wp1, while $E(S_n) \to \infty$: On average you will become infinitely rich even though with certainty you will go broke!!

   SOLUTION: In this case, $E(\ln(Y)) = (1/2)(\ln(u) + \ln(d)) = (1/2)\ln(ud) = (1/2)\ln(0.999) < 0$, so yet again $S_n \to 0$, wp1. But now $E(Y) > 1$ and so $E(S_n) = S_0 E(Y)^n \to \infty$.

3. Let $p = p^*$, the risk-neutral probability;

$$p^* = \frac{1 + r - d}{u - d}.$$

   Define $M_n = S_n/(1 + r)^n,\ n \geq 0$.

   (a) Argue that $\{M_n\}$ is a Markov chain.
   SOLUTION:
   $M_n = S_0 \times \overline{Y}_1 \times \cdots \times \overline{Y}_n,\ n \geq 0$, where the $\overline{Y}_i = Y_i/(1 + r)$ are iid; M_n satisfies the recursion $M_{n+1} = M_n \overline{Y}_{n+1}$, hence is a MC.

   (b) Prove that $\{M_n\}$ is a martingale; $E(M_{n+1} \mid M_n, \ldots, M_0) = M_n,\ n \geq 0$.

   SOLUTION: Under p^*, $E(Y) = 1+r$ and hence $E(\overline{Y}) = 1$. Thus M_n is a special case of such "product" martingales. We give the proof again for completeness. Let $\mathcal{G}_n = \sigma\{M_0, \ldots, M_n\}$. Since $\{M_n\}$ is a MC, we have $E(M_{n+1}|\mathcal{G}_n) = E(M_{n+1}|M_n)$, and hence it must be shown that $E(M_{n+1}|M_n) = M_n,\ n \geq 0$. To this end: $E(M_{n+1}|M_n) = E(M_n \overline{Y}_{n+1}\ |M_n) = M_n E(\overline{Y}_{n+1}\ |M_n) = M_n E(\overline{Y}_{n+1}) = M_n \times 1 = M_n$.

   (c) Prove that p^* is the unique probability making $\{M_n\}$ into a martingale. (e.g., no other $0 < p < 1$ will work.)

   SOLUTION: As shown above in (b), $E(M_{n+1}|M_n) = M_n E(\overline{Y}_{n+1})$. Thus we get a MG if and only if $E(Y/(1+r)) = 1$. This is equivalent to solving for the value of p for which $pu + (1-p)d = 1 + r$; the unique solution is p^*.

4. Consider a process with state space $(0,1)$ defined via

$$X_{n+1} = \begin{cases} a + bX_n, & \text{with prob. } X_n \ , \\ bX_n & \text{with prob. } 1 - X_n. \end{cases}$$

where $0 < a < 1, \ 0 < b < 1$ and $a + b = 1$. (And assume that $0 < X_0 < 1$.)

(a) Show that $\{X_n\}$ forms a MG.
SOLUTION:
(Here, as usual, $\mathcal{F}_n = \sigma\{X_0, \ldots, X_n\}$.)
C1: Since $0 < X_n < 1$, we have $E|X_n| < 1 < \infty, \ n \geq 0$.
C2: $E(X_{n+1}|\mathcal{F}_n)) = (a + bX_n)X_n + (bX_n)(1 - X_n)$
$= (a + b)X_n$
$= X_n$.

(b) Explain why $E(X_\tau) = E(X_0)$ holds here for ANY stopping time τ.
SOLUTION: Since $0 < X_n < 1$, we have $0 < X_{n\wedge\tau} < 1, \ n \geq 0$ for any stopping time; so $\{X_{n\wedge\tau} : n \geq 0\}$ is bounded hence UI. Thus the optional stopping theorem holds.

5. Consider a process $\{X_n : n \geq 0\}$ such that $E|X_n| < \infty, \ n \geq 0$, and

$$E(X_{n+1}|\mathcal{F}_n) = aX_n + bX_{n-1}, \ n \geq 1$$

where $0 < a < 1, \ 0 < b < 1$ and $a + b = 1$. (Here, as usual, $\mathcal{F}_n = \sigma\{X_0, \ldots, X_n\}$.) Find a value of c such that the process $M_n = cX_n + X_{n-1}$ forms a MG with respect to $\mathcal{F}_n$: $E(M_{n+1}|\mathcal{F}_n) = M_n, \ n \geq 0$.

SOLUTION: $E(cX_{n+1} + X_n|\mathcal{F}_n) = c(aX_n + bX_{n-1}) + X_n$, so we need to find c such that $c(aX_n + bX_{n-1}) + X_n = cX_n + X_{n-1}$; equivalently find c such that $ca + 1 = c$, and $cb = 1$. We conclude that $c = b^{-1}$ (recall that $b = 1 - a$).

6. Let $\{X_n\}$ be a MG, and define the increments $\Delta_n = X_n - X_{n-1}, \ n \geq 1$. We thus can re-write $X_n = \Delta_1 + \cdots + \Delta_n, \ n \geq 1$. Prove that these increments are uncorrelated: $E(\Delta_n\Delta_m) = 0$ for $m < n$. (Hint: Condition on $\mathcal{F}_m$.)

SOLUTION:

First note that the MG property generalizes to: $E(X_{n+k}|\mathcal{F}_n) = X_n, \ k \geq 1$. To see this, consider $k = 2$. Since $\mathcal{F}_n \subset \mathcal{F}_{n+1}$, we have

$$E(X_{n+2}|\mathcal{F}_n) = E[E(X_{n+2}|\mathcal{F}_{n+1})|\mathcal{F}_n] = E[X_{n+1}|\mathcal{F}_n)] = X_n.$$

The general $k \geq 1$ case is then completed by induction. This implies that $E(\Delta_n|\mathcal{F}_m) = X_m - X_m = 0$ for any $n > m$. We are now ready to prove the result:

$$\begin{aligned} E(\Delta_n\Delta_m|\mathcal{F}_m) &= \Delta_m E(\Delta_n|\mathcal{F}_m) \text{ (since } \Delta_m \in \mathcal{F}_m) \\ &= \Delta_m E(X_n - X_{n-1}|\mathcal{F}_m) \\ &= \Delta_m(X_m - X_m) \text{ (MG property since } m \leq n-1) \\ &= \Delta_m \cdot 0 \\ &= 0. \end{aligned}$$

Now we use the general fact, for any rv X, that $E(X) = E[E(X|\mathcal{F}_m)]$; here $X = \Delta_n \Delta_m$; we conclude that $E(\Delta_n \Delta_m) = 0$.

7. Consider the Markov chain $\{X_n\}$ on $\{0, 1, 2, 3, 4\}$ with transition matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 0 & 1/3 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Suppose that $X_0 = i$ with $0 < i < 4$. Find $p(i) =$ the probability that the chain hits 4 before hitting 0.

SOLUTION:

First observe that $\{X_n\}$ is a MG: It's bounded, so C1 is satisfied. C2: Since $\{X_n\}$ is also a Markov chain, we know that $E(X_{n+1}|\mathcal{F}_n) = E(X_{n+1}|X_n)$. So we must verify that $E(X_{n+1}|X_n = i) = i,\ i \in \{0, 1, 2, 3, 4\}$. But note that for each row i, the mean of the distribution forming row i is in fact i. For example, row 1 has mean $(2/3)0 + (1/3)3 = 1$ and row 2 has mean $(1/5)(0 + 1 + 2 + 3 + 4) = 2$. In other words, for each i,

$$E(X_{n+1}|X_n = i) = \sum_{j=0}^{4} jP_{i,j} = i.$$

Thus it is a MG.

Let $\tau = \min\{n \geq 0 : X_n \in \{0, 4\}\}$, a stopping time. The optional stopping theorem applies (the stopping time is proper and the process is bounded, hence UI is satisfied): Using $E(X_\tau) = E(X_0) = i$ and expanding $E(X_\tau)$ then yields $p(i)4 + (1 - p(i))0 = i$, yielding

$p(i) = i/4$.

In other words, the answer is the same as for the gambler's ruin problem when $p = 0.5$ (e.g., when the underlying Markov chain is the simple symmetric walk.)

8. Suppose that $\{X_n : n \geq 0\}$ is a SUBMG such that $E(X_n) = c,\ n \geq 0$ (e.g., the means are all identical). Prove that in fact, $\{X_n : n \geq 0\}$ must be a martingale.

SOLUTION:

By the SUBMG property, $E(X_{n+1}|\mathcal{F}_n) \geq X_n$, the random variable $B_n = E(X_{n+1} - X_n \mid \mathcal{F}_n)) = E(X_{n+1} \mid \mathcal{F}_n)) - X_n$ is non-negative; $B_n \geq 0$ wp1, for each $n \geq 0$. But $E(B_n) = E(X_{n+1} - X_n) = E(X_{n+1}) - E(X_n) = c - c = 0$. A non-negative rv with mean 0 must be 0 wp1.; thus $B_n = 0$, wp1, $n \geq 1$, and indeed we have a MG.

9. Consider a random walk $R_n = \sum_{j=1}^{n} \Delta_j$, $R_0 = 0$, with iid Δ_j such that $P(\Delta \neq 0) > 0$ and such that

$$m(\theta) = E(e^{\theta\Delta}) < \infty,$$

for all $\theta \in (-\delta, \delta)$ some $\delta > 0$. That is, Δ has a finite moment generating function in a neighborhood of the origin.

(a) For $m(\theta) < \infty$, prove that $X_n = m(\theta)^{-n}e^{\theta R_n},\ n \geq 0$ forms a martingale.
SOLUTION: It is of the product form $X_n = Y_1 \times \cdots \times Y_n$, where the iid $Y_i = \frac{e^{\theta \Delta_i}}{m(\theta)}$ satisfy $E(Y) = 1$.

(b) Assume that there exists a $\theta^* \in (-\delta, \delta)$ such that $m(\theta^*) = 1$, so that then $X_n = e^{\theta^* R_n}$ forms a MG. Let $a, b > 0$ and define a stopping time (easily shown to be proper)

$$\tau = \min\{n \geq 1 : R_n \notin (-b, a)\},$$

the first time that the random walk either goes $\geq a$ or $\leq -b$. Since we are not assuming that this random walk is simple, there typically will be an *overshoot* when the random walk crosses a or $-b$. If a is crossed first, then $X_\tau = e^{a\theta^*}e^{H_a\theta^*}$, where $H_a = R_\tau - a$, the overshoot above a. Similarly, if $-b$ is crossed first, then $X_\tau = e^{-b\theta^*}e^{-H_b\theta^*}$, where $H_b = |R_\tau - (-b)|$, the overshoot below $-b$. Letting $p(a) = P(R_\tau \geq a)$ and $p(b) = 1 - p(a) = P(R_\tau \leq -b)$. Assuming the appropriate UI condition holds[1] use the MG optional stopping theorem to obtain an expression for $p(a)$.
SOLUTION: Using $E(X_\tau) = E(X_0) = 1$ and expanding the expected value $E(X_\tau)$ yields

$$1 = e^{a\theta^*}E(e^{H_a\theta^*})p(a) + e^{-b\theta^*}E(e^{-H_b\theta^*})(1 - p(a)).$$

We then solve for $p(a)$:

$$p(a) = \frac{1 - e^{-b\theta^*}E(e^{-H_b\theta^*})}{e^{a\theta^*}E(e^{H_a\theta^*}) - e^{-b\theta^*}E(e^{-H_b\theta^*})}. \tag{1}$$

(c) Assuming that both H_a and H_b are very very small (in comparison to a, b), use your expression in (b) to obtain the following explicit approximation to $p(a)$:

$$p(a) \approx \frac{1 - e^{-b\theta^*}}{e^{a\theta^*} - e^{-b\theta^*}}.$$

SOLUTION:
Treating $E(e^{H_a\theta^*}) \approx 1$ and $E(e^{-H_b\theta^*}) \approx 1$ and plugging into (3) then yields the approximation.

(d) (Assume here that $E(\Delta) \neq 0$.) Using a similar method as in (b) on R_n itself together with the above approximation for $p(a)$, derive the following approximation for $E(\tau)$:

$$E(\tau) \approx \frac{a(1 - e^{-b\theta^*}) - b(e^{a\theta^*} - 1)}{(e^{a\theta^*} - e^{-b\theta^*})E(\Delta)}.$$

[1]It does hold: $0 \leq X_{n\wedge\tau} = e^{\theta^* R_n}I\{\tau > n\} + e^{\theta^* R_\tau}I\{\tau \leq n\} \leq e^{|\theta^*|(a+b)} + e^{\theta^* R_\tau}$. Thus if we can show that $E(e^{\theta^* R_\tau}) < \infty$ then UI follows by the dominated convergence theorem. To this end we first recall that since $X_{n\wedge\tau}$ itself is a MG, we know that $E(X_{n\wedge\tau}) = 1,\ n \geq 0$. Moreover $\lim_n X_{n\wedge\tau} = X_\tau = e^{\theta^* R_\tau}$. Thus Fatou's Lemma yields $1 = \underline{\lim}\, E(X_{n\wedge\tau}) \geq E(\underline{\lim}\, X_{n\wedge\tau}) = E(e^{\theta^* R_\tau})$, and we conclude that $E(e^{\theta^* R_\tau}) \leq 1$; hence it's finite.

SOLUTION: (Note that $E|\Delta| < \infty$ since $m(\theta) < \infty$ by assumption.) Assuming that $E(\tau) < \infty$ (this can be proved), Wald's equation yields

$$E(R_\tau) = E(\tau)E(\Delta),$$

and so

$$E(\tau) = E(R_\tau)/E(\Delta).$$

Expanding $E(R_\tau)$ then yields

$$E(R_\tau) = (a + E(H_a))p(a) - (b + E(H_b))(1 - p(a)). \tag{2}$$

Using approximations $E(H_a) \approx 0$ and $E(H_b) \approx 0$ in (2) yields

$$E(R_\tau) \approx ap(a) - b(1 - p(a)).$$

Plugging in our approximation for $p(a)$ derived in (c) then yields the result.

(e) Suppose specifically that $\Delta = S - T$, where S and T are independent and $S \sim exp(\mu)$, $T \sim exp(\lambda)$ and $\lambda < \mu$.
In this case, find θ^* and then use it (via the expression in (b)) to compute $p(a)$ exactly.
Also, compute $E(\tau)$ exactly.
SOLUTION:
In this case, for any $\theta \in (-\lambda, \mu)$,

$$m(\theta) = E(e^{\theta S})E(e^{-\theta T}) = \frac{\mu}{\mu - \theta} \times \frac{\lambda}{\lambda + \theta}.$$

Solving $m(\theta) = 1$ yields (easily) that

$$\theta^* = \mu - \lambda.$$

By the memoryless property of the exponential distributions we have exponential overshoots: $H_a \sim exp(\mu)$ and $H_b \sim exp(\lambda)$ and thus we can compute exactly $E(H_a) = 1/\mu,\ E(H_b) = 1/\lambda,\ E(e^{H_a\theta^*}) = \mu/\lambda,\ E(e^{H_b\theta^*}) = \lambda/\mu.$
Plugging these into (3) while letting $\rho = \lambda/\mu$ then yields

$$p(a) = \frac{1 - \rho e^{-b(\mu-\lambda)}}{\rho^{-1}e^{a(\mu-\lambda)} - \rho e^{-b(\mu-\lambda)}}. \tag{3}$$

Note that by letting $b \to \infty$ we obtain a formula for $P(M \geq a)$ where $M = \max_n R_n$:

$$P(M \geq a) = \rho e^{-a(\mu-\lambda)}.$$

Computing $E(\tau)$ exactly: Plug (3) into (2) together with $E(H_a) = 1/\mu,\ E(H_b) = 1/\lambda$ and $E(\Delta) = 1/\mu - 1/\lambda$ to obtain the result.