IEOR 6711, HMWK 8, Professor Sigman

Throughout, we consider the binomial lattice model, $S_n = S_0 \times Y_1 \cdots \times Y_n,\ n \geq 0$, with given parameters $0 < d < 1 + r < u$, and $0 < p < 1$.

1. Consider the case $u = 1.1,\ d = 0.9,\ p = 0.5$; $P(Y = 1.1) = 0.5$ and $P(Y = 0.9) = 0.5$; thus $E(Y) = 1$. (Assume that $S_0 > 0$.) Prove (using straightforward basic probability theory) that $S_n \to 0$, wp1, even though $E(S_n) = S_0,\ n \geq 1$. (Hint: Take logarithms.)

2. *Continuation:* Change u to 1.11 leaving all else the same. Prove that now, $S_n \to 0$, wp1, while $E(S_n) \to \infty$: On average you will become infinitely rich even though with certainty you will go broke!!

3. Let $p = p^*$, the risk-neutral probability;
$$p^* = \frac{1 + r - d}{u - d}.$$
Define $M_n = S_n/(1 + r)^n,\ n \geq 0$.
   (a) Argue that $\{M_n\}$ is a Markov chain.
   (b) Prove that $\{M_n\}$ is a martingale; $E(M_{n+1} \mid M_n, \ldots, M_0) = M_n,\ n \geq 0$.
   (c) Prove that p^* is the unique probability making $\{M_n\}$ into a martingale. (e.g., no other $0 < p < 1$ will work.)

4. Consider a process with state space $(0, 1)$ defined via
$$X_{n+1} = \begin{cases} a + bX_n, & \text{with prob. } X_n\ , \\ bX_n & \text{with prob. } 1 - X_n. \end{cases}$$
where $0 < a < 1,\ 0 < b < 1$ and $a + b = 1$. (And assume that $0 < X_0 < 1$.)
   (a) Show that $\{X_n\}$ forms a MG.
   (b) Explain why $E(X_\tau) = E(X_0)$ holds here for ANY stopping time τ.

5. Consider a process $\{X_n : n \geq 0\}$ such that $E|X_n| < \infty,\ n \geq 0$, and
$$E(X_{n+1}|\mathcal{F}_n) = aX_n + bX_{n-1},\ n \geq 1$$
where $0 < a < 1,\ 0 < b < 1$ and $a + b = 1$. (Here, as usual, $\mathcal{F}_n = \sigma\{X_0, \ldots, X_n\}$.)
Find a value of c such that the process $M_n = cX_n + X_{n-1}$ forms a MG with respect to $\mathcal{F}_n$: $E(M_{n+1}|\mathcal{F}_n) = M_n,\ n \geq 0$.

6. Let $\{X_n\}$ be a MG, and define the increments $\Delta_n = X_n - X_{n-1},\ n \geq 1$. We thus can re-write $X_n = \Delta_1 + \cdots + \Delta_n,\ n \geq 1$. Prove that these increments are uncorrelated: $E(\Delta_n \Delta_m) = 0$ for $m < n$. (Hint: Condition on $\mathcal{F}_n$.)

7. Consider the Markov chain $\{X_n\}$ on $\{0, 1, 2, 3, 4\}$ with transition matrix
$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 0 & 1/3 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Suppose that $X_0 = i$ with $0 < i < 4$ and let $\tau = \min\{n \geq 0 : X_n \in \{0, 4\}\}$. By using Martingale theory, find $p(i) =$ the probability that the chain hits 4 before hitting 0.

8. Suppose that $\{X_n : n \geq 0\}$ is a SUBMG such that $E(X_n) = c,\ n \geq 0$ (e.g., the means are all identical). Prove that in fact, $\{X_n : n \geq 0\}$ must be a martingale.

9. Consider a random walk $R_n = \sum_{j=1}^{n} \Delta_j$, $R_0 = 0$, with iid Δ_j such that $P(\Delta \neq 0) > 0$ and such that
$$m(\theta) = E(e^{\theta\Delta}) < \infty),$$
for all $\theta \in (-\delta, \delta)$ some $\delta > 0$. That is, Δ has a finite moment generating function in a neighborhood of the origin.

   (a) For $m(\theta) < \infty$, Prove that $X_n = m(\theta)^{-n} e^{\theta R_n},\ n \geq 0$ forms a martingale.

   (b) Assume that there exists a $\theta^* \in (-\delta, \delta)$ such that $m(\theta^*) = 1$, so that then $X_n = e^{\theta R_n}$ forms a MG. Let $a, b > 0$ and define a stopping time (easily shown to be proper)
$$\tau = \min\{n \geq 1 : R_n \notin (-b, a)\},$$
the first time that the random walk either goes $\geq a$ or $\leq -b$. Since we are not assuming that this random walk is simple, there typically will be an *overshoot* when the random walk crosses a or $-b$. If a is crossed first, then $X_\tau = e^{a\theta^*} e^{H_a\theta^*}$, where $H_a = R_\tau - a$, the overshoot above a. Similarly, if $-b$ is crossed first, then $X_\tau = e^{-b\theta^*} e^{-H_b\theta^*}$, where $H_b = |R_\tau - (-b)|$, the overshoot below $-b$. Letting $p(a) = P(R_\tau \geq a)$ and $p(b) = 1 - p(a) = P(R_\tau \leq -b)$. Assuming the appropriate UI condition holds, use the MG optional stopping theorem to obtain an expression for $p(a)$.

   (c) Assuming that both H_a and H_b are very very small (in comparison to a, b), use your expression in (b) to obtain the following explicit approximation to $p(a)$:
$$p(a) \approx \frac{1 - e^{-b\theta^*}}{e^{a\theta^*} - e^{-b\theta^*}}.$$

   (d) (Assume here that $E(\Delta) \neq 0$.) Using a similar method as in (b) on R_n itself together with the above approximation for $p(a)$, derive the following approximation for $E(\tau)$:
$$E(\tau) \approx \frac{a(1 - e^{-b\theta^*}) - b(e^{a\theta^*} - 1)}{(e^{a\theta^*} - e^{-b\theta^*})E(\Delta)}.$$

   (e) Suppose specifically that $\Delta = S - T$, where S and T are independent and $S \sim exp(\mu)$, $T \sim exp(\lambda)$ and $\lambda < \mu$.
   In this case, find θ^* and then use it (via the expression in (b)) to compute $p(a)$ exactly.
   Also, compute $E(\tau)$ exactly.