For a PP(λ)

$| N(t) = n$,

$$(t_1, \dots t_n) \overset{dist}{=} (U_{(1)}, \dots, U_{(n)})$$

order stats

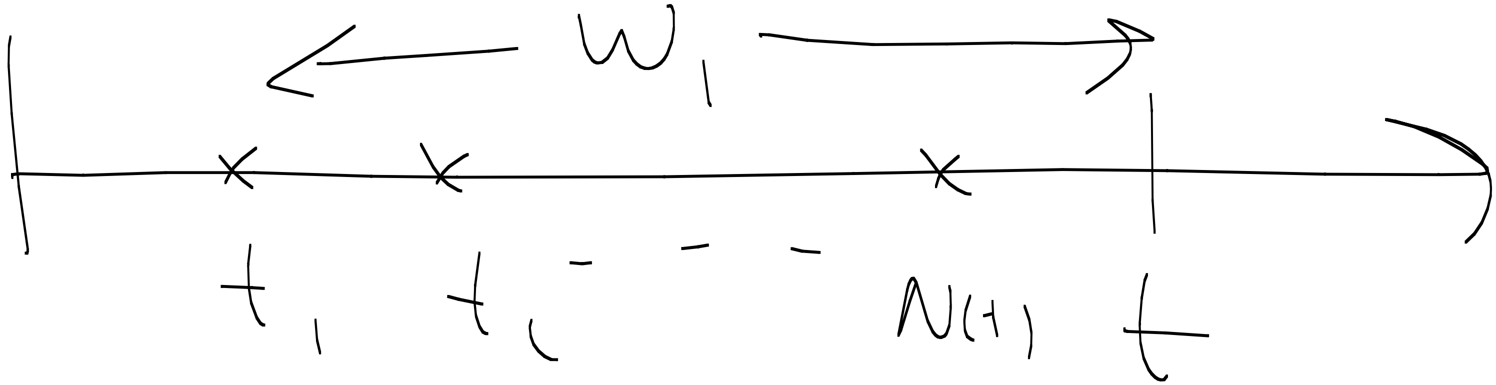
of iid unif$(0,t)$ rvs $U_1, \dots, U_n$

Examples

1) Bus departs a platform at fixed time t.

Passengers arrive (waiting) $\sim PP(\lambda)$

$$W_i = t - t_i$$



ON average, what is $E(W_1)$?

$| N(t) = 1$

$t_1 \sim U_1$

$W_1 = t - t_1 \sim U$

$E(W_1) = E(U) = \frac{t}{2}$

$$| N(t) = 2$$

$$W_1 = t - U_{(1)}$$

$$W_2 = t - U_{(2)}$$

$$\frac{W_1 + W_2}{2} = \frac{2t - (U_1 + U_2)}{2}$$

$$E(W) = t/2$$

$$\sum_{i=1}^{N(t)} t - U_{(i)} = \sum_{i=1}^{N(t)} t - U_i$$

$$E(W) = \frac{t}{2} \checkmark$$

2) $M|G|\infty$ queue

$E(S) = \frac{1}{\mu}$. $G(x) = \mathbb{P}(S \le x)$

iid

Service times $\{S_n : n \ge 1\}$

$\{t_n\} \sim PP(\lambda)$

$\tilde{t}_n = t_n + S_n$ = departure time of n^{th} customer

$L(t) = \#$ in service at t

$L(0) = 0$ $\quad \big(L(t) : t \geq 0 \big)$

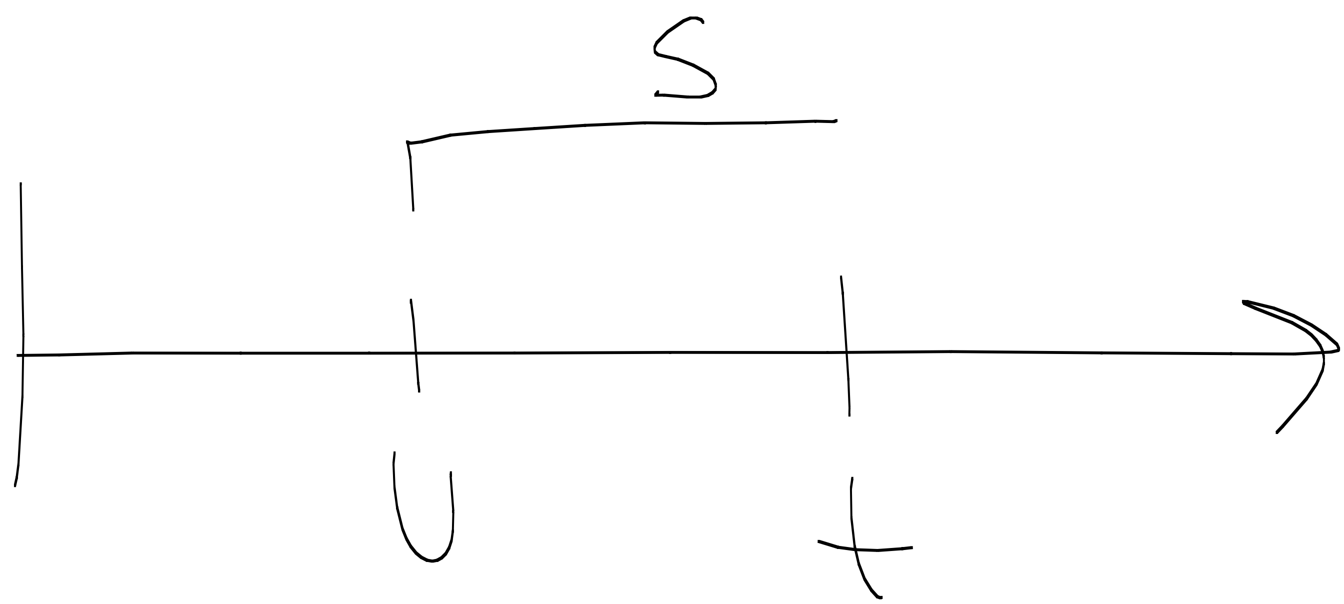
objective

Fix $t > 0$ and figure out the dist. of $L(t)$

$$
\begin{tikzcd}
 & L(t) \sim \text{Poisson} \quad \alpha(t) = \lambda t\, p(t) \\
N(t) \arrow[ur, "p(t)"] \arrow[dr, "1-p(t)"'] & \text{independent} \\
 & D(t) \text{ departures by } t \quad \text{Poisson } \lambda t\,(1-p(t))
\end{tikzcd}
$$

$| \; N(t)$

We treat a "typical" arrival time as $U \sim \text{unif}(0,t)$



$$p(t) = \mathbb{P}(S > t - U) = \mathbb{P}(S > \underset{\uparrow}{U}) = \frac{1}{t}\int_0^t \mathbb{P}(S > u)\,du$$

$$\alpha(t) = \lambda t\, p(t) = \lambda \int_0^t \mathbb{P}(S > u)\, du$$

$$\xrightarrow[t \to \infty]{} \lambda E(S) = \frac{\lambda}{\mu} = \rho$$

$$\mathbb{P}(L(t) = n) = e^{-\alpha(t)} \frac{(\alpha(t))^n}{n!}, \quad n \geq 0$$

$$\xrightarrow[t \to \infty]{} e^{-\rho} \frac{\rho^n}{n!}, \quad n \geq 0$$

" limiting distr. of $(L(t))$ is poisson(ρ) "

$M/M/\infty$ model

$$G(x) = 1 - e^{-\mu x}$$

$$\mathbb{P}(S > u) = e^{-\mu u}$$

$$\lambda \int_0^t \mathbb{P}(S > u)\,du = \rho(1 - e^{-\mu t}) \xrightarrow[t \to \infty]{} \rho$$

$$\mathbb{P}(S>x) = \begin{cases} \frac{2-x}{2}, & 0<x<2 \\ 0, & x \geq 2 \end{cases}$$

$\mu = 1$

$\alpha(t) = \lambda(1-e^{-t})$

$S \sim \text{unif}(0,2) \qquad E(S) = 1 = \mu$

$\rho = \lambda$

$$\alpha(t) = \begin{cases} \lambda \int_0^t \frac{2-x}{x}\, dx & t<2 \\ \lambda \int_0^2 \frac{2-x}{x}\, dx = \rho = \lambda & t \geq 2 \end{cases}$$

$$S \sim \text{unif}(1,2)$$

$$\mathbb{P}(S>x) = \begin{cases} 1, & 0 \le x < 1 \\ 2-x & 1 \le x < 2 \\ 0 & x \ge 2 \end{cases}$$

$$\int_0^t \mathbb{P}(S>x) = \int_0^1 1\,dx + \int_1^t (2-x)\,dx$$

$$1 + \int_1^t (2-x)\,dx \qquad 1 \le t < 2$$

iid Service (S_i)

$X(t) = 2$

pp(X)

$L(t) + I\{S_1 > t\} + I\{S_2 > t\}$

M|M|1 $\quad | L(\infty) = 1$

$| L(\infty) = n$

$$e^{-\rho} \frac{\rho^n}{n!}, \quad n \geq 0$$

$$L(\infty) \stackrel{dist}{=} \lim_{t \to \infty} L(t)$$

$$\mathbb{P}(L(\infty) = n) = \lim_{t \to \infty} \mathbb{P}(L(t) = n)$$

M|M|∞

$\Big|\, L(t) = n$

$(L(t))$ has the Markov Property

CTMC

$(X(t))$ state space discrete

$\mathcal{S} = \mathbb{Z}$ (subset or)

$$\mathbb{P}\big(X(s+t)=j \mid X(s)=i, \{X(u) : 0 \le u < s\}\big)$$

$$= \mathbb{P}\big(X(s+t)=j \mid X(s)=i\big)$$

$$= P_{ij}(t)$$

For each $t > 0$
we have a transition matrix

$$\left(P_{ij}(t) \right)_{i,j \in \mathcal{S}}$$

H_i = holding time in state i $\sim \exp(a_i)$

$E(H_i) = 1/a_i$

$P_{ij} = \mathbb{P}($ Next transition is into j | now in i)

X_n = nth state visited

embedded MC

(discrete-time)

A CTMC $(X(t), t \geq 0)$

is completely determined by

$P = (P_{ij})$ (embedded)

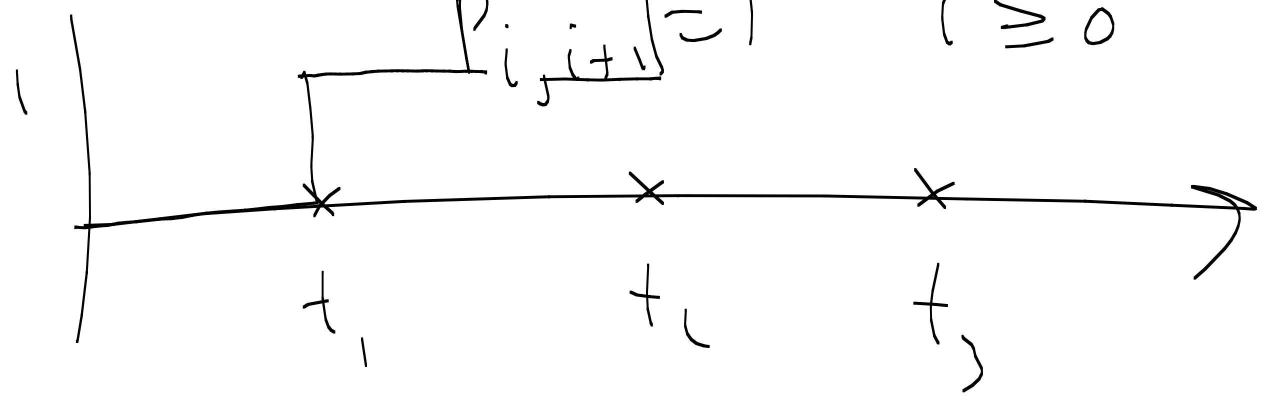
$\{a_i : i \in S\}$ rates

Examples

1) Poisson counting process $(N(t))$

$\mathcal{S} = \mathbb{N}$

$P_{i,i+1} = 1 \quad i \geq 0$

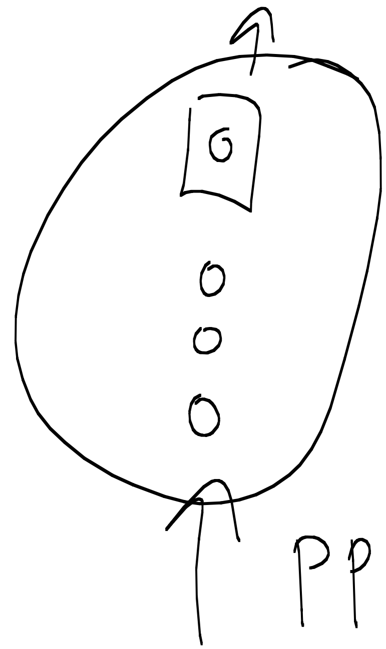


$a_i = \lambda, \quad i \geq 0$

$P_{0,j}(t)$

$= \mathbb{P}(N(t) = j) = e^{-\lambda t} (\lambda t)^j / j!$

2) M|M|1



Service times (S_n) iid $\sim \exp(\mu)$

pp(λ)

$X(t) = \#$ in "system" at t

$S_r \sim \exp(\mu) \mid X(t) = 0)$

$\mid X(t) = 1$

$T \sim$ next interarrival time $\sim \exp(\lambda)$

$\boxed{0}$ S_r

0 S_1

0 S_2

0 S_3

$\uparrow$ $\mathbb{PP}(\lambda)$

$X(t) = y$

$P_{01} = 1$

$P_{1,2} = \mathbb{P}(T < S_r) = \dfrac{\lambda}{\lambda+\mu}$

$\boxed{0}$ $P_{1,0} = \mathbb{P}(S_r < T) = \dfrac{\mu}{\lambda+\mu}$

$P_{n,n+1} = \lambda / \lambda+\mu$

$P_{n,n-1} = \mu / \lambda+\mu$

$n \geq \cancel{0}$

The embedded chain is a simple random walk

$$p = \lambda/(\lambda+\mu), \quad q = \frac{\mu}{\lambda+\mu}$$

$$(p_{0,1} = 1)$$

$$H_0 \sim \exp(\lambda) \quad a_0 = \lambda$$

$$H_1 \sim \exp(\lambda+\mu) \quad a_1 = \lambda+\mu$$

$$= \min(S_r, T) \quad ; \quad a_n = \lambda+\mu \quad n \geq 1$$

$M|M|\infty$

$a_0 = \lambda$

$a_1 = \lambda + \mu$

$L(t) = n \geq 1$

$a_2 = 2\mu + \lambda$

$\vdots$

$(\underbrace{S_{r_1}}, \ldots, S_{r_n})$ iid $\sim \exp(\mu)$

$a_n = n\mu + \lambda$
$n \geq 0$

$T \sim \exp(\lambda)$

$H_2 = \min(\underbrace{S_{r_1}, S_{r_2}, T}_{})$

$\sim \exp(2\mu + \lambda)$

$$p_j = \lim_{t \to \infty} \frac{1}{t} \int_0^t I\{X(s) = j) \, ds$$

wp1

$$p_j = \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{P}(X(s) = j) \, ds$$

Balance equations

"rate out of j = rate into j" (P_j)

$j \geq 0$

M/M/1 $\qquad \rho = \lambda/\mu$

$$\lambda P_0 = \mu P_1$$

$$(\lambda+\mu)P_1 = \lambda P_0 + \mu P_2$$

$$\vdots$$

$$(\lambda+\mu)P_n = \lambda P_{n-1} + \mu P_{n+1}$$

$$\lambda P_n = \mu P_{n+1}$$

$n \geq 0$

$$P_n = \rho^n P_0$$

$n \geq 0$

$$p_n = \rho^n p_0 = \rho^n (1-\rho), \quad n \geq 0$$

$$\sum_{n=0}^{\infty} p_n = 1$$

$$= p_0 (1 + \rho + \rho^2 + \cdots) < \infty$$

$$\Longleftrightarrow \rho < 1 \qquad \boxed{p_0 = 1 - \rho}$$