IEOR 6711, Solutions to Midterm Exam II. Professor K. Sigman

1. (30 points)

   (a) (10 points) Passengers arrive to a train platform according to a Poisson process at rate λ and the train uses an (N, t) policy before departing: For fixed integer $N \geq 1$, the train waits until the N^{th} passenger arrives, and then (from that point) waits an additional t units of time for any more passenger arrivals where $t > 0$ is a fixed constant. Then the train departs taking all passengers. Then independently, the next train uses the same (N, t) policy and so on forever. The cost incurred by the train company is as follows: There is a cost c per passenger per unit of time waiting (e.g., for a passenger who arrives at time s the cost is $c(d - s)$ where d denotes the time at which the train departs). Give an expression for the long-run cost rate that only involves the parameters λ, c and t.

   SOLUTION: Renewal reward theorem: A cycle length X is the length of time it takes for a train to depart. Letting $\{t_n\}$ denote the passenger arrival times, we have $X = t_N + t$. Thus $E(X) = (N/\lambda) + t$. During the t_N part, we can compute the waiting times of each of the N arrivals. Denote these customers by $C_1, \ldots, C_N$, and let $T_n = t_{n+1} - n,\ 1 \leq n \leq N - 1$ denote the interarrival times. C_1 arrives at time t_1 waits T_1 units of time until C_2 arrives and then both of them wait T_2 units of time until C_3 arrives and so on until $N - 1$ of them all wait together during T_{N-1}. So the waiting cost during t_N is

$$c(T_1 + 2T_2 + \cdots + (N-1)T_{N-1}),$$

   and the mean is

$$(c/\lambda)(1 + 2 + \cdots + N - 1) = cN(N-1)/2\lambda.$$

   During the final t time units, all N of the arrivals must wait an additional t time units, hence there is an additional waiting cost of cNt for them. Meanwhile during these t time units there will be $N(t)$ new arrivals, $E(N(t)) = \lambda t$ on average. We can condition on $N(t) = n$ to see that we can treat these new arrive times as iid uniforms over $(0, t)$ and hence each arrival on average waits $t/2$. (Since we are summing up *all* $N(t)$ of them, the order does not matter; the sum of the order statistics is the same as the sum of the non-ordered ones.) Hence the additional cost is $c\lambda t \times t/2 = c\lambda t^2/2$. So the total expected cost over the cycle is

$$E(R) = cN(N-1)/2\lambda + cNt + c\lambda t^2/2.$$

$$E(R)/E(X) = \frac{cN(N-1)/2\lambda + cNt + c\lambda t^2/2}{t + N/\lambda}.$$

   (b) (20 points) Consider a positive recurrent renewal process with interarrival time distribution $F(x) = P(X \leq x),\ x \geq 0$, and arrival rate $0 < \lambda < \infty$. Let $S^* = X_s$ denote a rv distributed as the limiting distribution for *spread*.

      i. (10 points) Do *not* assume that F has a density. Prove that $E(1/S^*) = \lambda$.
      SOLUTION: The spread process, $\{S(t)\}$, is positive recurrent regenerative and so we can use the general result that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(S(t))ds = E(f(S^*)),\ wp1.$$

Here $f(x) = 1/x$, and we will compute the left hand side via the renewal reward theorem (or basic regenerative process results): Letting $X = X_1$, the first cycle, $R = \int_0^X f(S(s))ds = \int_0^X (1/X)ds = 1$. Thus $E(R)/E(X) = E(X)^{-1} = \lambda$.

ii. (10 points) Prove that if $P(X > x) > 0,\ x \geq 0$, then it always holds that

$$\lim_{x\to\infty} \frac{P(S^* > x)}{P(X > x)} = \infty,$$

that is, S^* always has a heavier tail than X (its tail tends to 0 slower).

SOLUTION: We recall the formula (Lecture Notes on Renewal Theory I)

$$P(S^* > x) = \lambda x \overline{F}(x) + \overline{F}_e(x).$$

Thus

$$\frac{P(S^* > x)}{\overline{F}(x)} = \lambda x + \frac{\overline{F}_e(x)}{\overline{F}(x)} \geq \lambda x \to \infty.$$

2. (20 points) Let $\psi = \{t_n : n \geq 1\}$ denote a non-delayed positive recurrent renewal process at rate λ with iid interarrival times distributed as $F(x)$. Let $\psi^* = \{t_n^* : n \geq 1\}$ denote a time-stationary version. In particular, we know that $m^*(t) \stackrel{\text{def}}{=} E(N^*(t)) = \lambda t,\ t \geq 0$.

(a) (10 points) Set up a renewal equation for $H_0(t) = P(A(t) > x)$ (for the non-delayed version) and set up another one for $H(t) = P(A^*(t) > x)$ (the time-stationary version).

SOLUTION: Recalling that $t_1 \sim F$ and $t_1^* \sim F_e$, we condition on the first renewal:

$$\begin{aligned} P(A(t) > x) &= P(A(t) > x; t_1 > t) + P(A(t) > x; t_1 \leq t) && (1)\\ &= P(t_1 - t > x) + \int_0^t P(A(t-s) > x)dF(s) && (2)\\ &= \overline{F}(x+t) + \int_0^t P(A(t-s) > x)dF(s). && (3) \end{aligned}$$

Similarly

$$\begin{aligned} P(A^*(t) > x) &= P(A^*(t) > x; t_1^* > t) + P(A^*(t) > x; t_1^* \leq t) && (4)\\ &= P(t_1^* - t > x) + \int_0^t P(A(t-s) > x)dF_e(s) && (5)\\ &= \overline{F}_e(x+t) + \int_0^t P(A(t-s) > x)dF_e(s). && (6) \end{aligned}$$

The $A(t-s)$ piece in the two integrals (3) and (6) are the same non-delayed ones.

(b) (10 points) Write out the form of the solution for $H_0(t)$ and then use it to prove that the solution to the renewal equation for $H(t)$ is $\overline{F}_e(x)$.

SOLUTION: The non-delayed case can be written as $H_0(t) = Q_0(t) + (H_0 \star F)(t)$ where $Q_0(t) = \overline{F}(x+t)$.

The delayed case can be written as $H(t) = Q(t) + (H_0 \star F_e)(t)$ where $Q(t) = \overline{F}_e(x+t)$.

The solution to the non-delayed case is $H_0(t) = Q_0(t) + (Q_0 \star m)(t)$, where $m(t) = E(N(t)) = \sum_{j=1}^{\infty} F^{(*j)}(t)$. Plugging this solution for H_0 into $H(t) = Q(t) + (H_0 \star F_e)(t)$ yields solution $H(t) = Q(t) + (Q_0 \star m^*)(t)$, or

$$H(t) = Q(t) + \int_0^t Q_0(t-s)dm^*(s),$$

where we are using the fact that
$m^*(t) = (F_e + F_e * m)(t) = \sum_{j=0}^{\infty} F_e * F^{(*j)}(t) = E(N^*(t))$. But $E(N^*(t)) = \lambda t$, and so $\frac{d}{dt}m^*(t) = \lambda$ or $dm^*(s) = \lambda ds$; the solution is thus

$$\begin{aligned} H(t) &= \overline{F}_e(x+t) + \int_0^t \overline{F}(x+t-s)\lambda ds && (7)\\ &= \overline{F}_e(x+t) + \lambda \int_x^t \overline{F}(u)\lambda du && (8)\\ &= \overline{F}_e(x+t) + \overline{F}_e(x) - \overline{F}_e(x+t) && (9)\\ &= \overline{F}_e(x). && (10) \end{aligned}$$

3. (20 points) Consider a stochastic process $\{X(t) : t \geq 0\}$ with a *finite* state space $\mathcal{S} \subset \mathbb{Z}$, a finite subset of the integers. Assume that it changes state according to an irreducible discrete-time Markov chain with transition matrix $P = (P_{i,j})$ ($P_{i,i} = 0,\ i \in \mathcal{S}$ for simplicity). Assume further that whenever it enters state $i \in \mathbb{Z}$, it remains there, independent of the past, for a positive amount of time H_i distributed as F_i (some general distribution, not necessarily exponential). Assume that each F_i has finite first moment $0 < E(H_i) = 1/a_i < \infty$. Let

$$P_i \stackrel{\text{def}}{=} \lim_{t\to\infty} \frac{1}{t}\int_0^t P(X(s) = i)ds,\ i \in \mathcal{S}, \qquad (11)$$

denote the limiting distribution.

(a) (10 points) Prove that $\{P_i\}$ exists (e.g., $P_i > 0$ and $\sum_i P_i = 1$).
SOLUTION: If we fix a state i, we can use consecutive visits to state i as regeneration points. We need to show positive recurrence though. First observe that the embedded discrete-time Markov chain $\{X_n\}$ (say) is positive recurrent since it is an irreducible finite state space chain; thus letting $\tau = \tau_{ii}$ the discrete return time to state i, we have $E(\tau) < \infty$. Meanwhile, letting $T = T_{ii}$ denote the first (continuous) return time to state i, given we started with $X(0) = i$ (and starting with a new H_i), we have

$$T = H_i + \sum_{n=1}^{\tau} H_{X_n}.$$

Now let $b = \min\{a_i : i \in \mathcal{S}\}$. Then $E(H_i) \leq 1/b,\ i \in \mathcal{S}$. So, for example, $E(T \mid \tau = k) \leq 1/a_i + k/b$, and thus, more generally, $E(T \mid \tau) \leq 1/a + \tau/b$; thus $E(T) \leq 1/a + E(\tau)/b < \infty$. Positive recurrence.

(b) (10 points) Let $\{Y(t) : t \geq 0\}$ denote the same process (same transition matrix P) except with exponential holding times with the same rates a_i; $F_i(t) = 1 - e^{-a_i t}$, the CTMC case. Show that the limiting distribution for $\{X(t)\}$ given in (12) must be the same as the solution to the balance equations for $\{Y(t)\}$. In other words the limiting distribution is insensitive to the distributions F_i except through their means.

SOLUTION:

Fix i. Assume that $X(0) = i$ and a new H_i is started. P_i is the long-run proportion of time spent in state i, and from renewal reward can be expressed as

$$P_i = (1/a_i)/E(T_{ii}). \tag{12}$$

Let $T = T_{i,i}$, and $\tau = \tau_{i,i}$. Then

$$T = H_i + \sum_{n=1}^{\tau} H_{X_n}.$$

To compute $E(T)$ we observe that each time state $j \neq i$ is visited, the process spends on average $1/a_j$ amount of time and then moves to another state. Thus if we let $R(j) = \sum_{n=1}^{\tau} I\{X_n = j\}$, the total number of visits to j during T, $(R(i) = 1)$, then

$$E(T) = \sum_{j \in \mathcal{S}} E(R(j))/a_j.$$

We need to compute $E(R(j))$. Since $\{X_n\}$ is positive recurrent it has a unique stationary distribution, the solution to $\pi = \pi P$. Now $\pi_i = 1/E(\tau_{ii})$ and visits to state i are positive recurrent regeneration points for $\{X_n\}$. From the renewal reward theorem we can express each π_j as "the expected number of visits to state j between visits to state i"; $\pi_j = E(R(j))/E(\tau_{ii}) = \pi_i E(R(j))$. Thus $E(R(j)) = \pi_j/\pi_i,\ i \in \mathcal{S}$. We conclude from (12) that

$$P_i = \frac{\frac{\pi_i}{a_i}}{\sum_j \frac{\pi_j}{a_j}}.$$

These (P_i) only depend on the (π_i) and the (a_i), and must be the solution to the balance equations since such limiting distributions are unique.