A simple stochastic model for close U.S. presidential elections

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Abstract

In the context of a close U.S. presidential election, we present a probability model for computing such things as the probability that one presidential candidate wins the popular vote while losing the election (due to losing the Electoral College vote), and the probability of a *tie* (both candidates receive the same number of Electoral College votes). Computations can easily be carried out by Monte Carlo simulation, and yield, for example, the total number of ways that a tie in the Electoral College can occur (approximately 18 trillion). As an application, we illustrate how to use the model with polling data to handle a given specific presidental election.

1 Introduction

As laid down in the U.S. Constitution, the method by which a president wins an election is determined not by the national popular vote, but by the Electoral College. Each state is assigned a number of Electors equal to their total number of members in the U.S. Senate (2) and the U.S. House (this depends on the state's population). The candidate receiving the majority[1] of Electoral votes wins the election, and if there is a tie, then the final decision is determined by a vote in the U.S. House of Representatives (one vote per state). The Constitution gives states substantial power in determining how their Electors are chosen, and starting sometime in the 1820's, states began to use the "winner takes all" method: a candidate who wins the highest proportion of the popular vote in a state receives all the Electoral votes from that state. While this has reduced the chance of a tie, there have been some very interesting elections in which a candidate, while winning the popular vote, lost the election by losing the Electoral vote.[2] Motivated by this, we present a probability model for close U.S. Presidential elections so as to compute such things as the probability that one presidential candidate wins the popular vote while losing the election. Throughout, we use Monte Carlo simulation to obtain the results. By "close" we mean (as is usually meant by the media) the case when the national popular vote for each of the leading candidates is almost identical.

Our main model is presented in Section 2, where we use the current values for each state's number of Electors (yielding the current total of 538 in the Electoral College), and thus a candidate must collect at least 270 Electoral votes to win. As a first go, we consider a symmetric case in Section 3 and carry out some computations which yield the number of ways there can be a tie. Then in Sections 4 and 5, we show how to modify the model by utilizing polling data from each state and transforming the race into a close one if it is not already so.

[1] At least half of the total: Even when there are three (or more) strong candidates, to win the election a candidate must receive more than half of the Electoral votes; otherwise the election gets sent to the US House of Representatives: In 1824, John Quincy Adams won only 32% of the popular vote and only 84 out of a total of 261 Electoral votes, but was elected president by the House of Representatives. This was so even though Andrew Jackson won 42% of the popular vote and 99 Electoral votes.

[2] In 1876, Rutherford B. Hayes (Republican) became President by beating Samuel J. Tilden (Democrat) by only one Electoral vote (185 versus 184), while losing the popular vote by 3% (48% versus 51%). In 1888, Benjamin Harrison (Republican) became President by beating incumbent Grover Cleveland (Democrat) by 65 Electoral votes (233 versus 168), while losing the popular vote by 1% (49% versus 48%). In 1960, this almost happened again when John F. Kennedy (Democrat) beat Richard M. Nixon (Republican) by 84 Electoral votes even though the popular vote was almost identical (49.7% versus 49.5%). See for example [1].

2 Main model

For all $i \in \{1, 2, \ldots, 51\}$, we let n_i denote the number of Electors in state i (including the District of Columbia);

$$\sum_{i=1}^{51} n_i = 538,$$

a full list is given in Table 1, Column 2.

We consider for simplicity, a two-candidate race (candidates 1,2), and let $P_i \in [0, 1]$ denote the popular vote for candidate 1 in state i ($100P_i$ thus represents the percentage). Candidate 1 wins if and only if receiving at least 270 Electoral votes; a tie being the case when both candidates receive 269.

Since the number of Electors of a state is equal to the number of its members in the U.S. Congress[3] and since the number of members in the House of Representatives is proportional to the states' population, we conclude (by subtracting the 2 Senators from each n_i) that the national popular vote for candidate 1 can be represented by a weighted average[4]

$$V_1 = \frac{\sum_{i=1}^{51}(n_i - 2)P_i}{436}. \tag{2.1}$$

Since the winner takes all in each state[5], the total number of Electoral College votes won by candidate 1 is given by

$$C_1 = \sum_{i=1}^{51} n_i I\{P_i > 0.5\}, \tag{2.2}$$

where $I\{A\}$, the *indicator* for the event A, is defined to be equal to 1 if the event A ocurrs, 0 if not. (For candidate 2, $V_2 = 1 - V_1$ and $C_2 = 538 - C_1$.)

Apriori we do not know the values of the P_i, so we treat them as random variables (rvs). Of intrinsic interest then is to compute such things as

$$E(V_1), \tag{2.3}$$

the *expected popular vote*,

$$E(C_1) \tag{2.4}$$

the *expected Electoral vote*,

$$P(V_1 > 0.5,\ C_1 < C_2), \tag{2.5}$$

[3]Except the District of Columbia which is assigned 3 Electors anyhow.

[4]In the 1996 Presidentail race, for example, the difference between this weighted average and the unweighted popular vote was 1.5 hundredth of one percent.

[5]Except in Maine (4 Electors) and Nebraska (5 Electors) where the winner gets 2 Electors with the remaining selected by popular vote within each Congressional district. We ignore this distincton in our model.

the *probability of winning the popular vote but losing the election*, and

$$P(C_1 = C_2), \tag{2.6}$$

the *probability of a tie*.

3 Uniform symmetric case

If we have no prior knowledge about the candidates then a reasonable first approximation is to assume that each P_i is an independent r.v. U_i uniformly distributed over the unit interval; $P(U_i \le x) = x,\ x \in [0, 1]$. Noting that $1 - U_i$ is also uniformly distributed over the unit interval, it follows by symmetry that V_1 and V_2 are identically distributed, as are C_1 and C_2; in particular $E(V_1) = E(V_2) = 0.5$ and $E(C_1) = E(C_2) = 269$.

Simulation yields:

$$P(V_1 > 0.5,\ C_1 < C_2) = 0.081 \tag{3.7}$$

$$P(C_1 = C_2) = 0.008. \tag{3.8}$$

By symmetry $P(C_1 > C_2) = P(C_2 > C_1) = 0.5 - P(C_1 = C_2)/2$
so that we also get the conditional probabilities

$$P(V_1 > 0.5 \mid C_1 < C_2) = 0.163. \tag{3.9}$$

$$P(C_1 < C_2 \mid 0.5 < V_1 \le 0.51) = 0.439. \tag{3.10}$$

(3.10) is quite interesting for it means that if a candidate wins the popular vote by at most a margin of 1%, then there is about a 44% chance of losing the election.

The number of ways there can be a tie in the Electoral College

Let us note in passing that from the fact that $P(C_1 = C_2) = 0.008$, we can compute the number of ways that there can be a tie in the Electoral College: Let N_T denote this number. Observe that there are 2^{51} ($\approx 2.25 \times 10^{15}$) ways that the states can be divided into two sets. Under our uniform symmetric assumption, each such way is equally likely to be chosen; thus

$$P(C_1 = C_2) = \frac{N_T}{2^{51}}.$$

Solving for N_T yields

$$N_T \approx 1.8 \times 10^{13}, \tag{3.11}$$

18 trillion.

4 Modifications for a particular race

Here we will replace the uniform U_i by (truncated) normally distributed rvs with mean and variance determined by each states' polling data. p_i denotes the estimate for candidate 1's popular vote in state i, and σ_i^2 the variance.[6]

We then let X_i , $1 \leq i \leq 51$, be independent normally distributed rvs, with mean p_i and variance σ_i^2 respectively. To keep the proportions within $[0, 1]$, we truncate[7] by defining

$$Y_i = \begin{cases} X_i; & \text{if } 0 < X_i < 1 \\ 0, & \text{if } X_i \leq 0 \\ 1, & \text{if } X_i \geq 1. \end{cases}$$

Then

$$V_1 = \frac{\sum_{i=1}^{51}(n_i - 2)Y_i}{436} \tag{4.12}$$

$$C_1 = \sum_{i=1}^{51} n_i I\{Y_i > 0.5\} \tag{4.13}$$

$$C_2 = 538 - C_1. \tag{4.14}$$

5 Transformation to a close race: A dead heat

Consider the case when $E(V_1) > 0.5$, candidate 1 has a popular vote edge. (Use V_2 in what follows if instead $E(V_2) > 0.5$.) Let $b = E(V_1) - 0.50$. Now replace the polling means p_i by $p_i - b$ and re-define the X_i to have these new means (but the same variances as before). Define the truncated Y_i as before using the new X_i. Do all the computations over again.

The idea here is to make the race close by taking away a candidate's national popular edge, and then see which is predicted to win the Electoral College. This can show, as is the case as on November 4, 2000 (G.W. Bush versus Al Gore), that one candidate (Gore in this case) has as Electoral College edge when there is a dead heat in the popular vote.

5.1 Further modification: finding the critical b-value

One can modify even further by icrementally increasing the value of b from Section 5 by (say) tenths of a percent (or smaller) - further reducing Candidates 1's lead little by little

[6] In practice, when no strong third party candidate is present, we renormalize the p_i so that the two candidate's estimates sum to 1, but we then reduce the sample size of each state by the proportion of third party and undecided voters in that state (hence increasing the sample variance).

[7] In a close race in which the variances are small, truncation is not really needed, and V_1 has a normal distribution since it is a weighted sum of independent normals (so simulation is not needed). But even then, the joint distribution of V_1 and C_1 is not normally distributed; hence our use of simulation instead of a direct calculation.

- until the winner is the other candidate. For each value of b, redo the computations and graphs all the results with b on the $X-$axis, and $E(C_2)$ (or $P(C_2 > C_1)$) on the $Y-$ axis. Find the smallest value of b for which $E(V_2) > 269$, and $P(C_2 > C_1) > P(C_1 > C_2)$, for example.

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References

[1] *The Complete Book of U.S. Presidents.* W. A. Degregorio (1993), 4th ed., Barricade Books INC., New York.