1 Interest rates, and risk-free investments

1.1 Interest and compounded interest

Suppose that you place x_0 (\$) in an account that offers a fixed (never to change over time) annual *interest rate* of $r > 0$ (\$ per year). x_0 is called the *principle*, and one year later at time $t = 1$ you will have the amount $y_1 = x_0(1 + r)$ (your payoff) having made a profit of rx_0. Everything here is deterministic, in that the amount y_1 is not random; there is no uncertainty – hence no risk – associated with this investment. This is an example of a *risk-free investment*[1]. Formally, the variance of the payoff is zero; $Var(y_1) = 0$. If you reinvest y_1 for another year, then at time $t = 2$ you would have the amount $y_2 = y_1(1+r) = x_0(1+r)^2$. In general, if you keep doing this year after year, then at time $t = n$ years, your payoff would be $y_n = x_0(1+r)^n$, $n \geq 0$.

Reinvesting the payoff at the end of one time period for yet another is known as *compounding* the interest. In our example interest was compounded annually, but compounding could be done biannually, quarterly, monthly, daily and so on. For example, if done biannually, then after a half a year, at time $t = 0.5$, we would earn interest at rate $0.5r$ yielding $x_0(1 + 0.5r)$. Then a half a year later at time $t = 1$ interest would be earned again at rate $0.5r$ on the amount $x_0(1 + 0.5r)$ finally yielding a payoff at time $t = 1$ of the amount

$$
\begin{aligned}
y_1 &= x_0(1 + 0.5r)(1 + 0.5r) \\
&= x_0(1 + 0.5r)^2 \\
&= x_0(1 + 0.25r^2 + r) \\
&> x_0(1 + r).
\end{aligned}
$$

Thus as expected, biannual compounding is more profitable than annual compounding, and as is easily seen, the profit increases as the time period for compounding get smaller. For example, compounding quarterly yields payoff $x_0(1 + 0.25r)^4 > x_0(1 + 0.5r)^2$.

In general, if the time period is of size $1/k$ ($k > 0$ an integer), compounding takes place k times per year at a rate r/k per period and yields a payoff at time $t = 1$ of the amount

$$
y_1(k) = x_0(1 + \frac{r}{k})^k. \tag{1}
$$

If this payoff is reinvested for yet another year, then the payoff at time $t = 2$ is $x_0(1 + \frac{r}{k})^k(1 + \frac{r}{k})^k = x_0(1 + \frac{r}{k})^{2k}$ and in general, after n years yields $x_0(1 + \frac{r}{k})^{nk}$. This is important for monthly compounding in which the monthly rate is $r/12$ and the payoff after n years is $x_0(1 + \frac{r}{12})^{12n}$.

Over an arbitrary interval of time, $(0, t]$, compounding a total of k times would mean compounding every t/k units of time at rate rt/k and yielding a payoff at time t of

$$
y_t(k) = x_0(1 + \frac{rt}{k})^k. \tag{2}
$$

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[1]Investing in stocks on the other hand is an example of a *risky investment* because the price of stocks evolves over time in a random way, apriori unknown to us and not possible to predict with certainty. The stock price at a later time thus has a non-zero variance. As with gambling, when investing in stock there is even a risk of losing some or all of your principle.

Letting the compounding interval get smaller and smaller is equivalent to letting $k \to \infty$, in which case the compounding becomes *continuous compounding* yielding a payoff at time t of

$$\lim_{k \to \infty} y_t(k) = \lim_{k \to \infty} x_0(1 + \frac{rt}{k})^k$$
$$= x_0 e^{rt}, \tag{3}$$

where l'Hospital's rule is used to show that $(1 + \frac{rt}{k})^k \to e^{rt}$ as follows: First take the natural logarithm so that equivalently, we need to show that as $k \to \infty$,

$$\frac{\ln(1 + \frac{rt}{k})}{\frac{1}{k}} = rt.$$

Taking the derivative wrt k of numerator and denominator yields (after some cancellation)

$$rt(1 + \frac{rt}{k})^{-1},$$

which indeed converges to rt as $k \to \infty$.

Thus:

An initial investment of x_0 at time $t = 0$, under continuous compounded interest at rate r, is worth $x_0 e^{rt}$ at time $t \geq 0$.

1.2 Doubling your money

If the annual interest rate is r, and you invest x_0 under continuous compounding, then how long will you have to wait until you have doubled your money?

We wish to find the value of t (years) for which $x_0 e^{rt} = 2x_0$, or, equivalently, for which $e^{rt} = 2$. Solving we conclude $t = \ln(2)/r = (.6931)/r$. For example, if $r = 0.04$, then $t \approx 17$ years.

Notice how then, we could double yet again in another 17 years and so on, concluding that x_0 would increase by a factor of 4 in 34 years; a factor of 8 in 51 years and so on. This is simply an example of *geometric* growth, and corresponds to the steep rise in the graph of the function $f(t) = e^{rt}$, $t \geq 0$.

1.3 Effective and nominal interest rates

The annual rate r discussed previously is called the *nominal* interest rate. As we have seen, by compounding we can do better, and this motivates computing the *effective* interest rate, that is, the rate required without compounding to yield the same payment, at the end of one year, with compounding at rate r. For example, if compounding is done biannually, then the effective rate is the solution r' to the equation $1 + r' = (1 + 0.5r)^2$, or $r' = r + 0.25r^2$. So if $r = 0.07$, then $r' = 0.0712$.

In general, when compounding k times per year, it is the solution r' to $1 + r' = (1 + r/k)^k$, or $r' = (1 + r/k)^k - 1$, and when compounding continuously it is the solution to $1 + r' = e^r$, or $r' = e^r - 1$. For example if $r = 0.07$ and the compounding is monthly, then $r' = (1 + (0.07)/12)^{12} - 1 = 0.0723$. If compounding is continuous, then $r' = e^{0.07} - 1 = 0.0725$.

1.4 Present value and the value of money

Any given amount of money, x, is worth more to us at present (time $t = 0$) than being promised it at some time t in the future since we can always use x now as principle in a risk-free investment at rate r guaranteeing the amount $x(1+e^{rt}) > x$ at time t (we are using continuous compounding here for illustrative purposes). Stated differently, an amount x promised to us later at time t is only worth $x_0 = xe^{-rt} < x$ at present, since using this x_0 as principle yields exactly the amount x at time t: $x_0 e^{rt} = xe^{-rt}e^{rt} = x$. We thus call xe^{-rt} the *present value* (PV) of x. It is less than x and represents the value to us now of being promised x at a later time t in the future, if the interest rate is r. Sometimes this is referred to as *discounting* the amount x by the discount rate r, and the factor (always less than 1) by which we multiply x to obtain its present value is called the *discount factor*. Under the continuous compounding assumed above, the discount factor is e^{-rt}. As an immediate application, consider buying a certain kind of *Certificate of Deposit (CD)* from your bank. You pay now for this which ensures you a payoff of (say) $10,000$ two years later, using fixed interest rate $r = 0.05$. This means that you buy the CD now at a price equal to the present value $10,000e^{-(0.05)2} = 9,048$. Interest rates generally change over time and different interest rates are used for different types of investments, but a CD is an example where the rate is fixed over a period of time agreed upon in the contract (6 months, 1 year, 2 years, 5 years, etc.). Other such examples are treasury bonds purchased from the USA government. Your regular bank account, checking or savings, on the other hand, has an interest rate that changes over long periods of time. Notice how if after you buy your CD, a week later the bank's interest rate goes up from your $r = 0.05$ to 0.060, then the price of the same CD bought a week later goes down to $10,000e^{-(0.060)2} = 8,869$, and your CD has lost value. Although a CD is an example of you lending money to the bank, present value has an analogous meaning in the context of borrowing: if you borrow x_0 now, then you must pay back $x = x_0 e^{rt} > x_0$ at time t.

Present value can of course be computed whatever the compounding situation is. For example if there is no compounding then the present value of x at time $t = 1$ is $x/(1+r)$, and so the discount factor is $1/(1+r)$. In general when compounding k times during $(0, t]$ at annual rate r the present value of x at time t is $x/(1 + rt/k)^k$, and so the discount factor is $1/(1 + rt/k)^k$.

Examples for PV

1. You are to receive $10,000$ one year from now and another $10,000$ two years from now. If the annual nomimal interest rate is $r = 0.05$, compounded yearly, what is the present value of the total? What if compounded monthly? Continuously?

$$PV = \frac{10,000}{(1+r)} + \frac{10,000}{(1+r)^2},$$

derived as follows: $10,000(1+r)^{-1}$ is the present value of the first payment (to be received at time $t = 1$ year), and $10,000(1+r)^{-2}$ is the present value of the second payment (to be received at time $t = 2$ years). Plugging in $r = 0.05$ yields $PV = 18,594$.

If compounded monthly, then

$$PV = \frac{10,000}{(1+r/12)^{12}} + \frac{10,000}{(1+r/12)^{24}}.$$

Plugging in $r = 0.05$ yields $PV = 18,564$.

If compounded continuously, then

$$PV = 10,000e^{-r} + 10,000e^{-2r}.$$

Plugging in $r = 0.05$ yields $PV = 18,561$.

2. Suppose today is January 1, and you move into a new apartment in which you must pay a monthly rent of $1000 on the first of every month, starting January 1. You plan to move out after exactly one year. Then at the end of your year you will have payed a total of $12,000$. But suppose that you want to pay off all this 12 month future debt now on January 1. How much should you pay assuming that the nominal annual interest rate is $r = 0.06$ and interest will be compounded monthly? What if $r = 0.10$? Repeat when compounding is continuous. We need to compute the present value of the 12 month debt, where the monthly interest rate is $r/12$:

$$
\begin{aligned}
PV &= 1000\Big(1 + \sum_{i=1}^{11} \frac{1}{(1+r/12)^i}\Big) \\
&= 1000\Big(\sum_{i=0}^{11} \frac{1}{(1+r/12)^i}\Big),
\end{aligned}
$$

derived as follows:

You must pay 1000 on January 1, followed by 1000 on the first of every month for the next 11 months. On February 1, the 1000 to be payed has present value $1000/(1+r/12)$ since this is one month in the future. On March 1, the 1000 to be payed has present value $1000/(1+r/12)^2$ since this is two months in the future. Continuing in this fashion we finally get to the last monthly payment on December 1 (11 months from now) with present value $1000/(1+r/12)^{11}$. Summing up all 12 present values yields PV.

In effect, you could take the amount $PV < 12000$, pay your landlord 1000 from it now (January 1), then place the rest, $PV - 1000$, in the bank at interest rate r (compounded monthly): At the beginning of each month you would withdraw 1000 and pay your landlord. Then on December 1 there would be exactly 1000 left in the account, you would pay this last month's rent to your landlord and close the account.

Recalling the geometric series formula

$$\sum_{i=0}^{n} a^i = \frac{1 - a^{n+1}}{1 - a},$$

we can simplify our formula for PV by letting $a = 1/(1+r/12)$, $n = 11$;

$$
\begin{aligned}
PV &= 1000\Big(\frac{1 - \frac{1}{(1+r/12)^{12}}}{1 - \frac{1}{(1+r/12)}}\Big) \\
&= 1000\Big(\frac{1 - \frac{1}{(1+r/12)^{12}}}{\frac{r/12}{(1+r/12)}}\Big) \\
&= 1000\Big(\frac{12}{r} + 1\Big)\Big(1 - \frac{1}{(1+r/12)^{12}}\Big).
\end{aligned}
$$

Plugging in $r = 0.06$ yields $1000(201)(.05809) = 11,677$.

If $r = 0.10$ this amount drops to $1000(121)(.07664) = 11469$.

When compounding is continuous, 1000 at time i months from now has present value $1000e^{rt}$ with $t = i/12$; $1000e^{-\frac{ri}{12}}$. Thus

$$
\begin{aligned}
PV &= 1000\Big(\sum_{i=0}^{11} e^{-\frac{ri}{12}}\Big) \\
&= 1000\Big(\frac{1 - e^{-r}}{1 - e^{-\frac{r}{12}}}\Big).
\end{aligned}
$$

Plugging in $r = 0.06$ yields $PV = 11,676$. Plugging in $r = 0.10$ yields $PV = 11,467$.

3. You win a very special lottery which agrees to pay you $100 each year forever starting today. Even after you die, the payments will continue to your relatives/friends. Assuming an interest rate of $r = 0.08$ compounded annually (and assumed constant forever), what is the present value of this lottery? What if compounded continuously?

For any interest rate r compounded annually, the present value of this lottery is given by the infinite sum

$$
\begin{aligned}
PV &= \sum_{i=0}^{\infty} \frac{100}{(1+r)^i} \\
&= 100(1 + 1/r).
\end{aligned}
$$

via the sum of a geometric series (see The Appendix.) Thus $PV = 100/(0.08) + 100 = 1350$.

The point here is that $100 to be earned i years from now has a present value of only $100/(1 + r)^i$. (Note how the present value of the $100 becomes worthless as i tends to infinity.)

This means that if you are given the amount $x_0 = 1350$ now all at once, you could arrange to receive the $100 lottery amounts each year by simply placing x_0 in an account earning interest at rate $r = 0.08$ and withdrawing $100 each year. The account would never go broke.

Under continuous compounding

$$
\begin{aligned}
PV &= \sum_{i=0}^{\infty} 100e^{-ri} \\
&= 100/(1 - e^{-r}),
\end{aligned}
$$

via the sum of a geometric series. Thus $PV = 100/(1 - e^{-0.08}) = 1300$.

Contracts like this with a *forever into the future* payment scheme are called *perpetual* annuities. For obvious reasons, such contracts are not common.

1.5   Net present values

Assessing the present value of an investment, for purposes of comparison with another investment, involves allowing for negative amounts (money spent). In this case we call the present value the *net present value (NPV)*. The investment with the largest NPV (but of course positive) is the better investment using this criterion. Letting x_i denote the money earned/spent at year i yields a vector $(x_0, x_1, x_2, \ldots, x_n)$ representing the investment up to time $t = n$, and its NPV (when interest is compounded yearly at rate r) is given by

$$NPV = \sum_{i=0}^{n} x_i (1+r)^{-i}.$$

For example $(-5, 1, 3, 0, -1)$ means you spent 5 initially (at time $t = 0$) then earned 1 at $t = 1$, 3 at $t = 2$, nothing at $t = 4$ and spent 1 at $t = 5$. The NPV for this is $-5 + (1+r)^{-1} + 3(1+r)^{-2} - (1+r)^{-4}$.

Examples for NPV

1. (*Opening a winery:*) Suppose you plan to open your own winery. An initial investment of 1 (millions of dollars), is required. There are two possibilities to choose from: wait 2 years later to sell your first batch of wine earning 3, or earn 5 by waiting until 4 years later (aging the wine makes it more valuable). Annual interest rate is $r = .10$ compounded yearly. What should you do?

   $1 + r = 1.1$, and thus $(-1, 0, 3)$ versus $(-1, 0, 0, 0, 5)$ in terms of NPV yields:

   $NPV_1 = -1 + 3/(1.1)^2 = 1.479$.

   $NPV_2 = -1 + 5/(1.1)^4 = 2.41$. Thus it is better to wait 4 years, since this has a higher NPV.

APPENDIX: Geometric series

For any number a, and any integer $n \geq 0$,

$$\sum_{i=0}^{n} a^i = \frac{1 - a^{n+1}}{1 - a}, \tag{4}$$

as is easily derived via algebra; $(1 + a + a^2 + \cdots + a^n)(1 - a) = 1 - a^{n+1}$.
    If $|a| < 1$, then $\lim_{n \to \infty} a^n = 0$, thus

$$\sum_{i=0}^{\infty} a^i = \frac{1}{1 - a}. \tag{5}$$

In our interest rate examples, $a = 1/(1+r)$ in which case $\frac{1}{1-a} = 1 + 1/r$, yielding

$$\sum_{i=0}^{n} \frac{1}{(1+r)^i} = \left(1 + \frac{1}{r}\right)\left(1 - \frac{1}{(1+r)^{n+1}}\right), \tag{6}$$

and

$$\sum_{i=0}^{\infty} \frac{1}{(1+r)^i} = 1 + \frac{1}{r}. \tag{7}$$

6

In many applications, n starts at $n = 1$ instead of $n = 0$, the difference being

$$\sum_{i=1}^{n} a^i = -1 + \sum_{i=0}^{n} a^i.$$

This leads to the modified formulas

$$\sum_{i=1}^{n} \frac{1}{(1+r)^i} = \frac{1}{r}\left(1 - \frac{1}{(1+r)^n}\right), \tag{8}$$

and

$$\sum_{i=1}^{\infty} \frac{1}{(1+r)^i} = \frac{1}{r}. \tag{9}$$

1.6 *Loan Application; Annuity formulas*

As an application of the previous section, suppose you wish to take out a loan for \$$P$, at monthly interest rate r requiring you to make payments each month (starting 1 month from now), for n months. How much must you pay per month? Denoting this by A. Since the sum of the PV's of your n monthly payments must equal \$$P$ we conclude that

$$P = \sum_{i=1}^{n} A\frac{1}{(1+r)^i} = \frac{A}{r}\left(1 - \frac{1}{(1+r)^n}\right),$$

so that we can solve for A

$$A = \frac{r(1+r)^n P}{(1+r)^n - 1}.$$

In a *perpetual* situation, where $n = \infty$ (e.g., payments must be made each month forever), then $P = A/r$, so $A = rP$.