1 Geometric Brownian motion

Note that since BM can take on negative values, using it directly for modeling stock prices is questionable. There are other reasons too why BM is not appropriate for modeling stock prices. Instead, we introduce here a non-negative variation of BM called *geometric Brownian motion*, $S(t)$, which is defined by

$$S(t) = S_0 e^{X(t)}, \tag{1}$$

where $X(t) = \sigma B(t) + \mu t$ is BM with drift and $S(0) = S_0 > 0$ is the intial value. Taking logarithms yields back the BM; $X(t) = \ln(S(t)/S_0) = \ln(S(t)) - \ln(S_0)$. $\ln(S(t)) = \ln(S_0) + X(t)$ is normal with mean $\mu t + \ln(S_0)$, and variance $\sigma^2 t$; thus, for each t, $S(t)$ has a *lognormal* distribution.

As we will see in Section 1.4: letting $\overline{r} = \mu + \frac{\sigma^2}{2}$,

$$E(S(t)) = e^{\overline{r}t} S_0 \tag{2}$$

the expected price grows like a fixed-income security with continuously compounded interest rate $\overline{r}$.

In practice, $\overline{r} >> r$, the real fixed-income interest rate, that is why one invests in stocks. But unlike a fixed-income investment, the stock price has variability due to the randomness of the underlying Brownian motion and could drop in value causing you to lose money; there is risk involved here.

1.1 Lognormal distributions

If $Y \sim N(\mu, \sigma^2)$, then $X = e^Y$ is a non-negative r.v. having the *lognormal distribution*; called so because its natural logarithm $Y = \ln(X)$ yields a normal r.v.

X has density

$$f(x) = \begin{cases} \frac{1}{x\sigma\sqrt{2\pi}} e^{\frac{-(\ln(x)-\mu)^2}{2\sigma^2}}, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases}$$

This is derived via computing $\frac{d}{dx}F(x)$ for

$$F(x) = P(X \leq x) = P(Y \leq \ln(x)) = \Theta((\ln(x) - \mu)/\sigma),$$

where $\Theta(x)$ denotes the c.d.f. of $N(0, 1)$.

Observing that $E(X) = E(e^Y)$ and $E(X^2) = E(e^{2Y})$ are simply the moment generating function (MGF) $M_Y(s) = E(e^{sY})$ of $Y \sim N(\mu, \sigma^2)$ evaluated at $s = 1$ and $s = 2$ respectively yields

$$\begin{aligned} E(X) &= e^{\mu + \frac{\sigma^2}{2}} \\ E(X^2) &= e^{2\mu + 2\sigma^2} \\ Var(X) &= e^{2\mu + \sigma^2}(e^{\sigma^2} - 1). \end{aligned}$$

As with the normal distribution, the c.d.f. $F(x) = P(X \leq x) = \Theta((\ln(x) - \mu)/\sigma)$ does not have a closed form, but it can be computed from the unit normal cdf $\Theta(x)$. Thus computations for $F(x)$ are reduced to dealing with $\Theta(x)$.

We denote a lognormal μ, σ^2 r.v. by

$$X \sim lognorm(\mu, \sigma^2).$$

1.2 Back to our study of geometric BM, $S(t) = S(0)e^{X(t)}$

For $0 = t_0 < t_1 < \cdots < t_n = t$, the ratios $L_i \stackrel{\text{def}}{=} S(t_i)/S(t_{i-1}),\ 1 \leq i \leq n$, are independent lognormal r.v.s. which reflects the fact that it is the percentage of changes of the stock price that are independent, not the actual changes $S(t_i) - S(t_{i-1})$. For example

$$\begin{aligned} L_1 &\stackrel{\text{def}}{=} \frac{S(t_1)}{S(t_0)} = e^{X(t_1)}, \\ L_2 &\stackrel{\text{def}}{=} \frac{S(t_2)}{S(t_1)} = e^{X(t_2)-X(t_1)}, \end{aligned}$$

are independent and lognormal due to the normal independent increments property of BM; $X(t_1)$ and $X(t_2) - X(t_1)$ are independent and normally distributed. Note how therefore we can re-write

$$S(t) = S_0 L_1 L_2 \cdots L_n, \tag{3}$$

an independent product of n lognormal r.v.s. For example, suppose we wish to sample the stock prices at the end of each day. Then we could choose $t_i = i$ so that $L_i = S(i)/S(i-1)$, the percentage change over one day, and then realize (3) as the independent product of such daily changes. In this case the L_i are also identically distributed since $t_i - t_{i-1} = 1$ for each i: $\ln(L_i)$ is normal with mean μ and variance σ^2.

Geometric BM not only removes the negativity problem but can (in a limited and approximate sense) be justified from basic economic principles as a reasonable model for stock prices in an "ideal" non-arbitrage world. Roughly speaking, no one should be able to make a profit with certainty, by observing the past values $\{S(u) : 0 \leq u \leq t\}$ of the stock, and this forces us to consider non-negative models possessing this property. The idea is to force a "level playing field", in which the evolution of the stock prices must be such that the activity of buying or selling stock offers no arbitrage opportunities.

1.3 Geometric BM is a Markov process

Just as BM is a Markov process, so is geometric BM: *the future given the present state is independent of the past.*

$S(t+h)$ (the future, h time units after time t) is independent of $\{S(u) : 0 \leq u < t\}$ (the past before time t) given $S(t)$ (the present state now at time t). To see that this is so we note that

$$\begin{aligned} S(t+h) &= S_0 e^{X(t+h)} \\ &= S_0 e^{X(t)+X(t+h)-X(t)} \\ &= S_0 e^{X(t)} e^{X(t+h)-X(t)} \\ &= S(t) e^{X(t+h)-X(t)}. \end{aligned}$$

Thus given $S(t)$, the future $S(t+h)$ only depends on the future increment of the BM, $X(t+h) - X(t)$. But BM has independent increments, so this future is independent of the past; we get the Markov property.

Also note that $\{X(t+h)-X(t) : h \geq 0\}$ is yet again BM with the same drift and variance. This means that given $S(t)$, the future process $\{S(t)e^{X(t+h)-X(t)} : h \geq 0\}$ defines (in distribution) the same geometric BM but with new initial value $S(t)$. (So the Markov process has time stationary transition probabilities.)

1.4 Computing moments for Geometric BM

Recall that the moment generating function of a normal r.v. $X \sim N(\mu, \sigma^2)$ is given by

$$M_X(s) = E(e^{sX}) = e^{\mu s + \frac{\sigma^2 s^2}{2}}, \ -\infty < s < \infty.$$

Thus for BM with drift, since $X(t) \sim N(\mu t, \sigma^2 t)$,

$$M_{X(t)}(s) = E(e^{sX(t)}) = e^{\mu t s + \frac{\sigma^2 t s^2}{2}}, \ -\infty < s < \infty.$$

This allows us to immediately compute the moments and variance of geometric BM, by using the values $s = 1, 2$ and so on. For example, $E(S(t)) = E(S_0 e^{X(t)}) = S_0 M_{X(t)}(1)$, and $E(S^2(t)) = E(S_0^2 e^{2X(t)}) = S_0^2 M_{X(t)}(2)$:

$$\begin{aligned} E(S(t)) &= S_0 e^{(\mu + \frac{\sigma^2}{2})t} && (4) \\ E(S^2(t)) &= S_0^2 e^{2\mu t + 2\sigma^2 t} && (5) \\ Var(S(t)) &= S_0^2 e^{2\mu t + \sigma^2 t}(e^{\sigma^2 t} - 1). && (6) \end{aligned}$$

Similarly, any ratio, $S(t)/S(s) = e^{X(t)-X(s)}, \ s < t$, being lognormal (since $X(t) - X(s) \sim N(\mu(t-s), \sigma^2(t-s))$) has mean and variance

$$\begin{aligned} E\{S(t)/S(s)\} &= e^{(\mu + \frac{\sigma^2}{2})(t-s)} && (7) \\ E\{S^2(t)/S^2(s)\} &= e^{2\mu(t-s) + 2\sigma^2(t-s)} && (8) \\ Var\{S(t)/S(s)\} &= e^{2\mu(t-s) + \sigma^2(t-s)}(e^{\sigma^2(t-s)} - 1). && (9) \end{aligned}$$

Letting $\overline{r} = \mu + \frac{\sigma^2}{2}$, we see that

$$E(S(t)) = e^{\overline{r}t} S_0,$$

and more generally

$$E\{S(t)/S(s)\} = e^{\overline{r}(t-s)}.$$

1.5 The Binomial model as an approximation to geometric BM

The binomial lattice model (BLM) that we used earlier is in fact an approximation to geometric BM, and we proceed here to explain the details.

Recall that for BLM, $S_n = S_0 Y_1 Y_2 \cdots Y_n, \ n \geq 0$ where the Y_i are i.i.d. r.v.s. distributed as $P(Y = u) = p, \ P(Y = d) = 1-p$. Besides the initial value S_0, the parameters $0 < d < 1+r < u$, and $0 < p < 1$ completely determine this model. Our objective here is to estimate what these parameters should be in order for this BLM to nicely approximate geometric BM over a given time interval $(0, t]$.

From (3) we can quickly see that for any fixed t we can re-write $S(t)$ as a similar i.i.d. product, by dividing the interval $(0, t]$ into n equally sized subintervals $(0, t/n]$, $(t/n, 2t/n]$, $\ldots$, $((n-1)t/n, t]$, defining $t_i = it/n,\ 0 \le i \le n$ and defining $L_i = S(t_i)/S(t_{i-1})$. Each $\ln(L_i)$ has a normal distribution with mean $\mu t/n$ and variance $\sigma^2 t/n$. Thus we can approximate geometric BM over the fixed time interval $(0, t]$ by the BLM if we appoximate the lognormal L_i by the simple Y_i. To do so we will just match the mean and variance so as to produce appropriate values for u, d, p:

Find u, d, p such that $E(Y) = E(L)$ and $Var(Y) = Var(L)$. This is equivalent to matching the first two moments; $E(Y) = E(L)$ and $E(Y^2) = E(L^2)$.

Noting that $E(Y) = pu + (1-p)d$ and $E(Y^2) = pu^2 + (1-p)d^2$, and (from Section 1.4) $E(L) = e^{\mu t/n + \sigma^2 t/2n}$ and $E(L^2) = e^{2\mu t/n + 2\sigma^2 t/n}$, we must solve the following two equations for u, d, p:

$$pu + (1-p)d = e^{\mu t/n + \sigma^2 t/2n} \tag{10}$$

$$pu^2 + (1-p)d^2 = e^{2\mu t/n + 2\sigma^2 t/n}. \tag{11}$$

Since we have only two equations, there is no unique solution; we have one degree of freedom in the sense that we can apriori force one variable to take on a certain value ($p = 0.5$ for example), and then solve for the other two. The most common relationship to force is

$$ud = 1,$$

which says that $u = 1/d$, and has the effect of making the stock price in the BLM have the nice property that an up followed by a down (or vice versa) leaves the price alone: $udS_0 = duS_0 = S_0$. We shall assume this.

Then, letting $\nu = \mu + \sigma^2/2$, we can re-write the equations as

$$ud = 1, \tag{12}$$

$$pu + (1-p)d = e^{\nu(t/n)}, \tag{13}$$

$$pu^2 + (1-p)d^2 = e^{(2\nu+\sigma^2)(t/n)}. \tag{14}$$

(13) allows us to solve for p in terms of u and d,

$$p = \frac{e^{\nu(t/n)} - d}{u - d}. \tag{15}$$

Then using this formula for p together with $ud = 1$ to plug into the (14) allows us to solve for u (and hence d) (see derivation below):

$$u = \frac{1}{2}(e^{-\nu(t/n)} + e^{(\nu+\sigma^2)(t/n)}) + \frac{1}{2}\sqrt{(e^{-\nu(t/n)} + e^{(\nu+\sigma^2)(t/n)})^2 - 4} \tag{16}$$

When n is large, so that t/n is small, the solution can be approximated by the more simple

$$u = e^{\sigma\sqrt{t/n}}, \tag{17}$$

$$d = e^{-\sigma\sqrt{t/n}}. \tag{18}$$

(in the sense that the ratio of the two formulas for u tends to one as $n \to \infty$). This is nice because this formula does not depend upon knowing the true value of μ; only σ. Thus when using the BLM to price an option, we only need to estimate σ for the stock in question (via looking at past data) in order to get the necessary parameters (recall that the risk-neutral probability, $p^* = (1 + r - d)/(u - d)$, does not depend at all on the actual value of p).

Derivation of u in (16)

Multiplying (13) by d yields

$$pud + (1-p)d^2 = de^{\nu \frac{t}{n}}.$$

But since $ud = 1$,

$$(1-p)d^2 = de^{\nu \frac{t}{n}} - p.$$

Thus from (14)

$$pu^2 + (1-p)d^2 = pu^2 + de^{\nu \frac{t}{n}} - p = e^{(2\nu+\sigma^2)\frac{t}{n}},$$

or

$$p(u^2 - 1) + \frac{1}{u}e^{\nu \frac{t}{n}} = e^{(2\nu+\sigma^2)\frac{t}{n}}. \tag{19}$$

But

$$p = \frac{e^{\nu \frac{t}{n}} - d}{u - d} = \frac{e^{\nu \frac{t}{n}} - \frac{1}{u}}{u - \frac{1}{u}} = \frac{e^{\nu \frac{t}{n}}u - 1}{u^2 - 1}.$$

Plugging this formula for p into (19) yields

$$e^{\nu \frac{t}{n}}u - 1 + \frac{1}{u}e^{\nu \frac{t}{n}} = e^{(2\nu+\sigma^2)\frac{t}{n}}.$$

Multiplying by u yields

$$e^{\nu \frac{t}{n}}u^2 - u + e^{\nu \frac{t}{n}} = ue^{(2\nu+\sigma^2)\frac{t}{n}}.$$

Dividing by $e^{\nu \frac{t}{n}}$ and rearanging terms yields the following quadratic equation in u,

$$u^2 - u(e^{-\nu \frac{t}{n}} + e^{(\nu+\sigma^2)\frac{t}{n}}) + 1 = 0,$$

with solution ($u > 1$) given by (16).

1.6 Justification for the BLM approximation

The main idea throughout the BLM approximation is that when n is large,

$$\ln(Y_1 Y_2 \cdots Y_n) = \sum_{i=1}^{n} \ln(Y_i) \approx X(t) \sim N(\mu t, \sigma^2 t),$$

due to the central limit theorem (CLT). Thus (raising both sides to the e power, and multiplying by S_0),

$$S_n = S_0 Y_1 Y_2 \cdots Y_n \approx S_0 e^{X(t)} = S(t).$$

It can be shown that as $n \to \infty$, the approximation becomes exact (in distribution): the geometric BM can be obtained as a limit of the BLM approximation as the interval size gets smaller and smaller: $S_n \longrightarrow S(t)$ *in distribution as* $n \to \infty$.

The argument is based on the CLT and the fact that $E(\ln(Y_1 \cdots Y_n)) = nE(\ln(Y)) \longrightarrow \mu t$, and $Var(\ln(Y_1 \cdots Y_n)) \longrightarrow \sigma^2 t$. (e.g., the first two moments of $\ln(S_n/S_0)$ converge to those of $X(t) = \ln(S(t)/S_0)$.)

1.7 Option pricing for geometric BM: Black-Scholes

Consider a European call option with expiration date $t = T$, strike price K; the payoff is given by $C_T = (S(T) - K)^+$. Our objective in this section is to determine the option's price when the stock price, in continuous time, follows a geometric BM. We do so by trying to extend to continuous time the risk-neutral approach developed for the BLM. The formula derived is the Black-Scholes option pricing formula, Theorem 1.1 below.

Recall that under the BLM, the price of the option (with expiration date $t = n$)is given by a discounted expected value of payoff:

$$C_0 = \frac{1}{(1+r)^n} E^*(S_n - K)^+, \tag{20}$$

where E^* denotes expected value under the risk-neutral probability p^* for the up and down movement of the stock price. Under p^*, the expected rate of return of the stock equals the risk-free interest rate r; $E(S_1) = (1+r)S_0$, or $(pu + (1-p)d) = 1 + r$. The solution is $p = p^* = (1 + r - d)/(u - d)$. Another (more advanced) way of describing this is that under p^* the discounted stock prices $\{(1+r)^{-n} S_n : n \geq 0\}$ are "fair", that is, form a *martingale*.[1] If stock prices in continuous time follow a geometric BM, then we should (due to its derivation as a limit of the BLM) expect a similar discounted-expected-value form for the option price:

$$C_0 = e^{-rT} E^*(S(T) - K)^+, \tag{21}$$

where E^* denotes an appropriate risk-neutral expected value. This turns out to be so and we proceed next to sketch the corresponding results.

Risk-neutral version of $S(t)$

Letting $S(t) = S_0 e^{X(t)}$, where $X(t) = \sigma B(t) + \mu t$ is BM with drift μ, and variance σ^2, we solve for new values for μ and σ (denoted by μ^*, σ^*), under which the pricing is "fair", that is, such that the discounted prices $\{e^{-rt} S(t) : t \geq 0\}$ form a martingale,[2] which here means that $E(S(t)) = e^{rt} S_0,\ t \geq 0$. But we know that $E(S(t)) = e^{\overline{r}t} S_0$, where $\overline{r} = \mu + \sigma^2/2$ so we conclude that we need

$$\mu + \sigma^2/2 = r.$$

This is accomplished by leaving σ alone, $\sigma^* = \sigma$, but changing the drift term μ to

$$\mu^* = r - \sigma^2/2 \quad \textit{(the risk-neutral drift).} \tag{22}$$

In other words, when pricing the option we need to replace $S(t)$ by its risk-neutral version $S^*(t) = S_0 e^{X^*(t)}$, where

$$\begin{aligned} X^*(t) &= \sigma B(t) + \mu^* t \\ &= \sigma B(t) + (r - \sigma^2/2)t. \end{aligned}$$

[1] A discrete-time stochastic process $\{X_n : n \geq 0\}$ is a martingale if $E(X_{n+1} \mid X_0, \ldots X_n) = X_n,\ n \geq 0$. On average, conditional on all values up to now (n), the value one unit of time later is the same as now: It neither goes up or down. This is best understood in the context of gambling, where X_n denotes your total fortune after gambling n times in a "fair" game. When $X_n = (1+r)^{-n} S_n$ the martingale property reduces to $E(S_1) = (1+r)S_0$ which is equivalent to having $p = p^*$.

[2] A continuous-time stochastic process $\{X(t) : t \geq 0\}$ is a martingale if $E(X(t+h)|X(s) : 0 \leq s \leq t) = X(t),\ h \geq 0,\ t \geq 0$. When $X(t) = e^{-rt} S(t)$, the martingale property becomes $E(e^{-r(t+h)} S(t+h)|e^{-rs} S(s) : 0 \leq s \leq t) = e^{-rt} S(t),\ h \geq 0,\ t \geq 0$.

(21) then becomes

$$\begin{aligned} C_0 &= e^{-rT}E^*(S(T)-K)^+ \\ &= e^{-rT}E(S^*(T)-K)^+ \\ &= e^{-rT}E(S_0e^{\sigma B(T)+(r-\sigma^2/2)T}-K)^+, \end{aligned} \quad (23)$$

and notice how it does not depend upon the real μ, but does depend on the real variance term σ^2, the *volatility* of the stock.

When μ is replaced by μ^* we say that the geometric Brownian motion is being considered under its risk-neutral measure. μ^* serves us in continuous time the same way that the risk-neutral probability p^* does in discrete time.

The expected value in (23) can be computed in terms of the standard normal c.d.f. $\Theta(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{\frac{-y^2}{2}}dy$. The formula obtained is the famous

Theorem 1.1 (Black-Scholes Option pricing formula) *When the stock price follows a geometric BM, $S(t) = S_0e^{\sigma B(t)+\mu t},\ t \geq 0$, the price of a European call option with expiration date $t = T$ and strike price K is given by*

$$C_0 = S_0\Theta(c+\sigma\sqrt{T}) - e^{-rT}K\Theta(c),$$

where

$$\begin{aligned} c &= -\frac{\ln(K/S_0)-(r-\sigma^2/2)T}{\sigma\sqrt{T}} \\ &= \frac{\ln(S_0/K)+(r-\sigma^2/2)T}{\sigma\sqrt{T}}, \end{aligned}$$

and r is the risk-free interest rate.

Notice how, in order to compute our option price, the only parameters we need are: r, σ, K, and S_0. Of these the only one we need to estimate (from past stock data) is σ; the others would be known.

The pricing formula immediately extends to the price C_t of the same option at any time $0 \leq t \leq T$: just replace S_0 by $S(t)$ and T by $T-t$. For at time t we would know the stock price $S(t)$, and with $T-t$ time units remaining until the expiration date, the price would be the same as for an option with initial price $S(t)$ and expiration date $T-t$. More formally, the future evolution of the stock price h time units after time t would follow $\tilde{S}(h) = S(t)e^{X(t+h)-X(t)},\ h \geq 0$. From the Markov property (recall Section 1.3) $\tilde{S}(h)$ is the same geometric BM but with initial price $S(t)$ instead of S_0. The original expiration date occurs when $h = T-t$. Thus computing C_t is the same as computing C_0 in which T is changed to $T-t$ and S_0 is changed to $S(t)$.

This makes perfect sense: Suppose now is time $t < T$. You could buy a new call option with expiration date $T-t$ time units from now, namely at time T, and same strike price K. Clearly now at time t this new option is equivalent to your old one: they both have the same payoff at time T. Thus they must have the same price (or an arbitrage opportunity would arise).

The proof of (23) (that is, the proof that C_0 can indeed be computed as a discounted expected value when μ is changed to μ^*) is not trivial, but there are two methods of deriving it. The first method takes limits in (20) (as $n \to \infty$) when the BLM is used to approximate geometric BM during $(0, T]$; the limit converges exactly to (23). The second method deals with the geometric BM directly, and offers deep insight into continuous-time option pricing by using *stochastic calculus* as the main mathematical tool. The idea is to derive a partial differential equation that must be satisfied by $C_t = C_t(S(t), t)$, the cost of the option at time $0 \le t \le T$. It does so by trying to construct a "replicating portfolio" (of stock and risk-free asset) at each time $0 \le t \le T$ with value the same as C_t. This yields the famous Black-Scholes partial differential equation for the pricing of any derivative of the stock; the European call option being only a special case. We will sketch over the two methods in a later section.

1.8 Doing the integration in (23)

Here we carry out the integration to evaluate $E^*(S(T) - K)^+$ in (23) thereby yielding the Black-Scholes option pricing formula. Recalling that $X^*(T) \sim N(\mu^* T, \sigma^2 T)$ and hence has density

$$\frac{1}{\sigma\sqrt{2\pi T}} e^{\frac{-(x-\mu^*)^2}{2T\sigma^2}},$$

we compute

$$\begin{aligned}
E^*(S(T)-K)^+ &= E((S^*(T)-K)^+) \\
&= \int_{-\infty}^{\infty} (S_0 e^x - K)^+ \frac{1}{\sigma\sqrt{2\pi T}} e^{\frac{-(x-\mu^* T)^2}{2T\sigma^2}} dx \\
&= \int_{\ln(K/S_0)}^{\infty} (S_0 e^x - K) \frac{1}{\sigma\sqrt{2\pi T}} e^{\frac{-(x-\mu^* T)^2}{2T\sigma^2}} dx \\
&= \int_{\ln(K/S_0)}^{\infty} S_0 e^x \frac{1}{\sigma\sqrt{2\pi T}} e^{\frac{-(x-\mu^* T)^2}{2T\sigma^2}} dx - K \int_{\ln(K/S_0)}^{\infty} \frac{1}{\sigma\sqrt{2\pi T}} e^{\frac{-(x-\mu^* T)^2}{2T\sigma^2}} dx \\
&= \int_{\ln(K/S_0)}^{\infty} S_0 e^x \frac{1}{\sigma\sqrt{2\pi T}} e^{\frac{-(x-\mu^* T)^2}{2T\sigma^2}} dx - \frac{K}{\sqrt{2\pi}} \int_{\frac{\ln(K/S_0)-\mu^* T}{\sigma\sqrt{T}}}^{\infty} e^{\frac{y^2}{2}} dy \\
&= \int_{\ln(K/S_0)}^{\infty} S_0 e^x \frac{1}{\sigma\sqrt{2\pi T}} e^{\frac{-(x-\mu^* T)^2}{2T\sigma^2}} dx - K\overline{\Theta}\Big(\frac{\ln(K/S_0)-\mu^* T}{\sigma\sqrt{T}}\Big) \\
&= \int_{\ln(K/S_0)}^{\infty} S_0 e^x \frac{1}{\sigma\sqrt{2\pi T}} e^{\frac{-(x-\mu^* T)^2}{2T\sigma^2}} dx - K\Theta(c).
\end{aligned}$$

where $\overline{\Theta}(x) = 1 - \Theta(x)$ is the tail of the unit normal distribution, and by symmetry $\overline{\Theta}(x) = \Theta(-x)$. Note that we changed variables in the second integral, $y = (x - \mu^* T)/\sigma\sqrt{T}$ so as to reduce it in terms of the standard normal density. Notice that $\Theta(c) = \overline{\Theta}\big(\frac{\ln(K/S_0)-\mu^* T}{\sigma\sqrt{T}}\big) = P(S^*(T) > K)$ (the risk-neutral probability that you excercise the option) which is a nice way of remembering what the value for c is.

The same change of variables (together with algebra, etc.) yields the first integral as

$$e^{rT} S_0 \Theta(c + \sigma\sqrt{T}).$$

Thus we conclude that

$$e^{-rT} E^*(S(T)-K)^+ = S_0\Theta(c + \sigma\sqrt{T}) - K e^{-rT}\Theta(c),$$

the Black-Scholes option pricing formula.

1.9 Sketch of the proof of (23)

When the BLM, $S_n = S_0 Y_1 Y_2 \cdots Y_n$, is used to approximate $S(T)$, the interval $(0,T]$ is divided into n subintervals of size T/n, and the interest rate over each subinterval is $\frac{rT}{n}$ and compounded n times yielding the discount factor $\frac{1}{(1+\frac{rT}{n}))^n}$ at time T.

Thus (20) for the approximation becomes

$$C_0 = \frac{1}{(1+rT/n)^n} E^*(S_n - K)^+,$$

where E^* denotes expected value under the risk-neutral probability $p^* = p_n^*$, for the approximation.

But the discount factor $\frac{1}{(1+rT/n)^n} \longrightarrow e^{-rT}$ as $n \to \infty$, so we will obtain (23) in the limit if we can show that as $n \to \infty$,

$$E^*(S_n - K)^+ \longrightarrow E(S^*(T) - K)^+. \tag{24}$$

As pointed out in Section 1.6, $S_n \longrightarrow S(T)$ in distribution. This convergence, however, is under the actual probability $p = p_n$. Fortunately, under the risk-neutral probability p^*, the same convergence holds: Let $S_n^* = S_0 Y_1^* Y_2^* \cdots Y_n^*$ denote the BLM approximation when using the risk-neutral probability $p^* = p_n^*$; $P(Y^* = u) = p^*$, $P(Y^* = d) = 1 - p^*$. (In particular, $E^*(S_n - K)^+ = E(S_n^* - K)^+$.) *Then $S_n^* \longrightarrow S^*(T)$ in distribution, and so $(S_n^* - K)^+ \longrightarrow (S^*(T) - K)^+$ in distribution too.* The argument is once again based on the CLT, where one needs to instead verify that $E(\ln(Y_1^* \cdots Y_n^*)) = nE(\ln(Y^*)) \longrightarrow \mu^* T$ where $\mu^* = r - \sigma^2/2$, and $Var(\ln(Y_1^* \cdots Y_n^*)) \longrightarrow \sigma^2 T$. (e.g., the first two moments of $\ln(S_n^*/S_0)$ converge to those of $X^*(T) = \ln(S^*(T)/S_0)$.) Finally, from this it can be shown that (24) holds (straightforward uniform integrability argument).

The details involve going back, plugging in the approximating values $u = e^{\sigma\sqrt{T/n}}, d = u^{-1}$, $p^* = ((1 + \frac{rT}{n}) - d)/(u - d)$ for S_n^* and then taking the various limits.

1.10 Pricing other derivatives

As with the BLM, we can also express the price of other derivative as a discounted expected payoff under the risk-neutral measure:

$$C_0 = e^{-rT} E^*(C_T), \tag{25}$$

where C_T is the payoff to be received at the expiration time T. In general, the expected value $E^*(C_T)$ can not be computed explicitely as was the case for the European call, and numerical methods (such as Monte Carlo simulation, or using the BLM) must be employed.

1.11 Black-Scholes PDE

Let $f(t,x)$ denote the price of a derivative (of a stock with price $S(t)$ as geometric BM) at time t if $S(t) = x$. We assume that $t \in [0,T]$. Initial boundary conditions would have to be specified for each specific derivative. For example, $f(t,0) = 0,\ t \in [0,T]$ and $f(T,x) = (x-K)^+,\ x \geq 0$ are the initial conditions for the European call option.

Without proof we state the

Theorem 1.2 (Black-Scholes Partial Differential Equation (PDE)) *Let $f(t,x)$ denote the price at time t of a derivative of stock (such as a European call option) when $S(t) = x$. Then f must satisfy the partial differential equation:*

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} rx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2 x^2 = rf.$$

As a trivial example, suppose the derivative is the stock itself. Then we know that $f(t,x) = x$. Let's check: $\frac{\partial f}{\partial t} = 0$, $\frac{\partial f}{\partial x} = 1$, and $\frac{\partial^2 f}{\partial x^2} = 0$. The PDE becomes $rx = rf$ which indeed is correct.