1 Portfolio mean and variance

Here we study the performance of a one-period investment $X_0 > 0$ (dollars) shared among several different assets. Our criterion for measuring performance will be the mean and variance of its rate of return; the variance being viewed as measuring the risk involved. Among other things we will see that the variance of an investment can be reduced simply by diversifying, that is, by sharing the X_0 among more than one asset, and this is so even if the assets are uncorrelated. At one extreme, we shall find that it is even possible, under strong enough correlation between assets, to reduce the variance to 0, thus obtaining a risk-free investment from risky assets. We will also study the Markowitz optimization problem and its solution, a problem of minimizing the variance of a portfolio for a given fixed desired expected rate of return.

1.1 Basic model

You plan to invest a (deterministic) total of $X_0 > 0$ at time $t = 0$ in a *portfolio* of $n \geq 2$ distinct assets, and the payoff X_1 comes one period of time later (at time $t = 1$ for simplicity). Apriori you do not know how to distribute the amount X_0 among the n assets, your objective being to distribute X_0 in such a way as to give you the best performance. If X_{0i} is the amount to be invested in asset i, $i \in \{1, 2, \dots, n\}$, then $X_0 = X_{01} + X_{02} + \cdots + X_{0n}$. The portfolio chosen is described by the vector $(X_{01}, X_{02}, \dots, X_{0n})$ and its payoff is given by $X_1 = X_{11} + X_{12} + \cdots + X_{1n}$, where X_{1i} is the (random) payoff from investing X_{0i} in asset i, that is, the cash flow you receive at time $t = 1$. R_i, called the *total return*, is the payoff per dollar invested in asset i[1],

$$R_i = \frac{X_{1i}}{X_{0i}}.$$

We define the *rate of return* as the corresponding rate

$$r_i \stackrel{\text{def}}{=} R_i - 1 = \frac{X_{1i} - X_{0i}}{X_{0i}};$$

and it holds then that

$$X_{1i} = (1 + r_i)X_{0i}.$$

But note that unlike fixed-income securities, here the rate r_i is a random variable since X_{0i} is assumed so.

The *expected rate of return* (also called the mean or average rate of return) is given by $\overline{r}_i = E(r_i)$, and since X_{0i} is assumed deterministic (non-random) it also holds that

$$E(X_{1i}) = (1 + \overline{r}_i)X_{0i}.$$

Shorting is allowed, so some of the X_{0i} can be negative (as well as positive or zero), as long as $X_{01} + X_{02} + \cdots + X_{0n} = X_0 > 0$.

It is convenient to define *weights* (also called *proportions*),

$$\alpha_i = \frac{X_{0i}}{X_0} = \textit{ proportion of resources invested in asset } i,$$

[1] The point here is that the assets are bought/sold in shares and any proportion thereof. So if one dollar buys 0.4 shares of an asset, and yields payoff 6 dollars, then 10 dollars buys 4 shares and yields payoff 60 dollars.

and it follows that

$$\sum_{i=1}^{n} \alpha_i = 1,$$

and the portfolio can equivalently be described by $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ with rate of return, and expected rate of return given by

$$r = \sum_{i=1}^{n} \alpha_i r_i \quad (1)$$

$$\overline{r} = E(r) = \sum_{i=1}^{n} \alpha_i \overline{r}_i. \quad (2)$$

Letting $\sigma_i^2 = Var(r_i) = E(r_i^2) - \overline{r}_i^2$, and $\sigma_{ij} = Cov(r_i, r_j) = E(r_i r_j) - \overline{r}_i \overline{r}_j$, the variance of rate of return of the portfolio is given by

$$\sigma^2 = Var(r) = \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2 + 2 \sum_{1 \le i < j \le n} \alpha_i \alpha_j \sigma_{ij}. \quad (3)$$

σ^2 is a measure of the risk involved for this portfolio; it is a measure of how far from the mean $\overline{r}$ our true rate of return r could be. After all, $\overline{r}$ is an average, and the rate of return r is a random variable that may (with positive probability) take on values considerably smaller than $\overline{r}$.

Note that the value of X_0 is not needed in determining performance, only the proportions $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ are needed: Wether you invest a total of one dollar, or a million dollars, the values of r, $\overline{r}$ and σ^2 are the same when the proportions are the same. In effect, any portfolio can simply be described by a vector $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, where $\sum_{i=1}^{n} \alpha_i = 1$.

Clearly we could obtain $\sigma^2 = \min\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2\}$ by investing all of X_0 in the asset with the smallest variance; thus it is of interest to explore how, by investing in more than one asset, we can reduce the variance even further. We do so in the next sections.

1.2 Reducing risk by diversification

One of the main advantages of investing in more than one asset is the possible reduction of risk. Intuitively, by sharing your resources among several different assets, even if one of them has a disasterous (very low) payoff due to its variability, chances are the others will not. To illustrate this phenomina, let us consider n *uncorrelated* assets (e.g., $Cov(r_i, r_j) = 0$ for $i \neq j$) each having the same expected value and variance for rate of return; $\overline{r}_i = 0.20$, $\sigma_i^2 = 1$. If you invest all your resources in just one of them, then the performance of your investment is $(\overline{r}, \sigma^2) = (0.20, 1)$. Now suppose instead that you invest in all n assets in equal proportions, $\alpha_i = 1/n$. Then from (1) and (3) and the fact that $\sigma_{ij} = 0,\ i \neq j$, by the uncorrelated assumption, we conclude that the mean rate of return remains at $\overline{r} = 0.20$,

$$0.20 = 0.20 \sum_{i=1}^{n} \frac{1}{n},$$

but the variance for the portfolio drops to $\sigma^2 = 1/n$,

$$\frac{1}{n} = \sum_{i=1}^{n} \frac{1}{n^2} = \frac{n}{n^2}.$$

Thus the risk tends to 0 as the number of assets n increases while the rate of return remains the same. In essence, as n increases, our portfolio becomes risk-free.

Our example is important since it involves uncorrelated assets. But in fact by using correlated assets it is possible (theoretically) to reduce the variance to zero thus obtaining a risk-free investment!

To see this, consider two assets ($i = 1, 2$). Suppose further that the total return R_1 for asset 1 is governed by some random event A ("weather is great" for example) with $P(A) = 0.5$: If A occurs, then $R_1 = 2.5$; if A does not occur then $R_1 = 0$. Suppose that the total return for asset 2 is also governed by A but in the opposite way: If A occurs, then $R_2 = 0$; if A does not occur then $R_2 = 2.5$. In essence, asset 2 serves as insurance against the event "A does not occur". Letting $I\{A\}$ denote the indicator function for the event A ($= 1$ if A occurs; 0 if not), we see that $R_1 = 2.5I\{A\}$ and $R_2 = 2.5(1 - I\{A\})$. The rates of return can thus be expressed as $r_1 = 2.5I\{A\} - 1$, $r_2 = 2.5(1 - I\{A\}) - 1$, and it is easily seen that $\sigma_1^2 = \sigma_2^2 = (1.25)^2$.

Choosing equal weights $\alpha_1 = \alpha_2 = 0.5$, the rate of return becomes deterministic:

$$
\begin{aligned}
r = 0.5r_1 + 0.5r_2 &= 0.5\Big(2.5I\{A\} - 1 + 2.5(1 - I\{A\}) - 1\Big) \\
&= 0.5(2.5 - 2) \\
&= 0.5(0.5) \\
&= 0.25, \text{ w.p.1.}
\end{aligned}
$$

Thus $\sigma^2 = Var(r) = 0$ for this portfolio, and we see that this investment is equivalent to placing your funds in a risk-free account at interest rate $r = 0.25$.

The key here is the *negative correlation* between r_1 and r_2:

$$
\begin{aligned}
\sigma_{12} &= Cov(r_1, r_2) \\
&= (2.5)^2 Cov(I\{A\}, 1 - I\{A\}) \\
&= -(2.5)^2 Cov(I\{A\}, I\{A\}) \\
&= -(2.5)^2 Var(I\{A\}) \\
&= -(2.5)^2 P(A)(1 - P(A)) \\
&= -(1.25)^2,
\end{aligned}
$$

yielding a correlation coefficient $\rho = \sigma_{12}/\sigma_1\sigma_2 = -1$; perfect negative correlation. This method of making the investment risk-free is an example of *perfect hedging*; asset 2 was used to perfectly hedge against the risk in asset 1.

The above examples were meant for illustration only; assets are typically correlated in more complicated ways, as we know by watching stock prices fall all together at times. It thus is important to solve, for any given set of n assets (with given rates of return, variances and covariances), the weights corresponding to the *minimum-variance portfolio.* We start on this problem next.

1.3 Minimal variance when $n = 2$

When $n = 2$ the weights can be described by one number α where $\alpha_1 = \alpha$ and $\alpha_2 = 1 - \alpha$. Because shorting is allowed, one of these weights might be negative. For example $\alpha = -1$, $1 - \alpha = 2$ is possible if $X_{01} = -1$, and $X_{02} = 2$: short one dollar of asset 1 and buy two dollars of asset 2. The performance of our portfolio can then be described by

$$
\begin{aligned}
r &= \alpha r_1 + (1 - \alpha) r_2 &\quad (4)\\
\overline{r} &= E(r) = \alpha \overline{r}_1 + (1 - \alpha)\overline{r}_2 &\quad (5)\\
f(\alpha) &= Var(r) = \alpha^2\sigma_1^2 + (1 - \alpha)^2\sigma_2^2 + 2\alpha(1 - \alpha)\sigma_{12}. &\quad (6)
\end{aligned}
$$

denoting the (random) rate of return, expected rate of return, and variance of return respectively, when using weights α and $1-\alpha$.

Defining the correlation coefficient ρ between r_1 and r_2 via

$$\rho = \frac{\sigma_{12}}{\sigma_1\sigma_2},$$

we can rewrite

$$\sigma_{12} = \rho\sigma_1\sigma_2,$$

and $-1 \le \rho \le 1$.

The variance of the portfolio can thus be re-written as

$$f(\alpha) = \alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2 + 2\alpha(1-\alpha)\rho\sigma_1\sigma_2. \tag{7}$$

Our objective now is to find the value of α (denote this by α^*) yielding the minimum variance. This would tell us what proportions of the two assets to use (for any amount $X_0 > 0$ invested) to ensure the smallest risk. The portfolio $(\alpha^*, 1-\alpha^*)$ is called the *minimum-variance portfolio*. Our method is to solve $f'(\alpha) = 0$. Details are left to the reader who will carry out most of the analysis in a Homework Set 3. We assume here that both assets are risky, by which we mean that $\sigma_1^2 > 0$ and $\sigma_2^2 > 0$.

Theorem 1.1 *If both assets are risky (and the case $\sigma_1^2 = \sigma_2^2$ with $\rho = 1$ is not included)*[2] *then $f'(\alpha) = 0$ has a unique solution α^* and since $f''(\alpha) > 0$ for all α, $f(\alpha)$ is a strictly convex function and hence the solution α^* is the unique global minimum. This minimum and the corresponding mimimum value $f(\alpha^*)$ are given by the formulas*

$$\alpha^* = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}, \tag{8}$$

$$\sigma^2 = f(\alpha^*) = \frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}. \tag{9}$$

It follows that shorting is required for asset 1 if and only if $\rho > \sigma_2/\sigma_1$, whereas shorting is required for asset 2 if and only if $\rho > \sigma_1/\sigma_2$. (Both of these cases require positive correlation.)

Corollary 1.1 *If both assets are risky, then the variance for the minimal-variance portfolio is strictly smaller than either of the individual asset variances, $\sigma^2 < \min\{\sigma_1^2, \sigma_2^2\}$, unless*

$$\rho = \frac{\min\{\sigma_1, \sigma_2\}}{\max\{\sigma_1, \sigma_2\}},$$

in which case $\sigma^2 = \min\{\sigma_1^2, \sigma_2^2\}$. (This includes the case $\sigma_1^2 = \sigma_2^2$ and $\rho = 1$.) In particular, $\sigma^2 < \min\{\sigma_1^2, \sigma_2^2\}$ whenever $\rho < 0$.

Corollary 1.2 *If both assets are risky, then*

1. *if $\rho = 0$ (uncorrelated case), then*

$$\alpha^* = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}, \tag{10}$$

$$\sigma^2 = f(\alpha^*) = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}. \tag{11}$$

[2] If $\sigma_1^2 = \sigma_2^2$ and $\rho = 1$, then $f(\alpha) = \sigma_1^2 = \sigma_2^2$, for all α, and thus all portfolios have the same variance; there is no unique mimimum-variance portfolio.

2. *if* $\rho = -1$ *(perfect negative correlation), then the minimum variance portfolio is risk-free,* $\sigma^2 = f(\alpha^*) = 0$*, with deterministic rate of return given by (w.p.1.)*

$$r = \overline{r} = \frac{\sigma_2^2 + \sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2}\overline{r}_1 + \frac{\sigma_1^2 + \sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2}\overline{r}_2.$$

   In this case no shorting is required: both $\alpha^* > 0$ *and* $1 - \alpha^* > 0$.

3. *if* $\rho = 1$ *(perfect positive correlation) (and* $\sigma_1^2 \neq \sigma_2^2$*), then the minimum variance portfolio is risk-free,* $\sigma^2 = f(\alpha^*) = 0$*, with deterministic rate of return given by (w.p.1.)*

$$r = \overline{r} = \frac{\sigma_2^2 - \sigma_1\sigma_2}{(\sigma_1 - \sigma_2)^2}\overline{r}_1 + \frac{\sigma_1^2 - \sigma_1\sigma_2}{(\sigma_1 - \sigma_2)^2}\overline{r}_2.$$

   In this case shorting is required: $\alpha^* < 0$ *if* $\sigma_1 > \sigma_2$*;* $1 - \alpha^* < 0$ *if* $\sigma_1 < \sigma_2$.

1.4 Investing in two portfolios: treating a portfolio as an asset itself

Suppose you can invest in two different portfolios, where each portfolio has its own rate of return (that you have no control over). The idea here is that each portfolio is itself an "asset" with its own shares that you can buy/sell and short. We have in mind here for example large mutual fund portfolios such as the ones offered by TIAA-CREFF, or Vangard. Your objective is to choose the weights invested in each so as to minimize the variance of the rate of return. By treating each portfolio as an asset, our problem falls exactly in the $n = 2$ framework of the previous section; we can apply Theorem 1.1.

As a specific example, let us consider the case when the first asset is a pure stock portfolio and the second a less risky portfolio containing some bonds. The stock portfolio will have a higher variance and a higher rate of return than the bond portfolio and the two will be somewhat positively correlated. If you are very risk averse, then you might consider investing all in the bond portfolio; but, you can do a bit better by diversifying among the two portfolios. Data could be found to estimate $\overline{r}_1, \overline{r}_2, \sigma_1^2, \sigma_2^2, \rho$. Let us assume, for example, that

$$\begin{aligned} \overline{r}_1 &= 0.25 \\ \overline{r}_2 &= 0.05 \\ \sigma_1 &= 0.15 \\ \sigma_2 &= 0.05 \\ \rho &= 0.25 \end{aligned}$$

Plugging into formulas (8) and (9), we obtain

$\alpha^* = 0.0294$, $1 - \alpha^* = 0.9706$, $\sigma = 0.0498$, and $\overline{r} = ..05588$. So the variance went down very slightly and (of course) at the expense of yielding an average rate of return to about that of the bond portfolio.

Thus far in our study of portfolios, we have ignored our preference for a high average rate of return over a low one; we address this next.

1.5 The Markowitz Problem

Clearly, just as a rational investor wishes for a low variance on return, a high expected rate of return is also desired. For example, you can always keep variance down by investing in bonds over stocks, but you do so at the expense of a decent rate of return. Thus an investor's optimal portfolio could be best described by performing as $(\overline{r}, \sigma)$, where $\overline{r}$ is a desired (and feasible) average rate of return, and σ^2 the minimal variance possible for this given $\overline{r}$. (Put differently, an investor might wish to find the highest rate of return possible for a given acceptable level of risk.) Thus it is of interest to compute the weights corresponding to such an optimal portfolio.

This problem and its solution is originally due to Harry Markowitz in the 1950's.[3]

Using the notation from Section 1.1 for portfolios of n risky assets (and allowing for shorting) we want to find the solution to:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2 + 2 \sum_{1 \le i < j \le n} \alpha_i \alpha_j \sigma_{ij} \\
\text{subject to} \quad & \sum_{i=1}^{n} \alpha_i \overline{r}_i = \overline{r} \\
& \sum_{i=1}^{n} \alpha_i = 1.
\end{aligned}
$$

Here, $\overline{r}$ is a fixed pre-desired level for expected rate of return, and a solution is any portfolio $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ that minimizes the objective function (variance) and offers expected rate $\overline{r}$. This is an example of what is called a *quadratic program*, an optimization problem with a quadratic objective function, and linear constraints. Fortunately, our particular quadratic program can be reduced to a problem of merely solving linear equations, as we will see next.

Since the objective function is non-negative, it can be multiplied by any non-negative constant without changing the solution. Moreover, we can simplify notation by using the fact that $\sigma_{ii} = \sigma_i^2$. The following equivalent formulation is the most common in the literature:

$$
\begin{aligned}
\text{minimize} \quad & \frac{1}{2} \sum_{i,j=1}^{n} \alpha_i \alpha_j \sigma_{ij} && (12) \\
\text{subject to} \quad & \sum_{i=1}^{n} \alpha_i \overline{r}_i = \overline{r} && (13) \\
& \sum_{i=1}^{n} \alpha_i = 1. && (14)
\end{aligned}
$$

The solution is obtained by using the standard technique from calculus of introducing two more variables called *Lagrange multipliers*, λ and μ (one for each "subject to" constraint), and forming the *Lagrangian*

$$L = \frac{1}{2} \sum_{i,j=1}^{n} \alpha_i \alpha_j \sigma_{ij} - \lambda\Big(\sum_{i=1}^{n} \alpha_i \overline{r}_i - \overline{r}\Big) - \mu\Big(\sum_{i=1}^{n} \alpha_i - 1\Big). \quad (15)$$

Setting $\frac{\partial L}{\partial \alpha_i} = 0$ for each of the n weight variables α_i yields n equations;

$$\sum_{j=1}^{n} \alpha_j \sigma_{ij} - \lambda \overline{r}_i - \mu = 0, \; i \in \{1, 2, \ldots, n\}.$$

[3]Markowitz is one of three economists who won the Nobel Prize in Economics in 1990.The others are Merton Miller and William Sharpe.

Each such equation is linear in the $n+2$ variables $(\alpha_1, \alpha_2, \dots, \alpha_n, \lambda, \mu)$ and together with the remaining two "subject to" linear constraints, yields a set of $n+2$ linear equations with $n+2$ unknowns.

Thus a solution to the Markowitz problem is found by finding a solution $(\alpha_1, \alpha_2, \dots, \alpha_n, \lambda, \mu)$ to the set of $n+2$ linear equations,

$$\begin{aligned} \sum_{j=1}^{n} \alpha_j \sigma_{ij} - \lambda \overline{r}_i - \mu &= 0, \; i \in \{1, 2, \dots, n\} \\ \sum_{i=1}^{n} \alpha_i \overline{r}_i &= \overline{r} \\ \sum_{i=1}^{n} \alpha_i &= 1, \end{aligned} \tag{16}$$

and using the weights $(\alpha_1, \alpha_2, \dots, \alpha_n)$ as the solution.

In the end, the problem falls into the standard framework of linear algebra, and amounts to computing the inverse of a matrix: solve $\mathbf{Ax} = \mathbf{b}$; solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. The student will have had much practice of such methods in *Linear Programming (LP)* from Operations Research. There are various software packages for dealing with such computations.

We point out in passing that the Markowitz problem will of course only have a solution for values of $\overline{r}$ that are *feasible*, that is, can be achieved via

$$\sum_{i=1}^{n} \alpha_i \overline{r}_i = \overline{r},$$

from some portfolio $(\alpha_1, \alpha_2, \dots, \alpha_n)$.

Over all, we are considering the set of all feasible pairs $(\overline{r}, \sigma)$; those pairs for which there exists a portfolio $(\alpha_1, \alpha_2, \dots, \alpha_n)$ such that

$$\sum_{i=1}^{n} \alpha_i \overline{r}_i = \overline{r},$$

and

$$\sum_{i,j=1}^{n} \alpha_i \alpha_j \sigma_{ij} = \sigma^2.$$

The set of all feasible pairs is a subset of the two-dimensional $\sigma - \overline{r}$ plane, and is called the *feasible set.*

For each fixed feasible $\overline{r}$ the Markowitz problem yields that feasible pair $(\overline{r}, \sigma)$ with the smallest σ. As we vary $\overline{r}$ to obtain all such pairs, we obtain what is called the *minimum-variance set*, a subset of the feasible set. In general σ will increase as you increase your desired level of expected return $\overline{r}$.

1.6 Finding the minimum-variance portfolio

If we look at the set of all pairs $(\overline{r}, \sigma)$ in the minimum-variance set, we could find one with the smallest σ, that is, corresponding to the so-called *minimum-variance portfolio* that we considered in earlier sections. This pair denoted by $(\overline{r}^*, \sigma^*)$ is called the *minimum-variance point.*

We can modify the Markowitz problem to find the minimum-variance portfolio as follows: If we leave out our requirement that the expected rate of return be equal to a given level $\overline{r}$, then the Markowitz problem becomes

$$\begin{aligned} &\text{minimize} && \frac{1}{2}\sum_{i,j=1}^{n} \alpha_i \alpha_j \sigma_{ij} && (17)\\ &\text{subject to} && \sum_{i=1}^{n} \alpha_i = 1, && (18) \end{aligned}$$

and its solution yields the minimum-variance portfolio for n risky assets.

Lagrangian methods once again can be employed where now we need only introduce one new variable μ,

$$L = \frac{1}{2}\sum_{i,j=1}^{n} \alpha_i \alpha_j \sigma_{ij} - \mu\Big(\sum_{i=1}^{n} \alpha_i - 1\Big), \tag{19}$$

and the solution reduces to solving $n+1$ equations with $n+1$ unknowns:

$$\begin{aligned} \sum_{j=1}^{n} \alpha_j \sigma_{ij} - \mu &= 0,\ \ i \in \{1, 2, \ldots, n\} && (20)\\ \sum_{i=1}^{n} \alpha_i &= 1. && (21) \end{aligned}$$

1.7 Efficient Frontier

Suppose $(\overline{r}^*, \sigma^*)$ is the minimum variance point. Then we can go ahead and graph all pairs $(\overline{r}, \sigma)$ in the minimum-variance set satisfying $\overline{r} \geq \overline{r}^*$. This set of pairs is called the *efficient frontier* and corresponds to what are called *efficient portfolios*. As $\overline{r}$ increases, σ increases also: A higher rate of return involves higher risk. The efficient frontier traces out a nice increasing curve in the $\sigma - \overline{r}$ plane; see Figure 6.11 of the Text, Page 157.

We view the efficient frontier as corresponding to those portfolios considered by a rational investor.

When shorting is allowed, and there are at least two distinct values for $\overline{r}_i$ (e.g., not all are the same), then the efficient frontier is unbounded from above: You can obtain as high a expected return as is desirable. (The problem, however, is that you do so with increasing risk that tends to ∞.)

To see that it is unbounded: Select any two of the assets (say 1, 2) with different rates of return. Assume that $\overline{r}_1 > \overline{r}_2$. Invest only in these two yielding $\overline{r} = \alpha\overline{r}_1 + (1-\alpha)\overline{r}_2 = \alpha(\overline{r}_1 - \overline{r}_2) + \overline{r}_2$. As $\alpha \to \infty$ so does $\overline{r}$; any high rate is achievable no matter how large. Notice that to do this though, $(1-\alpha)$ becomes negative and large; we must short increasing amounts of asset 2.

1.8 Generating the efficient frontier from only two portfolios

Let $\mathbf{w}^1 = (\alpha_1^1, \alpha_2^1, \ldots, \alpha_n^1, \lambda^1, \mu^1)$ be a solution to the Markowitz problem for a given expected rate of return $\overline{r}^1$, and $\mathbf{w}^2 = (\alpha_1^2, \alpha_2^2, \ldots, \alpha_n^2, \lambda^2, \mu^2)$ be a solution to the Markowitz problem for a different given expected rate of return $\overline{r}^2$. From the linearity of the solution, it is immediate (reader should verify) that for any number α, the new point $\alpha\mathbf{w}^1 + (1-\alpha)\mathbf{w}^2$ is itself a solution to the Markowitz problem for expected rate of return $\alpha\bar{r}^1 + (1-\alpha)\bar{r}^2$. Here,

$$\begin{aligned}\alpha\mathbf{w}^1 &= (\alpha\alpha_1^1, \alpha\alpha_2^1, \ldots, \alpha\alpha_n^1, \alpha\lambda^1, \alpha\mu^1)\\(1-\alpha)\mathbf{w}^2 &= ((1-\alpha)\alpha_1^2, (1-\alpha)\alpha_2^2, \ldots, (1-\alpha)\alpha_n^2, (1-\alpha)\lambda^2, (1-\alpha)\mu^2),\end{aligned}$$

and thus $\alpha\mathbf{w}^1 + (1-\alpha)\mathbf{w}^2$ is of the form $(\alpha_1, \alpha_2, \ldots, \alpha_n, \lambda, \mu)$ with $\alpha_i = \alpha\alpha_i^1 + (1-\alpha)\alpha_i^2$, $\lambda = \alpha\lambda^1 + (1-\alpha)\lambda^2$, and $\mu = \alpha\mu^1 + (1-\alpha)\mu^2$.

This new point $(\alpha_1, \alpha_2, \ldots, \alpha_n, \lambda, \mu)$ is a solution to the $n+2$ linear equations following (16) for $\bar{r} = \alpha\bar{r}^1 + (1-\alpha)\bar{r}^2$.

We conclude that knowing two distinct solutions allows us to generate a whole collection of new solutions, and hence a whole bunch of points on the efficient frontier. It turns out that *the entire minimum-variance set can be generated from two such distinct solutions.* In particular, one can generate the entire efficient frontier from any two distinct solutions.

Treating each of the two fixed distinct solutions as portfolios and hence as "assets" in their own right, we conclude that we can obtain any desired investment performance by investing in these two "assets" only. The idea is to think of each of these two "assets" as mutual funds as in Section 1.4, and create your investment by investing in these two funds only; we are back to the $n = 2$ case.

If we imagine the entire asset marketplace as our potential investment opportunity, then we conclude that it suffices to only invest in two distinct (and excellent) mutual funds, in the sense that we can obtain any point on the efficient frontier by doing so.

1.9 Ruling out shorting

The Markowitz problem assumed shorting was allowed, but if shorting is not allowed, then the additional n constraints, $\alpha_i \geq 0,\ i \in \{1, 2, \ldots, n\}$, must be included as part of the "subject to". This complicates matters because now, instead of only equalities, there are inequalities in the constraints; the solution to this quadratic program is no longer obtained by simply inverting a matrix. But the problem can be handled by using the methods of LP, where n additional Lagrange multipliers must be utilized, and the problem becomes one of finding a feasible region for a LP. This problem will be discussed later.