1 Gambler's Ruin Problem

Let $N \geq 2$ be an integer and let $1 \leq i \leq N-1$. Consider a gambler who starts with an initial fortune of $\$i$ and then on each successive gamble either wins $1 or loses $1 independent of the past with probabilities p and $q = 1-p$ respectively. Let X_n denote the total fortune after the n^{th} gamble. The gambler's objective is to reach a total fortune of $\$N$, without first getting *ruined* (running out of money). If the gambler succeeds, then the gambler is said to *win* the game. In any case, the gambler stops playing after winning or getting ruined, whichever happens first.

$\{X_n\}$ yields a Markov chain (MC) on the state space $\mathcal{S} = \{0, 1, \ldots, N\}$. The transition probabilities are given by $P_{i,i+1} = p,\ P_{i,i-1} = q,\ 0 < i < N$, and both 0 and N are absorbing states, $P_{00} = P_{NN} = 1$.[1]

For example, when $N = 4$ the transition matrix is given by

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ 0 & q & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

While the game proceeds, this MC forms a simple random walk

$$X_n = i + \Delta_1 + \cdots + \Delta_n,\ n \geq 1,\ X_0 = i,$$

where $\{\Delta_n\}$ forms an i.i.d. sequence of r.v.s. distributed as $P(\Delta = 1) = p,\ P(\Delta = -1) = q = 1-p$, and represents the earnings on the successive gambles.

Since the game stops when either $X_n = 0$ or $X_n = N$, let

$$\tau_i = \min\{n \geq 0 : X_n \in \{0, N\} | X_0 = i\},$$

denote the time at which the game stops when $X_0 = i$. If $X_{\tau_i} = N$, then the gambler wins, if $X_{\tau_i} = 0$, then the gambler is ruined.

Let $P_i(N) = P(X_{\tau_i} = N)$ denote the probability that the gambler wins when $X_0 = i$.

> *$P_i(N)$ denotes the probability that the gambler, starting initially with $\$i$, reaches a total fortune of* **N** *before ruin; $1 - P_i(N)$ is thus the corresponding probably of ruin*

Clearly $P_0(N) = 0$ and $P_N(N) = 1$ by definition, and we next proceed to compute $P_i(N),\ 1 \leq i \leq N-1$.

Proposition 1.1 (Gambler's Ruin Problem)

$$P_i(N) = \begin{cases} \frac{1-(\frac{q}{p})^i}{1-(\frac{q}{p})^N}, & \text{if } p \neq q; \\ \frac{i}{N}, & \text{if } p = q = 0.5. \end{cases} \tag{1}$$

[1]There are three communication classes: $C_1 = \{0\},\ C_2 = \{1, \ldots, N-1\},\ C_3 = \{N\}$. C_1 and C_3 are recurrent whereas C_2 is transient.

Proof : For our derivation, we let $P_i = P_i(N)$, that is, we suppress the dependence on N for ease of notation. The key idea is to condition on the outcome of the first gamble, $\Delta_1 = 1$ or $\Delta_1 = -1$, yielding

$$P_i = pP_{i+1} + qP_{i-1}. \tag{2}$$

The derivation of this recursion is as follows: If $\Delta_1 = 1$, then the gambler's total fortune increases to $X_1 = i+1$ and so by the Markov property the gambler will now win with probability P_{i+1}. Similarly, if $\Delta_1 = -1$, then the gambler's fortune decreases to $X_1 = i-1$ and so by the Markov property the gambler will now win with probability P_{i-1}. The probabilities corresponding to the two outcomes are p and q yielding (2). Since $p + q = 1$, (2) can be re-written as $pP_i + qP_i = pP_{i+1} + qP_{i-1}$, yielding

$$P_{i+1} - P_i = \frac{q}{p}(P_i - P_{i-1}).$$

In particular, $P_2 - P_1 = (q/p)(P_1 - P_0) = (q/p)P_1$ (since $P_0 = 0$), so that $P_3 - P_2 = (q/p)(P_2 - P_1) = (q/p)^2 P_1$, and more generally

$$P_{i+1} - P_i = (\frac{q}{p})^i P_1, \ 0 < i < N.$$

Thus

$$\begin{aligned} P_{i+1} - P_1 &= \sum_{k=1}^{i}(P_{k+1} - P_k) \\ &= \sum_{k=1}^{i}(\frac{q}{p})^k P_1, \end{aligned}$$

yielding

$$\begin{aligned} P_{i+1} &= P_1 + P_1\sum_{k=1}^{i}(\frac{q}{p})^k = P_1\sum_{k=0}^{i}(\frac{q}{p})^k \\ &= \begin{cases} P_1\frac{1-(\frac{q}{p})^{i+1}}{1-(\frac{q}{p})}, & \text{if } p \neq q; \\ P_1(i+1), & \text{if } p = q = 0.5. \end{cases} \end{aligned} \tag{3}$$

(Here we are using the "geometric series" equation $\sum_{n=0}^{i} a^i = \frac{1-a^{i+1}}{1-a}$, for any number a and any integer $i \geq 1$.)

Choosing $i = N-1$ and using the fact that $P_N = 1$ yields

$$1 = P_N = \begin{cases} P_1\frac{1-(\frac{q}{p})^N}{1-(\frac{q}{p})}, & \text{if } p \neq q; \\ P_1 N, & \text{if } p = q = 0.5, \end{cases}$$

from which we conclude that

$$P_1 = \begin{cases} \frac{1-\frac{q}{p}}{1-(\frac{q}{p})^N}, & \text{if } p \neq q; \\ \frac{1}{N}, & \text{if } p = q = 0.5, \end{cases}$$

thus obtaining from (3) (after algebra) the solution

$$P_i = \begin{cases} \frac{1-(\frac{q}{p})^i}{1-(\frac{q}{p})^N}, & \text{if } p \neq q; \\ \frac{i}{N}, & \text{if } p = q = 0.5. \end{cases} \tag{4}$$

■

1.1 Becoming infinitely rich or getting ruined

In the formula (1), it is of interest to see what happens as $N \to \infty$; denote this by $P_i(\infty) = \lim_{N\to\infty} P_i(N)$. This limiting quantity denotes the probability that the gambler , if allowed to play forever unless ruined, will in fact never get ruined and instead will obtain an infinitely large fortune.

Proposition 1.2 *Define $P_i(\infty) = \lim_{N\to\infty} P_i(N)$. If $p > 0.5$, then*

$$P_i(\infty) = 1 - (\frac{q}{p})^i > 0. \tag{5}$$

If $p \leq 0.50$, then

$$P_i(\infty) = 0. \tag{6}$$

Thus, unless the gambles are strictly better than fair ($p > 0.5$), ruin is certain.

Proof : If $p > 0.5$, then $\frac{q}{p} < 1$; hence in the denominator of (1), $(\frac{q}{p})^N \to 0$ yielding the result. If $p < 0.50$, then $\frac{q}{p} > 1$; hence in the the denominator of (1), $(\frac{q}{p})^N \to \infty$ yielding the result. Finally, if $p = 0.5$, then $p_i(N) = i/N \to 0$. ■

Examples

1. John starts with \$2, and $p = 0.6$: What is the probability that John obtains a fortune of $N = 4$ without going broke?

   SOLUTION $i = 2$, $N = 4$ and $q = 1 - p = 0.4$, so $q/p = 2/3$, and we want

   $$P_2(4) = \frac{1-(2/3)^2}{1-(2/3)^4} = 0.91$$

2. What is the probability that John will become infinitely rich?

   SOLUTION

   $P_2(\infty) = 1 - (2/3)^2 = 5/9 = 0.56$

3. If John instead started with $i = \$1$, what is the probability that he would go broke?

   SOLUTION

   The probability he becomes infinitely rich is $P_1(\infty) = 1 - (q/p) = 1/3$, so the probability of ruin is $1 - P_1(\infty) = 2/3$.

1.2 Applications

Risk insurance business

Consider an insurance company that earns \$1 per day (from interest), but on each day, independent of the past, might suffer a *claim* against it for the amount \$2 with probability $q = 1 - p$. Whenever such a claim is suffered, \$2 is removed from the reserve of money. Thus on the n^{th} day, the net income for that day is exactly Δ_n as in the gamblers' ruin problem: 1 with probability p, -1 with probability q.

If the insurance company starts off initially with a reserve of \$$i \geq 1$, then what is the probability it will eventually get ruined (run out of money)?

The answer is given by (5) and (**??**): If $p > 0.5$ then the probability is given by $(\frac{q}{p})^i > 0$, whereas if $p \leq 0.5$ ruin will always ocurr. This makes intuitive sense because if $p > 0.5$, then the average net income per day is $E(\Delta) = p - q > 0$, whereas if $p \leq 0.5$, then the average net income per day is $E(\Delta) = p - q \leq 0$. So the company can not expect to stay in business unless earning (on average) more than is taken away by claims.

1.3 Random walk hitting probabilities

Let $a > 0$ and $b > 0$ be integers, and let $R_n = \Delta_1 + \cdots + \Delta_n,\ n \geq 1,\ R_0 = 0$ denote a simple random walk initially at the origin. Let

$$p(a) = P(\{R_n\} \text{ hits level } a \text{ before hitting level } -b).$$

By letting $i = b$, and $N = a + b$, we can equivalently imagine a gambler who starts with $i = b$ and wishes to reach $N = a + b$ before going broke. So we can compute $p(a)$ by casting the problem into the framework of the gambler's ruin problem: $p(a) = P_i(N)$ where $N = a + b$, $i = b$. Thus

$$p(a) = \begin{cases} \frac{1-(\frac{q}{p})^b}{1-(\frac{q}{p})^{a+b}}, & \text{if } p \neq q; \\ \frac{b}{a+b}, & \text{if } p = q = 0.5. \end{cases} \tag{7}$$

Examples

1. Ellen bought a share of stock for \$10, and it is believed that the stock price moves (day by day) as a simple random walk with $p = 0.55$. What is the probability that Ellen's stock reaches the high value of \$15 before the low value of \$5?

   SOLUTION

   We want "the probability that the stock goes up by 5 before going down by 5." This is equivalent to starting the random walk at 0 with $a = 5$ and $b = 5$, and computing $p(a)$.

   $$p(a) = \frac{1 - (\frac{q}{p})^b}{1 - (\frac{q}{p})^{a+b}} = \frac{1 - (0.82)^5}{1 - (0.82)^{10}} = 0.73$$

2. What is the probability that Ellen will become infinitely rich?

   SOLUTION

Here we equivalently want to know the probability that a gambler starting with $i = 10$ becomes infinitely rich before going broke. Just like Example 2 on Page 3:

$1 - (q/p)^i = 1 - (0.82)^{10} \approx 1 - 0.14 = 0.86.$

1.4 Maximums and minimums of the simple random walk

Formula (7) can immediately be used for computing the probability that the simple random walk $\{R_n\}$, starting initially at $R_0 = 0$, will ever hit level a, for any given positive integer $a \geq 1$: Keep a fixed while taking the limit as $b \to \infty$ in (7). The result depends on wether $p < 0.50$ or $p \geq 0.50$. A little thought reveals that we can state this problem as computing the tail $P(M \geq a),\ a \geq 0$, where $M \stackrel{\text{def}}{=} \max\{R_n : n \geq 0\}$ is the all-time maximum of the random walk; a non-negative random variable, because $\{M \geq a\} = \{R_n = a,\ \text{for some } n \geq 1\}$.

Proposition 1.3 *Let $M \stackrel{\text{def}}{=} \max\{R_n : n \geq 0\}$ for the simple random walk starting initially at the origin ($R_0 = 0$).*

1. *When $p < 0.50$,*
$$P(M \geq a) = (p/q)^a,\ a \geq 0;$$
M has a geometric distribution with "success" probability $1 - (p/q)$:
$$P(M = k) = (p/q)^k(1 - (p/q)),\ k \geq 0.$$
In this case, the random walk drifts down to $-\infty$, wp1, but before doing so reaches the finite maximum M.

2. *If $p \geq 0.50$, then $P(M \geq a) = 1,\ a \geq 0$: $P(M = \infty) = 1$; the random walk will, with probability 1, reach any positive integer a no matter how large.*

Proof : Taking the limit in (7) as $b \to \infty$ yields the result by considering the two cases $p < 0.5$ or $p \geq 0.5$: If $p < 0.5$, then $(q/p) > 1$ and so both $(q/p)^b$ and $(q/p)^{a+b}$ tend to ∞ as $b \to \infty$. But before taking the limit, multiply both numerator and denominator by $(q/p)^{-b} = (p/q)^b$, yielding
$$p(a) = \frac{(p/q)^b - 1}{(p/q)^b - (q/p)^a}.$$
Since $(p/q)^b \to 0$ as $b \to \infty$, the result follows.

If $p > 0.5$, then $(q/p) < 1$ and so both $(q/p)^b$ and $(q/p)^{a+b}$ tend to 0 as $b \to \infty$ yielding the limit in (7) as 1. If $p = 0.5$, then $p(a) = b/(b + a) \to 1$ as $b \to \infty$. ■

If $p < 0.5$, then $E(\Delta) < 0$, and if $p > 0.5$, then $E(\Delta) > 0$; so Proposition 1.3 is consistent with the fact that any random walk with $E(\Delta) < 0$ (called the *negative drift* case) satisfies $\lim_{n\to\infty} R_n = -\infty$, wp1, and any random walk with $E(\Delta) > 0$ (called the *positive drift* case) satisfies $\lim_{n\to\infty} R_n = +\infty$, wp1. [2]

But furthermore we learn that when $p < 0.5$, although wp1 the chain drifts off to $-\infty$, it first reaches a finite maximum M before doing so, and this rv M has a geometric distribution.

[2] From the strong law of large numbers, $\lim_{n\to\infty} \frac{R_n}{n} = E(\Delta)$, wp1, so $R_n \approx nE(\Delta) \to -\infty$ if $E(\Delta) < 0$ and $\to +\infty$ if $E(\Delta) > 0$.

Finally Proposition 1.3 also offers us a proof that when $p = 0.5$, the *symmetric* case, the random walk will wp1 hit any positive value, $P(M \geq a) = 1$.

By symmetry, we also obtain analogous results for the minimum, :

Corollary 1.1 *Let* $m \stackrel{\text{def}}{=} \min\{R_n : n \geq 0\}$ *for the simple random walk starting initially at the origin* ($R_0 = 0$)*.*

1. *If* $p > 0.5$*, then*
$$P(m \leq -b) = (q/p)^b, \ b \geq 0.$$
In this case, the random walk drifts up to $+\infty$*, but before doing so drops down to a finite minimum* $m \leq 0$*. Taking absolute values of* m *makes it non-negative and so we can express this result as* $P(|m| \geq b) = (q/p)^b, \ b \geq 0$; $|m|$ *has a geometric distribution with "success" probability* $1 - (q/p)$: $P(|m| = k) = (q/p)^k(1 - (q/p)), \ k \geq 0$.

2. *If* $p \leq 0.50$*, then* $P(m \leq -b) = 1, \ b \geq 0$: $P(m = -\infty) = 1$*; the random walk will, with probability 1, reach any negative integer* a *no matter how small.*

Note that when $p < 0.5$, $P(M = 0) = 1 - (p/q) > 0$. This is because it is possible that the random walk will never enter the positive axis before drifting off to $-\infty$; with positive probability $R_n \leq 0, \ n \geq 0$. Similarly, if $p > 0.5$, then $P(m = 0) = 1 - (q/p) > 0$; with positive probability $R_n \geq 0, \ n \geq 0$.

Recurrence of the simple symmetric random walk

Combining the results for both M and m in the previous section when $p = 0.5$, we have

Proposition 1.4 *The simple symmetric* ($p = 0.50$) *random walk, starting at the origin, will wp1 eventually hit any integer* a*, positive or negative. In fact it will hit any given integer* a *infinitely often, always returning yet again after leaving; it is a recurrent Markov chain.*

Proof : The first statement follows directly from Proposition 1.3 and Corollary 1.1; $P(M = \infty) = 1 = P(m = -\infty) = 1$. For the second statement we argue as follows: Using the first statement, we know that the simple symmetric random walk starting at 0 will hit 1 eventually. But when it does, we can use the same result to conclude that it will go back and hit 0 eventually after that, because that is stochastically equivalent to starting at $R_0 = 0$ and waiting for the chain to hit -1, which also will happen eventually. But then yet again it must hit 1 again and so on (back and forth infinitely often), all by the same logic. We conclude that the chain will, over and over again, return to state 0 wp1; it will do so infinitely often; 0 is a recurrent state for the simple symmetric random walk. Thus (since the chain is irreducible) all states are recurrent. ■

Let $\tau = \min\{n \geq 1 : R_n = 0 \mid R_0 = 0\}$, the so-called *return time* to state 0. We just argued that τ is a proper random variable, that is, $P(\tau < \infty) = 1$. This means that if the chain starts in state 0, then, if we wait long enough, we will (wp1) see it return to state 0. What we will prove later is that $E(\tau) = \infty$; meaning that on average our wait is infinite. This implies that *the simple symmetric random walk forms a null recurrent Markov chain.*