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1 IEOR 6711: Introduction to Martingales in discrete time

Martingales are stochastic processes that are meant to capture the notion of a fair game in the context of gambling. In a fair game, each gamble on average, regardless of the past gambles, yields no profit or loss. But the reader should not think that martingales are used just for gambling; they pop up naturally in numerous applications of stochastic modeling. They have enough structure to allow for strong general results while also allowing for dependencies among variables. Thus they deserve the kind of attention that Markov chains do. Gambling, however, supplies us with insight and intuition through which a great deal of the theory can be understood.

1.1 Basic definitions and examples

Definition 1.1 *A stochastic process* $\mathbf{X} = \{X_n : n \geq 0\}$ *is called a martingale (MG) if*

C1: $E(|X_n|) < \infty,\ n \geq 0,$ *and*

C2: $E(X_{n+1}|X_0, \ldots, X_n) = X_n,\ n \geq 0.$

Notice that property C2 can equivalently be stated as

$$E(X_{n+1} - X_n | X_0, \ldots, X_n) = 0,\ n \geq 0. \tag{1}$$

In the context of gambling, by letting X_n denote your *total fortune* after the n^{th} gamble, this then captures the notion of a fair game in that on each gamble, regardless of the outcome of past gambles, your expected change in fortune is 0; on average you neither win or lose any money.

Taking expected values in C2 yields $E(X_{n+1}) = E(X_n),\ n \geq 0$, and we conclude that

$$E(X_n) = E(X_0),\ n \geq 0,\ \text{for any MG;}$$

At any time n*, your expected fortune is the same as it was initially.*

For notational simplicity, we shall let $\mathcal{G}_n = \sigma\{X_0, \ldots, X_n\}$ denote all the events determined by the rvs $X_0, \ldots, X_n$, and refer to it as the *information* determined by $\mathbf{X}$ up to and including time n. Note that $\mathcal{G}_n \subset \mathcal{G}_{n+1},\ n \geq 0$; information increases as time n increases.

Then the martingale property C2 can be expressed nicely as

$$E(X_{n+1}|\mathcal{G}_n) = X_n,\ n \geq 0.$$

A very important fact is the following which we will make great use of throughout our study of martingales:

Proposition 1.1 *Suppose that* $\mathbf{X}$ *is a stochastic process satisfying C1. Let* $\mathcal{G}_n = \sigma\{X_0, \ldots, X_n\},\ n \geq 0$. *Suppose that* $\mathcal{F}_n = \sigma\{U_0, \ldots, U_n\},\ n \geq 0$ *is information for some other stochastic process such that it contains the information of* $\mathbf{X}$*:*
$\mathcal{G}_n \subset \mathcal{F}_n,\ n \geq 0$. *Then if* $E(X_{n+1}|\mathcal{F}_n) = X_n,\ n \geq 0$, *then in fact* $E(X_{n+1}|\mathcal{G}_n) = X_n,\ n \geq 0$, *so* $\mathbf{X}$ *is a MG.*

$\mathcal{G}_n \subset \mathcal{F}_n$ implies that $\mathcal{F}_n$ also determines $X_0, \ldots, X_n$, but may also determine other things as well. So the above Proposition allows us to verify condition C2 by using more information than is necessary. In many instances, this helps us verify C2 in a much simpler way than would be the case if we directly used $\mathcal{G}_n$.
Proof :

$$\begin{aligned} E(X_{n+1}|\mathcal{G}_n) &= E(E(X_{n+1}|\mathcal{F}_n))|\mathcal{G}_n) \\ &= E(X_n|\mathcal{G}_n) \\ &= X_n. \end{aligned}$$

The first equality follows since $\mathcal{G}_n \subset \mathcal{F}_n$; we can always condition first on more information. The second equality follows from the assumption that $E(X_{n+1}|\mathcal{F}_n) = X_n$, and the third from the fact that X_n is determined by $\mathcal{G}_n$. ■

Because of the above, we sometimes speak of *a MG* $\mathbf{X}$ *with respect to* $\mathcal{F}_n,\ n \geq 0$, where $\mathcal{F}_n$ determines $X_0, \ldots, X_n$ but might be larger.

Examples

In what follows, we typically will define $\mathcal{F}_n$ from the start to be perhaps larger than is needed inorder to check C2.

1. *Symetric random walks.* Let $R_n = \Delta_1 + \cdots + \Delta_n,\ n \geq 1,\quad R_0 = 0$ where $\{\Delta_n : n \geq 1\}$ is i.i.d. with $E(\Delta) = 0$, and $E(|\Delta|) < \infty$. That $\mathbf{R}$ is a MG is easily verified. C1: $E(|R_n|) \leq nE(|\Delta|) < \infty$. C2: We choose $\mathcal{F}_n = \sigma\{\Delta_1, \ldots, \Delta_n\}$, which clearly determines all that we need. $R_{n+1} = R_n + \Delta_{n+1}$ yielding $E(R_{n+1}|\mathcal{F}_n) = R_n + E(\Delta_{n+1}|\mathcal{F}_n)$
   $= R_n + E(\Delta_{n+1})$
   $= R_n + 0$
   $= R_n$

   For simplicity we chose $R_0 = 0$, and so $E(R_n) = E(R_0) = 0,\ n \geq 0$; but any initial condition, $R_0 = x$, will do in which case $\{R_n\}$ is still a MG, and $E(R_n) = E(R_0) = x,\ n \geq 0$.

2. (*Continuation.*) Assume further that $\sigma^2 = Var(\Delta) < \infty$. Then

   $X_n = R_n^2 - n\sigma^2,\ n \geq 0$ forms a MG.
   C1: $E(|X_n|) \leq E(R_n^2) + n\sigma^2$
   $= Var(R_n) + n\sigma^2 = n\sigma^2 + n\sigma^2 = 2n\sigma^2 < \infty$.

   C2: $X_{n+1} = (R_n + \Delta_{n+1})^2 - (n+1)\sigma^2$
   $= R_n^2 + \Delta_{n+1}^2 + 2R_n\Delta_{n+1} - (n+1)\sigma^2$.

$E(X_{n+1}|\mathcal{F}_n) = R_n^2 + E(\Delta_{n+1}^2) + 2R_nE(\Delta_{n+1}) - (n+1)\sigma^2$
$= R_n^2 + \sigma^2 - (n+1)\sigma^2$
$= R_n^2 - n\sigma^2 = X_n.$

3. *More general symetric random walks.* The random walk from Example 1 can be generalized by allowing each increment Δ_n to have its own mean 0 distribution; the MG property C2 still holds: Letting $\{\Delta_n : n \geq 1\}$ be an independent sequence of rvs (that is not necessarily identically distributed) with $E(\Delta_n) = 0$ and $E(|\Delta_n|) < \infty,\ n \geq 1$, again yields a MG. If in addition, each Δ_n has the same variance $\sigma^2 < \infty$, then $R_n^2 - n\sigma^2$ from Example 2 also remains a MG.

   Starting with any MG $\{X_n\}$, by defining $\Delta_n \stackrel{\text{def}}{=} X_n - X_{n-1},\ n \geq 1$, and recalling from (1) that $E(\Delta_n) = 0,\ n \geq 1$, we see that any MG can be rewritten as a symetric random walk, $X_n = X_0 + \sum_{j=1}^{n} \Delta_j$, in which the increments, while not independent, are uncorrelated (you will prove this fact as a homework exercise on Homework 1): $E(\Delta_n\Delta_m) = 0,\ n \neq m$.

4. Let $X_n = Y_1 \times \cdots \times Y_n,\ X_0 = 1$ where $\{Y_n : n \geq 1\}$ is i.i.d. with $E(Y) = 1$ and $E(|Y|) < \infty$. Then $\mathbf{X}$ is a MG. C1: $E(|X_n|) \leq (E(|Y|))^n < \infty$.

   C2: We use $\mathcal{F}_n = \sigma\{Y_1, \ldots, Y_n\}$. $X_{n+1} = Y_{n+1}X_n$. $E(X_{n+1}|\mathcal{F}_n) = X_nE(Y_{n+1}|\mathcal{F}_n)$
   $= X_nE(Y_{n+1}) = X_n \times 1 = X_n.$

5. *Doob's Martingale:*

   Let X be any r.v. such that $E(|X|) < \infty$. Let $\{Y_n : n \geq 0\}$ be any stochastic process (on the same probability space as X), and let $\mathcal{F}_n = \sigma\{Y_0, Y_1, \ldots, Y_n\}$. Then $X_n \stackrel{\text{def}}{=} E(X|\mathcal{F}_n)$ defines a MG called a Doob's MG. C1: $E(|X_n|) = E(|E(X|\mathcal{F}_n)|) \leq E(E(|X|\ |\mathcal{F}_n)) = E(|X|) < \infty$. (Note that $E(X_n) = E(X),\ n \geq 0$.)

   C2: $E(X_{n+1}|\mathcal{F}_n) = E(E(X|\mathcal{F}_{n+1})|\mathcal{F}_n)$
   $= E(X|\mathcal{F}_n)$ (because $\mathcal{F}_n \subset \mathcal{F}_{n+1}$)
   $= X_n.$

   In essence, X_n approximates X, and as n increases the approximation becomes more refined because more information has been gathered and included in the conditioning. For example, if X is completely determined by

   $$\mathcal{F}_\infty = \lim_{n\to\infty} \mathcal{F}_n \stackrel{\text{def}}{=} \cup_{n=0}^{\infty}\mathcal{F}_n = \sigma\{Y_0, Y_1, Y_2, \ldots\}$$

   then it seems reasonable that $X_n \to X,\ n \to \infty$, wp1. This is so, and in fact, it can be shown that in general, $X_n \to E(X|\mathcal{F}_\infty),\ n \to \infty$ wp1. The idea here is that $\mathcal{F}_\infty$ is the most information available, and so $E(X|\mathcal{F}_\infty)$ is the best approximation to X possible, given all we know.

1.2 Optional Stopping

Here we study conditions ensuring that the MG property $E(X_n) = E(X_0),\ n \geq 0$ can be extended to stopping times τ, $E(X_\tau) = E(X_0)$. As we will see, the conditions involve uniform integrability (UI), and the main result Theorem 1.1 has many applications.

1.2.1 *Stopped Martingales*

Recall that a *stopping time* τ with respect to a stochastic process $\{X_n : n \geq 0\}$ is a discrete r.v. with values in $\{0, 1, 2, \ldots\}$ such that for each $n \geq 0$, the event $\{\tau = n\}$ is determined by (at most) $\{X_0, \ldots, X_n\}$, the information up to and including time n. Equivalently, the event $\{\tau > n\}$ is determined by (at most) $\{X_0, \ldots, X_n\}$. Unless otherwise stated, we shall always assume that all stopping times in question are proper, $P(\tau < \infty) = 1$. (Of course, in some examples, we must first prove that this is so.)

Let $a \wedge b = \min\{a, b\}$.

Proposition 1.2 *If $\mathbf{X} = \{X_n : n \geq 0\}$ is a MG, and τ is a stopping time w.r.t. $\mathbf{X}$, then the stopped process $\overline{\mathbf{X}} = \{\overline{X}_n : n \geq 0\}$ is a MG, where*

$$\begin{aligned} \overline{X}_n &\stackrel{\text{def}}{=} \begin{cases} X_n & \textit{if } \tau > n\ , \\ X_\tau & \textit{if } \tau \leq n. \end{cases} \\ &= X_{n \wedge \tau}, \end{aligned}$$

where

$$a \wedge b = \min\{a, b\}.$$

Since $\overline{X}_0 = X_0$, we conclude that $E(\overline{X}_n) = E(X_0)$, $n \geq 0$: Using any stopping time as a gambling strategy yields at each fixed time n, on average, no benefit; the game is still fair.

Proof : (C1:) Since

$$|\overline{X}_n| \leq \max_{0 \leq k \leq n} |X_k| \leq |X_0| + \cdots + |X_n|,$$

we conclude that $E|\overline{X}_n| \leq E(|X_0|) + \cdots + E(|X_n|) < \infty$, from C1 for $\mathbf{X}$.

(C2:) It is sufficient to use $\mathcal{F}_n = \sigma\{X_0, \ldots, X_n\}$ since $\sigma\{\overline{X}_0, \ldots, \overline{X}_n\} \subset \mathcal{F}_n$ by the stopping time property that $\{\tau > n\}$ is determined by $\{X_0, \ldots, X_n\}$. Noting that both $\overline{X}_n = X_n$ and $\overline{X}_{n+1} = X_{n+1}$ if $\tau > n$, and $\overline{X}_{n+1} = \overline{X}_n$ if $\tau \leq n$ yields

$$\overline{X}_{n+1} = \overline{X}_n + (X_{n+1} - X_n) I\{\tau > n\}.$$

Thus

$$\begin{aligned} E(\overline{X}_{n+1} | \mathcal{F}_n) &= \overline{X}_n + E((X_{n+1} - X_n) I\{\tau > n\} | \mathcal{F}_n) \\ &= \overline{X}_n + I\{\tau > n\} E((X_{n+1} - X_n) | \mathcal{F}_n) \\ &= \overline{X}_n + I\{\tau > n\} \cdot 0 \\ &= \overline{X}_n. \end{aligned}$$

■

1.2.2 Martingale Optional Stopping Theorem

Since $\lim_{n \to \infty} n \wedge \tau = \tau$, wp1., we conclude that $\lim_{n \to \infty} \overline{X}_n = X_\tau$, wp1. It is therefore of interest to know when we can interchange the limit with expected value:

$$\textit{When does} \quad \lim_{n \to \infty} E(\overline{X}_n) = E(X_\tau)\ ? \tag{2}$$

For if (2) holds, then since $E(\overline{X}_n) = E(X_0),\ n \geq 0$, we would conclude that

$$E(X_\tau) = E(X_0). \tag{3}$$

We well know that uniform integrability (UI) of $\{\overline{X}_n\}$ is the needed condition for the desired interchange, so we at first state this important result, and then give some reasonable sufficient conditions (useful in many applications) ensuring the UI condition.

Theorem 1.1 (Martingale Optional Stopping Theorem) *If $\mathbf{X} = \{X_n : n \geq 0\}$ is a MG and τ is a stopping time w.r.t. $\mathbf{X}$ such that the stopped process $\overline{\mathbf{X}}$ is UI, then (3) holds: Your expected fortune when stopping is the same as when you started; the stopping strategy does not help to increase your expected fortune.*

Proposition 1.3 *If $\mathbf{X}$ is a MG and τ is a stopping time w.r.t. $\mathbf{X}$, then each of the following conditions alone ensures that $\overline{\mathbf{X}}$ is (UI) and hence that (3) holds:*

1. $\sup_{n\geq 0} |\overline{X}_n| \leq Y$, wp1., *where Y is a r.v. such that $E(Y) < \infty$.*
2. *The stopping time τ is bounded: $P(\tau \leq k) = 1$ for some $k \geq 1$.*
3. $E(|X_\tau|) < \infty$ *and* $E(|X_n|; \tau > n) \to 0,\ n \to \infty$.
4. $E(\tau) < \infty$ *and* $\sup_{n\geq 0} E(|X_{n+1} - X_n|\ |\mathcal{F}_n) \leq B$, *some* $B < \infty$.
5. *There exists a $\delta > 0$ and a $B > 0$ such that $\sup_{n\geq 0} E(|\overline{X}_n|^{1+\delta}) \leq B$. (e.g., $\overline{\mathbf{X}}$ is bounded in L^p for some $p > 1$).*

Proof :

1. The dominated convergence theorem.
2. $|\overline{X}_n| \leq Y \stackrel{\text{def}}{=} \max\{|X_1|, \ldots, |X_k|\}$, and thus the dominated convergence theorem applies since $E(Y) \leq E(|X_1|) + \cdots + E(|X_k|) < \infty$.
3. $$|\overline{X}_n| = |X_\tau| I\{\tau \leq n\} + |X_n| I\{\tau > n\}.$$
   Thus if $E(|X_n|; \tau > n) \to 0$, then $\lim_{n\to\infty} E(|\overline{X}_n|) = \lim_{n\to\infty} E(|X_\tau| I\{\tau \leq n\})$.

   But $|X_\tau| I\{\tau \leq n\} \to |X_\tau|$ and $|X_\tau| I\{\tau \leq n\} \leq |X_\tau|$ with $E(|X_\tau|) < \infty$ by assumption. Thus from the dominated convergence theorem we obtain $\lim_{n\to\infty} E(|\overline{X}_n|) = E(|X_\tau|) = E(\lim_{n\to\infty} |\overline{X}_n|)$ which is equivalent to UI.

   (We note in passing that the condition $E(|X_n|; \tau > n) \to 0$ is satisfied if $\{X_n\}$ is UI.)
4. Follows from (3):

$$
\begin{aligned}
E(|X_\tau|) &\le E(|X_0|) + \sum_{n=1}^{\infty} E(|X_n - X_{n-1}| I\{\tau > n-1\}) \\
&= E(|X_0|) + \sum_{n=1}^{\infty} E\{E(|X_n - X_{n-1}| I\{\tau > n-1\} | \mathcal{F}_{n-1})\} \\
&= E(|X_0|) + \sum_{n=1}^{\infty} E\{I\{\tau > n-1\} E(|X_n - X_{n-1}| \, |\mathcal{F}_{n-1})\} \\
&\le E(|X_0|) + B \sum_{n=1}^{\infty} P(\tau > n-1) \\
&= E(|X_0|) + BE(\tau) \\
&< \infty.
\end{aligned}
$$

Similarly, $E(|X_n|; \tau > n) \le E(|X_0| + Bn; \tau > n) \le E(|X_0| + B\tau; \tau > n) \to 0$.

5. From Markov's inequality,

$$
P(|\overline{X}_n| > x) \le x^{-(1+\delta)} E(|\overline{X}_n|^{1+\delta}) \le Bx^{-(1+\delta)},
$$

yielding

$$
\begin{aligned}
\sup_{n \ge 0} E(|\overline{X}_n| I\{|\overline{X}_n| > x\}) &= \sup_{n \ge 0} \Big\{ xP(|\overline{X}_n| > x) + \int_x^{\infty} P(|X_n| > y) dy \Big\} \\
&\le Bx^{-\delta} + \int_x^{\infty} By^{-(1+\delta)} dy = 2Bx^{-\delta},
\end{aligned}
$$

which tends to 0, as $x \to \infty$, uniformly in n.

■

1.3 Applications

1. *Wald's equation.* Let $\{Y_n : n \ge 1\}$ be i.i.d. with finite mean $\mu = E(Y)$. Let $\Delta_n = Y_n - \mu$, so that the Δ_n are i.i.d. with mean 0. Now let $R_n = \Delta_1 + \cdots + \Delta_n$, $n \ge 1$, $R_0 = 0$, denote the associated *symetric random walk*, which we know is a MG. Let τ be any stopping time w.r.t. $\{Y_n\}$ such that $E(\tau) < \infty$. If the required UI condition is met, then we conclude from Theorem 1.1 that $E(R_\tau) = 0 = E(R_0)$. Since

$$
R_\tau = -\tau\mu + \sum_{j=1}^{\tau} Y_j,
$$

taking expected values then yields the well-known Wald's equation,

$$
E\Big\{ \sum_{j=1}^{\tau} Y_j \Big\} = E(\tau)E(Y).
$$

The UI condition is met via (4) of Proposition 1.3: $E(\tau) < \infty$ is assumed, and

$$E(|R_{n+1} - R_n| \, |\mathcal{F}_n) = E(|\Delta|) < \infty; \ B = E(|\Delta|).$$

2. *Hitting times for the simple symetric random walk.* Let $\{R_n\}$ denote the simple symetric random walk, with $R_0 = 0$; the increment distribution is $P(\Delta = 1) = P(\Delta = -1) = 0.5$. For fixed integers $a > 0$ and $b > 0$, let

$$\tau = \min\{n \geq 0 : R_n \in \{a, -b\}\}, \tag{4}$$

the first passage time of the random walk to level a or $-b$. (We already know from basic random walk theory that $P(\tau < \infty) = 1$.) If the required UI condition is met, then $E(R_\tau) = 0$. But by definition of τ, $R_\tau = a$ or $R_\tau = -b$.
Letting $p_a = P(R_\tau = a)$ and $p_{-b} = P(R_\tau = -b) = 1 - p_a$, we conclude that $0 = E(R_\tau) = ap_a - b(1 - p_a)$, or

$$p_a = \frac{b}{a+b}, \tag{5}$$

and we have computed the probability that the random walk goes up by a units before dropping down by b units. This gives the solution to the *gambler's ruin problem* when $p = 0.5$.

UI: Noting that up to time τ, the random walk is restricted within the bounded interval $[-b, a]$, the UI condition is obtained via (1) of Proposition 1.3:

$\sup_{n \geq 0} |\overline{R}_n| \leq \max\{a, b\}$.

3. *Continuation.* $X_n = R_n^2 - n$ defines yet another MG since $\sigma^2 = Var(\Delta) = 1$. Let τ be as in (4). If UI holds, then $0 = E(X_0) = E(X_\tau) = E(R_\tau^2) - \tau$, or $E(\tau) = E(R_\tau^2)$. Using p_a and p_{-b} from (5), we conclude that

$$E(\tau) = \frac{b}{a+b}a^2 + \frac{a}{a+b}b^2 = ab. \tag{6}$$

UI:

$$\begin{aligned} |X_{n\wedge\tau}| &= |R_{n\wedge\tau}^2 - n \wedge \tau| \\ &\leq R_{n\wedge\tau}^2 + n \wedge \tau \\ &\leq Y \stackrel{\text{def}}{=} (a+b)^2 + \tau. \end{aligned}$$

Thus the UI condition can obtained via (1) of Proposition 1.3 if we can show that $E(\tau) < \infty$. Since $E(X_{n\wedge\tau}) = E(X_0) = 0$, we have $E(\tau \wedge n) = E(R_{n\wedge\tau}^2) \leq (a+b)^2, \ n \geq 0$. The monotone convergence theorem then yields $E(\tau \wedge n) \uparrow E(\tau) \leq (a+b)^2$.

Another way to deduce that $E(\tau) < \infty$: Consider the random walk with $R_0 = 0$ but restricted to the states $-b, \ldots, a$, where the transition probabilities for the boundaries are changed to $P(X_1 = -b | X_0 = -b) = 1 = P(X_1 = a | X_0 = a)$; a and $-b$ are now absorbing states. Then τ can be interpreted for the Markov chain as the time until absorption when initially starting at the origin; thus from finite state space Markov chain theory, $E(\tau) < \infty$.

4. *Hitting times for the simple non-symetric random walk.* We now consider the simple random walk $\{R_n\}$ in which $P(\Delta = 1) = p,\ P(\Delta = -1) = q$ and $p \neq q$; $E(\Delta) = p - q$. Although $\{R_n\}$ is no longer a MG, the transformed process $X_n = (q/p)^{R_n}$ is readily verfied to be a MG, with $X_0 = 1$. (It is a special case of a MG of the form $X_n = Y_1 \times \cdots \times Y_n$ in which $\{Y_n\}$ is i.i.d. with $E(Y) = 1$; here $Y_n = (q/p)^{\Delta_n}$.) Let τ be defined as (4), and observe that $0 \leq X_{n\wedge\tau} \leq (q/p)^a$, if $p < q$, and $0 \leq X_{n\wedge\tau} \leq (p/q)^b$, if $p > q$; yielding the fact that $X_{n\wedge\tau}$ is bounded hence UI. Let p_a and p_{-b} defined as before. Thus we conclude that

$$1 = E\{(q/p)^{R_\tau}\},$$

yielding

$$1 = (q/p)^a p_a + (q/p)^{-b} p_{-b}$$

or

$$p_a = \frac{1 - (q/p)^{-b}}{(q/p)^a - (q/p)^{-b}}. \tag{7}$$

This gives the solution to the *gambler's ruin problem* when $p \neq 0.5$.

If $q > p$ then random walk $\{R_n\}$ is negative drift transient; $lim_{n\to\infty} R_n = -\infty$ wp1., and R_n reaches a finite maximum $M \stackrel{\text{def}}{=} \max_n R_n$ before drifting towards $-\infty$. Thus p_a in (7) increases to $P(M \geq a)$ as $b \to \infty$ yielding

$$P(M \geq a) = \lim_{b\to\infty} p_a = (p/q)^a,\ a \geq 0, \tag{8}$$

and we conclude that

The maximum, M, of the simple random walk with negative drift has a geometric distribution with parameter p/q.

Note that $P(M = 0) = 1 - P(M \geq 1) = 1 - p/q > 0$; there is a positive probability that the random walk will never go above the origin.

1.4 Sub and super martinagles

Relaxing the equality in C2 for the definition of a MG allows for superfair and subfair games, yielding the notions of submartingales and supermartingales:

Definition 1.2 *A stochastic process* $\mathbf{X} = \{X_n : n \geq 0\}$ *is called a submartingale (SUBMG) if*

C1: $E(|X_n|) < \infty,\ n \geq 0,$ *and*

(SUB)C2: $E(X_{n+1}|X_0, \ldots, X_n) \geq X_n,\ n \geq 0.$

Similarly, $\mathbf{X}$ *is called a supermartingale (SUPMG) if C1 holds and the inequality in (SUB)C2 is replaced by* $\leq$*, referred to as Condition (SUP)C2.*