1 Time-reversible Markov chains

In these notes we study positive recurrent Markov chains \( \{X_n : n \geq 0\} \) for which, when in steady-state (stationarity), yield the same Markov chain (in distribution) if time is reversed. The fundamental condition required is that for each pair of states \( i, j \) the long-run rate at which the chain makes a transition from state \( i \) to state \( j \) equals the long-run rate at which the chain makes a transition from state \( j \) to state \( i \); \( \pi_i P_{i,j} = \pi_j P_{j,i} \).

1.1 Two-sided stationary extensions of Markov chains

For a positive recurrent Markov chain \( \{X_n : n \in \mathbb{N}\} \) with transition matrix \( P \) and stationary distribution \( \pi \), let \( \{X_n^* : n \in \mathbb{N}\} \) denote a stationary version of the chain, that is, one in which \( X_0 \sim \pi \). It turns out that we can extend this process to have time \( n \) take on negative values as well, that is, extend it to \( \{X_n^* : n \in \mathbb{Z}\} \). This is a way of imagining/assuming that the chain started off initially in the infinite past, and we call this a two-sided extension of our process. To get such an extension\(^1\) we start by shifting the origin to be time \( k \geq 1 \) and extending the process \( k \) time units into the past: Define \( X_n^*(k) = X_{n+k}^*, \ -k \leq n < \infty \). Note how \( \{X_n^*(k) : n \in \mathbb{N}\} \) has the same distribution as \( \{X_n^* : n \in \mathbb{N}\} \) by stationarity, and in fact this extension on \( -k \leq n < \infty \) is still stationary too. Now as \( k \to \infty \), the process \( \{X_n^*(k) : -k \leq n < \infty\} \) converges (in distribution) to a truly two-sided extension and it remains stationary; we get the desired two-sided stationary extension \( \{X_n^* : n \in \mathbb{Z}\} \). And for each time \( n \in \mathbb{Z} \) it holds that \( P(X_n^* = j) = \pi_j, \ j \in S \).

1.2 Time-reversibility: Time-reversibility equations

Let \( \{X_n^* : n \in \mathbb{Z}\} \) be a two-sided extension of a positive recurrent Markov chain with transition matrix \( P \) and stationary distribution \( \pi \). The Markov property is stated as “the future is independent of the past given the present state”, and thus can be re-stated as “the past is independent of the future given the present state”. But this means that the process \( X_n^{(r)} = X_{-n}^*, \ n \in \mathbb{N} \) denoting the process in reverse time, is still a (stationary) Markov chain. (By reversing time, the future and past are swapped.) In fact it has transition probabilities that can be exactly computed in terms of \( \pi \) and \( P \): letting \( P(r) = (P_{i,j}(r)) \) denote the time reversed transition probabilities,

\[ P_{i,j}(r) = P(X_1^{(r)} = j \mid X_0^{(r)} = i) = P(X_0^* = j \mid X_1^* = i) = P(X_1^* = j \mid X_0^* = i)P(X_0^* = j)/P(X_1^* = i) = \frac{\pi_j}{\pi_i} P_{j,i}. \]

So the time-reversed Markov chain is a Markov chain with transition probabilities given by

\[ P_{i,j}(r) = \frac{\pi_j}{\pi_i} P_{j,i}. \quad (1) \]

\(^1\)The mathematical justification for extending a stationary stochastic process to be two-sided stationary is called Kolmogorov’s extension theorem from probability theory.
Definition 1.1 A positive recurrent Markov chain with transition matrix \( P \) and stationary distribution \( \pi \) is called time reversible if the reverse-time stationary Markov chain \( \{X_n^{(r)} : n \in \mathbb{N}\} \) has the same distribution as the forward-time stationary Markov chain \( \{X_n^* : n \in \mathbb{N}\} \), that is, if \( P(r) = P; P_{i,j}(r) = P_{i,j} \) for all pairs of states \( i, j \). Equivalently this means that it satisfies the time-reversibility equations

\[
\pi_i P_{i,j} = \pi_j P_{j,i},
\]

for all pairs of states \( i, j \). In words: for each pair of states \( i, j \), the long-run rate at which the chain makes a transition from state \( i \) to state \( j \) equals the long-run rate at which the chain makes a transition from state \( j \) to state \( i \).

An inconvenience with our definition is that it requires us to have at our disposal the stationary distribution \( \pi \) in advance so as to check if the chain is time-reversible. But the following can help us avoid having to know \( \pi \) in advance and can even help us find \( \pi \):

Proposition 1.1 If for an irreducible Markov chain with transition matrix \( P \), there exists a probability solution \( \pi \) to the “time-reversibility” set of equations

\[
\pi_i P_{i,j} = \pi_j P_{j,i},
\]

for all pairs of states \( i, j \), then the chain is positive recurrent, time-reversible and the solution \( \pi \) is the unique stationary distribution.

Proof: It suffices to show that such a solution also satisfies \( \pi = \pi P \), for then (via Theorem 2.1 in Lecture Notes 4) it is the unique stationary distribution and since it satisfies the time-reversibility equations, the chain is also time reversible. To this end, fixing a state \( j \) and summing over all \( i \) yields

\[
\sum_i \pi_i P_{i,j} = \sum_i \pi_j P_{j,i}
= \pi_j \sum_i P_{j,i}
= \pi_j \times 1
= \pi_j,
\]

namely, \( \pi = \pi P \).

The importance of the above Proposition is that the time-reversibility equations are simpler to solve/check than are the \( \pi = \pi P \) equations. So if you suspect (via some intuition) that your chain is time-reversible, then you should first try to solve the time-reversibility equations. Similarly, if you think your chain is time-reversible and have a guess for \( \pi \) at hand, then you should check to see if it satisfies the time-reversibility equations.

Examples

1. Simple random walk on the non-negative integers: Here is an example where intuition quickly tells us that we have a time-reversible chain. Consider a negative drift simple random walk, restricted to be non-negative, in which \( P_{0,1} = 1 \) and otherwise \( P_{i,i-1} = p < 0.5, P_{i,i+1} = 1-p > 0.5 \). In this case, since the chain can only make a transition (change of state) of magnitude \( \pm1 \), we immediately conclude that for each state \( i \geq 0 \), “the rate from \( i \) to \( i+1 \) equals the rate from \( i+1 \) to \( i \)”. This is by the same elementary reasoning
as argued for why “the rate out of state $i$ equals the rate into state $i$, for each state $i$”, for any function/path and has nothing to do with Markov chains: every time there is a change of state from $i$ to $i + 1$ there must be (soon after) a change of state from $i + 1$ to $i$ because that is the only way the process can, yet again, go from $i$ to $i + 1$; there is a one-to-one correspondence. But “the rate from $i$ to $i + 1$ equals the rate from $i + 1$ to $i$” is equivalent to (in words) the time-reversibility equations, since here a pair $i, j$ can only be of the form $i, i + 1$ or $i, i - 1$. Thus the time-reversibility equations are

$$
\pi_0 = (1 - p)\pi_1, \ p\pi_i = (1 - p)\pi_{i+1}, \ i \geq 1,$$

yielding $\pi_1 = \pi_0/(1 - p)$, $\pi_2 = p\pi_0/(1 - p)^2$, $\ldots$, $\pi_n = p^{n-1}\pi_0/(1 - p)^n$, $n \geq 1$. Since $\sum_n \pi_n = 1$ must hold, we get

$$
\pi_0 \left(1 + (1 - p)^{-1} \sum_{n=0}^{\infty} \left[ \frac{p}{1 - p} \right]^n \right) = 1,
$$

and since $\frac{p}{1 - p} < 1$, the geometric series converges and we can solve explicitly for the stationary distribution:

$$
\pi_0 = (1 + \frac{1}{1 - 2p})^{-1}, \ \pi_n = (1 - p)^{-1}\left[ \frac{p}{1 - p} \right]^{n-1}\pi_0, \ n \geq 1,
$$

which simplifies to

$$
\pi_0 = \frac{1 - 2p}{2(1 - p)}, \ \pi_n = \left( \frac{1 - p}{2} \right)\left[ \frac{p}{1 - p} \right]^{n-1}, \ n \geq 1.
$$

As we will see later, this Markov chain is the embedded discrete-time chain for an $M/M/1$ queue in which $p = \lambda/(\lambda + \mu)$, where $\lambda$ is the Poisson arrival rate of customers, and $\mu$ is the exponential service time rate.

2. **Random walk on a connected graph**: Consider a finite connected graph with $n \geq 2$ nodes, labeled $1 - n$, and positive weights $w_{i,j} = w_{j,i} > 0$ for any pair of nodes $i, j$ for which there is an arc ($w_{i,j} = 0$ if there is no arc). We can define a Markov chain random walk (with state-dependent transition probabilities) on the nodes of the graph via

$$
P_{i,j} = \frac{w_{i,j}}{\sum_k w_{i,k}}.
$$

This chain is irreducible (by definition of connected graph). Moreover, because of the symmetry $w_{i,j} = w_{j,i}$, and the way in which the $P_{i,j}$ are defines, we expect that this chain should be time-reversible too. We can explicitly solve the time-reversibility equations, which are in this case (since $w_{i,j} = w_{j,i}$)

$$
\frac{\pi_i}{\sum_k w_{i,k}} = \frac{\pi_j}{\sum_k w_{j,k}},
$$

for each pair $i, j$, or equivalently that for a constant $C$

$$
\frac{\pi_i}{\sum_k w_{i,k}} = C, \text{ for all } i,
$$

for each pair $i, j$, or equivalently that for a constant $C$
or equivalently

$$\pi_i = C \sum_k w_{i,k}, \text{ for all } i.$$ 

Since it must hold that $\sum_i \pi = 1$, we conclude that

$$C = \left[ \sum_i \sum_k w_{i,k} \right]^{-1},$$

yielding the solution as

$$\pi_i = \frac{\sum_k w_{i,k}}{\sum_i \sum_k w_{i,k}}.$$ 

So the chain is time-reversible and we have solved for the stationary distribution.

Note that when all positive weights are defined to be 1, then the chain always moves to a next node by choosing it uniformly from among all possible arcs out: if there is an arc from $i$ to $j$, then $P_{i,j} = 1/b$, where $b = b(i,j)$ denotes the total number of arcs from $i$ to $j$. 

4