1 Time-reversible Markov chains

In these notes we study positive recurrent Markov chains $\{X_n : n \geq 0\}$ for which, when in steady-state (stationarity), yield the same Markov chain (in distribution) if time is reversed. The fundamental condition required is that for each pair of states i, j the long-run rate at which the chain makes a transition from state i to state j equals the long-run rate at which the chain makes a transition from state j to state i; $\pi_i P_{i,j} = \pi_j P_{j,i}$.

1.1 Two-sided stationary extensions of Markov chains

For a positive recurrent Markov chain $\{X_n : n \in \mathbb{N}\}$ with transition matrix P and stationary distribution π, let $\{X_n^* : n \in \mathbb{N}\}$ denote a *stationary version* of the chain, that is, one in which $X_0 \sim \pi$. It turns out that we can extend this process to have time n take on negative values as well, that is, extend it to $\{X_n^* : n \in \mathbb{Z}\}$. This is a way of imagining/assuming that the chain started off initially in the infinite past, and we call this a *two-sided extension* of our process. To get such an extension[1] we start by shifting the origin to be time $k \geq 1$ and extending the process k time units into the past: Define $X_n^*(k) = X_{n+k}^*, \ -k \leq n < \infty$. Note how $\{X_n^*(k) : n \in \mathbb{N}\}$ has the same distribution as $\{X_n^* : n \in \mathbb{N}\}$ by stationarity, and in fact this extension on $-k \leq n < \infty$ is still stationary too. Now as $k \to \infty$, the process $\{X_n^*(k) : \ -k \leq n < \infty\}$ converges (in distribution) to a truly two-sided extension and it remains stationary; we get the desired two-sided stationary extension $\{X_n^* : n \in \mathbb{Z}\}$. And for each time $n \in \mathbb{Z}$ it holds that $P(X_n^* = j) = \pi_j, \ j \in \mathcal{S}$.

1.2 Time-reversibility: Time-reversibility equations

Let $\{X_n^* : n \in \mathbb{Z}\}$ be a two-sided extension of a positive recurrent Markov chain with transition matrix P and stationary distribution π. The Markov property is stated as "the future is independent of the past given the present state", and thus can be re-stated as "the past is independent of the future given the present state". But this means that the process $X_n^{(r)} = X_{-n}^*, \ n \in \mathbb{N}$ denoting the process in reverse time, is still a (stationary) Markov chain. (By reversing time, the future and past are swapped.) In fact it has transition probabilities that can be exactly computed in terms of π and P: letting $P(r) = (P_{i,j}(r))$ denote the time reversed transition probabilities,

$$\begin{aligned} P_{i,j}(r) = P(X_1^{(r)} = j \mid X_0^{(r)} = i) &= P(X_0^* = j \mid X_1^* = i) \\ &= P(X_1^* = i \mid X_0^* = j) P(X_0^* = j)/P(X_1^* = i) \\ &= \frac{\pi_j}{\pi_i} P_{j,i}. \end{aligned}$$

So the time-reversed Markov chain is a Markov chain with transition probabilities given by

$$P_{i,j}(r) = \frac{\pi_j}{\pi_i} P_{j,i}. \tag{1}$$

[1]The mathematical justification for extending a stationary stochastic process to be two-sided stationary is called *Kolmogorov's extension theorem* from probability theory.

Definition 1.1 *A positive recurrent Markov chain with transition matrix P and stationary distribution π is called time reversible if the reverse-time stationary Markov chain $\{X_n^{(r)} : n \in \mathbb{N}\}$ has the same distribution as the forward-time stationary Markov chain $\{X_n^* : n \in \mathbb{N}\}$, that is, if $P(r) = P$; $P_{i,j}(r) = P_{i,j}$ for all pairs of states i, j. Equivalently this means that it satisfies the* **time-reversibility equations**

$$\pi_i P_{i,j} = \pi_j P_{j,i},$$

for all pairs of states i, j. In words: for each pair of states i, j, **the long-run rate at which the chain makes a transition from state i to state j equals the long-run rate at which the chain makes a transition from state j to state i.**

An inconvenience with our definition is that it requires us to have at our disposal the stationary distribution π in advance so as to check if the chain is time-reversible. But the following can help us avoid having to know π in advance and can even help us find π:

Proposition 1.1 *If for an irreducible Markov chain with transition matrix P, there exists a probability solution π to the "time-reversibility" set of equations*

$$\pi_i P_{i,j} = \pi_j P_{j,i},$$

for all pairs of states i, j, then the chain is positive recurrent, time-reversible and the solution π is the unique stationary distribution.

Proof : It suffices to show that such a solution also satisfies $\pi = \pi P$, for then (via Theorem 2.1 in Lecture Notes 4) it is the unique stationary distribution and since it satisfies the time-reversibility equations, the chain is also time reversible. To this end, fixing a state j and summing over all i yields

$$\begin{aligned}\sum_i \pi_i P_{i,j} &= \sum_i \pi_j P_{j,i} \\ &= \pi_j \sum_i P_{j,i} \\ &= \pi_j \times 1 \\ &= \pi_j,\end{aligned}$$

namely, $\pi = \pi P$. ∎

The importance of the above Proposition is that the time-reversibility equations are simpler to solve/check than are the $\pi = \pi P$ equations. So if you suspect (via some intuition) that your chain is time-reversible, then you should first try to solve the time-reversibility equations. Similarly, if you think your chain is time-reversible and have a guess for π at hand, then you should check to see if it satisfies the time-reversibility equations.

Examples

1. *Simple random walk on the non-negative integers:* Here is an example where intuition quickly tells us that we have a time-reversible chain. Consider a *negative drift* simple random walk, restricted to be non-negative, in which $P_{0,1} = 1$ and otherwise $P_{i,i+1} = p < 0.5$, $P_{i,i-1} = 1 - p > 0.5$. In this case, since the chain can only make a transition (change of state) of magnitude ± 1, we immediately conclude that for each state $i \geq 0$, "the rate from i to $i + 1$ equals the rate from $i + 1$ to i". This is by the same elementary reasoning

as argued for why "the rate out of state i equals the rate into state i, for each state i", for any function/path and has nothing to do with Markov chains: every time there is a change of state from i to $i+1$ there must be (soon after) a change of state from $i+1$ to i because that is the only way the process can, yet again, go from i to $i+1$; there is a one-to-one correspondence. But "the rate from i to $i+1$ equals the rate from $i+1$ to i" is equivalent to (in words) the time-reversibility equations, since here a pair i, j can only be of the form $i, i+1$ or $i, i-1$. Thus the time-reversibility equations are

$$\pi_0 = (1-p)\pi_1, \ p\pi_i = (1-p)\pi_{i+1}, \ i \geq 1,$$

yielding $\pi_1 = \pi_0/(1-p), \ \pi_2 = p\pi_0/(1-p)^2, \ldots, \pi_n = p^{n-1}\pi_0/(1-p)^n, \ n \geq 1$. Since $\sum_n \pi_n = 1$ must hold, we get

$$\pi_0\Big(1 + (1-p)^{-1}\sum_{n\geq 0}\Big[\frac{p}{1-p}\Big]^n\Big) = 1,$$

and since $\frac{p}{1-p} < 1$, the geometric series converges and we can solve explicitly for the stationary distribution:

$$\pi_0 = (1 + \frac{1}{1-2p})^{-1}, \ \pi_n = (1-p)^{-1}\Big[\frac{p}{1-p}\Big]^{n-1}\pi_0, \ n \geq 1,$$

which simplifies to

$$\begin{aligned} \pi_0 &= \frac{1-2p}{2(1-p)} \\ \pi_n &= (\frac{1}{2} - p)\Big[\frac{p}{1-p}\Big]^{n-1}, \ n \geq 1. \end{aligned}$$

As we will see later, this Markov chain is the embedded discrete-time chain for an M/M/1 queue in which $p = \lambda/(\lambda+\mu)$, where λ is the Poisson arrival rate of customers, and μ is the exponential service time rate.

2. *Random walk on a connected graph:* Consider a finite connected graph with $n \geq 2$ nodes, labeled $1-n$, and positive weights $w_{i,j} = w_{j,i} > 0$ for any pair of nodes i, j for which there is an arc ($w_{i,j} \stackrel{\text{def}}{=} 0$ if there is no arc). We can define a Markov chain random walk (with state-dependent transition probabilities) on the nodes of the graph via

$$P_{i,j} = \frac{w_{i,j}}{\sum_k w_{i,k}}.$$

This chain is irreducible (by definition of *connected* graph). Moreover, because of the symmetry $w_{i,j} = w_{j,i}$, and the way in which the $P_{i,j}$ are defines, we expect that this chain should be time-reversible too. We can explicitly solve the time-reversibility equations, which are in this case (since $w_{i,j} = w_{j,i}$)

$$\frac{\pi_i}{\sum_k w_{i,k}} = \frac{\pi_j}{\sum_k w_{j,k}},$$

for each pair i, j, or equivalently that for a constant C

$$\frac{\pi_i}{\sum_k w_{i,k}} = C, \ \text{for all } i,$$

or equivalently

$$\pi_i = C\sum_k w_{i,k}, \text{ for all } i.$$

Since it must hold that $\sum_i \pi = 1$, we conclude that

$$C = \Big[\sum_i \sum_k w_{i,k}\Big]^{-1},$$

yielding the solution as

$$\pi_i = \frac{\sum_k w_{i,k}}{\sum_i \sum_k w_{i,k}}.$$

So the chain is time-reversible and we have solved for the stationary distribution.

Note that when all positive weights are defined to be 1, then the chain always moves to a next node by choosing it uniformly from among all possible arcs out: if there is an arc from i to j, then $P_{i,j} = 1/b$, where $b = b(i,j)$ denotes the total number of arcs from i to j.