3^{rd}-order Intercept Point

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These derivations are for my documentation.

I. IIP_3 of a Differential Amplifier

We want to characterize the non-linearity of a differential amplifier. Its 3^{rd}-order nonlinearity is of most significance.

Assume that the amplifier output characteristic is given by

$$v_{out}(t) = \alpha_1 v_{in} + \alpha_3 v_{in}^3 \quad (1)$$

where higher order non-linearities have been neglected.

The 3^{rd}-order intercept point is defined as the point on P_{out} vs P_{in} plane where the lines corresponding to fundamental and the third order inter-modulation product intersect. It can also be defined on V_{out} vs V_{in} plane.

A. Inter-modulation Product: IM

Consider a two-tone test where you have applied

$$v_{in} = A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t) \quad (2)$$

to the amplifier. For the given amplifier transfer characteristics, this will generate tones at the fundamental frequencies (ω_1, ω_2), their 3^{rd} harmonics ($3\omega_1$, $3\omega_3$) and also two tones at the inter-modulation frequencies — $2\omega_1 - \omega_2$ and $2\omega_2 - \omega_1$.

$$\begin{aligned} v_{out} &= \left(\alpha_1 A_1 + \frac{3}{4}\alpha_3 A_1^3 + \frac{3}{2}\alpha_3 A_1 A_2^2\right)\cos(\omega_1 t) + \left(\alpha_1 A_2 + \frac{3}{4}\alpha_3 A_2^3 + \frac{3}{2}\alpha_3 A_1^2 A_2\right)\cos(\omega_2 t) \\ &+ \frac{\alpha_3}{4}A_1^3 cos(3\omega_1 t) + \frac{\alpha_3}{4}A_2^3 cos(3\omega_2 t) \\ &+ \frac{3}{4}\alpha_3 A_1^2 A_2 \cos((2\omega_1 - \omega_2)t) + \frac{3}{4}\alpha_3 A_1 A_2^2 \cos\left(\left(2\omega_2 - \omega_1\right)t\right) \\ &+ \frac{3}{4}\alpha_3 A_1^2 A_2 \cos\left(\left(2\omega_1 + \omega_2\right)t\right) + \frac{3}{4}\alpha_3 A_1 A_2^2 \cos\left(\left(2\omega_2 + \omega_1\right)t\right) \end{aligned} \tag{3}$$

The following relationships have been used.

$$\cos^3(\omega t) = \frac{1}{4}\left(3\cos(\omega t) + \cos(3\omega t)\right) \tag{4}$$

$$\sin^3(\omega t) = \frac{1}{4}\left(3\sin(\omega t) - \sin(3\omega t)\right) \tag{5}$$

$$\cos^2(\omega t) = \frac{1}{2}\left(1 + \cos(2\omega t)\right) \tag{6}$$

$$\sin^2(\omega t) = \frac{1}{2}\left(1 - \cos(2\omega t)\right) \tag{7}$$

Assuming,

1) $\alpha_3 \ll \alpha_1$
2) A_1 and A_2 are not large enough to compress the gain at ω_1 and ω_2
3) Tones at $2\omega_1+\omega_2$, $2\omega_2+\omega_1$, $3\omega_1$ and $3\omega_2$ are far away from the band of the amplifier.

then (3) can be approximated as

$$\begin{aligned} v_{out} &= \alpha_1 A_1 \cos(\omega_1 t) + \alpha_1 A_2 \cos(\omega_2 t) \\ &+ \frac{3}{4}\alpha_3 A_1^2 A_2 \cos\left(\left(2\omega_1 - \omega_2\right)t\right) + \frac{3}{4}\alpha_3 A_1 A_2^2 \cos\left(\left(2\omega_2 - \omega_1\right)t\right) \end{aligned} \tag{8}$$

Inter-modulation distortion, IM_3 is defined as the ratio of amplitude of output fundamental tone to the amplitude of the 3^{rd} order product in the output. If $A_1 = A_2 = A$, then

$$IM_3 = \frac{\alpha_1 A}{\frac{3}{4}\alpha_3 A^3} = \frac{4}{3}\frac{\alpha_1}{\alpha_3 A^2} \tag{9}$$

B. *Third Order Intercept Points*

IM_3 is dependent on the amplitude (A) of the input tones. Input 3^{rd}-order intercept point (IIP_3) is defined to characterize the linearity, independent of the input amplitude.

$$IIP_3 = A|_{IM_3=1} = \sqrt{\frac{4\alpha_1}{3\alpha_3}} \tag{10}$$

$$= IM_3^2 A^2 \tag{11}$$

Output Intercept Point OIP_3 is given by

$$OIP_3 = \alpha_1 IIP_3 = \alpha_1\sqrt{\frac{4\alpha_1}{3\alpha_3}} \tag{12}$$

For SpectreRF *pss* simulation, having $A_1 = A_2 = A$ means doing large signal simulation for both the tones. This becomes time-consuming. See SpectreRF user-manual. Instead, assume that $A_1 \gg A_2$, then set the tone at ω_1 as large signal and that at ω_2 as small signal. Then do *pss* and *pac* simulations.

C. IP_3 for modified two-tone test

Assuming $A_1 \gg A_2$, (8) can be further simplified to

$$v_{out} = \alpha_1 A_1 \cos(\omega_1 t) + \alpha_1 A_2 \cos(\omega_2 t) + \frac{3}{4}\alpha_3 A_1^2 A_2 \cos\left((2\omega_1 - \omega_2)\, t\right) \tag{13}$$

The amplitude corresponding to each of the tones in the output can be simulated or measured using a spectrum analyzer. Let

$$V_{L1} = \alpha_1 A_1 \tag{14}$$

$$V_{S1} = \alpha_1 A_2 \tag{15}$$

$$V_{S3} = \frac{3}{4}\alpha_3 A_1^2 A_2 \tag{16}$$

Using (14) and (15) in (16), we get

$$\begin{aligned} V_{S3} &= \frac{3}{4}\alpha_3 \frac{V_{L1}^2}{\alpha_1^2}\frac{V_{S1}}{\alpha_1} = \frac{3}{4}\frac{\alpha_3}{\alpha_1}\frac{V_{L1}^2}{\alpha_1^2} V_{S1} \\ &= \frac{1}{\alpha_1^2 IIP_3^2} V_{L1}^2 V_{S1} = \frac{V_{L1}^2 V_{S1}}{OIP_3^2} \\ OIP_3 &= V_{L1}\sqrt{\frac{V_{S1}}{V_{S3}}} \end{aligned} \tag{17}$$

The assumption, $A_1 \gg A_2$, is used in the following ways:

1) Second signal does not “cause” the nonlinearities. So the approximations made earlier are even more valid!
2) We need not bother about other harmonics/IM products.
3) We can use PSS and PAC for the simulation.

So, in my opinion, this would be a better way even in measurements. But there are other factors also which cause non-linearities/cancellation of non-linearities. So we would **(**not) use very large A_1, A_2 anyway and would extrapolate.

II. EFFECT OF NEGATIVE FEEDBACK ON DISTORTION [1]

Assume a single-ended amplifier (Fig. 1) with the following characteristics.

$$y(t) = \beta_1 u(t) + \beta_2 u^2(t) + \beta_3 u^3(t) \tag{18}$$

The coefficients β_n are given by

$$\beta_n = \frac{1}{n!} \left. \frac{d^n y}{du^n} \right|_{u=0} \tag{19}$$

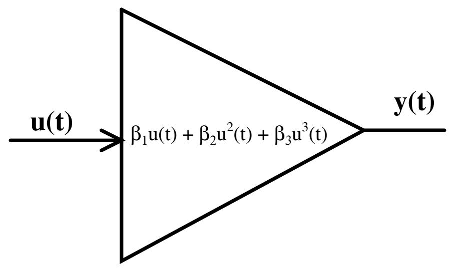
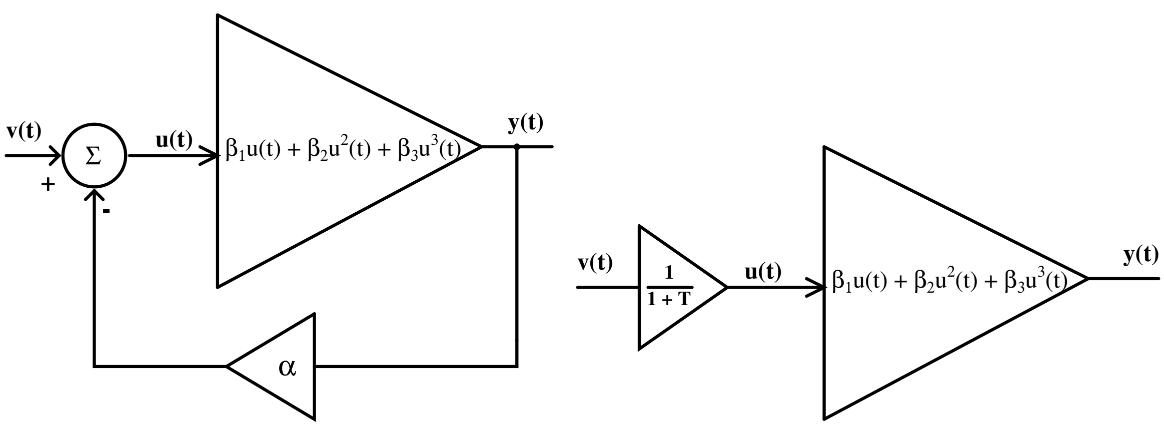


Fig. 1. Amplifier with weak non-linearity.

In the presence of feedback, the system looks like fig. 2(a). This is different from the case where the input is scaled down in which case the IIP_3 of the system will just scale up by the inverse of the attenuation factor. When there is a feedback the input signal seen by



(a) (b)

Fig. 2. (a) Amplifier with weak non-linearity now in a negative feedback system (b) Attenuated input applied to the same amplifier.

the amplifier is given by

$$u(t) = v(t) - \alpha y(t) \tag{20}$$

The output in terms of the input signal $v(t)$ can hence be given by

$$y(t) = \kappa_1 v(t) + \kappa_2 v^2(t) + \kappa_3 v^3(t)$$

where,

$$\kappa_n = \frac{1}{n!} \frac{d^n y}{dv^n}\bigg|_{v=0} \tag{21}$$

We want to derive the relationship between κ_n and β_n as a function of the loop factor $T = \beta_1 \alpha$ (where α is the feedback factor as shown in fig. 2(a)).

Calculation of κ_1

$$\kappa_1 = \frac{dy}{dv}\bigg|_{v=0} = \frac{dy}{du} \frac{du}{dv}\bigg|_{v=0} \quad \text{By defn.}$$

Note that $v = 0 \Leftrightarrow u = 0$, so

$$\begin{aligned}
\kappa_1 &= \frac{dy}{du}\bigg|_{u=0} \times \frac{d}{dv}(v - \alpha y)\bigg|_{v=0} \\
&= \beta_1 \left(1 - \alpha \kappa_1\right) \\
\kappa_1 &= \frac{\beta_1}{(1+T)} \quad \text{where, } T = \alpha\beta_1
\end{aligned} \tag{22}$$

We also make note of these relations which we found above:

$$\frac{du}{dv} = 1 - \alpha \frac{dy}{dv} \tag{23}$$

$$\frac{du}{dv}\bigg|_{v=0} = \frac{1}{1+T} \tag{24}$$

$$\frac{dy}{du}\bigg|_{u=0} = \beta_1 \tag{25}$$

Calculation of κ_2

$$\begin{aligned}
\kappa_2 &= \frac{1}{2!} \frac{d^2 y}{dv^2}\bigg|_{v=0} \quad \text{By defn.} \\
2\kappa_2 &= \frac{d}{dv}\left(\frac{dy}{dv}\right)\bigg|_{v=0} \\
&= \frac{du}{dv} \frac{d}{du}\left(\frac{dy}{dv}\right)\bigg|_{v=0} \\
&= \frac{1}{1+T} \frac{d}{du}\left(\frac{du}{dv} \frac{dy}{du}\right)\bigg|_{v=0} \quad \text{Using eqn. 24}
\end{aligned}$$

Expanding derivative of a product, we can write

$$
\begin{aligned}
2(1+T)\kappa_2 &= \left\{ \frac{dy}{du}\ \frac{d}{du}\left(\frac{du}{dv}\right) + \frac{du}{dv}\ \frac{d^2y}{du^2} \right\}\Bigg|_{v=0} \\
&= \left\{ \frac{dy}{du}\ \frac{d}{du}\left(\frac{du}{dv}\right) + \frac{2\beta_2}{1+T} \right\}\Bigg|_{u=0} \quad \text{Using eqn. 19, 24} \\
&= \beta_1\ \frac{d}{du}\left(\frac{du}{dv}\right)\Bigg|_{u=0} + \frac{2\beta_2}{1+T} \qquad (26)
\end{aligned}
$$

To find $\frac{d}{du}\left(\frac{du}{dv}\right)\Big|_{u=0}$

$$
\begin{aligned}
\frac{d}{du}\left(\frac{du}{dv}\right)\Bigg|_{u=0} &= \frac{d}{du}\left(1-\alpha\frac{dy}{dv}\right)\Bigg|_{u=0} \quad \text{Using eqn. 23} \\
&= -\alpha\frac{d}{du}\left(\frac{dy}{dv}\right)\Bigg|_{v,u=0} \\
&= -\alpha\frac{d}{du}\left\{\frac{d}{dv}\left(\beta_1 u + \beta_2 u^2 + \beta_3 u^3 + \ldots\right)\right\}\Bigg|_{v,u=0} \quad \text{Using eqn. 18} \\
&= -\alpha\frac{d}{du}\left\{\beta_1\frac{du}{dv} + 2\beta_2 u\frac{du}{dv} + \ldots\right\}\Bigg|_{v,u=0} \\
&\approx -\alpha\beta_1\frac{d}{du}\left(\frac{du}{dv}\right) - 2\alpha\beta_2\frac{d}{du}\left(u\frac{du}{dv}\right)\Bigg|_{v,u=0} \\
(1+\alpha\beta_1)\frac{d}{du}\ \frac{du}{dv} &= -2\alpha\beta_2\frac{d}{du}\left(u\frac{du}{dv}\right)\Bigg|_{v,u=0} \\
&= -2\alpha\beta_2\left(\frac{du}{dv} + u\frac{d^2u}{dv^2}\right)\Bigg|_{v,u=0} \\
&= \frac{-2\alpha\beta_2}{1+T} \quad \text{Using eqn. 24} \\
\frac{d}{du}\ \frac{du}{dv}\Bigg|_{u=0} &= \frac{-2\alpha\beta_2}{(1+T)^2} \qquad (27)
\end{aligned}
$$

Coming back to eqn. 26 we get,

$$
\begin{aligned}
2(1+T)\kappa_2 &= \beta_1\frac{-2\alpha\beta_2}{(1+T)^2} + \frac{2\beta_2}{1+T} \\
(1+T)\kappa_2 &= \frac{-\alpha\beta_1\beta_2}{(1+T)^2} + \frac{\beta_2}{1+T} \\
&= \frac{-T\beta_2}{(1+T)^2} + \frac{\beta_2}{1+T} = \frac{\beta_2}{1+T}\left(1-\frac{T}{1+T}\right) \\
\kappa_2 &= \frac{\beta_2}{(1+T)^3} \qquad (28)
\end{aligned}
$$

Calculation of κ_3

$$\begin{aligned}
\kappa_3 &= \frac{1}{3!}\left.\frac{d^3y}{dv^3}\right|_{v=0} \quad \text{By defn.} \\
6\kappa_3 &= \frac{d}{dv}\left\{\frac{d}{dv}\left(\frac{dy}{dv}\right)\right\}\Big|_{v=0} \\
&= \frac{d}{dv}\left\{\frac{d}{dv}\left(\frac{d}{dv}(\beta_1 u + \beta_2 u^2 + \beta_3 u^3)\right)\right\}\Big|_{v=0} \\
&= \frac{d}{dv}\left\{\frac{d}{dv}\left((\beta_1 + 2\beta_2 u + 3\beta_3 u^2)\frac{du}{dv}\right)\right\}\Big|_{u,v=0} \\
&= \frac{d}{dv}\left\{(\beta_1 + 2\beta_2 u + 3\beta_3 u^2)\frac{d^2u}{dv^2} + \left(2\beta_2\frac{du}{dv} + 6\beta_3 u\frac{du}{dv}\right)\frac{du}{dv}\right\} \\
&= \frac{d}{dv}\left\{(\beta_1 + 2\beta_2 u + 3\beta_3 u^2)\frac{d^2u}{dv^2} + (2\beta_2 + 6\beta_3 u)\left(\frac{du}{dv}\right)^2\right\} \\
&= (\beta_1 + 2\beta_2 u + 3\beta_3 u^2)\frac{d^3u}{dv^3} + \left(2\beta_2\frac{du}{dv} + 6\beta_3 u\frac{du}{dv}\right)\frac{d^2u}{dv^2} \\
&\quad + 6\beta_3\left(\frac{du}{dv}\right)^3 + (2\beta_2 + 6\beta_3 u)2\frac{du}{dv}\;\frac{d^2u}{dv^2}
\end{aligned}$$

Putting $u = 0$ above, we get

$$\begin{aligned}
6\kappa_3 &= \beta_1\frac{d^3u}{dv^3} + 2\beta_2\frac{du}{dv}\;\frac{d^2u}{dv^2} + 6\beta_3\left(\frac{du}{dv}\right)^3 + 4\beta_2\;\frac{du}{dv}\;\frac{d^2u}{dv^2} \\
&= \left\{\beta_1\frac{d^3u}{dv^3} + 6\beta_3\left(\frac{du}{dv}\right)^3 + 6\beta_2\;\frac{du}{dv}\;\frac{d^2u}{dv^2}\right\}\Big|_{u=0} \\
6\kappa_3 &= \beta_1\frac{d^3u}{dv^3} + \frac{6\beta_3}{(1+T)^3} + \frac{6\beta_2}{1+T}\;\frac{d^2u}{dv^2} \quad \text{Using eqn. 24} \qquad (29)
\end{aligned}$$

To find, $d^2u/dv^2|_{u,v=0}$:

$$\begin{aligned}
\left.\frac{d^2u}{dv^2}\right|_{u,v=0} &= \frac{d}{dv}\;\frac{du}{dv} = -\alpha\left.\frac{d^2y}{dv^2}\right|_{u,v=0} \quad \text{Using eqn. 23} \qquad (30) \\
&= -2\alpha\kappa_2 \quad \text{Using eqn. 21} \\
&= \frac{-2\alpha\beta_2}{(1+T)^3} \quad \text{Using eqn. 28} \qquad (31)
\end{aligned}$$

To find $d^3u/dv^3|_{u,v=0}$ use eqns. 30 and 21:

$$\left.\frac{d^3u}{dv^3}\right|_{u,v=0} = -\alpha\left.\frac{d^3y}{dv^3}\right|_{u,v=0} = -6\alpha\kappa_3 \qquad (32)$$

So, eqn. (29) becomes,

$$
\begin{aligned}
6\kappa_3 &= -6\alpha\beta_1\kappa_3 + \frac{6\beta_3}{(1+T)^3} + \frac{6\beta_2}{1+T}\,\frac{-2\alpha\beta_2}{(1+T)^3} \\
&= -6T\kappa_3 + \frac{6\beta_3}{(1+T)^3} + \frac{-12\alpha\beta_2^2}{(1+T)^4} \\
(1+T)\kappa_3 &= \frac{\beta_3}{(1+T)^3} + \frac{-2\alpha\beta_2^2}{(1+T)^4} \\
\kappa_3 &= \frac{\beta_3}{(1+T)^4} + \frac{-2\alpha\beta_2^2}{(1+T)^5} \\
\kappa_3 &= \frac{(1+T)\beta_3 - 2\alpha\beta_2^2}{(1+T)^5}
\end{aligned}
$$

So, the IIP_3 for the feedback system is,

$$
\begin{aligned}
IIP_{3fbk} &= \sqrt{\frac{4\kappa_1}{3\kappa_3}} \\
&= \sqrt{\frac{4\beta_1}{1+T} / \frac{3((1+T)\beta_3 - 2\alpha\beta_2^2)}{(1+T)^5}} \\
&= \sqrt{\frac{4\beta_1(1+T)^4}{3\left((1+T)\beta_3 - 2\alpha\beta_2^2\right)}} \\
&\approx \sqrt{\frac{4\beta_1(1+T)^3}{3\beta_3}} \quad \text{for } (\beta_3(1+T) \gg 2\alpha\beta_2^2) \\
IIP_{3fbk} &= IIP_3 \times (1+T)^{\frac{3}{2}} \qquad (33)
\end{aligned}
$$

III. How to simulate IIP_3 using SpectreRF?

A. Low frequency case: Where you do not need 50 ohm source and input matching.

Let's begin with the simplest example — a common-source amplifier with resistive load with no degeneration and no input matching. We will deal in units of dBV.

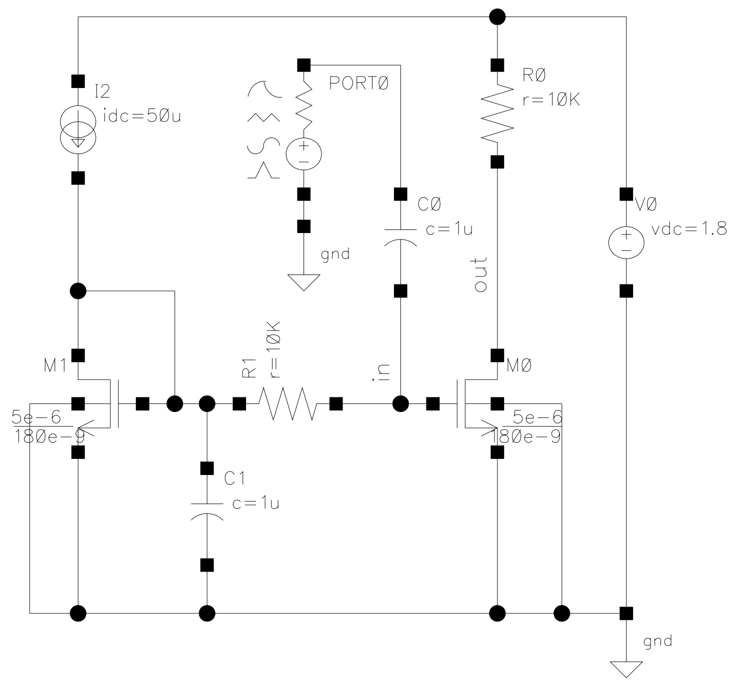
Fig. 3(a) presents a test setup for measuring the IIP3 of my amplifier. First I set the two-tones needed for the test as 10 MHz and 12 MHz (see Fig. 4(a) and 4(b)). The input is specified in terms of peak amplitude. Even though I am using a port in the simulation, it is not required *here*. In the results you will note that I have taken the V_{GS} voltage as my input reference. With ports, unless there is a perfect matching with its input resistance, the voltage you get from it is dependent on the load. For example, since the port is driving a capacitive load here, the voltage at the gate will be 40 mVpp for Vpk set to 20 mVpp in the port properties. I can now run pss to find the steady state response and then proceed to find the 3^{rd}-order amplitude in units of dBV. The pss results are shown in fig. 5. I tabulated the results for different V_{GS} cases and plotted them using MATLAB. The intercept point is shown in fig. 8(a). The IIP_3 for this Common-Source amplifier is about –7.5 dBV.

Next I simulated an amplifier with same nFET but with a source degeneration resistance of 500 Ω and load of 5 kΩ. Figs. 3(b), 6, 7 shows the test setup, the properties of the $vsin$ source used in the setup and the SpectreRF results when two-tone signals of amplitude 350 mV are applied, respectively. The $IIP3$ result is shown in Fig. 8(b). It has improved to about –2.5 dBV.

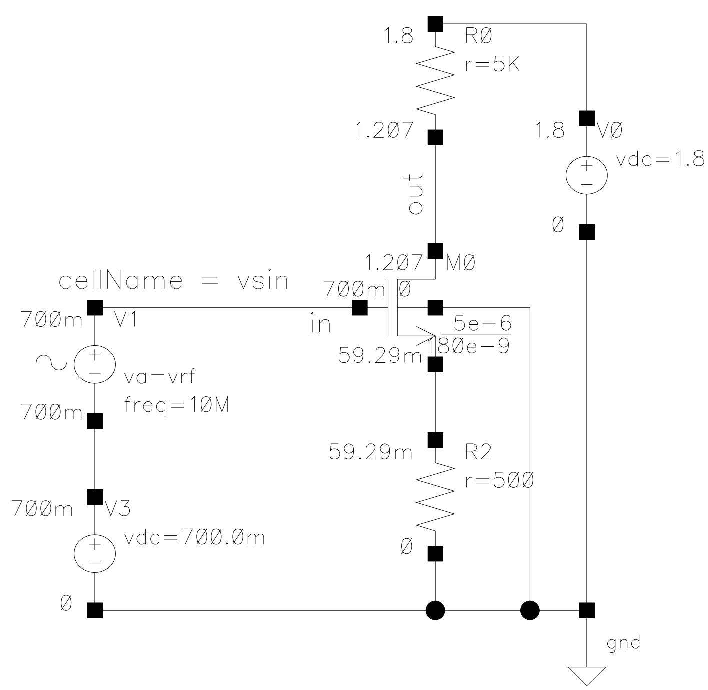
The data and MATLAB code for Common-Source amplifier is attached below.

```
% Results of two-tone test of Common-Source Amplifier
% Circuit: /home/anu/cadence/ee6314_F05/CSampl

vrf =    [1 2 5 10 20 30 40 50 75 100 150]*1e-3;
in =     [-54.02 -48 -40.04 -34.02 -28 -24.48 -21.98 ...
          -20.04 -16.52 -14.02 -10.5];
out10M = [-36.54 -30.52 -22.57 -16.6 -10.79 -7.663 ...
          -5.99 -5.09 -4.064 -3.63 -3.262];
out8M  = [-127.8 -109.74 -85.9 -67.98 -50.28 -38.52 ...
          -28.71 -23.59 -18.3 -16.09 -14.35];
```

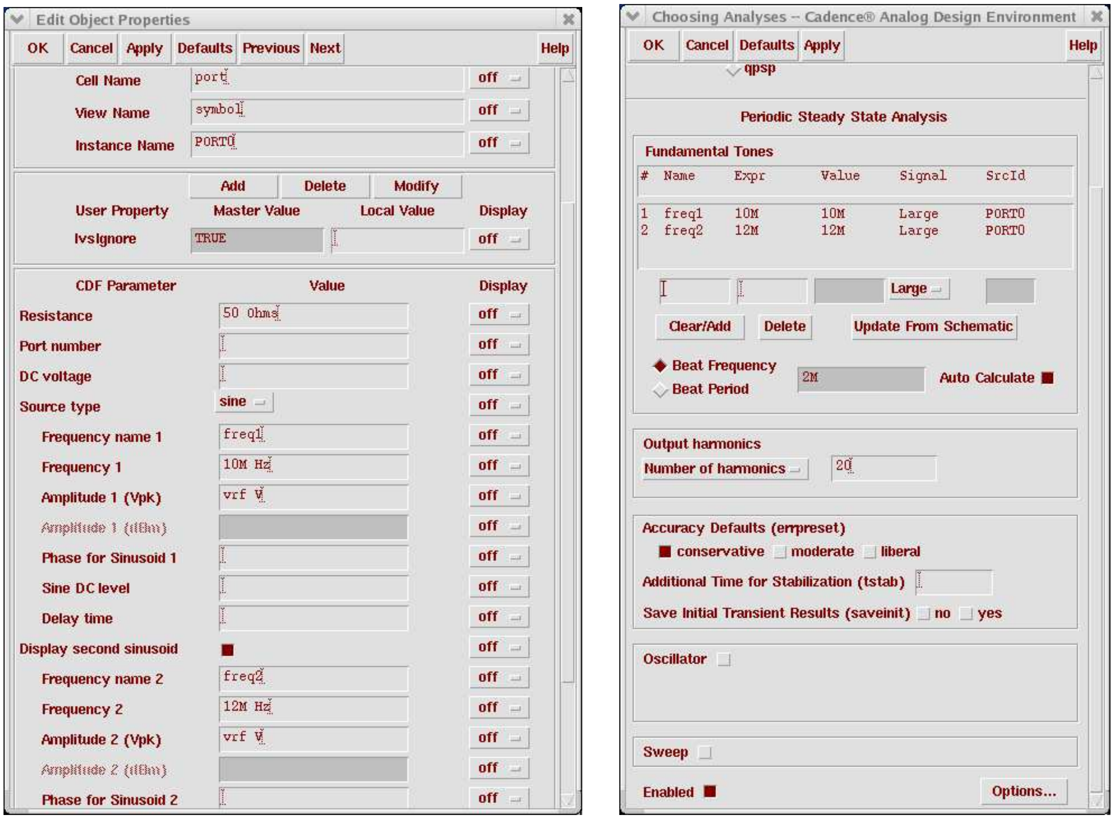


(a)



(b)

Fig. 3. Test setup for IIP3 measurement of (a) Common-Source Amplifier (b) Source-degenerated amplifier.



(a) (b)

Fig. 4. (a) Properties of the port used in the test setup of fig. 3(a) (b) PSS analysis form.

```
im3 = out10M - out8M;

inp =  [-54.02 -48 -40.04 -34.02 -28 -24.48 -21.98 ...
        -20.04 -16.52 -14.02 -10.5 -7.5];
int1 = out10M(1) + (inp-inp(1));
int3 = out8M(1) + 3*(inp-inp(1));


figure;
plot(in,out10M,'k-x','linewidth',2,'markersize',12); hold on;
plot(in,out8M,'k-o','linewidth',2,'markersize',12);
plot(inp,int1,'r-x','linewidth',2,'markersize',8);
plot(inp,int3,'g-o','linewidth',2,'markersize',8); hold off;
```

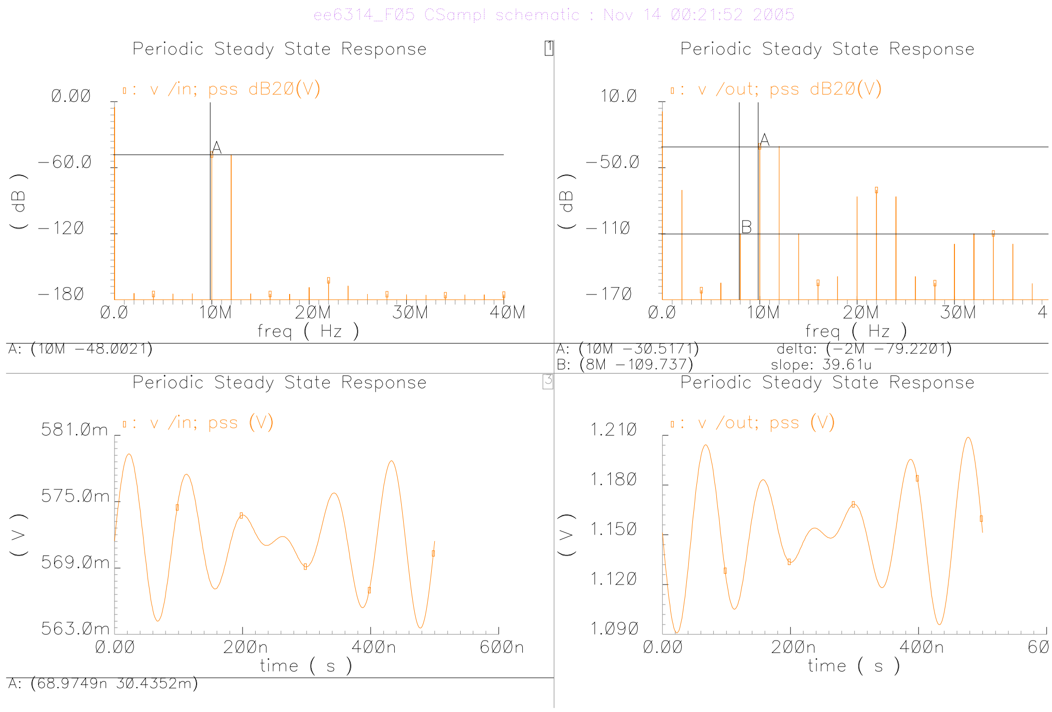


Fig. 5. SpectreRF results for smallest applied V_{GS}.

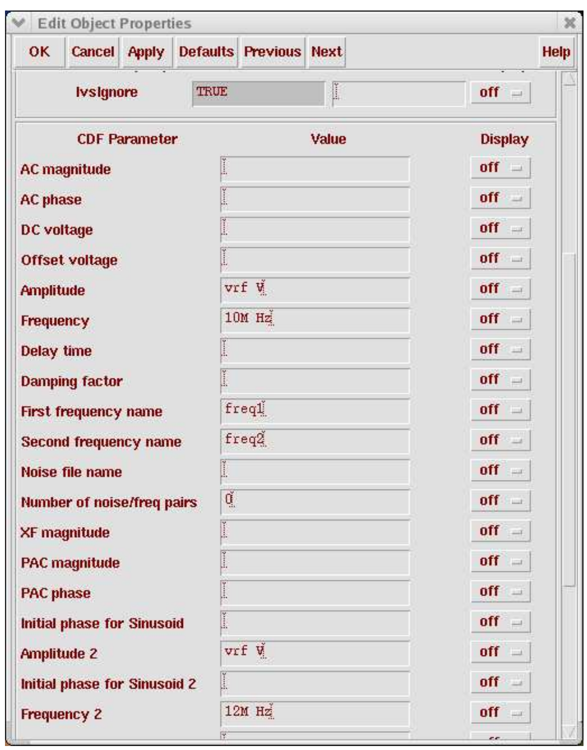


Fig. 6. Properties of the $vsin$ source used in the test setup.

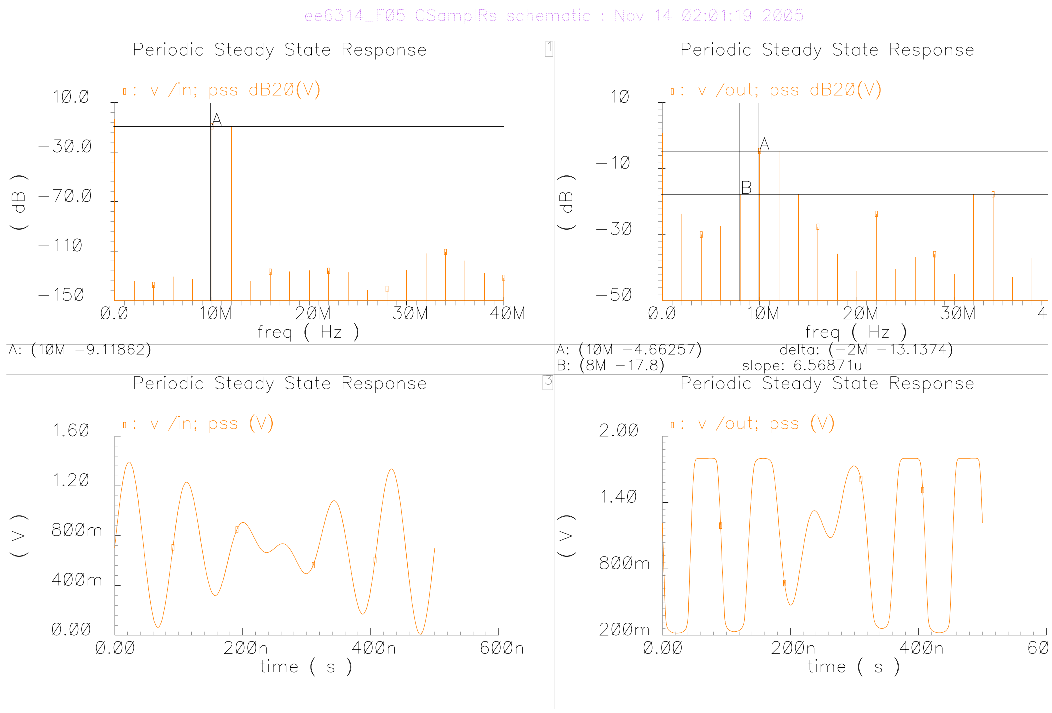
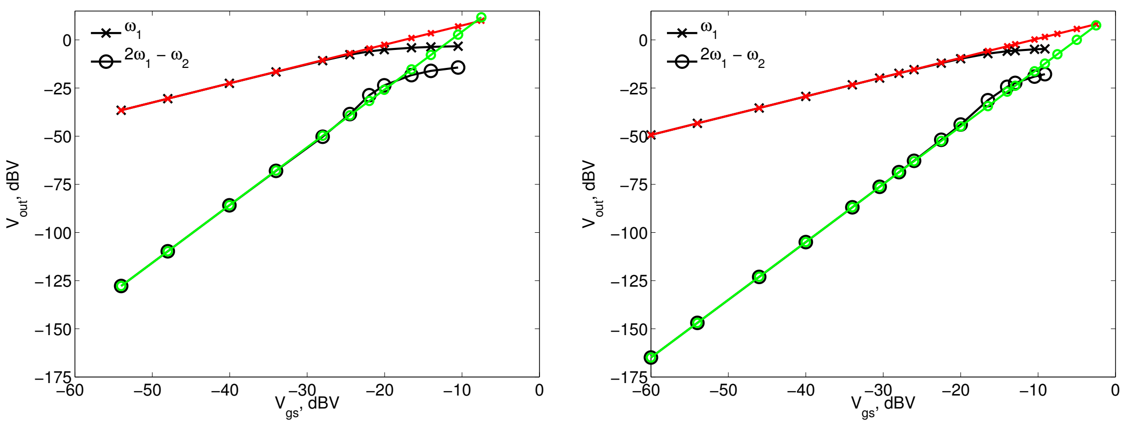


Fig. 7. SpectreRF results for largest applied V_{GS} for fig. 3(b)



(a)

(b)

Fig. 8. IIP_3 comparison for (a) Common-Source amplifier and (b) Source degenerated amplifier.

```
set(gca,'ylim',[-175 15],'ytick',[-175:25:0],'fontsize',16);
L = legend('\omega_1','2\omega_1 - \omega_2');
set(L,'box','off','fontsize',16,'location','northwest');
xlabel('V_{gs}, dBV','fontsize',16);
ylabel('V_{out}, dBV','fontsize',16);
% Results in an IIP3 of -7.5 dBV
print -depsc CSamplIIP3.eps
```

B. RF case: Where you do need 50 ohm source and input matching.

REFERENCES

[1] W. Sansen, "Distortion in Elementary transistor circuits," *IEEE Trans. on Circuits and Systems - II: Analog and Digital Signal Processing*, vol. 46, pp. 315–324, Mar. 1999.