$3^{rd}$-order Intercept Point

Anuranjan Jha,
Columbia Integrated Systems Lab,
Department of Electrical Engineering,
Columbia University, New York, NY

Last Revised: September 12, 2006

These derivations are for my documentation.

I. $IIP_3$ OF A DIFFERENTIAL AMPLIFIER

We want to characterize the non-linearity of a differential amplifier. Its $3^{rd}$-order non-linearity is of most significance.

Assume that the amplifier output characteristic is given by

$$v_{out}(t) = \alpha_1 v_{in} + \alpha_3 v_{in}^3$$  \hspace{1cm} (1)

where higher order non-linearities have been neglected.

The $3^{rd}$-order intercept point is defined as the point on $P_{out}$ vs $P_{in}$ plane where the lines corresponding to fundamental and the third order inter-modulation product intersect. It can also be defined on $V_{out}$ vs $V_{in}$ plane.

A. Inter-modulation Product: IM

Consider a two-tone test where you have applied

$$v_{in} = A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t)$$  \hspace{1cm} (2)

to the amplifier. For the given amplifier transfer characteristics, this will generate tones at the fundamental frequencies ($\omega_1$, $\omega_2$), their $3^{rd}$ harmonics ($3\omega_1$, $3\omega_2$) and also two tones
at the inter-modulation frequencies — $2\omega_1 - \omega_2$ and $2\omega_2 - \omega_1$.

$$v_{\text{out}} = \left( \alpha_1 A_1 + \frac{3}{4} \alpha_3 A_1^3 + \frac{3}{2} \alpha_3 A_1 A_2^2 \right) \cos(\omega_1 t) + \left( \alpha_1 A_2 + \frac{3}{4} \alpha_3 A_2^3 + \frac{3}{2} \alpha_3 A_1^2 A_2 \right) \cos(\omega_2 t)$$

$$+ \frac{\alpha_3}{4} A_1^3 \cos(3\omega_1 t) + \frac{\alpha_3}{4} A_2^3 \cos(3\omega_2 t)$$

$$+ \frac{3}{4} \alpha_3 A_1^2 A_2 \cos((2\omega_1 - \omega_2) t) + \frac{3}{4} \alpha_3 A_1 A_2^2 \cos((2\omega_2 - \omega_1) t)$$

$$+ \frac{3}{4} \alpha_3 A_1^2 A_2 \cos((2\omega_1 + \omega_2) t) + \frac{3}{4} \alpha_3 A_1 A_2^2 \cos((2\omega_2 + \omega_1) t)$$

(3)

The following relationships have been used.

$$\cos^3(\omega t) = \frac{1}{4} (3 \cos(\omega t) + \cos(3\omega t))$$

(4)

$$\sin^3(\omega t) = \frac{1}{4} (3 \sin(\omega t) - \sin(3\omega t))$$

(5)

$$\cos^2(\omega t) = \frac{1}{2} (1 + \cos(2\omega t))$$

(6)

$$\sin^2(\omega t) = \frac{1}{2} (1 - \cos(2\omega t))$$

(7)

Assuming,

1) $\alpha_3 \ll \alpha_1$

2) $A_1$ and $A_2$ are not large enough to compress the gain at $\omega_1$ and $\omega_2$

3) Tones at $2\omega_1 + \omega_2$, $2\omega_2 + \omega_1$, $3\omega_1$ and $3\omega_2$ are far away from the band of the amplifier.

then (3) can be approximated as

$$v_{\text{out}} = \alpha_1 A_1 \cos(\omega_1 t) + \alpha_1 A_2 \cos(\omega_2 t)$$

$$+ \frac{3}{4} \alpha_3 A_1^2 A_2 \cos((2\omega_1 - \omega_2) t) + \frac{3}{4} \alpha_3 A_1 A_2^2 \cos((2\omega_2 - \omega_1) t)$$

(8)

Inter-modulation distortion, $IM_3$ is defined as the ratio of amplitude of output fundamental tone to the amplitude of the $3^{rd}$ order product in the output. If $A_1 = A_2 = A$, then

$$IM_3 = \frac{\alpha_1 A}{\frac{3}{4} \alpha_3 A^2} = \frac{4}{3} \frac{\alpha_1}{\alpha_3 A^2}$$

(9)

B. Third Order Intercept Points

$IM_3$ is dependent on the amplitude ($A$) of the input tones. Input $3^{rd}$-order intercept point ($IIP_3$) is defined to characterize the linearity, independent of the input amplitude.

$$IIP_3 = A_{IM_3=1} = \sqrt{\frac{4\alpha_1}{3\alpha_3}}$$

(10)

$$= IM_3^2 A^2$$

(11)
Output Intercept Point $OIP_3$ is given by

$$OIP_3 = \alpha_1 IIP_3 = \alpha_1 \sqrt{\frac{4\alpha_1}{3\alpha_3}}$$  \hspace{1cm} (12)

For SpectreRF $pss$ simulation, having $A_1 = A_2 = A$ means doing large signal simulation for both the tones. This becomes time-consuming. See SpectreRF user-manual. Instead, assume that $A_1 \gg A_2$, then set the tone at $\omega_1$ as large signal and that at $\omega_2$ as small signal. Then do $pss$ and $pac$ simulations.

**C. $IP_3$ for modified two-tone test**

Assuming $A_1 \gg A_2$, (8) can be further simplified to

$$v_{out} = \alpha_1 A_1 \cos(\omega_1 t) + \alpha_1 A_2 \cos(\omega_2 t) + \frac{3}{4} \alpha_3 A_1^2 A_2 \cos((2\omega_1 - \omega_2) t)$$  \hspace{1cm} (13)

The amplitude corresponding to each of the tones in the output can be simulated or measured using a spectrum analyzer. Let

$$V_{L1} = \alpha_1 A_1$$  \hspace{1cm} (14)

$$V_{S1} = \alpha_1 A_2$$  \hspace{1cm} (15)

$$V_{S3} = \frac{3}{4} \alpha_3 A_1^2 A_2$$  \hspace{1cm} (16)

Using (14) and (15) in (16), we get

$$V_{S3} = \frac{3}{4} \alpha_3 \frac{V_{L1}^2 V_{S1}}{\alpha_1} = \frac{3}{4} \alpha_3 \frac{V_{L1}^2 V_{S1}}{\alpha_1}$$

$$= \frac{1}{\alpha_1^2 IIP_3^2} V_{L1} V_{S1} = \frac{V_{L1}^2 V_{S1}}{OIP_3^2}$$

$$OIP_3 = V_{L1} \sqrt{\frac{V_{S1}}{V_{S3}}}$$  \hspace{1cm} (17)

The assumption, $A_1 \gg A_2$, is used in the following ways:

1) Second signal does not “cause” the nonlinearities. So the approximations made earlier are even more valid!

2) We need not bother about other harmonics/IM products.

3) We can use PSS and PAC for the simulation.

So, in my opinion, this would be a better way even in measurements. But there are other factors also which cause non-linearities/cancellation of non-linearities. So we would (not) use very large $A_1$, $A_2$ anyway and would extrapolate.

©Anuranjan Jha - Columbia University – 2006 3
II. Effect of Negative Feedback on Distortion [1]

Assume a single-ended amplifier (Fig. 1) with the following characteristics.

\[ y(t) = \beta_1 u(t) + \beta_2 u^2(t) + \beta_3 u^3(t) \]  \hspace{1cm} (18)

The coefficients \( \beta_n \) are given by

\[ \beta_n = \frac{1}{n!} \left. \frac{d^n y}{du^n} \right|_{u=0} \]  \hspace{1cm} (19)

In the presence of feedback, the system looks like fig. 2(a). This is different from the case where the input is scaled down in which case the \( IIP_3 \) of the system will just scale up by the inverse of the attenuation factor. When there is a feedback the input signal seen by the amplifier is given by

\[ u(t) = v(t) - \alpha y(t) \]  \hspace{1cm} (20)

Fig. 1. Amplifier with weak non-linearity.

Fig. 2. (a) Amplifier with weak non-linearity now in a negative feedback system (b) Attenuated input applied to the same amplifier.
The output in terms of the input signal $v(t)$ can hence be given by

$$y(t) = \kappa_1 v(t) + \kappa_2 v^2(t) + \kappa_3 v^3(t)$$

where,

$$\kappa_n = \frac{1}{n!} \left. \frac{d^n y}{dv^n} \right|_{v=0}$$  \hspace{2cm} (21)

We want to derive the relationship between $\kappa_n$ and $\beta_n$ as a function of the loop factor $T = \beta_1 \alpha$ (where $\alpha$ is the feedback factor as shown in fig. 2(a)).

**Calculation of $\kappa_1$**

$$\kappa_1 = \left. \frac{dy}{dv} \right|_{v=0} = \left. \frac{dy}{du} \frac{du}{dv} \right|_{v=0}$$  \hspace{2cm} By defn.

Note that $v = 0 \iff u = 0$, so

$$\kappa_1 = \left. \frac{dy}{du} \right|_{u=0} \times \left. \frac{d}{dv} (v - \alpha y) \right|_{v=0}$$

$$= \beta_1 \left( 1 - \alpha \kappa_1 \right)$$

$$\kappa_1 = \frac{\beta_1}{(1 + T)}$$  \hspace{2cm} where, $T = \alpha \beta_1$ \hspace{2cm} (22)

We also make note of these relations which we found above:

$$\left. \frac{du}{dv} \right|_{v=0} = 1 - \alpha \frac{dy}{dv}$$  \hspace{2cm} (23)

$$\left. \frac{du}{dv} \right|_{v=0} = \frac{1}{1 + T}$$  \hspace{2cm} (24)

$$\left. \frac{dy}{du} \right|_{u=0} = \beta_1$$  \hspace{2cm} (25)

**Calculation of $\kappa_2$**

$$\kappa_2 = \frac{1}{2!} \left. \frac{d^2 y}{dv^2} \right|_{v=0}$$  \hspace{2cm} By defn.

$$2\kappa_2 = \left. \frac{d}{dv} \left( \frac{dy}{dv} \right) \right|_{v=0}$$

$$= \left. \frac{du}{dv} \frac{d}{dv} \left( \frac{dy}{dv} \right) \right|_{v=0}$$

$$= \frac{1}{1 + T} \left. \frac{d}{du} \left( \frac{du}{dv} \frac{dy}{dv} \frac{du}{dv} \right) \right|_{v=0}$$  \hspace{2cm} Using eqn. 24
Expanding derivative of a product, we can write

\[
2(1 + T) \kappa_2 = \left\{ \frac{dy}{du} \frac{d}{du} \left( \frac{du}{dv} \right) + \frac{du}{dv} \frac{d^2 y}{du^2} \right\} \bigg|_{v=0} = \left\{ \frac{dy}{du} \frac{d}{du} \left( \frac{du}{dv} \right) + \frac{2 \beta_2}{1 + T} \right\} \bigg|_{u=0} \quad \text{Using eqn. 19, 24}
\]

\[
= \beta_1 \frac{d}{du} \left( \frac{du}{dv} \right) \bigg|_{u=0} + \frac{2 \beta_2}{1 + T} \quad (26)
\]

To find \( \frac{d}{du} \left( \frac{du}{dv} \right) \bigg|_{u=0} \)

\[
\frac{d}{du} \left( \frac{du}{dv} \right) \bigg|_{u=0} = \frac{d}{du} \left( 1 - \alpha \frac{dy}{dv} \right) \bigg|_{u=0} \quad \text{Using eqn. 23}
\]

\[
= -\alpha \frac{d}{du} \left( \frac{dy}{dv} \right) \bigg|_{v,u=0}
\]

\[
= -\alpha \frac{d}{du} \left\{ \frac{d}{dv} \left( \beta_1 u + \beta_2 u^2 + \beta_3 u^3 + \ldots \right) \right\} \bigg|_{v,u=0} \quad \text{Using eqn. 18}
\]

\[
= -\alpha \frac{d}{du} \left\{ \beta_1 \frac{du}{dv} + 2 \beta_2 \frac{du}{dv} + \ldots \right\} \bigg|_{v,u=0}
\]

\[
\approx -\alpha \beta_1 \frac{d}{du} \left( \frac{du}{dv} \right) - 2 \alpha \beta_2 \frac{d}{du} \left( \frac{du}{dv} \right) \bigg|_{v,u=0}
\]

\[
(1 + \alpha \beta_1) \frac{d}{du} \frac{du}{dv} = -2 \alpha \beta_2 \frac{d}{dv} \left( \frac{du}{dv} \right) \bigg|_{v,u=0}
\]

\[
= -2 \alpha \beta_2 \frac{d}{dv} \left( \frac{du}{dv} + \frac{d^2 u}{dv^2} \right) \bigg|_{v,u=0}
\]

\[
= -\frac{2 \alpha \beta_2}{1 + T} \quad \text{Using eqn. 24}
\]

\[
\frac{d}{du} \frac{du}{dv} \bigg|_{u=0} = -\frac{2 \alpha \beta_2}{(1 + T)^2} \quad (27)
\]

Coming back to eqn. 26 we get,

\[
2(1 + T) \kappa_2 = \beta_1 \frac{-2 \alpha \beta_2}{(1 + T)^2} + \frac{2 \beta_2}{1 + T}
\]

\[
(1 + T) \kappa_2 = -\alpha \beta_1 \beta_2 \frac{1}{(1 + T)^2} + \frac{\beta_2}{1 + T}
\]

\[
= -\frac{T \beta_2}{(1 + T)^2} + \frac{\beta_2}{1 + T} = \frac{\beta_2}{1 + T} \left( 1 - \frac{T}{1 + T} \right)
\]

\[
\kappa_2 = \frac{\beta_2}{(1 + T)^3} \quad (28)
\]
Calculation of $\kappa_3$

$$
\kappa_3 = \frac{1}{3!} \left. \frac{d^3 y}{dv^3} \right|_{v=0} \quad \text{By defn.}
$$

$$
6\kappa_3 = \left. \frac{d}{dv} \left\{ \frac{d}{dv} \left( \frac{dy}{dv} \right) \right\} \right|_{v=0}
$$

$$
= \left. \frac{d}{dv} \left\{ \frac{d}{dv} \left( \beta_1 u + \beta_2 u^2 + \beta_3 u^3 \right) \right\} \right|_{v=0}
$$

$$
= \left. \frac{d}{dv} \left\{ \left( \beta_1 + 2\beta_2 u + 3\beta_3 u^2 \right) \frac{du}{dv} \right\} \right|_{u,v=0}
$$

$$
= \left. \frac{d}{dv} \left\{ (\beta_1 + 2\beta_2 u + 3\beta_3 u^2) \frac{d^2 u}{dv^2} + \left( 2\beta_2 \frac{du}{dv} + 6\beta_3 u \frac{du}{dv} \right) \frac{du}{dv} \right\} \right|_{v=0}
$$

$$
= (\beta_1 + 2\beta_2 u + 3\beta_3 u^2) \frac{d^3 u}{dv^3} + \left( 2\beta_2 \frac{du}{dv} + 6\beta_3 u \frac{du}{dv} \right) \frac{d^2 u}{dv^2}
$$

$$
+ 6\beta_3 \left( \frac{du}{dv} \right)^3 + (2\beta_2 + 6\beta_3 u) \frac{du}{dv} \frac{d^2 u}{dv^2}
$$

Putting $u = 0$ above, we get

$$
6\kappa_3 = \beta_1 \frac{d^3 u}{dv^3} + 2\beta_2 \frac{du}{dv} \frac{d^2 u}{dv^2} + 6\beta_3 \left( \frac{du}{dv} \right)^3 + 4\beta_2 \frac{du}{dv} \frac{d^2 u}{dv^2}
$$

$$
= \left\{ \beta_1 \frac{d^3 u}{dv^3} + 6\beta_3 \left( \frac{du}{dv} \right)^3 + 6\beta_2 \frac{du}{dv} \frac{d^2 u}{dv^2} \right\} \bigg|_{u=0}
$$

$$
6\kappa_3 = \beta_1 \frac{d^3 u}{dv^3} + \frac{6\beta_3}{(1+T)^3} + \frac{6\beta_2}{1+T} \frac{d^2 u}{dv^2} \quad \text{Using eqn. 24} \quad (29)
$$

To find, $\frac{d^2 u}{dv^2} \big|_{u,v=0}$:

$$
\left. \frac{d^2 u}{dv^2} \right|_{u,v=0} = \left. \frac{d}{dv} \frac{du}{dv} \right|_{u,v=0} = -\alpha \left. \frac{d^2 y}{dv^2} \right|_{u,v=0} \quad \text{Using eqn. 23} \quad (30)
$$

$$
= -2\alpha \kappa_2 \quad \text{Using eqn. 21}
$$

$$
= -2\alpha \beta_2 \quad \text{Using eqn. 28} \quad (31)
$$

To find $\frac{d^3 u}{dv^3} \big|_{u,v=0}$ use eqns. 30 and 21:

$$
\left. \frac{d^3 u}{dv^3} \right|_{u,v=0} = -\alpha \left. \frac{d^3 y}{dv^3} \right|_{u,v=0} = -6\alpha \kappa_3 \quad (32)
$$
So, eqn. (29) becomes,

\[
6\kappa_3 = -6\alpha\beta_1\kappa_3 + \frac{6\beta_3}{(1+T)^3} + \frac{6\beta_2}{1+T} \frac{-2\alpha\beta_2}{(1+T)^3}
\]

\[
= -6T\kappa_3 + \frac{6\beta_3}{(1+T)^3} + \frac{-12\alpha\beta_2^2}{(1+T)^4}
\]

\[
(1+T)\kappa_3 = \frac{\beta_3}{(1+T)^3} + \frac{-2\alpha\beta_2^2}{(1+T)^4}
\]

\[
\kappa_3 = \frac{\beta_3}{(1+T)^4} + \frac{-2\alpha\beta_2^2}{(1+T)^5}
\]

\[
\kappa_3 = \frac{(1+T)\beta_3 - 2\alpha\beta_2^2}{(1+T)^5}
\]

So, the \(IIP_3\) for the feedback system is,

\[
IIP_{3fbk} = \sqrt{\frac{4\kappa_1}{3\kappa_3}}
\]

\[
= \sqrt{\frac{4\beta_3((1+T)\beta_3 - 2\alpha\beta_2^2)}{1+T}}
\]

\[
= \sqrt{\frac{4\beta_3(1+T)^4}{3((1+T)\beta_3 - 2\alpha\beta_2^2)}}
\]

\[
\approx \sqrt{\frac{4\beta_3(1+T)^3}{3\beta_3}} \text{ for } (\beta_3(1+T) \gg 2\alpha\beta_2^2)
\]

\[
IIP_{3fbk} = IIP_3 \times (1+T)^{\frac{3}{2}}
\]  
(33)
III. How to simulate $IIP_3$ using SpectreRF?

A. Low frequency case: Where you do not need 50 ohm source and input matching.

Let’s begin with the simplest example — a common-source amplifier with resistive load with no degeneration and no input matching. We will deal in units of dBV.

Fig. 3(a) presents a test setup for measuring the $IIP_3$ of my amplifier. First I set the two-tones needed for the test as 10 MHz and 12 MHz (see Fig. 4(a) and 4(b)). The input is specified in terms of peak amplitude. Even though I am using a port in the simulation, it is not required here. In the results you will note that I have taken the $V_{GS}$ voltage as my input reference. With ports, unless there is a perfect matching with its input resistance, the voltage you get from it is dependent on the load. For example, since the port is driving a capacitive load here, the voltage at the gate will be 40 mVpp for $V_{pk}$ set to 20 mVpp in the port properties. I can now run pss to find the steady state response and then proceed to find the 3rd-order amplitude in units of dBV. The pss results are shown in fig. 5. I tabulated the results for different $V_{GS}$ cases and plotted them using MATLAB. The intercept point is shown in fig. 8(a). The $IIP_3$ for this Common-Source amplifier is about $-7.5$ dBV.

Next I simulated an amplifier with same nFET but with a source degeneration resistance of 500 $\Omega$ and load of 5 k$\Omega$. Figs. 3(b), 6, 7 shows the test setup, the properties of the $v_{sin}$ source used in the setup and the SpectreRF results when two-tone signals of amplitude 350 mV are applied, respectively. The $IIP_3$ result is shown in Fig. 8(b). It has improved to about $-2.5$ dBV.

The data and MATLAB code for Common-Source amplifier is attached below.

```matlab
vrf = [1 2 5 10 20 30 40 50 75 100 150]*1e-3;
in = [-54.02 -48 -40.04 -34.02 -28 -24.48 -21.98 ... -20.04 -16.52 -14.02 -10.5];
out10M = [-36.54 -30.52 -22.57 -16.6 -10.79 -7.663 ... -5.99 -5.09 -4.064 -3.63 -3.262];
out8M = [-127.8 -109.74 -85.9 -67.98 -50.28 -38.52 ... -28.71 -23.59 -18.3 -16.09 -14.35];
```

©Anuranjan Jha - Columbia University – 2006
Fig. 3. Test setup for IIP3 measurement of (a) Common-Source Amplifier (b) Source-degenerated amplifier.
Fig. 4. (a) Properties of the port used in the test setup of fig. 3(a) (b) PSS analysis form.

\[ im3 = \text{out10M} - \text{out8M}; \]

\[ \text{inp} = [-54.02 -48 -40.04 -34.02 -28 -24.48 -21.98 \ldots -20.04 -16.52 -14.02 -10.5 -7.5]; \]

\[ \text{int1} = \text{out10M}(1) + (\text{inp}-\text{inp}(1)); \]

\[ \text{int3} = \text{out8M}(1) + 3*(\text{inp}-\text{inp}(1)); \]

\[ \text{figure}; \]
\[ \text{plot(in, out10M,'k-x','linewidth',2,'markersize',12); hold on;} \]
\[ \text{plot(in, out8M,'k-o','linewidth',2,'markersize',12);} \]
\[ \text{plot(inp,int1,'r-x','linewidth',2,'markersize',8);} \]
\[ \text{plot(inp,int3,'g-o','linewidth',2,'markersize',8); hold off;} \]
Fig. 5. SpectreRF results for smallest applied $V_{GS}$.

Fig. 6. Properties of the $v_{sin}$ source used in the test setup.
Fig. 7. SpectreRF results for largest applied $V_{GS}$ for fig. 3(b)

Fig. 8. $IIP_3$ comparison for (a) Common-Source amplifier and (b) Source degenerated amplifier.
B. RF case: Where you do need 50 ohm source and input matching.

REFERENCES