Differentiable Manifolds Lectures

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March 22, 2013

1 First Lecture

Definition 1.1. A function f defined on $\mathbb{R}^n$ is $\mathcal{C}^k$ for a positive integer k if

$$\frac{\partial^l f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_l}}$$

exists and is ontinuous for any positive integer $l \leq k$. A function is $\mathcal{C}^\infty$ if f is $\mathcal{C}^k$ $\forall k \in \mathbb{N}$

Definition 1.2. A *homeomorphism* is a continuous function with continuous inverse.

Definition 1.3. A coordinate chart (U, φ) on a topological space X is an open set $U \subset X$ with a map $\varphi : U \to \mathbb{R}^n$ such that φ is a homeomorphism onto $\varphi(U)$.

Let X be a topological space with a coordinate charts (U, φ), (V, ψ) and a function $f : I \to \mathbb{R}$. Consider $f \circ \varphi^{-1}$ as a function defined on $\varphi(U)$ and differentiable $f \circ \varphi^{-1}$. **Question** : Is $f \circ \varphi^{-1}$ differentiable the same as $f \circ \psi^{-1}$ differentiable?

$$f \circ \varphi^{-1} = f \circ \psi^{-1} \circ (\psi \circ \varphi^{-1}) \qquad f \circ \psi^{-1} = f \circ \varphi^{-1} \circ (\varphi \circ \psi^{-1}).$$

From this we see that the important thing is that we need to make sire that both $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are differentiable.

Definition 1.4. The coordinate charts (U, φ) and (V, ψ) are said to be compatible if the transition maps are *diffeomorphisms*.

Definition 1.5. An atlas $\mathcal{A}$ for a topological space X is a collection of coordinate charts such that any two coordinate charts in $\mathcal{A}$ are smoothly compatible.

2 Second Lecture

Given a topological space X, a coordinate chart (U, φ) is a double such that $U \subset X$ is open, $\varphi(U)$ is open, and $\varphi : U \to \mathbb{R}^n$ is a homeomorphism.

Definition 2.1. We say that the coordinate charts (U, φ) and (V, ψ) are *smoothly compatible* if $U \cap V = \emptyset$ or $\psi \circ \varphi^{-1}$ is a diffeomorphism from $\varphi(U \cap V)$ to $\psi(U \cap V)$.

Definition 2.2. A *diffeomorphism* is a homeomorphism between two manifolds that is differentiable ($\mathcal{C}^\infty$) whose inverse is also differentiable.

Definition 2.3. An atlas $\mathcal{A}$ for a topological space X is a collection of coordinate charts that cover X such that any two coordinate charts in $\mathcal{A}$ are smoothly compatible.

Example 2.4. Consider the set of all lines through the origin in $\mathbb{R}^3$. This set is called a projective space $\mathbb{RP}^2$. Since we can represent each line with more than one vector, this set is the same as the quotient space $\mathbb{R}^3 \backslash \{(0,0,0)\}/\sim \qquad (x_1, x_2, x_3) \sim \lambda(x_1, x_2, x_3)$. This space is equipped with the quotient topology. i.e., $\pi : \mathbb{R}^3\backslash\{(0,0,0)\} \to \mathbb{RP}^2$ is continuous. We use $[x_1, x_2, x_3]$ to denote the equivalence class of vectors (x_1, x_2, x_3). On the set $U_1 := \{x_1 \neq 0\}$, we can use the coordinate chart

$$\begin{gathered} \varphi_1 : U_1 \to \mathbb{R}^2 \\ (x_1, x_2, x_3) \mapsto \left(\tfrac{x_2}{x_1}, \tfrac{x_3}{x_1}\right) \end{gathered} .$$

Consider the similar, corresponding coordinate charts $(U_2, \varphi_2), (U_3, \varphi_3)$. **Claim:** $\mathcal{A} = \{(U_i, \varphi_i)\}_{i=1,2,3}$ is an atlas for $\mathbb{RP}^2$. We need to check that $\varphi_2 \circ \varphi_1^{-1}$ is a diffeomorphism.

$$\begin{array}{ccc} (x_1, x_2, x_3) & \xrightarrow{\varphi_1} & (x_2/x_1, x_3/x_1) \\ \Big\downarrow{\scriptstyle \varphi_2} & & \\ (x_1/x_2, x_3/x_2) & & \end{array}$$

From this we see that $\varphi_2 \circ \varphi_1^{-1}(x_2/x_1, x_3/x_1) = (x_1/x_2, x_3/x_2)$. We choose coordinates $(u, v) \in \mathbb{R}^2$ and write $\varphi_2 \circ \varphi_1^{-1}$ in terms of (u, v). Note that $U_1 \cap U_2 = \{x_1 \neq 0, x_2 \neq 0\}$. Using $x_2/x_1 = u$ and $x_3/x_1 = v$, we obtain

$$\varphi_2 \circ \varphi_1^{-1}(u, v) = \left(\frac{x_1}{x_2}, \frac{v}{u}\right).$$

This is, in fact, differentiable and homeomorphic.

Definition 2.5. Two atlases are *compatible (or equivalent)* if their union is another atlas.

Definition 2.6. A *differentiable (or smooth) structure* on a topological space is an equivalence class of atlases. This is also called a *maximal atlas.*

Definition 2.7. A topological space X is said to be *Hausdorff* if any two points can be separated by disjoint open sets. $\forall p, q \in X$, $\exists U, V$ open s.t. $p \in U, q \in V$ and $U \cap V = \emptyset$.

Definition 2.8. A topological space X is said to be *2nd countable* if it has a countable basis $\mathcal{B}$ of open sets.

Definition 2.9. A *basis* $\mathcal{B}$ is a subset of the collection of all open sets (sometimes the topology) such that any open set can be written as a union of elements of $\mathcal{B}$.

Example 2.10. $\mathbb{R}^n$ is 2nd countable because we can take $\mathcal{B}$ to be the collation of balls of rational radii centered at rational points.

Example 2.11. $\mathbb{R}^n$ is Hausdorff (for two points $p, q \in \mathbb{R}^n$, take balls of radii $1/3|p - q|$).

Remark. A subspace of a topological space that is Hausdorff and 2nd countable is one itself.

Definition 2.12. A *differentiable manifold* is a topological space that is Hausdorff, 2nd countable, and has a differentiable structure.

3 Third Lecture

Definition 3.1. A *level set* of a real-valued function f of n variables is a set of the form

$$L_c(f) = \{(x_1, \ldots, x_n) | f(x_1, \ldots, x_n) = c\}.$$

Example 3.2. Define $F(x, y, z) = x^2 + y^2 + z^2$ and $X = \{(x, y, z) | x^2 + y^2 + z^2 = 1\}$. Then X is a level set of F.

Let $F : U \to \mathbb{R}$ be a smooth map, where $U \subset \mathbb{R}^n$ is open. In fact, $F^{-1}(c)$ is always closed, Hausdorff, and 2nd countable. The question is when is $F^{-1}(c)$ a smooth manifold? In other words, when can we solve one variable in terms of the other? $F(x, y, z) = c \Longrightarrow z = f(x, y)$.

Theorem 3.3. *Let $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a smooth function. Let $\mathbb{R}^{n+m}$ have coordinates (x, y) such that $x \in \mathbb{R}^n, y \in \mathbb{R}^m$. Fix a point (a, b) such that $F(a, b) = c$ and $c \in \mathbb{R}^m$. If $\nabla_y F(a, b)$ is invertible (i.e.* **rank** *$D_F(a, b) = m$), then there exists open sets U, V such that $a \in U, b \in V$, and a unique, bijective, and smooth function $g : U \to V$ such that $g(a) = b$ and*

$$\{(x, g(x)) | x \in U\} = \{(x, y) \in U \times V | f(x, y) = c\}.$$

Theorem 3.4. *$F : \mathbb{R}^n \to \mathbb{R}$ is a smooth function, $c \in \mathbb{R}$, and suppose $\nabla F(x) \neq 0, \forall x \in F^{-1}(c)$. $F^{-1}(c) \neq \emptyset$. Then $F^{-1}(c)$ is a smooth $(n-1)$-dimensional manifold.*

Remark. If a smooth manifold is connected, then any coordinate chart is homeomorphic to an open subset of $\mathbb{R}^n$ for a fixed n. This n is called the dimension of said manifold.

Proof of Theorem 3.4. First of all, $F^{-1}(c)$ is Hausdorff and 2nd countable. Let

$$\tilde{U}_i = \left\{ \frac{\partial F}{\partial x_i} \neq 0 \right\} \quad U_i = \tilde{U}_i \cap F^{-1}(c).$$

We know that U_i is open in $F^{-1}(c)$ because of the induced topology of $\mathbb{R}^n$. $F^{-1}(c)$ is covered by these U_i's because of our assumption that $\nabla F(x) \neq 0 \ \forall x \in F^{-1}(c)$. In other words,

$$F^{-1}(c) \subset \bigcup_{i=1}^{n} U_i.$$

By the implicit function theorem, $\forall a \in U_i$, there exists a neighborhood U_a of a which we may assume to be in $\tilde{U}_i$ *such that* $U_a \cap F^{-1}(c)$ is the graph of a function

$$x_i = f_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

This f_i is unique. We take the collection

$$\bigcup_{a \in F^{-1}(c)} U_a$$

as coordinate charts. There are two cases when two coordinate charts overlap: either they belong to the same U_i, or not. If they belong to the same U_i, the transition map is the identity map, which is a diffeomorphism. If they belong to different U_i's, we see that the transition maps satisfy the criterion that make $F^{-1}(c)$ a manifold:

$$\varphi_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$
$$\varphi_j(x_1, \ldots, x_n) = (x_1, \ldots, x_{j-1}, x_{j-1}, \ldots, x_n)$$
$$\varphi_j \circ \varphi_i^{-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = (x_1, \ldots, x_{i-1}, f_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), x_{i+1}, \ldots, x_{j-1}, x_{j+1}, x_n)$$

□

Theorem 3.5. *Let $U \subset \mathbb{R}^{n+m}$ be open. $F : U \to \mathbb{R}^m$ is a smooth function. Let $c \in \mathbb{R}^m$, and $F^{-1}(c) \neq \emptyset$. If $\forall a \in F^{-1}(c)$, $DF_a : \mathbb{R}^{n+m} \to \mathbb{R}^m$, $F^{-1}(c)$ is an n-dimensional smooth manifold.*

4 Fourth Lecture

Let M be a smooth manifold.

Definition 4.1. $f : M \to \mathbb{R}$ is a smooth function if $\forall p \in M$, $\exists (U, \varphi)$ coordinate chart containing p such that $f \circ \varphi^{-1}$ is $\mathcal{C}^\infty$.

Definition 4.2. Let N be a smooth manifold$f : M \to N$ is said to be smooth if $\forall p \in M$, $\exists (U, \varphi)$ coordinate chart containing p and (V, ψ) coordinate chart containing $f(p)$ such that $\psi \circ f \circ \varphi^{-1}$ is $\mathcal{C}^\infty$.

Definition 4.3. $f : M \to N$ is a diffeomorphism if f and f^{-1} are $\mathcal{C}^\infty$.

Consider the space of smooth functions on M.

Question. Can we approximate a characteristic function by smooth functions?

Question. Suppose we are given k points on M. Is there a smooth function that has a given value at each of these points?

Remark. If an analytic function is zero on an open set of M, then it is equal to zero if M is connected.

There are smooth functions that are not analytic!

Example 4.4. Let $f(x) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0 \end{cases}$, $f : \mathbb{R} \to \mathbb{R}$. We see that $f^{(n)}(0) = 0\ \forall n \in \mathbb{N}$, and thus the taylor expansion is identically zero in a neighborhood of 0. However, $f \not\equiv 0$ in a neighborhood of $t = 0$.

Lemma 4.5. *Existence of cut-off and bump functions.*
(1) $\exists h \in \mathcal{C}^\infty$, $h : \mathbb{R} \to \mathbb{R}$ *such that* $h(t) = \begin{cases} 1 & t \leq 1 \\ 0 < h(t) < 1 & 1 < t < 2 \\ 0 & t \geq 2 \end{cases}$
(2) $\exists H \in \mathcal{C}^\infty$, $H : \mathbb{R}^n \to \mathbb{R}$ *such that* $H(t) = \begin{cases} 1 & |x| \leq 1 \\ 0 < H(t) < 1 & 1 < |x| < 2 \\ 0 & |x| \geq 2 \end{cases}$

Proof. Consider $f(2-t)$ and $f(1-t)$. Then, we see that

$$h(t) = \frac{f(2-t)}{f(2-t) + f(t-1)}.$$

Also, $H(x) = h(|x|)$. We have that $H(x)$ is the *bump* function supported in $B_2(0)$. □

Definition 4.6. The support of a function g is $\mathrm{supp}(g) = \{x | g(x) \neq 0\}$.

Theorem 4.7. *Existence of partition of unity. Let* M *be a smooth manifold, and* $\mathcal{O}$ *is an open cover of* M. *Then* $\exists \varphi_\alpha \in \mathcal{C}^\infty$ *such that* $\varphi : M \to [0,1]$ *where* $\alpha \in A$, *any index set and:*
(1) The set of supports $\{supp(\varphi_\alpha)\}_{\alpha \in A}$ *is locally finite. i.e.,* $\forall p \in M$, *there exists a neighborhood of* p *that intersects a finite amount of* $supp(\varphi_\alpha)$.
(2)

$$\sum_\alpha \varphi_\alpha(p) = 1 \qquad \forall\ p \in M.$$

Note that this sum is finite $\forall p$ *because of (1), and thus, we were able to choose an arbitrary index set* A.
(3) $\forall \alpha \in A$, $\exists U \in \mathcal{O}$ *such that* $supp(\varphi_\alpha) \subset U$.

Example 4.8. $\mathbb{R} = (-\infty, 2.5) \cup (0.5, \infty)$. Take

$$\varphi_1(t) = \frac{f(2-t)}{f(2-t) + f(t-1)}, \quad \varphi_2(t) = \frac{f(t-1)}{f(2-t) + f(t-1)}.$$

Then we see that this holds all the properties from Theorem 4.7 because $\mathrm{supp}(\varphi_1) \subset (0.5, \infty)$ and $\mathrm{supp}(\varphi_2) \subset (-\infty, 2.5)$.

Proposition 4.9. *Let M be a smooth manifold. Then for any closed set $A \subset M$ and any open set U containing A, there exists a smooth function $\varphi : M \to \mathbb{R}$ such that $\varphi \equiv 1$ on A and supp$(\varphi) \subset U$.*

Proof. Let $U_0 = U$, and $U_1 = M \backslash A$. Then (U_0, U_1) is an open cover. Then, from Theorem 4.7, there exists a partition of unity $\{\varphi_\alpha\}$ such that $\text{supp}(\varphi_\alpha) \subset U_0$ or $\text{supp}(\varphi_\alpha) \subset U_1$ $\forall \alpha$. Consider

$$\varphi_1 = \sum_{\text{supp}(\varphi_\alpha) \subset U_1} \varphi_\alpha \Longrightarrow \varphi_1 \equiv 0 \text{ on} A.$$

I claim that $\varphi_0 = 1 - \varphi_1$ is the desired function. We see that $\varphi_0 \equiv 1$ on A. Also,

$$\varphi_0 = \sum_{\text{supp}(\varphi_\alpha) \subset U_0} \varphi_\alpha \Longrightarrow \text{supp}(\varphi_0) \subset U_0 = U.$$

□

Theorem 4.10. *Suppose $A \subset U \subset M$, where A is closed, U open, and M is a smooth manifold. $f : A \to \mathbb{R}$ is a function that can be extended to a smooth function in a neighborhood of A. Then $\exists \tilde{f} : M \to \mathbb{R}$ smooth such that $\tilde{f}|_A = f$ and supp$(\tilde{f}) \subset U$.*

5 Fifth Lecture

Example 5.1. $F : \mathbb{R}^n \to \mathbb{R}$ is smooth and $c \in \mathbb{R}$ is a regular value for f. Then the tangent space of $f^{-1}(C)$ at $a \in f^{-1}(c)$ is the $(n-1)$ dimensional affine space passing through a and orthogonal to $\nabla \cdot f(a)$. The equation for this tangent space is

$$\nabla f(a) \cdot (x - a) = 0.$$

We can view directional derivatives as tangent vectors. Given $v \in \mathbb{R}^n$, and a smooth function f defined near a (a is on v), the directional derivative is given by

$$D_v|_a f = \left.\frac{d}{dt}\right|_{t=0} f(a + tv).$$

The directional derivative assigns a number to each smooth function.

$$D_v|_a f = \left.\frac{d}{dt}\right|_{t=0} f(a + tv) = \sum_{i=1}^{n} v_i \left.\frac{\partial}{\partial x^i}\right|_a f = \sum_{i=1}^{n} v \cdot \nabla f(a).$$

We consider the directional derivatives as operators which form a vector space spanned by $\left.\frac{\partial}{\partial x^i}\right|_a$.

Definition 5.2. X is a derivation at a if X is a linear map from $\mathcal{C}^\infty(\mathbb{R}^n) \to \mathbb{R}$ that suffices

$$X(cf) = cX(f)$$
$$X(f + g) = X(f) + X(g)$$
$$X(fg) = f(a)X(g) + g(a)X(f)$$

Remark. A directional derivative at a is a derivation at a.

Remark. The set of derivations at a forms a real vector space.

Theorem 5.3. *The vector space of directional derivatives and the vector space of derivations at a are isomorphic to each other as real vector spaces.*

Remark. Only the local behavior of a smooth function at a matters. We should define a derivation as a linear map from the space of *germs* of functions at a.

Definition 5.4. Given $a \in \mathbb{R}^n$, a germ of a function at a is a pair (f, U) with $a \in U$ such that f is differentiable in U. We define an equivalence relation $(f, U) \sim (g, v)$ if $f \equiv g$ on $U \cap V$.

Lemma 5.5. *Let c denote a constant function with value c. Then $X(c) = 0$ for any derivation.*

Proof. $X(c) = X(c \cdot 1) = cX(1) \Longrightarrow$

$$X(1) = X(1 \cdot 1) = 1X(1) + 1X(1) = 2X(1) \iff X(1) = 0$$

□

Lemma 5.6. *Suppose f is differentiable in some neighborhood U of $a \in \mathbb{R}^n$. Then, $\exists \varepsilon > 0$ such that $B_\varepsilon(a) \subset U$ and differentiable functions g_i in $B_\epsilon(a)$ such that*

$$f(x) = f(a) + \sum_{i=1}^n g_i(x)(x^i - a^i) \text{ in } B_\varepsilon(a) \text{ and } g_i(a) = \left.\frac{\partial}{\partial x^i}\right|_a f.$$

Proof of Theorem 5.3. Any directional derivative is a derivation. Thus, it suffices to show that the inclusion map is an isomorphism. It is obvious to see that this inclusion map is linear. We want to show that if $\forall f$,

$$\sum_{i=1}^n v^i \left.\frac{\partial}{\partial x^i}\right|_a f = 0,$$

then $v^i = 0$ for each i. This would show that the kernel is zero, and thus the inclusion map is injective. Note: $\frac{\partial x^j}{\partial x^i} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$. Let $f = x^j$. Then we see that

$$0 = \sum_{i=1}^n v^i \left.\frac{\partial x^j}{\partial x^i}\right|_a = v^j.$$

Since is true for any j, we have that indeed, $v = 0$, and the inclusion map is injective. Now to get subjectiveness, we need to ask ourselves the question: If a derivation X corresponds to a directional derivative

$$\sum_{i=1}^n v^i \left.\frac{\partial}{\partial x^i}\right|_a,$$

what are the v^i's with respect to X? We define them in the following way:

$$\sum_{i=1}^n v^i \left.\frac{\partial}{\partial x^i}\right|_a x^j = v^j = X(x^j).$$

Taking these $X(x^j)$'s, I claim that

$$X(f) = \sum_{i=1}^n X(x^i) \left.\frac{\partial}{\partial x^i}\right|_a f \quad \forall f \in \mathcal{C}^\infty.$$

From Lemma 5.6, We have that

$$X(f) = X\left(f(a) + \sum_{i=1}^n g_i(x)(x^i - a^i)\right).$$

From Lemma 5.5 and the product rule that we defined, we have

$$\begin{aligned} X(f) &= X\left(f(a) + \sum_{i=1}^{n} g_i(x)(x^i - a^i)\right) \\ &= \sum_{i=1}^{n} X(g_i(x)(x^i - a^i)) \\ &= \sum_{i=1}^{n} (a^i - a^i) X(g_i(x)) + g_i(a)(X(x^i) - X(a^i)) \\ &= \sum_{i=1}^{n} X(x^i) \left.\frac{\partial}{\partial x^i}\right|_a f \end{aligned}$$

□

Definition 5.7. Let M be a differentiable manifold, and $p \in M$. The tangent space of M at p, T_pM, is the vector space of all derivations X at p, where $X : \mathcal{C}^\infty(M) \to \mathbb{R}$ and $X(fg) = f(a)X(g) + g(a)X(f)$.

Remark. Let $F : M \to N$ be a differentiable map, and $f : N \to \mathbb{R}$ be a differentiable function. Then, we can pull-back f by F, i.e. $f \circ F = F^*(f) : M \to \mathbb{R}$. Suppose $X \in T_pM$ is a derivation at $p \in M$. Then we can push-forward X by F, i.e. $F_*(X)$ is a derivation at $F(p) \Longrightarrow F_*(X) \in T_{F(p)}N$, where

$$F_*(X)(f) := X(F^*(f)) = X(f \circ F).$$

6 Sixth Lecture

Recall that for M a differentiable manifold, $p \in M$, we define T_pM as the real vector space of derivations at p. X is a derivation at p if $X : \mathcal{C}^\infty \to \mathbb{R}$ is linear and

$$X(fg) = f(p)X(g) + g(p)X(f).$$

For $\mathbb{R}^n$, $T_p\mathbb{R}^n$ is the real vector space spanned by $\left.\frac{\partial}{\partial x^i}\right|_p$. For a coordinate chart (U, φ) on M, we can identify U with $\varphi(U)$ and take $\left.\frac{\partial}{\partial x^i}\right|_p = (\varphi^{-1})_* \left(\left.\frac{\partial}{\partial x^i}\right|_p\right)$ as a basis for T_pM.

Question. What if two coordinate charts (U, φ) and (V, ψ) overlap?

Let x^i be the coordinates for the image of φ and $\tilde{x}^j$ be the coordinates of the image of ψ. We have already established that at p, we can identify $\left.\frac{\partial}{\partial x^i}\right|_p$. Notice however, that we can write $\tilde{x}^j = \left(\psi \circ \varphi^{-1}\right)^j (x^1, \ldots, x^n)$. Then, from the chain rule,

$$\begin{aligned} \left.\frac{\partial}{\partial x^i}\right|_p &= \sum_{j=1}^{n} \frac{\partial \left(\psi \circ \varphi^{-1}\right)^j}{\partial x^i} \left.\frac{\partial}{\partial \tilde{x}^j}\right|_p \\ &= \frac{\partial \left(\psi \circ \varphi^{-1}\right)^j}{\partial x^i} \left.\frac{\partial}{\partial \tilde{x}^j}\right|_p \quad \text{(Einstein - Summation Convetion).} \end{aligned}$$

Definition 6.1. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable. If $g : \mathbb{R}^m \to \mathbb{R}$ and X is a derivation at $p \in \mathbb{R}^n$, then F_*X is a derivation at $F(p) \in \mathbb{R}^m$. This is defined as

$$(F_*X)(g) = X(\underbrace{F^*g}_{\text{Pullback}}) = X(g \circ F).$$

Question. In terms of local derivations, what is $F_*\left(\left.\frac{\partial}{\partial x^i}\right|_p\right)(g)$?

Denote $y^\alpha = F^\alpha(x^1, \dots, x^n)$Well from definition, we know that

$$\begin{aligned} F_*\left(\left.\frac{\partial}{\partial x^i}\right|_p\right)(g) &= \left.\frac{\partial}{\partial x^i}\right|_p g(F) \\ &= \left.\frac{\partial}{\partial x^i}\right|_p g(F^1(x^1, \dots, x^n), \dots, F^m(x^1, \dots, x^m)) \\ &= \frac{\partial g(F(p))}{\partial y^\alpha} \left.\frac{\partial F^\alpha}{\partial x^i}\right|_p . \end{aligned}$$

This then implies that

$$F_*\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \left.\frac{\partial F^\alpha}{\partial x^i}\right|_p \left.\frac{\partial}{\partial y^\alpha}\right|_{F(p)} .$$

In the case where (U, φ) and (V, ψ) are coordinate charts, consider $F : (U, \varphi) \to (V, \psi)$. Then we can write it as

$$F_*\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \left.\frac{\partial\left(\psi \circ F \circ \varphi^{-1}\right)^\alpha}{\partial x^i}\right|_p \left.\frac{\partial}{\partial y^\alpha}\right|_{F(p)} .$$

Definition 6.2. TM (called the tangent bundles of M) as a set is the disjoint union of tangent spaces:

$$TM = \bigsqcup_{p \in M} T_pM.$$

Question. What is the topology on TM?

Example 6.3. On $\mathbb{R}^2$ there is the usual topology. However, we can find a different topology from the metric

$$d((x_0, y_0), (x_1, y_1)) = |y_0| + |x_1 - x_0| + |y_1|.$$

Then consider the sequence $(1/n, 1)$. Then in this converges in this metric to $(0, 1)$!

Example 6.4. Consider $\mathbb{S}^2 = \{(x, y, z) | x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. Then we see that

$$T\mathbb{S}^2 = \left\{ (x, y, z, u, v, w) \middle| \begin{array}{c} x^2 + y^2 + z^1 = 1 \\ ux + vy + wz = 0 \end{array} \right\}.$$

Example 6.5. If c is a regular value of f, and $f^{-1}(c) \subset \mathbb{R}^n$, then we have that

$$Tf^{-1}(c) = \left\{ (a, x) \middle| \begin{array}{c} a \in f^{-1}(c) \\ \nabla f(a) \cdot x = 0 \end{array} \right\} \subset \mathbb{R}^{2n}.$$

Note. We need to find an intrinsic definition of tangent bundles! Note that there is always a projection map $\pi : TM \to M$. We will define a differentiable structure to make π smooth.

Theorem 6.6. *Let M be an n-dimensional manifold. There exists a differentiable structure on TM such that the projection map $\pi : TM \to M$ is differentiable, and TM is a $2n$-dimensional manifold.*

Proof. Suppose (U, φ) is a coordinate chart on M, and $x^i, i = 1, \dots, n$ are coordinates on $\varphi(U)$. Take $(\pi^{-1}(U), \tilde{\varphi})$ as a coordinate chart for TM. Any element in $\pi^{-1}(U)$ is a derivation at $p \in U$. But we know that any derivation is of the form $v^i \frac{\partial}{\partial x^i}\Big|_p$. We define the map $\tilde{\varphi}$ as:

$$v^i \frac{\partial}{\partial x^i}\Big|_p \underset{\tilde{\varphi}}{\mapsto} (\underbrace{x^1(p), \dots, x^n(p)}_{\in \varphi(U)}, \underbrace{v^1, \dots, v^n}_{\in \mathbb{R}^n}) \in \varphi(U) \times \mathbb{R}^n.$$

$\tilde{\varphi} : \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n \subset \mathbb{R}^{2n}$. It is a bijection onto its image because its inverse can be written explicitly by

$$(x, v) \mapsto v^i \frac{\partial}{\partial x^i}\Big|_{\varphi^{-1}(x)}.$$

When two coordinate charts $(U, \varphi), (V, \psi)$ overlap, then $(\pi^{-1}(U), \tilde{\varphi})$ and $(\pi^{-1}(V), \tilde{\psi})$ overlap. We want to show that $\tilde{\psi} \circ \tilde{\varphi}^{-1}$ is a smooth map. The sets $\tilde{\varphi}\left(\pi^{-1}(U) \cap \pi^{-1}(V)\right) = \varphi(U \cap V) \times \mathbb{R}^n$ and $\tilde{\psi}\left(\pi^{-1}(U) \cap \pi^{-1}(V)\right) = \psi(U \cap V) \times \mathbb{R}^n$ are both open in $\mathbb{R}^{2n}$. We see that the transition map

$$\begin{aligned} \tilde{\psi} \circ \tilde{\varphi}^{-1}(x^1, \dots, x^n, v^1, \dots, v^n) &= \tilde{\psi}\left(v^i \frac{\partial}{\partial x^i} = v^i \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j}\right) \\ &= \left(\tilde{x}^1, \dots, \tilde{x}^n, v^1 \frac{\partial \tilde{x}^i}{\partial x^i}(x), \dots, v^j \frac{\partial \tilde{x}^j}{\partial x^j}(x)\right). \end{aligned}$$

Is clearly smooth. Choosing a countable cover $\{U_i\}$ of M by coordinate domains, we obtain a countable cover of TM by coordinate domains. Note that two points in the same fiber of π lie in one chart. If X at p and Y at q lie in different fibers, there exist disjoint coordinate domains U_i, U_j for M such that $p \in U_i$ and $q \in U_j$ (because M is Hausdorff), and then the sets $\pi^{-1}(U_i)$ and $\pi^{-1}(U_j)$ are disjoint neighborhoods containing (p, X) and (q, Y) respectively. Thus, TM is a $2n$-dimensional manifold. We see that π is smooth because its coordinate representation with respect to charts (U, φ) for M and $(\pi^{-1}(U), \tilde{\phi})$ for TM is $\pi(x, v) = x$. ☐

7 Seventh Lecture

Recap of last lecture: Let M be a $\mathcal{C}^\infty$ manifold, and let $TM = \bigsqcup_{p \in M} T_pM$ be the tangent bundle at p. Then from last class, we established that there exists a differentiable structure on TM that makes TM a $\mathcal{C}^\infty$ manifold.

Definition 7.1. A $\mathcal{C}^\infty$ vector field Y at p is a $\mathcal{C}^\infty$ map $Y : M \to TM$ such that $\pi \circ Y = \mathrm{Id}_M$. i.e., $Y(p) \in T_pM$. In local coordinate charts, we have that $Y(p) = Y^i(p) \frac{\partial}{\partial x^i}\Big|_p$. And Y^i are $\mathcal{C}^\infty$ functions on coordinate charts. Consider $p \in U \cap V$ where $(U, \varphi), (V, \psi)$ are the usual coordinate charts on M. Then we have that

$$Y(p) = Y^i(p) \left.\frac{\partial}{\partial x^i}\right|_p = \tilde{Y}^j(p) \left.\frac{\partial}{\partial \tilde{x}^j}\right|_p.$$

Then, from the chain rule, we have that

$$Y^i(p) \frac{\tilde{x}^j}{\partial x^i}(p) = \tilde{Y}^j(p).$$

Example 7.2. On $\mathbb{R}^2\backslash(0,0)$, Let

$$\begin{aligned} Y_1 &= x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x} \\ Y_2 &= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial y} \\ \tilde{Y}_1 &= \frac{1}{\sqrt{x^2+y^2}}\left(x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}\right) \\ \tilde{Y}_2 &= \frac{1}{\sqrt{x^2+y^2}}\left(x\frac{\partial}{\partial y} - x\frac{\partial}{\partial y}\right). \end{aligned}$$

We see that Y_1, Y_2 can be extended into $(0,0)$ and become vector fields on $\mathbb{R}^2$, but not $\tilde{Y}_1, \tilde{Y}_2$. Consider the stereographic projection $\varphi : \mathbb{S}^2\backslash(1,0,0) \to \mathbb{R}^2$. We can use φ^{-1} to push forward Y_1 and Y_2 so that they become vector fields on $\mathbb{S}^2\backslash(1,0,0)$. We notice that $\varphi_*^{-1}(Y_1)$ is tangent to a longitude line. We also see that $\varphi_*^{-1}(Y_2)$ is tangent to a latitude line.

Let $\mathcal{T}(M)$ be the set of all $\mathcal{C}^\infty$ vector fields on M. Again, this is a real vector space. We can also multiply a $\mathcal{C}^\infty$ vector field by a $\mathcal{C}^\infty$ funtion. $\mathcal{C}^\infty(M)$ has a ring structure. Then we see that $\mathcal{T}(M)$ can be considered as a module over $\mathcal{C}^\infty(M)$. i.e., a vector space over rings. All of this means that if

$$Y \in \mathcal{T}(M), f \in \mathcal{C}^\infty(M) \Longrightarrow fY \in \mathcal{T}(M).$$

We say that $\mathcal{T}(M)$ can be identified with the vector space of global derivations on M. A global derivation is a linear map $Y : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ such that

$$Y(fg) = fY(g) + gY(f).$$

Any global derivation comes from a global $\mathcal{C}^\infty$ vector fields. Now suppose X, Y are $\mathcal{C}^\infty$ vector fields and thus global derivations.

Question. Is YX a global derivation?

$$\begin{aligned} YX(fg) &\stackrel{?}{=} f * YX(g) + g * YX(f) \\ &= Y(gX(f) + fX(g)) = (Y(g))(X(f)) + g * (YX(f)) + (Y(f))(X(g)) + f * YX(g) \end{aligned}$$

But since we have that

$$XY(fg) = (X(g))(Y(f)) + g * (YX(f)) + (X(f))(Y(g)) + f * (XY(g)),$$

We notice that

$$[X,Y]fg = (XY - YX)fg = g(XY - YX)(f) + f(XY - YX)(g).$$

And thus we see that if $X, Y \in \mathcal{T}(M)$, then $X, Y \notin \mathcal{T}(M)$ but $[X,Y] = XY - YX \in \mathcal{T}(M)$. This is a binary operation on $\mathcal{T}(M)$ that satisfies

(a) Linearity: $[aX + bY, Z] = a[X,Z] + b[Y,z]$.

(b) Skew symmetry: $[X,Y] = [Y,X]$.

(c) Jacobi identity: for $X, Y, Z \in \mathcal{T}(M)$, then

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

This makes $\mathcal{T}(M)$ a lie algebra.

Local Formula: For $X = v^i\frac{\partial}{\partial x^i}$, and $Y = w^j\frac{\partial}{\partial x^j}$, we see that

$$[X,Y] = \left(v^i\frac{\partial w^j}{\partial x^i} - w^j\frac{\partial v^i}{\partial x^i}\right)\frac{\partial}{\partial x^i}.$$

8 Eighth Lecture

Recall that for a manifold M, X is a smooth vector field if and only if X is a global derivation: $X : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ linear, and $X(fg) = fX(g) + gX(f)$. $XY - YX = [X,Y]$ is a smooth vector field. $[\cdot,\cdot]$ makes $\mathcal{J}(M)$ a lie algebra. Suppose $F : M \to N$ is smooth. Recall that $F_* : T_pM \to T_{F(p)}$. $M \xrightarrow{F} N \xrightarrow{f} \mathbb{R}$. For $Y \in T_pM$, we see that $F_*(Y)(f) = Y(f \circ F) \in T_{F(p)}N$.

Example 8.1. $\gamma : (-\varepsilon, \varepsilon) \to M$. If we have a tangent vector in $(-\varepsilon,\varepsilon)$, $\dfrac{\partial}{\partial t}$, then we see that $\gamma_*\left(\dfrac{\partial}{\partial t}\right) = \dfrac{\partial \gamma}{\partial t}$ is a tangent vector of $\gamma(t)$. We see that $\dfrac{\partial \gamma}{\partial t}$ is only defined in the image of γ.

Definition 8.2. Y is a vector field on M. Z is a vector field on N. Y and Z are said to be F-related at p if $F_*(Y(p)) = Z(F(p))$.

Theorem 8.3. *Suppose $V_1, V_2 \in \mathcal{T}(M)$ and $W_1, W_2 \in \mathcal{T}(N)$. Suppose V_i and W_i are F-related for $i = 1, 2$. Then $[V_1, V_2]$ is F-related to $[W_1, W_2]$. In particular, if F is a diffeomorphism, then F_* can be defined on $\mathcal{T}(N)$. Then $W_i = F_*(V_i)$. and the statement is equivalent to $F_*[V_1, V_2] = [F_*(V_1), F_*(V_2)] \implies F_*$ is an isomorphism.*

Question. What is the geometric meaning of $[X,Y]$?
We are going to interpret it as the lie derivative of Y along X.

Example 8.4. $M = \mathbb{R}^n$, $X = V$ is a constant vector field. Y is another smooth vector field on $\mathbb{R}^n$. Then we see that

$$D_v X(p) = \lim_{t\to 0} \frac{Y(p+tv) - Y(p)}{t} = \lim_{t\to\infty} \frac{Y(p) - Y(p - tV)}{t}.$$

Definition 8.5. A $\mathcal{C}^\infty$ map $\varphi : (-\varepsilon,\varepsilon) \times M \to M$ is a flow generated by a vector field X if $\varphi(t,\cdot) : M \to M$ is a diffeomorphism if $\forall t \in (-\varepsilon, \varepsilon)$ and $\varphi(0,p) = p$. i.e., $\varphi(0,\cdot) : M \to M$ is the identity. We see that

$$\varphi_*\left(\frac{\partial}{\partial t}\right)(t,p) = X(\varphi(t,p)).$$

We write $\varphi_t(\cdot) = \varphi(t,\cdot)$.

Example 8.6. Let $M = \mathbb{R}^2$. $X(x,y) = -y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y}$. What is the flow generated by X? Take $p = (a,b)$ on $\mathbb{R}^2$. We look for a curve $\varphi(t,p)$ where p is fixed. We write it as $\gamma(t) = (x(t), y(t))$. Then we have that $\varphi(0,p) \longrightarrow x(0) = a, y(0) = b$. Also we see that $\gamma'(t) = (x'(t), y'(t)) = (-y(t), x(t))$.

$$\begin{cases} x'(t) = y(t) & x(0) = a \\ y'(t) = x(t) & y(0) = b \end{cases} \implies \begin{array}{l} x(t) = a\cos(t) - b\sin(t) \\ y(t) = b\cos(t) + a\sin(t) \end{array}$$

This is for a fixed (a,b). Since we want this for an arbitrary p, we have that

$$\begin{aligned} \varphi(t,x,y) &= (x\cos(t) - y\sin(t), y\cos(t) + x\sin(t)) \\ \varphi_t\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \\ \varphi_t(x+iy) &= e^{it}(x+iy) \end{aligned}$$

We see that this corresponds with rotation! We can check that φ_t is a diffeomorphism.

Example 8.7. $X(x,y) = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$

Theorem 8.8. *A $\mathcal{C}^\infty$ vector field X of compact support gives a smooth family of diffeomorphisms that are defined for $t \in \mathbb{R}$.*

$$suppX = \overline{\{p \in M | X(p) \neq 0\}}.$$

Here, the Existence and uniqueness depend on the system of first order ODE. Smooth dependence of the ODE solutions depends on the initial data and the vector field. Uniqueness of solution of the ODE system implies $\varphi_t \circ \varphi_s = \varphi_{t+s}$. φ_t forms a 1-parameter group of diffeomorphisms. We see that $\varphi_t(\varphi_{-t}(p)) = \varphi_0(p) = p$. in particular,

$$(\varphi_t)_* : T_{\varphi_{-t}(p)}M \to T_pM.$$

Definition 8.9. We define the *Lie Derivative* as

$$\mathcal{L}_X Y(p) = \lim_{t \to 0} \frac{Y(p) - (\varphi_t)_*(Y(\varphi_{-t}(p)))}{t}.$$

Proposition 8.10. $[X, Y] = \mathcal{L}_X Y$.

9 Ninth Lecture

Where do vector fields come from? They come from velocity of smooth flows! If $\varphi : (-\varepsilon, \varepsilon) \times M \to M$, is a smooth map, then we have that $\varphi_* \left(\frac{\partial}{\partial t} \right)$ is a vector field on M.

Recall. On $\mathbb{R}^n$, there exists a gradient vector field associated with any smooth function f:

$$\nabla f = \left(\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n} \right).$$

Question. Given smooth functions f on a smooth manifold M, is ∇f a smooth vector field?

Answer. No!

Proof. Recall that given local coordinates of two overlapping charts x^i and $\tilde{x}^j$, on the overlap we can write a derivation as

$$\begin{aligned} v = v^i \frac{\partial}{\partial x^i} &= \tilde{v}^j \frac{\partial}{\partial \tilde{x}^j} \\ &= \tilde{v}^j \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial}{\partial x^i}. \end{aligned}$$

Here, we got the last line from the chain rule. Then, this implies that on the overlap, we can write

$$v^i = \tilde{v}^j \frac{\partial x^i}{\partial \tilde{x}^j}.$$

Recall that v is a derivation if and only if it is a smooth vector field. Now on $\mathbb{R}^n$, this implies we need

$$\frac{\partial f}{\partial x^i} = \frac{\partial f}{\partial \tilde{x}^j} \frac{\tilde{x}^j}{\partial x^i} \quad i = 1, \ldots, n.$$

However, notice that

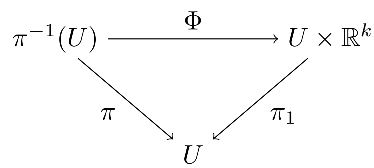
$$\frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial x^i}{\partial \tilde{x}^k} = \frac{\partial \tilde{x}^j}{\partial \tilde{x}^k} = \delta^j_k.$$

Thus, we have that ∇f is not a smooth vector field in a general manifold. □

Definition 9.1. A real vector bundle of rank k over a smooth manifold M is a pair (E, π) where E is a smooth manifold of dimension $n + k$, and π is a smooth projection map $\pi : E \to M$, such that

(1) $\forall p \in M, \pi^{-1}(p)$ is a k-dimensional real vector space.

(2) There is existence of a *local trivialization*: $\forall p \in M$, there exists a neighborhood U of p and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that the diagram commutes.



Here we have that $\pi = \pi_1 \circ \Phi \iff \pi^{-1}(p)$ is sent to $\{p\} \times \mathbb{R}^k$ by Φ. i.e., it sends fiber to fiber.

(3) $\forall q \in U$, the restriction of Φ to $\pi^{-1}(q)$ is a linear isomorphism between $\pi^{-1}(q)$ and $\{q\} \times \mathbb{R}^k$.

Example 9.2. Consider $TM =$ the tangent bundle. Then we have that $\Phi : \pi^{-1}(U) \to (U \times \mathbb{R}^k)$ where Φ is defined by $v^i \frac{\partial}{\partial x^i}\Big|_p \mapsto (p, v_i)$. Furthermore, we have that when changing coordinates, $v^i = \tilde{v}^j \frac{\partial x^i}{\partial \tilde{x}^j}$. The Jacobian for this transformation is a linear isomorphism.

Definition 9.3. A section of (E, π) is a smooth map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{Id}_M$. This corresponds to global sections. I can obviously define a local section by $\sigma : U \subset M \to E$.

Note. A local trivialization gives local sections of the bundle.

Example 9.4. Consider $E = TM$. Then we have that $\Phi : \pi^{-1}(U) \to (U \times \mathbb{R}^n)$ where Φ is defined by $v^i \frac{\partial}{\partial x^i}\Big|_p \mapsto (p, v_i)$. Then I claim $\forall v_0 \in \mathbb{R}^n$, $\Phi^{-1}(p, v_0)$ gives a local section. In fact, if $v_0 = (1, 0, \dots, 0)$, then we see that $\Phi^{-1}(p, v_0) = \frac{\partial}{\partial x^1}\Big|_p$.

Definition 9.5. A vector bundle (E, π) is trivial if a global trivialization exists. i.e., there exists a diffeomorphism $\Phi : E \to M \times \mathbb{R}^k$ defined by $\pi^{-1}(q) \mapsto \{q\} \times \mathbb{R}^k$ such that $\Phi|_{\pi^{-1}(q)}$ is a linear isomorphism between $\pi^{-1}(q)$ and $\{q\} \times \mathbb{R}^k$.

Example 9.6 (Product Bundles)**.** Consider a smooth manifold M. Then an example of a rank-k vector bundle over M is $M \times \mathbb{R}^k$. Here, we have $\pi = \pi_1 : M \times \mathbb{R}^k \to M$ as its smooth projection. We see that $M \times \mathbb{R}^k$ is trivial because we have the identity map $\mathrm{Id} : M \times \mathbb{R}^k \to M \times \mathbb{R}^k$ defined as the global trivialization. Another example of a product bundle follows: Take $\mathbb{S}^1 \times \mathbb{R}$. We have this is a trivial bundle over $\mathbb{S}^1$.

Example 9.7. The total space of an infinite Mobius strip is a non-trivial rank 1 vector bundle over $\mathbb{S}^1$.

Note. If a bundle is trivial, then there exists a global section that is nowhere zero. Take any non-zero vector $v_0 \in \mathbb{R}^k$, and consider $\Phi^{-1}(p, v_0)$. Since Φ is a linear isomorphism on each fiber, $\Phi^{-1}(p, v_0) \neq 0 \in \pi^{-1}(p)$. But since we had that the bundle was trivial, then there exists a global section that takes $p \in M$ to this particular $\Phi^{-1}(p, v_0)$.

Definition 9.8. Suppose two local trivializations overlap (we have overlapping neighborhoods $U, V \subset M$). Then on $\pi^{-1}(U) \cap \pi^{-1}(V)$, we obtain $\Psi \circ \Phi^{-1} : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$. We have $\Psi \circ \Phi^{-1}$ restricts to a linear isomorphism between fibers. Notice that $\Psi \circ \Phi^{-1}(p, v) = \big(p, f(p, v)\big)$. Now, $f(p, v)$ is a linear isomorphism $\forall p \in U \cap V$. However, notice that $\Psi \circ \Phi^{-1}$ is smooth. This implies that $f(p, v) = \tau(p)v \Longrightarrow \tau(p)$ is smooth as p varies. The longshot is that we have $\tau(p)$ is a linear isomorphism from $\mathbb{R}^k \to \mathbb{R}^k$ that depends smoothly on $p \in U \cap V$. Thus, we get a smooth map from $U \cap V \to GL_k(\mathbb{R})$.

10 Tenth Lecture

Recall. A smooth vector bundle over a smooth manifold M of rank k is a smooth manifold (E, π) where π is a smooth projective map. M is called a base. Here, each fiber $E_p = \pi^{-1}(p)$ is a vector space of $\dim K$. Also, there exists a local trivialization. i.e., for each point p, there exists a local diffeomorphism on a neighborhood U of p defined by $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$. Here, Φ sends fiber $(\pi^{-1}(q), q \in U)$ to fiber $(\{q\} \times \mathbb{R}^k)$. Finally, $\Phi|_{\pi^{-1}(q)} : \pi^{-1}(q) \to \{q\} \times \mathbb{R}^k$ is a linear isomorphism.
A section σ is a smooth map $M \xrightarrow{\sigma} E$ such that $\pi \circ \sigma = \mathrm{Id}_M$.

Definition 10.1. We can define the zero section of E as the global section $\zeta : M \to E$ defined by $\zeta(p) = 0 \in E_p$ $\forall p \in M$.

Definition 10.2 (Transition Map)**.** When we have two overlapping local trivializations, $\pi^{-1}(U) \cap \pi^{-1}(V) \neq 0$, we have that $\Psi \circ \Phi^{-1} : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$ sends $\{q\} \times \mathbb{R}^k \mapsto \{q\} \times \mathbb{R}^k$ and is a linear isomorphism. Because $\Psi \circ \Phi^{-1}$ is a diffeomorphism, this linear isomorphism varies smoothly as q varies. This gives a smooth map $U \cap V \to GL_k(\mathbb{R})$.

Note. A local trivializations gives a local section. Take any $v_0 \in \mathbb{R}^k$ and consider $\Phi^{-1}(p, v_0)$. In the case of tangent spaces, we see that $v^i \left.\dfrac{\partial}{\partial x^i}\right|_p \overset{\Phi}{\mapsto} (p, v^i)$. Here, $\Phi^{-1}(p, (1, 0, \ldots, 0)) = \left.\dfrac{\partial}{\partial x^1}\right|_p$.

Definition 10.3. Consider the basis $e_1, \ldots, e_k$ for $\mathbb{R}^k$. Then the elements $\{\Phi^{-1}(p, e_i)\}_{i=1,\ldots,k}$ form a basis for each $E_p = \pi^{-1}(p)$. $\{\Phi^{-1}(\cdot, e_k)\}$ is a *local frame.*

Definition 10.4. Suppose $\pi : E \to M, \pi' : E' \to M'$ are two smooth vector bundles. A bundle map from E to E' is a pair of $\mathcal{C}^\infty$ maps $F : E \to E'$ and $f : M \to M'$ such that F covers f, or the diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{F} & E \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow{f} & M' \end{array}$$

and $F|_{E_p} : E_p \to E'_{f(p)}$ is a linear map. We can define a smooth bundle isomorphism by replacing $\mathcal{C}^\infty$ maps above with diffeomorphisms and linear maps with linear isomorphisms.

Question. Can we classify vector bundles up to bundle isomorphisms?

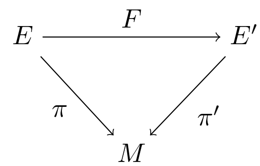
Example 10.5. A $\mathcal{C}^\infty$ map $f : M \to N$ induces a bundle map on the tangent bundle by f_*.

$$\begin{array}{ccc} TM & \xrightarrow{F} & TN \\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle\pi_2} \\ M & \xrightarrow{f} & N \end{array}$$

Then we know that $F(X_p) = f_*(X_p) \in T_{f(p)}N$. We know by the chain rule that

$$f_* \left(\frac{\partial}{\partial x^i} \right) = \frac{\partial f^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha}.$$

Example 10.6. In the case where $M = N$ and f is the identity, then any bundle map induces a map on smooth sections. Denote the smooth global sections of E by $\Gamma(E)$, and the ones of E' by $\Gamma(E')$. We have the following commutative diagram:



We see that F induces a map (analogous to the push forward) $\mathcal{F} : \Gamma(E) \to \Gamma(E')$ defined by $\mathcal{F}(\sigma)(p) = (F \circ \sigma)(p) = F(\sigma(p))$. We see that $F(\sigma)$ is a section of E', and it is smooth by composition. Because F is linear on fiber to fiber, the resulting map $\mathcal{F}$ is linear over $\mathbb{R}$. In fact, it satisfies a strong linearity property. A map $\mathcal{F} : \Gamma(E) \to \Gamma(E')$ is said to be linear over $\mathcal{C}^\infty(M)$ if for any smooth functions $u_1, u_2 \in \mathcal{C}^\infty(M)$, and smooth sections $\sigma_1, \sigma_2 \in \Gamma(E)$, then

$$\mathcal{F}(u_1\sigma_1 + u_2\sigma_2) = u_1\mathcal{F}(\sigma_1) + u_2\mathcal{F}(\sigma_2).$$

It is called a module over $\mathcal{C}^\infty(M)$.

Question. Does every linear map $\mathcal{F} : \Gamma(E) \to \Gamma(E')$ come from a bundle map $F : E \to E'$?

Theorem 10.7. *Suppose E and E' are smooth vector bundles over M. A linear map $\mathcal{F} : \Gamma(E) \to \Gamma(E')$ comes from the bundle map $F : E \to E'$ if and only if $\mathcal{F}$ is a linear over $\mathcal{C}^\infty(M)$. We consider $\Gamma(E)$ and $\Gamma(E')$ as modules over $\mathcal{C}^\infty(M)$. We require the $\mathcal{F}$ is a module morphism.*

Proof. If $\mathcal{F} : \Gamma(E) \to \Gamma(E')$ is from a bundle map F, we show that it is linear. We know that $\mathcal{F}\big(\sigma(p)\big) = F\big(\sigma(p)\big)$. Let u be a smooth function. Then we see

$$\begin{aligned} \mathcal{F}(u\sigma)(p) &= F(u\sigma(p)) = F(u(p)\sigma(p)) \\ &= u(p)F(\sigma(p)) = u(p)\mathcal{F}(\sigma)(p) \\ &= \left(u\mathcal{F}(\sigma)\right)(p). \end{aligned}$$

Thus, we see that $\mathcal{F}$ is linear. Now assume $\mathcal{F} : \Gamma(E) \to \Gamma(E')$ linear over $\mathcal{C}^\infty(M)$, we show that it must come from a bundle map $F : E \to E'$. To define F_p on an $v \in E_p$, take a section $\sigma \in \Gamma(E)$ such that $\sigma(p) = v \implies F_p(v) = F_p(v) = F_p(\sigma(p)) = \mathcal{F}(\sigma)(p)$. What if we choose a different $\tilde{\sigma}$ such that $\sigma(p) = \tilde{\sigma}(p)$. We need to show $\mathcal{F}(\sigma)(p) = \mathcal{F}(\tilde{\sigma})(p)$ if $\sigma(p) = \tilde{\sigma}(p)$, or if $\sigma(p) = 0 \implies \mathcal{F}(\sigma)(p) = 0$. If $\sigma(p) = 0$, we write it as $\sigma = u\tilde{\sigma}$ such that $u(p) = 0 \implies \mathcal{F}(\sigma)(p) = \mathcal{F}(u\tilde{\sigma})(p) = u\mathcal{F}(\sigma)(p) = u(p)\mathcal{F}(\sigma)(p) = 0$. Thus, we see that $\mathcal{F}$ is linear in $\mathcal{C}^\infty(M)$. ◻

11 Eleventh Lecture

In this lecture we're going to introduce dual spaces and work out some of their properties. We then apply this to vector bundles.

Definition 11.1. Let V be a finite dimensional real vector space. $V^* = \{\omega | \omega : V \to \mathbb{R} \text{ is linear}\}$. V^* is a real vector space. Take any basis $\{E_i\}_{i=1,\dots,n}$ of V. Consider the dual basis $\{\mathcal{E}^i\}_{i=1,\dots,n}$ of V^* defined by

$$\mathcal{E}^i(E_j) = \delta^i_j.$$

Proposition 11.2. *$\{\mathcal{E}^i\}_{i=1,\dots,n}$ is a basis.*

Proof. We first need to show that they span. Given any $\omega \in V^*$, take $X \in V$, and define it by $X = X^i E_i$. Then we see that $\omega(X) = \omega(X^i E_i) = X^i \omega(E_i)$ from linearity of ω. Notice that $\mathcal{E}^i(X) = \mathcal{E}^j(X^i E_i) = X^i \mathcal{E}^j(E_i) = X^i$. This kills off all the other terms. Then we see that $\omega(X) = \omega(E_i)\mathcal{E}^i(X) \implies \omega = \omega(E_i)\mathcal{E}^i(X) = \omega(E_i)\mathcal{E}^i$. So we see that it spans. Now we check that the elements in the set are linearly independent. Consider the case where

$$\sum_{i=1}^{n} a_i \mathcal{E}^i = 0.$$

Does this imply that $a_i = a_2 = \cdots = a_n = 0$? Apply this equation to each basis vector. Then we see that

$$\sum_{i=1}^{n} a_i \mathcal{E}^i(E_j) = a_j = 0.$$

$\square$

Definition 11.3. $V^{**} = (V^*)^*$.

Definition 11.4. If $A : V \to W$ is a linear map, the dual map $A^* : W^* \to V^*$ is a linear map defined as follows: for $\omega \in W^*$, we have $A^*\omega(X) = \omega(Ax)$. We can easily check that this is linear.

Definition 11.5 (Cobundle or Dual Bundle)**.** Consider a rank k vector bundle on a differentiable manifold M. We know $\pi^{-1}(p) = E_p$ is a k dimensional vector space. We define local trivializations $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ as a diffeomorphism. Φ restricts to a linear isomorphism between $\pi^{-1}(p)$ and $\{q\} \times \mathbb{R}^k$. Then we define

$$E^* = \bigsqcup_{p \in M} E_p^*,$$

where $\pi^* : E^* \to M$ is the obvious projection map. Given $U \subset M$, we consider $(\pi^*)^{-1}(U)$ and define local trivializations as follows:

$$\begin{gathered} (\pi^*)^{-1}(U) \to U \times \mathbb{R}^k \\ E_p^* \ni \omega_p \mapsto \big(p, \omega_p(E_1), \ldots, \omega_p(E_n)\big), \end{gathered}$$

where $\{E_1, \ldots, E_k\}$ is the local frame coming from local trivialization of E: $E_i(p) = \Phi^{-1}(p, e_i)$.

Example 11.6. We apply this procedure to the tangent bundle and denote the dual bundle by T^*M, and a section of T^*M is called a 1-form. The space of smooth sections of T^*M is denoted by $\Gamma(T^*M)$.

Note. Given a smooth function on M, the gradient

$$\nabla f = \left(\frac{\partial}{\partial x_i}, \ldots, \frac{\partial}{\partial x_n} \right)$$

is not a section of the tangent bundle. However, it *is* a section of the contangent bundle, or a 1-form. In fact, it is better to consider the total differential

$$df = \frac{\partial f}{\partial x^i} dx^i, \quad \text{i.e., } df(X_p) = (Xf)_p.$$

Remark. $D_v f = \nabla f \cdot v$. Since this operation is linear, it can be considered as a linear functional.

Question. How does a section of T^*M transform? Recall for the tangent bundle,

$$\begin{array}{ccc} & TM = \bigsqcup_{p \in M} T_p M & \\ \swarrow^{\Phi} & & \searrow^{\pi} \\ U \times \mathbb{R}^n & & M \end{array}$$

Here, we have that $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ is defined by $v^i \frac{\partial}{\partial x^i}\Big|_p \mapsto (p, v^1, \ldots, v^n)$, and $\Phi^{-1}(p, e_i) = \frac{\partial}{\partial x^i}\Big|_p$ as our local frame.

Question. What are the dual local frame to $\left.\frac{\partial}{\partial x^i}\right|_p$? We are looking for something of the form

$$\omega^j \left(\left.\frac{\partial}{\partial x^i}\right|_p \right) = \delta^j_i.$$

Take x^j and define $dx^j(X_p) = X(x^j)_p$. We apply this to $\left.\frac{\partial}{\partial x^i}\right|_p$. Then we have

$$dx^j \left(\left.\frac{\partial}{\partial x^i}\right|_p \right) = \delta^j_i \implies$$

$dx^i, \ldots, dx^n$ form a local frame of T^*M on U. Then we can define a local trivialization as follows:

$$\begin{gathered} (\pi^*)^{-1}(U) \to U \times \mathbb{R}^k \\ v_i \left. dx^j \right|_p \mapsto (p, v_1, \ldots, v_n). \end{gathered}$$

Notice that they transform differently from $\frac{\partial}{\partial x^i}$. So now we can define this abstract thing df as

$$df = \frac{\partial f}{\partial x^i} \, dx^i.$$

12 Twelfth Lecture

In this class we are going to go over tensor products!

Definition 12.1. Let $V_1, \ldots,_k, W$ be finite dimensional vector spaces. $F : V_1 \times \cdots \times V_k \to W$ is multilinear if $\forall (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k) \in V_1 \times \cdots \times V_{i-1} \times V_{i+1} \times \cdots \times V_k$, the map $V_i \to W$ defined by $v \mapsto (v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_k)$ is linear.

Remark. If $W = \mathbb{R}$, then we call F a multilinear function. If $V_1 = \cdots = V_k = V$, then the collection of multilinear functions $V_1 \times \cdots \times V_k \to \mathbb{R}$ is denoted by $T^k(V)$ and an element of which is called a covariant k-tensor.

Example 12.2. $T^1(V) = V^*$, $T^2(V)$ is just the collection of bilinear forms on V.

Example 12.3. On $\mathbb{R}^n$, the dot product is an element of $T^2(\mathbb{R}^n)$. The determinant function is $\det : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$, so it is an element of $T^n(\mathbb{R}^n)$. On $\mathbb{R}^3$, the cross product is not because it is vector valued.

Definition 12.4. If $\omega, m \in T^1(V)$, we define $\omega \otimes m \in T^2(V)$ by

$$\omega \otimes m(v_1, v_2) = \omega(v_1) m(v_2).$$

More generally, if $S \in T^k(V), T \in T^l(V)$, then $S \otimes T \in T^{k+l}(V)$ is defined by

$$(S \otimes T)(v_1, \ldots, v_{k+l}) = S(V_1, \ldots, V_k) T(V_{k+1}, \ldots, V_{k+l}).$$

Note. In general $S \otimes T \neq T \otimes S$. Let $\omega, \eta \in T^1(V)$. Then

$$\begin{gathered} \omega \otimes \eta(v_1, v_2) = \omega(v_1)\eta(v_2) \\ \eta \otimes \omega(v_1, v_2) = \eta(v_1)\omega(v_2). \end{gathered}$$

However, we do have associativity! i.e., $(S \otimes T) \otimes U = S \otimes (T \otimes U)$.

Theorem 12.5. *The set* $\{v^{i_1} \otimes \cdots \otimes v^{i_k} | 0 \leq i_1 \leq \cdots \leq i_k \leq n\}$ *is a basis for* $T^k(V)$.

Definition 12.6. We define the free product as

$$F(V \times W) = \left\{ \sum_{i=1}^{N} a_i(v_i, \omega_i) \middle| a_i \in \mathbb{R}, N \in \mathbb{N}, (v_i, \omega_i) \in V \times W \right\}.$$

Example 12.7. An element of $F(\mathbb{R}^2, \mathbb{R}^2)$ would be $3((1,2),(3,1)) + 7((1,2),(-1,2)) - 1((3,0),(5,10))$.

13 Thirteenth Lecture

Recall. The cotangent bundle T^*M is dual to the tangent bundle $TM \to M$. Recall that if $F : M \to N$ is a smooth map, then the push forward $F_* : T_pM \to T_{F(p)}N$ is defined for For $X \in T_pM$, we have that $(F_*X)f = X(f \circ F)$.

Recall. $A : V \to W$ is linear, then the dual map $A^* : W^* \to V^*$ is linear. So we consider the dual map $F^* : T^*_{F(p)}N \to T^*_pM$. Is defined by $\omega \in T^*_{F(p)}N \mapsto F^*\omega \in T^*_pM$. When we are acting on a vector field $X \in T_pM$, we have $F^*\omega(X) = \omega(F_*X)$ where F_* is the push forward.

Recall. $F_* : T_pM \to F_{F(p)}N$ can't be globalized in the sense that $\mathcal{F} : \Gamma(TM) \to \Gamma(TN)$ is not defined. *However,* $F^* : T^*_{F(p)}N \to T^*_pM$ can be globalized. i.e., $\mathcal{J}^* : \Gamma(T^*N) \to \Gamma(T^*M)$ is well defined. i.e., any one form ω on N can be defined as $\mathcal{J}^*\omega$ as a one form on M.

Example 13.1. $\gamma : [a,b] \to M$ and suppose ω is a one form on M in local coordinates $\omega = \omega_i\, dx^i$ and $\gamma(t) = \big(x^1(t), \ldots, x^n(t)\big)$. Then γ^* is a one-form on $[a,b]$. We choose coordinates on $[a,b]$. We basically have to answer $\gamma^*\omega = (?)\, dt$. Consider

$$\begin{aligned}
\gamma^*\omega\left(\frac{d}{dt}\right) &= \omega\left(\gamma_*\left(\frac{d}{dt}\right)\right) \\
&= \omega\left(\frac{\partial x^i}{\partial t}\frac{\partial}{\partial x^i}\right) \\
&= \omega_j\, dx^j\left(\frac{\partial x^i}{\partial t}\frac{\partial}{\partial x^i}\right) \\
&= w_i\frac{\partial x^i}{\partial t}.
\end{aligned}$$

Then we see that

$$\gamma^*(\omega) = \omega_i\left(x^1(t), \ldots, x^n(t)\right)\frac{\partial x^i}{\partial t}\, dt.$$

Consider in general, consider $F : M \to N$, with local coordinates $x^i_{i=1,\ldots,m}$ on M and $y^\alpha_{\alpha=1,\ldots,n}$ on N. Now suppose $\omega = \omega_\alpha(y^1, \ldots, y^n)\, dy^\alpha$, in local coordinates. Suppose F is given by $y^\alpha = F^\alpha(x^1, \ldots, x^m)$. Then we see that

$$F^*\omega = \omega_\alpha\left(F^1(x^1, \ldots, x^m), \ldots, F^n(x^1, \ldots, x^m)\right)\frac{\partial F^\alpha}{\partial x^j}\, dx^j. \tag{1}$$

Proposition 13.2. *The differential commutes with the pull back. i.e, consider* $F : M \to N$ *and let* f *be a smooth function on* N. *Then* $F^*\, df = d(f \circ F)$.

Proof. Take $X \in T_pM$. Then,

$$\begin{aligned}
(F^*d\,f)\,(X) &= df\,(F_*X) \\
&= (F_*X)f \\
&= d(f \circ F)(X).
\end{aligned}$$

But this is true for any X, so we get that $F^*df = d(f \circ F)$. □

Definition 13.3. Now we define *line integrals*! Suppose $[a,b] \subseteq \mathbb{R}$ is a compact interval, and ω is a smooth convector field on $[a,b]$ (this means that the component function of ω admits a smooth extension to some neighborhood of $[a,b]$. Let t denote the standard coordinate on $\mathbb{R}$, then ω can be written as $\omega_t = f(t)\,dt$, for some smooth coordinate function $f : [a,b] \to \mathbb{R}$. We define the integral of ω over $[a,b]$ as

$$\int_{[a,b]} \omega = \int_a^b f(t)\,dt. \tag{2}$$

Lets consider the more general case. If $\gamma : [a,b] \to M$ is a smooth curve segment and ω is a smooth convector field on M, we define the line integral of ω over γ to be

$$\begin{aligned}\int_\gamma \omega &= \int_a^b \gamma^*\omega \\ &= \int_a^b \omega_i(x^1(t), \ldots, x^n(t))\frac{\partial x^i}{\partial t}\,dt.\end{aligned}$$

We should check that this is independent of coordinate charts. i.e, $\omega_i = \dfrac{\partial x^i}{\partial t} = \tilde{\omega}_j \dfrac{\tilde{x}^j}{\partial t}$.

Example 13.4. Let ω be a smooth covector field on $[a,b] \subseteq \mathbb{R}$. Let $\varphi : [c,d] \to [a,b]$ be an *increasing* diffeomorphism (meaning that $t_1 < t_2$ implies $\varphi(t_1) < \varphi(t_2)$). Let s denote the standard coordinate on $[c,d]$ and t on $[a,b]$. Then we have $(\varphi^*)_s = f(\varphi(s))\varphi'(s)\,ds$. Then

$$\int_{[c,d]} \varphi^*\omega = \int_c^d f(\varphi(s))\varphi'(s)\,ds = \int_a^b f(t) = \int_{[a,b]} \omega.$$

Similarly, if φ was a decreasing diffeomorphism,

$$\int_{[c,d]} \varphi^*\omega = -\int_{[a,b]} \omega.$$

Proposition 13.5. *Let M be a smooth manifold, $\gamma : [a,b] \to M$ be a piecewise smooth curve segment, $F : M \to N$ be a smooth map, and $\eta \in \Gamma(N^*)$. Then*

$$\int_\gamma F^*\eta = \int_{F\circ\gamma} \eta.$$

Proof. The proof follows from the definitions. Assume γ is a smooth curve segment whose image is contained in the domain of a single smooth chart.

$$\begin{aligned}\int_\gamma F^*\eta = \int_{[a,b]} \gamma^*\left(F^*\eta\right) &= \int_{[a,b]} \gamma^*\left((\eta_j \circ F)dF^j\right) \\ &= \int_a^b (\eta_j \circ F \circ \gamma)\,\frac{\partial (F\circ\gamma)^j}{\partial t}\,dt.\end{aligned}$$

Let $F \circ \gamma = \tilde{\gamma}$ be a curve on N. Then we see that the last line above becomes

$$\begin{aligned}\int_a^b (\eta_j \circ F \circ \gamma)\,\frac{\partial (F\circ\gamma)^j}{\partial t}\,dt &= \int_a^b (\eta_j \circ \tilde{\gamma})\,\frac{\partial \tilde{\gamma}^j}{\partial t}\,dt \\ &= \int_a^b \tilde{\gamma}^*\eta = \int_{\tilde{\gamma}} \eta \\ &= \int_{F\circ\gamma} \eta.\end{aligned}$$

Recall that this was in the case where the image of γ was contained in a single smooth chart. Suppose now that γ is an arbitrary smooth curve segment. By compactness there exists a finite partition $a = a_0 < a_1 < \cdots < a_k = b$ of $[a, b]$ such that $\gamma\big([a_{i-1}, a_i]\big)$ is contained in the domain of a single smooth chart for each $i = 1, \dots, k$, so we apply the computation above on each sub interval. Finally, if γ is only piecewise smooth, we simply apply the same argument on each subinterval on which γ is smooth on and add everything up. □

Definition 13.6. If $\gamma : [a, b] \to M$ and $\tilde{\gamma} : [c, d] \to M$ are piecewise smooth curve segments, we say that $\tilde{\gamma}$ is a reparametrization of γ if $\tilde{\gamma} = \gamma \circ \varphi$ for some diffeomorphism $\varphi : [c, d] \to [a, b]$. If φ is an increasing function, we say that $\tilde{\gamma}$ is a forward reparametrization. If φ is decreasing, it is called a backwards one.

Proposition 13.7. *Suppose M is a smooth manifold, $\omega \in \Gamma(M^*)$, and γ is a piecewise smooth curve segment in M For any parametrization $\tilde{\gamma}$ of γ, we have*

$$\int_{\tilde{\gamma}} \omega = \begin{cases} \int_\gamma \omega & \text{if } \tilde{\gamma} \text{ is a forward reparametrization} \\ -\int_\gamma \omega & \text{if } \tilde{\gamma} \text{ is a backward reparametrization.} \end{cases}$$

Proposition 13.8. *If $\gamma : [a, b] \to M$ is a piecewise smooth curve segment, the line integral of ω over γ can also be expressed as the ordinary integral*

$$\int_\gamma \omega = \int_a^b \omega_{\gamma(t)}\big(\gamma'(t)\big)\, dt.$$

Proof. Suppose that γ is smooth and that its image is contained in the domain of a single smooth chart. Writing the coordinate representations of γ and ω on this smooth chart as $\big(x^1(t), \dots, x^n(t)\big)$ and $\omega_i\, dx^i$, we have

$$\omega_{\gamma(t)}\big(\gamma'(t)\big) = \omega_i\big(\gamma(t)\big)\, dx^i\big(\gamma'(t)\big) = \omega_i\big(\gamma(t)\big)\dot{\gamma}^i(t).$$

Recall the formula $F^*\omega = (\omega_j \circ F)\, dF^j$. Applying this on the pull back $\gamma^*\omega$, we have

$$(\gamma^*\omega)_t = (\omega_i \circ \gamma)(t)\, d(\gamma^i)_t = \omega_i\big(\gamma(t)\big)\dot{\gamma}^i(t)\, dt = \omega_{\gamma(t)}\big(\gamma'(t)\big)\, dt.$$

Then we have that by definition of our integral,

$$\int_\gamma \omega = \int_{[a,b]} \gamma^*\omega = \int_a^b \omega_{\gamma(t)}\big(\gamma'(t)\big)\, dt.$$

Recall that this was in the case where the image of γ was contained in a single smooth chart. Suppose now that γ is an arbitrary smooth curve segment. By compactness there exists a finite partition $a = a_0 < a_1 < \cdots < a_k = b$ of $[a, b]$ such that $\gamma\big([a_{i-1}, a_i]\big)$ is contained in the domain of a single smooth chart for each $i = 1, \dots, k$, so we apply the computation above on each sub interval. Finally, if γ is only piecewise smooth, we simply apply the same argument on each subinterval on which γ is smooth on and add everything up. □

Theorem 13.9 (Fundamental Theorem for Line Integrals)**.** *Suppose $\gamma : [a, b] \to M$ is a piecewise smooth curve segment on M. Let $f \in \mathcal{C}^\infty(M)$. Then*

$$\int_\gamma df = f\big(\gamma(b)\big) - f\big(\gamma(a)\big).$$

Proof. First suppose that γ is smooth. Then, by Proposition 13.8,

$$\int_\gamma df = \int_a^b df_{\gamma(t)}\big(\gamma'(t)\big)\, dt = \int_a^b (f \circ \gamma)'(t)\, dt = f \circ \gamma(b) - f \circ \gamma(a).$$

Now suppose that γ is merely piecewise smooth, let $a = a_0 < \cdots < a_k = b$ be the end points of the subintervals on which γ is smooth. We apply the calculation above to act segment and sum them up:

$$\int_\gamma df = \sum_{i=1}^k f(\gamma(a_i)) - f(\gamma(a_{i-1})) = f(\gamma(b)) - f(\gamma(a)).$$

□

Definition 13.10. A smooth covector field ω on a smooth manifold M is said to be *exact* on M if there is a function $f \in \mathcal{C}^\infty(M)$ such that $\omega = df$. f is said to be the potential for ω.

14 Fourteenth Lecture

Consider the tensor bundle and its following global sections $\Gamma(\otimes^k T^*M) \ni \omega$. We have that $\otimes^k T^*M$ is locally given by

$$A_{i_1,i_1,\ldots,i_k}\, dx^{i_1} \otimes \cdots \otimes dx^{i_k},$$

where $1 \leq i_1 \leq \cdots \leq i_k \leq n$, where n is the dimension of M.

Recall.

$$\otimes^k V^* = T^k(V) = \{L : \underbrace{V \times \cdots \times V}_{k \text{ copies}} \to \mathbb{R}\}.$$

The elements of this set are called covariant k-tensors. L is a multilinear function.

There are two distinguished classes of multilinear spaces: symmetric, and alternating.

Definition 14.1. A alement $\omega \in T^2(V)$ is symmetric if $\omega(X,Y) = \omega(Y,X)$. We say that $\omega \in T^k(V)$ is symmetric if

$$\omega\big(X_1,\ldots,X_k\big) = \omega\big(X_{\sigma(1)},\ldots,X_{\sigma(k)}\big),$$

where σ is an element in the group of permutations S_k.

Example 14.2. If A is an $n \times n$ matrix, we have that $(A^T + A)/2$ is symmetric. In fact,

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2}$$

is skew symmetric.

Definition 14.3. $\omega \in T^2(V)$. We define its *symmetrization* as $\mathrm{Sym}\omega(X,Y) = 1/2\big(\omega(X,Y) + \omega(Y,X)\big)$. For $\omega \in T^k(V)$, we define it as

$$\mathrm{Sym}\omega\big(X_1,\ldots,X_k\big) = \frac{1}{k!}\sum_{\sigma \in S_k} \omega\big(X_{\sigma(1)},\ldots,X_{\sigma(k)}\big).$$

Definition 14.4. $T^k(v) \supseteq \Sigma^k(V) = \{\omega \in T^k(V) | \omega \text{ is symmetric } \}$. It is easy to check that $\mathrm{Sym}\omega \in \Sigma^k(V)$.

Definition 14.5. We can also define the symmetric product, which is sort of the the symmetrization of the tensor product. Let $\omega \in T^k(V), \eta \in T^l(V)$. Then we define $\omega \cdot \eta = \mathrm{Sym}\ (\omega \otimes \eta)$. This implies that $\omega \cdot \eta \in \Sigma^{k+l}(V)$ because $\omega \otimes \eta \in T^{k+l}(V)$. In particular, lets look at the case when $\omega, \eta \in T^1(V)$. Then we have

$$\begin{aligned}\omega \cdot \eta(X,Y) = \mathrm{Sym}\ (\omega \otimes \eta)(X,Y)& \\ &= \frac{1}{2}\left(\omega \otimes \eta(X,Y) + \omega \otimes \eta(Y,X)\right) \\ &= \frac{1}{2}\left(\omega(X)\eta(Y) + \omega(Y)\eta(X)\right).\end{aligned}$$

So now, for $\omega \in V^*, \eta \in V^*$ we have that $\omega \cdot \eta = 1/2(\omega \otimes \eta + \eta \otimes \omega) \in \otimes^2 V$ is an element that corresponds with an element in $\Sigma^2(V)$.