Selected Problems from Evans

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Chapter 2

Problem 3

Modify the proof of the mean value formulas to show for $n \geq 3$ that

$$u(0) = \fint_{\partial B(0,r)} g\,dS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|y|^{n-2}} - \frac{1}{r^{n-2}} \right) f\,dy$$

provided

$$\begin{cases} -\Delta u = f & \text{in } B^0(0,r) \\ u = g & \text{on } \partial B(0,r) \end{cases}.$$

Proof. First we notice that the formula makes intuitive sense. The first term is just the MVP and the second term is contributed from the inhomogeneity. As per the hint to look at the proof for the MVP, I define

$$\phi(r) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(0,r)} u\,dS = \fint_{\partial B(0,r)} g\,dS.$$

Differentiating, the proof of Theorem 2 yields

$$\phi'(r) = \frac{r}{n} \fint_{B(0,r)} \Delta u\,dy = -\frac{1}{n\alpha(n)r^{n-1}} \int_{B(0,r)} f\,dy.$$

Fix $\epsilon > 0$. Then notice that

$$\phi(\epsilon) = \phi(r) - \int_{\epsilon}^{r} \phi'(t)\,dt. \tag{1}$$

In order to evaluate the second term of (1), I do integration by parts:

$$-\int_{\epsilon}^{r} \phi'(t)\,dt = \frac{1}{n\alpha(n)} \int_{\epsilon}^{r} \int_{B(0,t)} \frac{1}{t^{n-1}} f\,dy\,dt \tag{2}$$

$$= \frac{1}{n\alpha(n)} \left(\frac{1}{2-n} \frac{1}{t^{n-2}} \int_{B(0,t)} f\,dy \Bigg|_{\epsilon}^{r} - \int_{\epsilon}^{r} \int_{\partial B(0,t)} \frac{1}{n-2} \frac{1}{t^{n-2}} f\,dydt \right) \tag{3}$$

$$= \frac{1}{n(n-2)\alpha(n)} \left(\left(\int_{\epsilon}^{r} \int_{\partial B(0,t)} f \frac{1}{t^{n-2}}\,dy\,dt - \frac{1}{r^{n-2}} \int_{B(0,r)} f\,dy + \frac{1}{\epsilon^{n-2}} \int_{B(0,\epsilon)} f\,dy \right) \right). \tag{4}$$

It is obvious that the last term is bounded by $C \cdot \epsilon^2 \to 0$ as $\epsilon \to 0$. As $\epsilon \to 0$, the first term becomes

$$\lim_{\epsilon \to 0} \int_{\epsilon}^{r} \int_{\partial B(0,t)} \frac{1}{t^{n-2}} f\,dy\,dt = \int_{B(0,r)} \frac{1}{|y|^{n-2}} f\,dy, \tag{5}$$

as desired. So in the end, letting $\varepsilon \to 0$, (1) yields the desired

$$u(0) = \phi(0) = \fint_{\partial B(0,r)} g\,dS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|y|^{n-2}} - \frac{1}{r^{n-2}} \right) f\,dy$$

□

Problem 4

We say $v \in C^2(\bar{U})$ is *subharmonic* if $-\Delta v \leq 0 \iff \Delta v \geq 0$ in U.

(a) Prove for subharmonic v that

$$v(x) \leq \fint_{B(x,r)} v \, dS$$

for all $B(x,r) \subset U$.
(b) Prove that $\max_{\bar{U}} v = \max_{\partial U} v$.
(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove v is subharmonic.
(d) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic.

Proof. The proof starts out being essentially the same as the MVP for harmonic functions. I define

$$\phi(r) = \fint_{\partial B(x,r)} v \, dy.$$

From the assumption that $\Delta v \geq 0$, we have

$$0 \leq \int_{B(x,r)} \Delta v \, dy = \int_{\partial B(x,r)} \frac{\partial v}{\partial \nu} \, dS.$$

From the differentiation of ϕ (seen in Theorem 2), we have

$$\phi'(r) = \fint_{\partial B(x,r)} \frac{\partial v}{\partial \nu} \, dS \geq 0.$$

Hence, ϕ is increasing $\Longrightarrow v(x) = \phi(0) \leq \phi(r) = \fint_{\partial B(x,r)} v \, dS$. Transforming to polar coordinates in the same way as the proof on page 26 yields the desired results for (a).

The proof for (b) is identical to that of Theorem 4, except for one little inequality that doesn't matter in the proof.

(c) Recall Jensen's inequality: if $f : \mathbb{R} \to \mathbb{R}$ is convex, $U \subset R^n$ is open and bounded, and $u : U \to \mathbb{R}$ is summable, then

$$f\left(\fint_U u \, dx\right) \leq \fint_U f(u) \, dx.$$

Then from the MVP of harmonic functions, we have

$$v = \phi(u) = \phi\left(\fint_U u \, dx\right) \leq \fint_U \phi(u) \, dx = \fint_U v \, dx.$$

(d) We know that if u is harmonic, u_{x_i} is harmonic. Then from part (c), $(u_{x_i})^2$ is subharmonic. Obviously the sum of subharmonic functions is subharmonic, so $v = |Du|^2 = \sum_{i=1}^n (u_{x_i})^2$ is subharmonic. □

Problem 5

Prove that there exists a constant C, depending only on n, such that

$$\max_{B(0,1)} |u| \leq C \left(\max_{\partial B(0,1)} |g| + \max_{B(0,1)} |f| \right),$$

whenever u is a solution of

$$\begin{cases} -\Delta u = f & \text{in } B^0(0,1) \\ u = g & \text{on } \partial B(0,1) \end{cases}.$$

Proof. Let $M = \max_{B(0,1)} f$ and define $v(x) = u(x) + \frac{M}{2n}|x|^2$. Then we see that $-\Delta v = -\Delta u - M = f - M \leq 0 \implies v$ is subharmonic. Also notice that $u \leq v \; \forall x \in B(0,1)$. Then

$$\max_{B(0,1)} u \leq \max_{B(0,1)} v = \max_{\partial B(0,1)} v \leq \max_{\partial B(0,1)} |g| + \frac{1}{2n} M = \max_{\partial B(0,1)} |g| + \frac{1}{2n} \max_{B(0,1)} |f|,$$

where we have used part (b) from Problem 4. □

Problem 6

Use Poisson's formula for the ball to prove

$$r^{n-2} \frac{r - |x|}{(r + |x|)^{n-1}} u(0) \leq u(x) \leq r^{n-2} \frac{r + |x|}{(r - |x|)^{n-1}} u(0)$$

whenever u is positive and harmonic in $B^0(0,r)$. This is an explicit form of Harnack's inequality.

Proof. We recall Poisson's formula:

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g}{|x-y|^n} \, dS \tag{6}$$

solves $\Delta u = 0$ in $B^0(0,r)$, $u = g$ on $\partial B(0,r)$. Since we are integrating around the boundary, $y \in \partial B(0,r) \implies |x - y| \leq |x| + r$. Then we easily see

$$\begin{aligned} u(x) &= \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g}{|x-y|^n} \, dS = \frac{(r + |x|)(r - |x|)}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g}{|x-y|^n} \, dS \\ &\leq \frac{(r + |x|)(r - x)}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g}{(|x| + r)^n} \, dS = \frac{r - |x|}{(|x| + r)^{n-1}} \frac{r^{n-2}}{n\alpha(n)r^{n-1}} \int_{\partial B(0,r)} g \, dS \\ &= r^{n-2} \frac{r - |x|}{(|x| + r)^{n-1}} ⨍_{\partial B(0,r)} g \, dS = r^{n-2} \frac{r - |x|}{(|x| + r)^{n-1}} u(0). \end{aligned}$$

The other inequality is easily seen the same way by noticing that $|x - y| \geq |x| - r$. □

Problem 7

Prove Theorem 15 in section 2.2.4. (Hint: Since $u \equiv 1$ solves (44) for $g \equiv 1$, the theory automatically implies

$$\int_{\partial B(0,1)} K(x,y) \, dS = 1$$

for each $x \in B^0(0,1)$.)

Proof. First recall that we must prove that $u \in C^\infty(B^0(0,r)), \Delta u = 0$ in $B^0(0,r)$, and $\lim_{x \to x^0} u(x) = g(x^0)$ for each $x^0 \in \partial B(0,r)$. The book defines u by my (6) and

$$\frac{\partial G}{\partial \nu}(x,y) = K(x,y) = \frac{r^2 - |x|^2}{n\alpha(n)r} \frac{1}{|x-y|^n}$$

as Poisson's kernel. Let $G(x,y)$ be Green's function for the ball. Fixing x, we see that $y \mapsto G(x,y)$ is harmonic for $x \neq y$. Since $G(x,y) = G(y,x)$, we see that $x \mapsto G(x,y)$ is also harmonic for $x \neq y$. This implies that $x \mapsto K(x,y)$ is harmonic. Then since $K(x,y)$ is both smooth and harmonic, we have that $u \in C^\infty(B^0(0,r))$ and $\Delta u = 0$ by (6) since $\Delta_x K(x,y) = 0$.

Now we try to show that $\lim_{x\to x^0} u(x) = g(x^0)\ \forall x^0 \in \partial B(0,r)$. Fix $x^0 \in B(0,r)$. Fix $\epsilon > 0$. Choose $\delta > 0$ small such that $|g(y) - g(x^0)| < \epsilon$ if $|y - x^0| < \delta$. This just comes from continuity of g. Recall that $\int_{\partial B(0,1)} K = 1$. Then for $|x - x^0| < \delta$, we see

$$\begin{aligned}
|u(x) - g(x^0)| &= \left| \int_{\partial B(0,r)} K(x,y)g(y)\,dS(y) - g(x^0) \right| = \left| \int_{\partial B(0,r)} K(x,y)\big(g(y) - g(x^0)\big)\,dS(y) \right| \\
&\le \left| \int_{\partial B(0,r)\cap|x-x^0|\le\delta} K(x,y)\big(g(y) - g(x^0)\big)\,dS(y) \right| + \left| \int_{\partial B(0,r)\cap|x-x^0|>\delta} K(x,y)\big(g(y) - g(x^0)\big)\,dS(y) \right| \\
&\le \int_{\partial B(0,r)\cap|x-x^0|\le\delta} K(x,y)\left|g(y) - g(x^0)\right|\,dS(y) + \int_{\partial B(0,r)\cap|x-x^0|>\delta} K(x,y)\left|g(y) - g(x^0)\right|\,dS(y) \\
&= I + J.
\end{aligned}$$

Then

$$I \le \int_{\partial B(0,r)} K \cdot \epsilon\,dy = \epsilon.$$

For J, recall that g is continuous and bounded:

$$\begin{aligned}
J &\le \|g\|_{L^\infty} \int_{\partial B(0,r)\cap|x-x^0|>\delta} K\,dS(y) \\
&= \|g\|_{L^\infty} \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)\cap|x-x^0|>\delta} \frac{1}{|x-y|^n}\,dS(y) \\
&\to 0 \text{ as } |x| \to r.
\end{aligned}$$

Then we finally get that $|u(x) - g(x^0)| \le 2\epsilon$ for small $|x - x^0|$, and we are done proving the last part. ☐

Problem 9

Let U^+ denote the open half-ball $\{x \in R^n \| x| < 1, x_n > 0\}$. Assume $u \in C^2(\bar{U}^+)$ is harmonic in U^+ with $u = 0$ on $\partial U^+ \cap \{x_n = 0\}$. Set

$$v(x) := \begin{cases} u(x) & \text{if } x_n \ge 0 \\ -u(x_1, \dots, x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

for $x \in U = B^0(0,1)$. Prove v is harmonic in U.

Proof. If $x_n \ge 0, u_{x_ix_i} = v_{x_ix_i} \implies \Delta v = 0$. Let $x_n < 0$. Then notice that $v_{x_i} = -u_{x_i}(x_1, \dots, x_{n-1}, -x_n), i = 1, \dots, n-1$. Similarly, $v_{x_ix_i} = -u_{x_ix_i}(x_1, \dots, x_{n-1}, -x_n), i = 1, \dots, n-1$. For $i = n, v_{x_n} = u_{x_n}(x_1, \dots, x_{n-1}, -x_n)$ and $v_{x_nx_n} = -u_{x_nx_n}(x_1, \dots, x_{n-1}, -x_n)$. Since $u \in C^2(\bar{U}^+) \implies v$ is too $\implies$ all $v_{x_ix_i}$ extend continuously to $\partial U^+ \cap \{x_n = 0\}$. So we get that $\Delta v = -\Delta u = 0 \implies v$ is harmonic for all $x \in U$. ☐

Chapter 5

Problem 2

Let U, V be open sets, with $V \subset\subset U$. Show there exists a smooth function ζ such that $\zeta \equiv 1$ on $V, \zeta = 0$ near ∂U.

Proof. Take $V \subset\subset W \subset\subset U$, and consider

$$\chi_W(x) = \begin{cases} 1 & x \in W \\ 0 & x \notin W. \end{cases}$$

Let $\eta_\epsilon(x)$ be the standard mollifier, and define the following convolution:

$$\begin{aligned} w := \chi_W * \eta_\epsilon(x) &= \int_{\mathbb{R}^n} \chi_W * \eta_\epsilon(x)\, dx \\ &= \int_W \eta_\epsilon(x)\, dx. \end{aligned}$$

It is obvious that w is smooth and has support in $W \cap B(0, \epsilon)$. Since $\bar{W}$ is compact, we can cover it by open balls W_i such that

$$\bar{W} \subset \bigcup_{i=1}^{n} W_i.$$

Letting $\eta_{\epsilon_i}(x)$ be the standard mollifier for each respective ball W_i, define $w_i := \chi_W * \eta_{\epsilon_i}(x)$. Note that $\sum_i w_i$ is nowhere zero. Then

$$\zeta(x) := \sum_{i=1}^{N} \frac{w_i}{\sum_i w_i}(x)$$

is the desired smooth function such that $\zeta \equiv 1$ on V (because $V \subset\subset W$), and $\zeta = 0$ near ∂U (because $W \subset\subset U$). □

Problem 4

Assume U is bounded and $U \subset\subset \bigcup_{i=1}^N V_i$. Show there exists C^∞ functions ζ_i such that

$$\begin{cases} 0 \leq \zeta_i \leq 1, \quad \text{spt } \zeta_i \subset V_i \quad (i = 1, \ldots, N) \\ \sum_{i=1}^N \zeta_i = 1 \qquad\qquad \text{on } U. \end{cases}$$

The functions $\{\zeta_i\}_{i=1}^N$ form a partition of unity.

Proof. Take $U \subset\subset \bigcup_{i=1}^N W_i \subset\subset \bigcup_{i=1}^N V_i$. For each i, we apply Problem 2 and find a $w_i \equiv 1$ on W_i and 0 near ∂V_i. Note that $\sum w_i$ is nowhere zero. Hence

$$\zeta_i := \frac{w_i}{\sum_i w_i}$$

is smooth and the sum is obviously 1. Also supp $\zeta_i \subset V_i$ because $\bar{W}_i \subset V_i$ and $\{x|\zeta_i \neq 0\} = \{x|w_i \neq 0 \subset W_i$. □

Problem 6

Prove directly that if $u \in W^{1,p}(0,1)$ for some $1 < p < \infty$, then

$$|u(x) - u(y)| \leq |x - y|^{1-\frac{1}{p}} \left(\int_0^1 |u'|^p\, dt \right)^{1/p}$$

for a.e. $x, y \in [0, 1]$.

Proof. Let $v(x)$ be the weak derivative of u. I define

$$\tilde{u}(x) := \int_0^x v(x)\, dx.$$

Let $\phi \in C_c^\infty(0,1)$ be arbitrary. Note that since $u, v \in L^p(0,1)$, they are also in $L^1(0,1) \implies \tilde{u}$ is absolutely continuous $\implies \tilde{u}' = v$ a.e. Then,

$$\int_0^1 u\phi'\, dx = -\int_0^1 v\phi\, dx = -\int_0^1 \tilde{u}'\phi\, dx = \int_0^1 \tilde{u}\phi'\, dx,$$

where we used the definition of a weak derivative, fundamental theorem of calculus, and integration by parts, respectively. This implies that

$$\int_0^1 (u - \tilde{u})\phi' \, dx = 0 \quad \Longrightarrow \quad u = u' \text{ a.e.}$$

Without loss of generality, let $y < x$ and $x, y \in [0, 1]$.

$$\begin{aligned}
|u(x) - u(y)| &= |\tilde{u}(x) - \tilde{u}(y)| \\
&= \left| \int_y^x v \, dx \right| \\
&\leq \left(\int_y^x |v|^p \, dx \right)^{1/p} \left(\int_y^x 1^{1-\frac{1}{p}} \, dx \right)^{1-\frac{1}{p}} \\
&\leq |x - y|^{1-\frac{1}{p}} \left(\int_0^1 |v|^p \, dx \right)^{1/p},
\end{aligned}$$

as desired. □

Problem 8

Integrate by parts to prove the interpolation inequality:

$$\int_U |Du|^2 \, dx \leq C \left(\int_U u^2 \, dx \right)^{1/2} \left(\int_U |D^2 u|^2 \, dx \right)^{1/2}$$

for all $u \in C_c^\infty(U)$. Assume ∂U is smooth, and prove this inequality if $u \in H^2(U) \cap H_0^1(U)$.

Proof. Let $u \in C_c^\infty(U)$. Then integrating by parts and applying Holder's inequality for $p = 2$ yields

$$\begin{aligned}
\int_U |Du|^2 \, dx &= -\int_U u \Delta u \, dx \leq C \int_U |u||D^2 u| \, dx \\
&\leq C \left(\int_U |u^2| \, dx \right)^{1/2} \left(\int_U |D^2 u|^2 \right)^{1/2}.
\end{aligned}$$

There is no boundary term because u has compact support and so the integral along the boundary is zero. Now let $u \in H_0^1(U) \cap H^2(U)$. We will approximate by smooth functions: let $\{v_k\}_{k=1}^\infty \subset C_c^\infty(U)$ converge to u in $H_0^1(U)$ and $\{w_k\}_{k=1}^\infty \subset C^\infty(U)$ converge to v in $H^2(U)$. Then Green's formula implies

$$\begin{aligned}
\int_U Dw_k \cdot Dv_k \, dx &= -\int_U v_k \Delta w_k \, dx \\
&\leq C \int_U |v_k||D^2 w_k| \, dx \\
&\leq C \left(\int_U |v_k|^2 \, dx \right)^{1/2} \left(\int_U |D^2 w_k| \, dx \right)^{1/2}.
\end{aligned}$$

There is no boundary term because v_k has compact support. As $k \to \infty$, the RHS of the inequality above converges to $C\|u\|_{L^2(U)}\|D^2 u\|_{L^2(U)}$ by the definitions of our sequences v_k and w_k. All that is left is to show that the LHS of the inequality above converges to $\|Du\|^2_{L^2(U)}$ as $k \to \infty$:

$$\begin{aligned}
\int_U (Dw_k \cdot Dv_k - |Du|^2)\,dx &= \int_U (Dw_k \cdot (Dv_k - Du) + Du \cdot (Dw_k - Du))\,dx \\
&\leq \int_U (|Dw_k||Dv_k - Du| + |Du||Dw_k - Du|)\,dx \\
&\leq \|Dw_k\|_{L^2(U)}\|Dv_k - Du\|_{L^p(U)} + \|Du\|_{L^2(U)}\|Dw_k - Du\|_{L^2(U)} \\
&\to 0 \text{ as } k \to \infty.
\end{aligned}$$

So for $u \in H^1_0(U) \cap H^2(U)$ we are left with

$$\|Du\|^2_{L^2(U)} \leq \|u\|_{L^2(U)}\|D^2u\|_{L^2(U)},$$

as desired. □

Problem 10

Suppose U is connected and $u \in W^{1,p}(U)$ satisfies

$$Du = 0 \quad \text{a.e. in } U.$$

Prove u is constant a.e. in U.

Proof. Fix $\epsilon > 0$ such that $\text{dist}(\partial B(x,r), \partial U) < \epsilon$ and consider the convolution $u^\epsilon := \eta_\epsilon * u$. Recall that $\eta_\epsilon \in C^\infty_c(U)$ and we know that u^ϵ is smooth. Then I claim that the regular derivative of u^ϵ is equal to the convolution of η_ϵ and the weak derivative of u:

$$Du^\epsilon = \eta_\epsilon * Du.$$

We see this explicitly:

$$\begin{aligned}
Du^\epsilon(x) &= D\int_U \eta_\epsilon(x-y)u(y)\,dy = \int_U D_x\eta_\epsilon(x-y)u(y)\,dy \\
&= -\int_U D_y\eta_\epsilon(x-y)u(y)\,dy = (-1)\cdot(-1)\int_U Du(y)\eta_\epsilon(x-y)\,dy \\
&= [\eta_\epsilon * Du](x).
\end{aligned}$$

But from the given that $Du = 0$ a.e. in U, we conclude that u^ϵ is constant almost everywhere in our ball $\implies u$ is too because $u^\epsilon \to u$ in $W^{1,p}_{\text{loc}}(U)$. The fact that U is connected implies that any two points can be connected through a single path; a path that can be finitely covered by open balls. Applying the fact that u is constant a.e. in open balls inside U, and since any one of these balls has a non-empty intersection with its neighbor, u must be constant in all of U a.e.

□

Problem 14

Let U be bounded, with a C^1 boundary. Show that a 'typical' function $u \in L^p(U)$ $(1 \leq p < \infty)$ does not have a trace on ∂U. More precisely, prove that there does not exist a bounded linear operator

$$T : L^p(U) \to L^p(\partial U)$$

such that $Tu = u|_{\partial U}$ whenever $u \in C(\bar{U}) \cap L^p(U)$.

Proof. □

Chapter 6

Problem 3

Assume U is connected. A function $u \in H^1(U)$ is a weak solution of *Neumann's problem*

$$\begin{cases} -\Delta u = f & \text{in } U \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases} \tag{7}$$

if

$$\int_U Du \cdot Dv \, dx = \int_U fv \, dx \tag{8}$$

for all $v \in H^1(U)$. Suppose $f \in L^2(U)$. Prove that (7) has a weak solution if and only if

$$\int_U f \, dx = 0.$$

Proof. Suppose u is a weak solution. i.e., (8) holds. Then in particular, we can make $v \equiv 1 \implies Dv = 0 \implies$

$$0 = \int_U f \, dx.$$

Now assume that the integral of f is zero. Note that if u is a weak solution to (7), then $u + c$ is another one because the c becomes zero when we plug it into (8). Note that by modifying this constant, we can find a weak solution u satisfying

$$\int_U u \, dx = 0.$$

Define

$$A := \{f \in L^2(U) | \textstyle\int_U f \, dx = 0\}$$
$$A' := \{f \in H^1(U) | \textstyle\int_U f \, dx = 0\}.$$

Then A is a closed subspace because it can be defined as the kernel of a linear functional defined by $f \mapsto \int_U f \, dx$. Now I define the bilinear form on A' as

$$B[u, v] = \int_U Du \cdot Dv \, dx,$$

$\forall v, u \in A'$. By Holder's inequality, we have

$$|B[u, v]| \leq \|Du\|_{L^2(U)} \|Dv\|_{L^2(U)} \leq \|u\|_{A'} \|v\|_{A'}. \tag{9}$$

Also note that, by definition,

$$B[u, u] = \|Du\|^2_{L^2(U)}.$$

By our construction of A, we have that $\forall u \in A$,

$$(u)_U = \fint_U u \, dx = 0.$$

Then we can apply Poincare's inequality:

$$\|u\|^2_{L^2(U)} = \int_U u^2 \, dx = \int_U (u - (u)_U)^2 \, dx \leq C \int_U |Du|^2 \, dx = CB[u, u].$$

Putting these inequalities together gives us

$$\|u\|^2_{A'} = \int_U u^2 + |Du|^2 \, dx = \int_U u^2 \, dx + B[u, u] \leq (C + 1)B[u, u]. \tag{10}$$

Then by the Lax-Milgram theorem, we have that there is a unique $u \in A'$ such that $B[u, v] = \langle f, v\rangle$ for any $v \in A'$. But note that we haven't used the fact that the integral of f over U is zero! Applying this gives the required results. $\square$

Problem 6

Assume u is a smooth solution of

$$\begin{cases} Lu = -\sum_{i,j=1}^n a^{ij} u_{x_i x_j} = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

where f is bounded. Fix $x^0 \in \partial U$. A *barrier* at x^0 is a C^2 function w such that

$$Lw \geq 1 \text{ in } U, \quad w(x^0) = 0, \quad w \geq 0 \text{ on } \partial U.$$

We want to show that if w is a barrier at x^0, there exists a constant C such that

$$|Du(x^0)| \leq C \left| \frac{\partial w}{\partial \nu}(x^0) \right|.$$

Proof. Note that since $w(x^0) = 0, w \geq 0$ on ∂U and it is a supersolution, then the weakmaximum principle imples that

$$\min_{\overline{U}} w = \min_{\partial U} w,$$

so $w(x^0)$ is a minimum. Since f is bounded, we know that $\|f\|_{L^\infty(U)} < \infty$. Define $v_1 := u + \|f\|_{L^\infty(U)} w$ and $v_2 := u - \|f\|_{L^\infty(U)} w$. We now calculate Lv_1 and Lv_2:

$$\begin{aligned} Lv_1 &= f + \|f\|_{L^\infty(U)} Lw \geq f + \|f\|_{L^\infty(U)} \geq 0 \\ Lv_2 &= f - \|f\|_{L^\infty(U)} Lw \leq f - \|f\|_{L^\infty(U)} \leq 0. \end{aligned}$$

This shows that v_1 is a supersolution and v_2 is a subsolution. By the weak maximum principle, v_1 attains its minimum on ∂U, while v_2 attains its maximum there. Finally note that

$$\begin{aligned} v_1\big|_{\partial U} &= 0 + \|f\|_{L^\infty(U)} w\big|_{\partial U} = \|f\|_{L^\infty(U)} w\big|_{\partial U} \\ v_2\big|_{\partial U} &= 0 - \|f\|_{L^\infty(U)} w\big|_{\partial U} = -\|f\|_{L^\infty(U)} w\big|_{\partial U} \end{aligned}$$

This means that v_1 attains a minimum at x^0 and v_2 attains a maximum at x^0. Then we have

$$\begin{aligned} \frac{\partial v_1}{\partial \nu}(x^0) &= \frac{\partial u}{\partial \nu}(x^0) + \|f\|_{L^\infty(U)} \frac{\partial w(x^0)}{\partial \nu} \leq 0 \\ \frac{\partial v_2}{\partial \nu}(x^0) &= \frac{\partial u}{\partial \nu}(x^0) - 2\|f\|_{L^\infty(U)} \frac{\partial w(x^0)}{\partial \nu} \geq 0. \end{aligned}$$

Putting these two together yield

$$\left| \frac{\partial u}{\partial \nu}(x^0) \right| \leq \|f\|_{L^\infty(U)} \left| \frac{\partial w}{\partial \nu}(x^0) \right|.$$

□

Problem 7

Assume U is connected. Use (1) energy methods and (2) the maximum principle to show that the only smooth solutions of the Neumann boundary-value problem

$$\begin{cases} -\Delta u = 0 & \text{in } U \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases} \tag{11}$$

are $u \equiv C$, for some constant C.

Proof. 1. Let u, v be solutions to the general Neumann boundary-value problem:

$$\begin{cases} -\Delta u = f & \text{in } U \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial U \end{cases}, \tag{12}$$

and let $w = u - v$. Then w solves (11). If w attains its maximum on the interior of U, then it is a constant and we are done by the strong maximum principle. Assume that it does not attain its maximum in its interior. By the weak maximum principle, w has a maximum at ∂U, call it x^0. This implies that $w(x^0) > w(x)\ \forall x \in U$ because we assume that its maximum is not attained on the interior. If w is not a constant, then we can now use Hopf's Lemma:

$$\frac{\partial w}{\partial \nu} = \frac{\partial u}{\partial \nu} - \frac{\partial v}{\partial \nu} > 0.$$

However, since $\partial w/\partial \nu = 0$, we have that w is a constant. Note that this proves uniqueness of (12) up to a constant.

2. Let w solve (11). Then we can use green's formula:

$$\int_U |Dw|^2\, dx = -\int_U w\Delta w\, dx + \int_{\partial U} \frac{\partial w}{\partial \nu} w\, dS.$$

The RHS is equal to zero because w solves (11). Then since $\int_U |Dw|^2\, dx = 0$, $|Dw|^2$ is positive, and U is connected, we have that w is a constant.

□