1 Harnack Inequality

1.1 Introduction

We begin by defining our differential operator

$$Lu = D_i(a^{ij}D_j u + b^i u) + c^i D_i u + du$$

and force boundedness of the coefficients on $L$:

$$\sum |a^{ij}(x)|^2 \leq \Lambda^2, \quad \lambda^{-2} \left( \sum |b^i(x)|^2 + |c^i(x)|^2 \right) + \lambda^{-1} |d(x)| \leq \nu^2.$$ 

We also make this an elliptic equation by enforcing

$$a^{ij}(x)\xi_i \xi_j \geq \lambda |\xi|^2, \quad \forall x \in \Omega, \xi \in \mathbb{R}^n.$$ 

Then $u$ is a weak solution to $Lu = 0$ ($\geq 0$, $\leq 0$) in the domain $\Omega$ if

$$\int_\Omega \left\{ (a^{ij}D_j u + b^i u) D_i v - (c^i D_i u + du)v \right\} dx = 0$$ 

for all non-negative functions $v \in C^1_0(\Omega)$. Let $f^i, g$ be locally integrable functions in $\Omega$. Then $u$ is a weak solution of the inhomogeneous equation

$$Lu = g + D_i f^i$$

in $\Omega$ if it satisfies

$$\int_\Omega \left\{ (a^{ij}D_j u + b^i u) D_i v - (c^i D_i u + du)v \right\} dx = \int_\Omega (f^i D_i v - gv) \, dx$$

for all $v \in C^1_0(\Omega)$.

1.2 Structural Inequalities

We rewrite $Lu = g + D_i f^i$ as

$$D_i A^i(x, u, Du) + B(x, u, Du) = 0,$$

where

$$A^i(x, z, p) = a^{ij}p_j + b^i z - f^i$$

$$B(x, z, p) = c^i p_i + dz - g$$
for \((x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n\). Then we say that \(u\) is a weak subsolution (supersolution, solution) of (1) in \(\Omega\) if \(A^i(x, u, Du)\) and \(B(x, u, Du)\) are locally integrable and

\[
\int_{\Omega} (D_i v A^i(x, u, Du) - v B(x, u, Du)) \, dx \leq (\geq, =) 0
\]

for all non-negative \(v \in C_0^1(\Omega)\). Writing \(b = (b^1, \ldots, b^n), c = (c^1, \ldots, c^n), f = (f^1, \ldots, f^n)\) and using the Schwarz inequality, we have the inequalities

\[
p_i A^i(x, z, p) \geq \frac{\lambda}{2} |p|^2 - \frac{1}{\lambda} (|b z|^2 + |f|^2)
\]

\[
|B(x, z, p)| \leq |c||p| + |dz| + |g|.
\]

We can consolidate this further by writing

\[
\bar{\epsilon} = |z| + k, \quad \bar{b} = \lambda^{-2} (|b|^2 + |c|^2 + k^{-2} |f|^2) + \lambda^{-1} (|d| + k^{-1} |g|)
\]

for some \(k > 0\). Then for an \(0 < \epsilon < 1\), we finally have the following inequalities

\[
p_i A^i(x, z, p) \geq \frac{\lambda}{2} (|p|^2 - 2\bar{b} \bar{\epsilon}^2)
\]

\[
|\bar{\epsilon} B(x, z, p)| \leq \frac{\lambda}{2} \left( \epsilon |p|^2 + \frac{\bar{b}}{\epsilon} \bar{\epsilon}^2 \right).
\]

If we denote \(a^{ij}\) by \(a\), then we can write \(|A(x, z, p)| \leq |a||p| + |b z| + |f|\). Also note that we can divide (1) by \(\frac{\lambda}{2}\) to finally get the structural inequalities

\[
|A(x, z, p)| \leq |a||p| + 2b^{1/2} \bar{\epsilon}
\]

\[
p \cdot A \geq |p|^2 - 2\bar{b} \bar{\epsilon}^2
\]

\[
|\bar{\epsilon} B(x, z, p)| \leq \epsilon |p|^2 + \frac{1}{\epsilon} \bar{b} \bar{\epsilon}^2
\]

for some \(\epsilon \in (0, 1], \bar{\epsilon} = |z| + k; \bar{b} = \frac{1}{\lambda} (|b|^2 + |c|^2 + k^{-2} |f|) + \frac{1}{\lambda} (|d| + k^{-1} |g|)\). For the purposes of the following proof, we will let \(k = k(R) = \frac{1}{2} (R^{\delta} \|f\|_q + R^{2\delta} \|g\|_{q/2})\), for \(\delta = 1 - n/q\).

### 1.3 The Harnack Inequality

Before proving The Harnack Inequality, we must prove two theorems first.

**Theorem 1.1.** Let \(L\) be uniformly elliptic with bounded coefficients. \(f^i \in L^q(\Omega), g \in L^{q/2}(\Omega)\) for \(q > n\). Let \(u \in W^{1,2}(\Omega)\) be a subsolution in \(\Omega\). Then for any \(B_{2R}(y) \subset \Omega\), and \(p > 1\), we have

\[
\sup_{B_{R}(y)} u \leq C \left( R^{-n/p} \|u^+\|_{L^p(B_{2R}(y))} + k(R) \right).
\]

**Theorem 1.2.** Let \(L\) be uniformly elliptic with bounded coefficients and suppose that \(f^i \in L^q(\Omega), g \in L^{q/2}(\Omega)\) for \(q > n\). Let \(u \in W^{1,2}(\Omega)\) be a supersolution in \(\Omega\). If \(u\) is non-negative in \(B_{2R}(y) \subset \Omega\) and \(1 \leq p < n/(n-2)\), we have

\[
R^{-n/p} \|u\|_{L^p(B_{2R}(y))} \leq C \left( \inf_{B_{R}(y)} u + k(R) \right).
\]
Proof. It is convenient to prove these two theorems conjointly in the case where \( u \) is a bounded non-negative subsolution. We begin by assuming that \( R = 1, k > 0 \). The general case will be obtained from transforming \( x \mapsto x/R \) and letting \( k \to 0 \). Let \( \beta \neq 0, \eta \in C^2_0(B_1) \) be non-negative. We define \( v := \eta^2 \tilde{u}^\beta \). Recall that \( \tilde{u} = u + k \). Then we have that
\[
Dv = 2\eta D\eta \tilde{u}^\beta + \beta \eta^2 \tilde{u}^{\beta - 1}Du.
\]
Note that \( v \) is a valid test function. Then we plug this \( v \) into our definition of subsolutions
\[
\int_{\Omega} (D_i v A^i(x, u, Du) - v B(x, u, Du)) \, dx \leq 0
\]
and obtain
\[
\int_{\Omega} (2\eta D\eta \tilde{u}^\beta A(x, u, Du) + \beta \eta^2 \tilde{u}^{\beta - 1}Du A(x, u, Du)) \, dx \leq \int_{\Omega} \eta^2 \tilde{u}^\beta B(x, u, Du) \, dx. \tag{6}
\]
We will now attempt to apply our structural inequalities (3) into (6):
\[
\eta^2 \tilde{u}^{\beta - 1}Du A(x, u, Du) \geq \eta^2 \tilde{u}^{\beta - 1} |Du|^2 - 2\eta^2 \tilde{u}^{\beta - 1} B(x, u, Du) \leq \eta^2 \tilde{u}^{\beta - 1} \tilde{u}B(x, u, Du) \leq \eta^2 \tilde{u}^{\beta - 1} \left( \epsilon |Du|^2 + \frac{1}{\epsilon} \tilde{u} \right) = \epsilon \eta^2 |Du|^2 \tilde{u}^{\beta - 1} + \frac{1}{\epsilon} \eta^2 \tilde{u}^{\beta + 1}.
\]
The last structural inequality gives us this:
\[
|D\eta \cdot A(x, u, Du)\tilde{u}^\beta| \leq |a| |\eta| |D\eta|\tilde{u}^\beta |Du| + 2\beta \eta |D\eta|\tilde{u}^{\beta + 1} \leq \frac{\epsilon}{\eta^2 \tilde{u}^{\beta - 1} |Du|^2 + \frac{1}{\epsilon} |a|^2 |D\eta|^2 \tilde{u}^{\beta + 1} + \tilde{u} \tilde{u}^{\beta + 1} + \beta \eta^2 \tilde{u}^{\beta + 1} \leq \epsilon \eta^2 |Du|^2 \tilde{u}^{\beta - 1} + \frac{1}{\epsilon} \eta^2 \tilde{u}^{\beta + 1}.
\]
Applying this to (6) yields
\[
\int_{\Omega} (\beta \eta^2 \tilde{u}^{\beta - 1} - 2\beta \eta^2 \tilde{u}^{\beta + 1}) \, dx \leq \int_{\Omega} \left\{ \epsilon \eta^2 |Du|^2 \tilde{u}^{\beta - 1} + \frac{1}{\epsilon} \beta \eta^2 \tilde{u}^{\beta + 1} + \epsilon \eta^2 \tilde{u}^{\beta - 1} |Du|^2 + \left( 2 + \frac{|a|^2}{\epsilon} \right) |D\eta|^2 \tilde{u}^{\beta + 1} + 2\beta \eta^2 \tilde{u}^{\beta + 1} \right\} \, dx
\]
Combining like terms yields
\[
\int_{\Omega} (\beta - 2\epsilon) \eta^2 |Du|^2 \tilde{u}^{\beta - 1} \, dx \leq \int_{\Omega} \left\{ \left( 2\beta + 1 \right) \beta \eta^2 + \left( 2 + \frac{|a|^2}{\epsilon} \right) |D\eta|^2 \right\} \tilde{u}^{\beta + 1} \, dx.
\]
Let \( \epsilon = \min \left\{ 1, \frac{\beta}{\beta+1} \right\} \). Then we can further consolidate this into
\[
\int_{\Omega} \eta^2 |Du|^2 \tilde{u}^{\beta - 1} \, dx \leq C(\beta) \int_{\Omega} \left( \beta \eta^2 + (1 + |a|^2) |D\eta|^2 \right) \tilde{u}^{\beta + 1} \, dx. \tag{7}
\]
We introduce the function
\[
w := \begin{cases} \tilde{u}^{(\beta+1)/2} & \text{if } \beta \neq -1 \\ \log \tilde{u} & \text{if } \beta = -1 \end{cases}
\]
so that
\[ |\eta Dw| = \begin{cases} 
\eta \frac{\beta+1}{2} u^{(\beta-1)/2} |Du| & \text{if } \beta \neq -1 \\
\eta |Du| \bar{u}^{-1} & \text{if } \beta = -1 
\end{cases} \]

and let \( \gamma = \beta + 1 \). Then we can rewrite (7) as
\[
\int_{\Omega} |\eta Dw|^2 \, dx \leq \begin{cases} 
C \epsilon^2 \int_{\Omega} (\bar{b} |\eta|^2 + (1 + |a|^2) |D\eta|^2) \, w^2 \, dx & \text{if } \beta \neq -1 \\
C \int_{\Omega} (\bar{b} |\eta|^2 + (1 + |a|^2) |D\eta|^2) \, dx & \text{if } \beta = 1
\end{cases}
\tag{8}
\]

Before we move any further, we need to introduce a few results from analysis of Sobolev spaces.

**Lemma 1.3 (Interpolation Inequality).** Let \( p \leq q \leq r \). Then for \( u \in L^r(\Omega) \), we have
\[
\|u\|_q \leq \epsilon \|u\|_r + \epsilon^{-\mu} \|u\|_p,
\]
where
\[
\mu = \frac{1}{r} - \frac{1}{q} + \frac{1}{q} - \frac{1}{r}.
\]

**Theorem 1.4 (Sobolev Inequalities).**
\[
W^{1,p}_0(\Omega) \subset \begin{cases} 
L^{np/(n-p)}(\Omega) & \text{for } p < n \\
C^0(\Omega) & \text{for } p > n.
\end{cases}
\]

Furthermore, there exists a constant \( C = C(n, p) \) such that, for any \( u \in W^{1,p}_0(\Omega) \), we have
\[
\|u\|_{np/(n-p)} \leq C \|Du\|_p \quad \text{for } p < n
\]
\[
\sup_{\Omega} |u| \leq C |\Omega|^{1/n} \|Du\|_p \quad \text{for } p > n.
\]

From the Sobolev Inequalities, we have
\[
\|\eta w\|_{2\hat{n}/(\hat{n}-2)}^2 \leq C \int_{\Omega} (|D\eta w|^2 + |\eta Dw|^2) \, dx;
\]

for \( \hat{n} = n \) for \( n > 2 \), and \( 2 < \hat{2} < q \). Now we apply Hölder’s Inequality and the Interpolation Inequality to get
\[
\int_{\Omega} \bar{b}(\eta w)^2 \, dx \leq \|\bar{b}\|_{q/2} \|\eta w\|_{2q/(q-2)}^2
\]
\[
\leq \|\bar{b}\|_{q/2} (\epsilon \|\eta w\|_{2\hat{n}/(\hat{n}-2)} + \epsilon^{-\sigma} \|\eta w\|_2^2),
\]

where \( \sigma = \hat{n}/(q - \hat{n}) \). Now we attempt to plug in these estimates into (8). Let \( \xi = \hat{n}/(\hat{n} - 2) \). Now we add
Taking the lim sup yields

\[ \|\eta w\|_{2\chi} \leq C\gamma^2 \int_{\Omega} (\hat{b}(\eta w)^2 + |wD\eta|^2 + |a|^2 |wD\eta|^2) \, dx \]

\[ \leq C\gamma^2 \hat{b}_{q/2} (\epsilon \|\eta w\|_{2\chi} + \epsilon^{-\gamma} \|\eta w\|_{2\chi})^2 + C\gamma^2 \int_{\Omega} (1 + |a|^2 |wD\eta|^2) \, dx \]

\[ = C\gamma^2 \left( \epsilon \|\eta w\|_{2\chi} + \epsilon^{-\gamma} \|\eta w\|_{2\chi} \right)^2 + C\gamma^2 \int_{\Omega} (1 + |a|^2 |wD\eta|^2) \, dx \]

\[ \leq C\gamma^2 \left( \epsilon^2 \|\eta w\|_{2\chi}^2 + \epsilon^{1-\sigma} \|\eta w\|_{2\chi} \|\eta w\|_{2\chi} + \epsilon^{-2\sigma} \|\eta w\|_{2\chi}^2 + \|wD\eta\|_2^2 \right) \]

\[ \|\eta w\|_{2\chi}^2 (1 - C\gamma^2 \epsilon^2) \leq C\gamma^2 \left( \epsilon^{1-\sigma} + \epsilon^{-2\sigma} \right) (\|\eta w\|_{2\chi}^2 + \|wD\eta\|_2^2) \]

\[ \|\eta w\|_{2\chi}^2 \leq \frac{C\gamma^2 \left( \epsilon^{1-\sigma} + \epsilon^{-2\sigma} \right)}{1 - C\gamma^2 \left( \epsilon^{1-\sigma} + \epsilon^{-2\sigma} \right)} (\|\eta w\|_{2\chi}^2 + \|wD\eta\|_2^2) \]

\[ \leq C(1 + |\gamma|)^{\sigma+1} \|(\eta + |D\eta|)w\|_2^2 \]

Then we finally get

\[ \|\eta w\|_{2\chi} \leq C(1 + |\gamma|)^{\sigma+1} \|(\eta + |D\eta|)w\|_2, \quad (9) \]

where \( C = C(\hat{b}, \Lambda, \nu, q, |\beta|) \) is bounded when \( |\beta| \) is bounded away from zero. We will now get a better cutoff function \( \eta \). Let \( r_1, r_2 \) be such that \( 1 \leq r_1 < r_2 \leq 3, \eta \equiv 1 \) in \( B_{r_1}, \beta \equiv 0 \) in \( \Omega \setminus B_{r_2} \), with

\[ |D\eta| \leq \frac{2}{r_2 - r_1}. \]

Then we have from (9)

\[ \|w\|_{L^{2\chi}(B_{r_1})} \leq \frac{C(1 + |\gamma|)^{\sigma+1}}{r_2 - r_1} \|w\|_{L^2(B_{r_2})}, \quad (10) \]

Before we move on, lets make a quick backtrack to functional spaces. Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain. Then I claim that if \( u \) is a measurable function on \( \Omega \) such that \( |u|^p \in L^1(\Omega) \) for some \( p \in \mathbb{R} \), and define

\[ \phi_p(u) := \left( \frac{1}{|\Omega|} \int_{\Omega} |u|^p \, dx \right)^{1/p}, \]

then we have

\[ \lim_{p \to \infty} \phi_p(u) = \sup_{\Omega} |u|. \quad (11) \]

To start, note that

\[ \phi_p(u) = \left( \frac{1}{|\Omega|} \int_{\Omega} |u|^p \, dx \right)^{1/p} \leq \left( \frac{1}{|\Omega|} \int_{\Omega} \left( \sup_{\Omega} |u| \right)^p \, dx \right)^{1/p} = \sup_{\Omega} |u|. \]

Taking the lim sup yields

\[ \limsup_{p \to \infty} \phi_p(u) \leq \sup_{\Omega} |u|. \]
Now fix \( \epsilon \) and define \( A = \{ x \in \Omega : |u| \geq \sup_{\Omega} |u| - \epsilon \} \). Then we see that

\[
\phi_p(u) = \left( \frac{1}{|\Omega|} \int_{\Omega} |u|^p \, dx \right)^{1/p} \geq \left( \frac{1}{|A|} \int_A |u|^p \, dx \right)^{1/p} \geq \left( \frac{1}{|A|} \left( \sup_{\Omega} |u| - \epsilon \right)^p \, dx \right)^{1/p} = \frac{|A|}{|\Omega|} \left( \sup_{\Omega} |u| - \epsilon \right).
\]

Taking \( \epsilon \to 0 \) and then the \( \lim \inf \) yields

\[
\liminf_{p \to \infty} \phi_p(u) \geq \sup_{\Omega} |u|.
\]

This and our first inequality yield the claim (11). We go back to our proof. For \( r < 4 \), we define the function

\[
\phi(p, r) := \left( \int_{B_r} |\bar{u}|^p \, dx \right)^{1/p}.
\]

From what we just showed, we have that

\[
\phi(\infty, r) = \lim_{p \to \infty} \phi(p, r) = \sup_{B_r} \bar{u}.
\]

Now we can rewrite inequality (11) as

\[
\phi(\chi^{r_1}, r_1) \leq \left( \frac{C(1 + |\gamma|)^{\sigma + 1}}{r_2 - r_1} \right)^{2/|\gamma|} \phi(\gamma, r_2) \quad \text{if } \gamma > 0 \tag{12}
\]

\[
\phi(\gamma, r_2) \leq \left( \frac{C(1 + |\gamma|)^{\sigma + 1}}{r_2 - r_1} \right)^{2/|\gamma|} \phi(\chi^{r_1}, r_1) \quad \text{if } \gamma < 0 \tag{13}
\]

Now we are in the right position to start our iteration. Recall that when \( u \) is a subsolution, we have \( \beta > 0 \) and \( \gamma > 1 \). Taking \( p > 1 \) and setting \( \gamma = \gamma_m = \chi^{m_p} \) and \( r_m = 1 + 2^{-m} \). Then plugging this into our inequality (12) gives us

\[
\phi(\chi^{m_p}, 1) \leq \left( \frac{C(1 + |\chi^{m_p}|)^{\sigma + 1}}{r_2 - 1} \right)^{2/(\chi^{m_p})} \phi(\chi^{m_p}, r_2) \leq \left( \frac{C(1 + |\chi^{m_p}|)^{\sigma + 1}}{r_3 - r_2} \right)^{2/(\chi^{m_p})} \phi(\chi^{m_p}, r_4) \leq \cdots \leq (C\chi)^{2(1+\sigma)\Sigma m_p^{-m}} \phi(p, 2) = C \phi(p, 2).
\]

Letting \( m \to \infty \) in the above yields

\[
\sup_{B_{r_2}} |u| \leq C \| \bar{u} \|_{L^p(B_{r_2})}, \quad C = C(\hat{n}, \Lambda, \nu, q, p).
\]

Transforming \( x \to x/R \) we have the desired estimate (4) if \( u \) is a subsolution. Now in the cases when \( u \) is a supersolution, we need to approach the problem a bit differently. Recall that when \( u \) is a supersolution, \( \beta < 0 \) and \( \gamma < 1 \). Then for any \( p, p_0 \) such that \( 0 < p_0 < p < \chi \), we have

\[
\phi(p, 2) \leq C \phi(p_0, 3) \quad \phi(-p_0, 3) \leq C \phi(-\infty, 1).
\]

Then we will finish proving our theorem if we can show that \( \phi(p_0, 3) \leq C(-p_0, 3) \). This is done in an intricate way and will not be done here. Or it will be later, but for now we move on. \( \square \)
Putting our two theorems together give us the full Harnack Inequality:

**Theorem 1.5.** Let \( L \) be uniformly elliptic and have bounded coefficients. Let \( u \in W^{1,2}(\Omega) \) satisfy \( u \geq 0 \) in \( \Omega \) and \( Lu = 0 \) in \( \Omega \). Then for any ball \( B_{4R}(y) \subset \Omega \), we have

\[
\sup_{B_{R}(y)} u \leq C \inf_{B_{R}(y)} u,
\]

where \( C = C(n, \Lambda/\lambda, \nu R) \).

### 1.4 Applications of the Harnack Inequality

We can give a new proof of the strong maximum principle (instead of using Hopf’s Lemma) now:

**Theorem 1.6.** \( L \) uniformly elliptic, bounded coeff, \( u \in W^{1,2}(\Omega) \), \( Lu \geq 0 \) in \( \Omega \). Then if for some ball \( B \subset\subset \Omega \) we have

\[
\sup_{B} u = \sup_{\Omega} u \geq 0,
\]

then the function \( u \) must be constant in \( \Omega \).

**Proof.** We apply the Harnack inequality with \( p = 1 \) to the function \( v = M - u \) and show that \( M - u = 0 \).

Finally, we can use the Harnack Inequality to imply Hölder continuity.

**Theorem 1.7.** Let \( L \) be uniformly elliptic and have bounded coefficients. Then if \( u \in W^{1,2}(\Omega) \) is a solution of

\[
Lu = g + D_i f^i,
\]

then \( u \) is locally Hölder continuous in \( \Omega \), and for any ball \( B_0 = B_{R_0}(y) \subset \Omega \) and \( R \leq R_0 \) we have

\[
\text{osc}_{B_{R}(y)} u \leq CR^\alpha (R_0^{-\alpha} \sup_{B_0} |u| + k).
\]

**Proof.** Start out by defining \( M_0, M_1, M_4, m_1, m_4 \). Then apply Harnack inequality with \( p = 1 \) to \( M_4 - u \) and \( u - m_4 \) to end up with

\[
\omega(R) \leq \omega(4R) + k(R).
\]

Before we go on, we have to prove something else first. Suppose \( \omega \) is non-decreasing on \((0, R_0], R \leq R_0 \) satisfies

\[
\omega(\tau R) \leq \gamma \omega(R) + \sigma(R),
\]

where \( \sigma \) is also non-decreasing, \( 0 < \gamma, \tau, < 1 \). Then for any \( \mu \in (0, 1) \), we have

\[
\omega(R) \leq C \left( \left( \frac{R}{R_0} \right)^\alpha \omega(R_0) + \sigma(R^\mu) \right).
\]

After doing some iterations, the key step is

\[
\omega(R) \leq \gamma^{m-1} \omega(R_0) + \frac{\sigma(R_1)}{1 - \gamma}
\]

\[
\leq \frac{1}{\gamma} \left( \frac{R}{R_1} \right)^{\log \gamma/\log \tau} \omega(R_0) + \frac{\sigma(R_1)}{1 - \gamma}.
\]

\[ \Box \]