

A Brief Summary of Sobolev Spaces

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1 Definitions of Sobolev Spaces and Elementary Properties

First lets talk about some motivation for Sobolev Spaces. While looking for solutions for PDE's, it might be difficult to find nice and smooth solutions. We remedy this problem by introducing the notion of a weak derivative. Let $U \subset \mathbb{R}^n$ be open, $u \in C^1(U)$, and $\phi \in C_c^\infty(U)$. Then from integration by parts,

$$\int_U u \phi_{x_i} dx = - \int_U u_{x_i} \phi dx.$$

Notice that we do not have a boundary term because ϕ has compact support. Let $u \in C^k(U)$, $\phi \in C_c^\infty(U)$. Then similarly, by summing up through all our coordinates, we have

$$\int_U u D^\alpha \phi dx = (-1)^k \int_U D^\alpha u \phi dx,$$

for all multiindex $|\alpha| = k$.

Definition 1.1. Let $u, v \in L_{loc}^p(U)$ and α is a multi index. We say that v is the α^{th} - weak derivative of u if

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v \phi dx$$

for all $\phi \in C_c^\infty(U)$.

Note. It is quite easy to show that if v is a weak derivative of u , it is unique up to a set of measure zero.

Definition 1.2. The Sobolev Space $W^{k,p}(U)$ consists of all locally summable functions $u : U \rightarrow \mathbb{R}$ such that for all multiindex α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.

Note. If $p = 2$, we write $H^k(U) = W^{k,2}(U)$.

Definition 1.3. If $u \in W^{k,p}(U)$, we define its *Sobolev norm* by

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha| \leq k} \int_U |D^\alpha u|^p dx \right)^{1/p} & 1 \leq p < \infty \\ \sum_{|\alpha| \leq k} \text{esssup}_U |D^\alpha u| & p = \infty. \end{cases}$$

Theorem 1.4 (Elementary Properties). *Let $u, v \in W^{k,p}(U)$, $|\alpha| \leq k$. Then*

1. $D^\alpha u \in W^{k-|\alpha|,p}(U)$ and $D^\beta(D^\alpha u) = D^{\alpha+\beta}u$ for all multiindexes α, β .
2. $\forall a, b \in \mathbb{R}$, we have $au + bv \in W^{k,p}(U)$ and

$$D^\alpha(au + bv) = aD^\alpha u + bD^\alpha b.$$

3. Let $\phi \in C_c^\infty(U)$. Then $\phi u \in W^{k,p}(U)$ and

$$D^\alpha(\phi u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \phi D^{\alpha-\beta} u.$$

Proof. The proof of the first two are trivial. The proof of 3. requires induction. \square

Theorem 1.5. $\forall k$ and $1 \leq p \leq \infty$, $W^{k,p}(U)$ is a Banach Space.

Proof. Well first we have to show that the Sobolev norm is indeed a norm. It is trivial to show that $\|\lambda u\|_{W^{k,p}(U)} = |\lambda| \|u\|_{W^{k,p}(U)}$ and $\|u\|_{W^{k,p}(U)} = 0 \iff u \equiv 0$ a.e. The triangle inequality is also easy to prove. We need only prove that $W^{k,p}(U)$ is complete. Let $\{u_m\}_{m=1}^\infty$ be a cauchy sequence in $W^{k,p}(U)$. We need to show that it converges in $W^{k,p}(U)$. First note that by definition, $D^\alpha u_m$ is a cauchy sequence in $L^p(U)$. i.e., there exists a function u_α such that $D^\alpha u_m \rightarrow u_\alpha$ in $L^p(U)$ as $m \rightarrow \infty \forall |\alpha| \leq k$ because $L^p(U)$ is complete. In particular, $u_m \rightarrow u_{(0,\dots,0)} =: u$. Now I claim that $u \in W^{k,p}(U)$ and $D^\alpha u = u_\alpha$ in the weak sense. Let $\phi \in C_c^\infty(U)$. Then,

$$\begin{aligned} \int_U u D^\alpha \phi \, dx &= \lim_{m \rightarrow \infty} \int_U u_m D^\alpha \phi \, dx \\ &= \lim_{m \rightarrow \infty} (-1)^{|\alpha|} \int_U D^\alpha u_m \phi \, dx \\ &= (-1)^{|\alpha|} \int_U u_\alpha \phi \, dx. \end{aligned}$$

We have shown $u \in W^{k,p}(U)$, $D^\alpha u = u_\alpha$. Since $D^\alpha u_m \rightarrow D^\alpha u$ in $L^p(U) \implies u_m \rightarrow u$ in $W^{k,p}(U)$. \square

2 Approximation Theorems

Recall that $U_\epsilon = \{x \mid \text{dist}(x, \partial U) > \epsilon\}$.

Theorem 2.1. Let $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$. Set $u^\epsilon = \eta_\epsilon * u$ in U_ϵ . Then, $u^\epsilon \in C^\infty(U_\epsilon)$ and $u^\epsilon \rightarrow u$ in $W_{\text{loc}}^{k,p}(U)$ as $\epsilon \rightarrow 0$.

Proof. The first part of the theorem is done in the appendix. It is literally just taking derivatives and using integration by parts. For the second part of the proof, we must show that the regular derivative of u^ϵ is the mollification of the weak derivative of u . i.e., $D^\alpha u^\epsilon = \eta_\epsilon * D^\alpha u$ for all multiindexes $|\alpha| \leq k$.

$$\begin{aligned} D^\alpha u^\epsilon &= D^\alpha \int_U \eta_\epsilon(x-y) u \, dy = \int_U D_x \eta_\epsilon(x-y) u \, dy \\ &= (-1)^{|\alpha|} \int_U D_y \eta_\epsilon(x-y) u(y) \, dy = (-1)^{|\alpha|+|\alpha|} \int_U \eta_\epsilon(x-y) D^\alpha u \, dy \\ &= \eta_\epsilon * [D^\alpha u]. \end{aligned}$$

Note that as $\epsilon \rightarrow 0$, $D^\alpha u^\epsilon \rightarrow D^\alpha u$. Then for $V \subset\subset U$,

$$\|u^\epsilon - u\|_{W^{k,p}(V)}^p = \sum_{|\alpha| \leq k} \|D^\alpha u^\epsilon - D^\alpha u\|_{L^p(V)}^p \rightarrow 0.$$

\square

Theorem 2.2. Let U be bounded and $\partial U \in C^1$. Let $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$. Then $\exists u_m \in C^\infty(\bar{U})$ such that $u_m \rightarrow u$ in $W^{k,p}(U)$.

Proof. The trick in this proof is to cover the boundary with open balls (which we can do because ∂U is compact). In each ball, we will move just high enough to mollify within U . Then we show that the mollification converges to u in an compactly supported subset of each ball, and pick a $V_0 \subset\subset U$ cleverly to make U covered by such balls. Then we apply partition of unity to make a smooth function that converges to u in the whole U . \square

3 Extensions

Our goal is to extend functions from $W^{1,p}(U)$ to $W^{1,p}(\mathbb{R}^n)$, which can be subtle because extending them to 0 on $\mathbb{R}^n \setminus \{U\}$ might make a discontinuity along ∂U which leads to no longer having a weak partial derivative.

Theorem 3.1. *Assume U is bounded and ∂U is C^1 . Select a bounded open set V such that $U \subset\subset V$. Then There exists a bounded linear operator*

$$E : W^{1,p}(U) \rightarrow W^{1,p}(\mathbb{R}^n)$$

such that for each $u \in W^{1,p}(U)$,

1. $Eu = u$ a.e. in U ,
2. Eu has support within V , and
- 3.

$$\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)},$$

the constant C depending only on p, U , and V .

Proof. First we assume that $u \in C^1(U)$ and ∂U is flat near $x_0 \in \partial U$. Take $B_r(x_0)$ and define

$$\bar{u} := \begin{cases} u(x) & x \in B^+ \\ -3u(x_1, \dots, x_{n-1}, -x_n) + 4u(x_1, \dots, x_{n-1}, -\frac{x_n}{2}) & x \in B^- \end{cases}$$

We then show $\bar{u} \in C^1(U)$ by differentiating directly. Now assume that ∂U is not flat near x_0 . Then we can just apply the straightening out strategy that was introduced in the appendix. We henceforth have the estimate

$$\|\bar{u}\|_{W^{1,p}(B)} \leq C\|u\|_{W^{1,p}(B^+)}.$$

Now we just apply a partition of unity and make our estimate

$$\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}.$$

We simply define $Eu := \bar{u}$ and notice that it is the linear operator that we desire. Now recall that u is still assumed to be smooth. By our approximation theorems, we have $u_m \in C^\infty(\bar{U})$ that converge to u in $W^{1,p}(U)$. Then we can apply our new estimate on these u_m functions

$$\|Eu_m - Eu_l\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u_m - u_l\|_{W^{1,p}(U)}.$$

Since $W^{1,p}(U)$ is a Banach space, the sequence Eu_m is a cauchy sequence that converges to $\bar{u} =: Eu$. \square

4 Traces

Theorem 4.1. *Assume U is bounded and ∂U is C^1 . Then there exists a bounded linear operator*

$$T : W^{1,p}(U) \rightarrow L^p(\partial U)$$

such that

1. $Tu = u|_{\partial U}$ if $u \in W^{1,p}(U) \cap C(\bar{U})$,

2.

$$\|Tu\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)},$$

for each $u \in W^{1,p}(U)$, with the constant C depending only on p and U .

Proof. The proof is merely computations. □

5 Sobolev Inequalities

5.1 Gagliardo-Nirenberg-Sobolev Inequality

First let $1 \leq p < n$. Then we want to show that if $\exists C > 0, 1 \leq q < \infty$ such that

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)}$$

holds, then C and q do not depend on u . To do this, let $u \in C_c^\infty(\mathbb{R}^n)$ and define $u_\lambda(x) := u(\lambda x)$. Applying our desired inequality to our new function yields

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} \leq C\|Du_\lambda\|_{L^p(\mathbb{R}^n)}.$$

Now we explicitly find the LHS and RHS of the inequality above:

$$\int_{\mathbb{R}^n} |u_\lambda|^q dx = \int_{\mathbb{R}^n} |u(\lambda x)|^q dx = \frac{1}{\lambda^n} \int_{\mathbb{R}^n} |u|^q dx,$$

and

$$\int_{\mathbb{R}^n} |Du_\lambda|^p dx = \lambda^p \int_{\mathbb{R}^n} |Du(\lambda x)|^p dx = \frac{\lambda^p}{\lambda^n} \int_{\mathbb{R}^n} |Du|^p dx.$$

Putting these into our inequality yields

$$\begin{aligned} \frac{1}{\lambda^{n/q}} \|u\|_{L^q(\mathbb{R}^n)} &\leq C \left(\frac{\lambda^p}{\lambda^n} \right)^{1/p} \|Du\|_{L^p(\mathbb{R}^n)} \\ \|u\|_{L^q(\mathbb{R}^n)} &\leq C \lambda^{1 - \frac{n}{p} + \frac{n}{q}} \|Du\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

Now consider the case where $1 - \frac{n}{p} + \frac{n}{q} \neq 0$. Then sending $\lambda \rightarrow 0, \infty$ would lead to a contradiction $\implies 1 - \frac{n}{p} + \frac{n}{q} = 0 \implies$

$$q = p^* = \frac{np}{n-p}.$$

This is called the *Sobolev conjugate* of p .

Theorem 5.1 (Gagliardo-Nirenberg-Sobolev Inequality). *Assume $1 \leq p < n$. There exists a constant C depending only on p and n , such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)},$$

for all functions $u \in C_c^1(\mathbb{R}^n)$.

Proof. Computations. □

Theorem 5.2. *Let U be a bounded open subset of \mathbb{R}^n , and ∂U is C^1 . Assume $1 \leq p < n$ and $u \in W^{1,p}(U)$. Then $u \in L^{p^*}(U)$, with the estimate*

$$\|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)},$$

the constant C depending only on p, n , and U .

Proof. There are a few key steps in this proof. First note that since ∂U is C^1 , we can apply our extension theorem to get a linear operator $Eu = \bar{u} \in W^{1,p}(\mathbb{R}^n)$. The function \bar{u} has compact support in U , $\bar{u} = u$ in U , and

$$\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}.$$

Now we approximate \bar{u} with $u_m \in C_c^\infty(\mathbb{R}^n)$. Applying the Gagliardo-Nirenberg-Sobolev Inequality yields

$$\|u_m - u_l\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du_m - Du_l\|_{L^p(\mathbb{R}^n)}.$$

The latter converges $\implies u_m \rightarrow \bar{u}$ in $L^{p^*}(\mathbb{R}^n)$. So then we have $\|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|D\bar{u}\|_{L^p(\mathbb{R}^n)}$. This completes the proof. \square

Note. This implies that $W^{1,p}(U) \subset L^{p^*}(U)$, i.e., $W^{1,p}(U)$ is embedded into $L^{p^*}(U)$.

5.2 Morrey's Inequality

Now we suppose $n < p < \infty$. We will show that if $u \in W^{1,p}(U)$, then u is Holder continuous.

Theorem 5.3. *Assume $n < p \leq \infty$. Then $\forall u \in C^1(\mathbb{R}^n)$, there exists a constant C , depending only on p and n such that*

$$\|u\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

Proof. Computations. \square

Theorem 5.4. *Let U be a bounded, open subset of \mathbb{R}^n and suppose ∂U is C^1 . Assume $n < p \leq \infty$ and $u \in W^{1,p}(U)$. Then u has a version $u^* \in C^{0,\gamma}(\bar{U})$, for $\gamma = 1 - \frac{n}{p}$, with the estimate*

$$\|u^*\|_{C^{0,1-\frac{n}{p}}(\bar{U})} \leq C\|u\|_{W^{1,p}(U)}.$$

Proof. The proof is very similar to the proof of Theorem 5.2. \square

6 Compactness

We already know from the Gagliardo-Nirenberg-Sobolev Inequality that $W^{1,p}(U) \subset L^{p^*}(U)$. Now we show that it is compactly embedded.

Definition 6.1. Let X, Y be Banach Spaces with $X \subset Y$. We say that X is compactly embedded in Y (written $X \subset\subset Y$) if $\forall x \in X$,

$$\|x\|_Y \leq C\|x\|_X$$

and every bounded sequence in X is precompact in Y (there exists a convergent subsequence).

Theorem 6.2. *Let U be a bounded open subset of \mathbb{R}^n and $\partial U \in C^1$. Suppose $1 \leq p < n$. Then $W^{1,p}(U) \subset\subset L^q(U)$ for any $1 \leq q < p^*$.*

Proof. The proof for this theorem is very long. First note that the Gagliardo-Nirenberg-Sobolev Inequality finishes the first two requirements for $W^{1,p}(U)$ to be compactly embedded in $L^q(U)$. All that we need to show now is that for any bounded $\{u_m\}_{m=1}^\infty$ in $W^{1,p}(U)$, $\exists \{u_{m_j}\}_{j=1}^\infty$ that converges in $L^q(U)$. From our extension theorem, we may assume that $U = \mathbb{R}^n$ and prove the theorem for $V \subset \mathbb{R}^n$ bounded and open. We can also assume that all the u_m have compact support. Fix $\epsilon > 0$ and define $u_m^\epsilon := \eta_\epsilon * u_m$. Our job is to show that $u_m^\epsilon \rightarrow u_m$ in $L^q(V)$ uniformly as $\epsilon \rightarrow 0$! Then we show that our sequence u_m^ϵ is actually uniformly bounded and equicontinuous (by showing they have bounded derivatives). Now our main tool is to use the Arzela-Ascoli compactness theorem: *Suppose $\{f_k\}_{k=1}^\infty$ is a uniformly bounded and uniformly equicontinuous sequence of functions. Then there exists a subsequence $\{f_{k_j}\}_{j=1}^\infty$ and a continuous function f such that $f_{k_j} \rightarrow f$ iniformly on compact subsets of \mathbb{R}^n .* We then manipulate the Arzela-Ascoli theorem, convergence of $u_m^\epsilon \rightarrow u_m$, and the triangle inequality, to extract a converging subsequence from u_m . \square

Note. For $n < p \leq \infty$, we can apply the same proof by using Morrey's inequality and the Arzela-Ascoli compactness theorem.

7 Additional Topics

7.1 Poincare's Inequalities

We can use the theorem we just proved to find a very strong inequality involving the average of a function in $W^{1,p}(U)$.

Definition 7.1. We define

$$(u)_U := \int_U u \, dx.$$

Theorem 7.2 (Poincare's Inequality). *Let U be a bounded, connected open subset of \mathbb{R}^n , with a C^1 boundary ∂U . Assume $1 \leq p \leq \infty$. Then*

$$\|u - (u)_U\|_{L^p(U)} \leq C \|Du\|_{L^p(U)}$$

for all functions $u \in W^{1,p}(U)$.

Proof. Proof is done by contradiction, tricky computations, and Theorem 6.2. \square

Note. There is a Poincare's inequality for a ball. It follows from the previous theorem and changing variables and letting U be a ball.

7.2 Difference Quotients

Now we will study difference quotient approximations to weak derivatives. From now on, assume $u : U \rightarrow \mathbb{R}$ is a locally summable function and $V \subset\subset U$. $D^h u := (D_1^h u, \dots, D_n^h u)$.

Theorem 7.3 (Difference quotients and weak derivatives). *Assume $V \subset\subset U$.*

1. *Suppose $1 \leq p < \infty$ and $u \in W^{1,p}(U)$. Then,*

$$\|D^h u\|_{L^p(V)} \leq C \|Du\|_{L^p(U)}.$$

2. *Assume $1 < p < \infty$, $u \in L^p(V)$, and the difference quotient is bounded:*

$$\|D^h u\|_{L^p(V)} \leq C.$$

Then $u \in W^{1,p}(V)$ with the same bound

$$\|Du\|_{L^p(V)} \leq C.$$

Proof. For (1), assume temporarily that u is smooth. Then explicitly compute the integrals we want. Then approximate into $W^{1,p}(U)$. For (2) we use the boundedness to make a sequence with a convergent subsequence (applying the compactness theorem). \square

Theorem 7.4. *Let U be open and bounded with $\partial U \in C^1$. Then $u : U \rightarrow \mathbb{R}$ is Lipschitz continuous if and only if $u \in W^{1,\infty}(U)$.*

Proof. The proof is strictly mechanical. \square

Theorem 7.5. *Let $u \in W_{\text{loc}}^{1,p}(U)$ for some $n < p \leq \infty$. Then u is differentiable almost everywhere in U and its gradient equals its weak gradient a.e.*

7.3 Fourier Transform Methods

8 Other Spaces of Functions

8.1 The Space H^{-1}