Geophys. Astrophys. Fluid Dynamics, Vol. 51, pp. 87–102
Reprints available directly from the publisher

THE TRIPOLE: A NEW COHERENT VORTEX STRUCTURE OF INCOMPRESSIBLE TWO-DIMENSIONAL FLOWS

LORENZO M. POLVANI

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.

XAVIER J. CARTON

Center for Meteorology and Physical Oceanography, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.

(Received 12 February 1988; in final form 20 March 1989)

Using a contour dynamical algorithm, we have found rotating tripolar V-state solutions for the inviscid Euler equations in two-dimensions. We have studied their geometry as a function of their physical parameters. Their stability was investigated with the aid of contour surgery, and most of the states were found to be stable. Under finite-amplitude perturbations, tripoles are shown to either fission into two asymmetric dipoles or to evolve into a shielded axisymmetric vortex, demonstrating the existence of two new "reversible transitions" between topologically distinct coherent vortex structures. These dynamical results are confirmed by pseudo-spectral simulations, with which we also show how continuous tripolar long-lived coherent vortex structures can be generated in a variety of ways.

KEY WORDS: Coherent vortices, Euler equations.

1. INTRODUCTION

In recent years it has become apparent that coherent vortex structures play an important role in the dynamics of two- and three-dimensional incompressible flows. From geophysical observations and laboratory experiments, it is well known that rotating and/or stratified flows can be realistically described by two-dimensional systems. For this reason much effort has gone into both analytical and numerical investigations of the generation, evolution and interaction of two-dimensional coherent structures.

These have most often been observed as the final stage of unstable laminar flows, or of initially random vorticity fields. A by now classical example is the study of McWilliams (1984) on the spontaneous emergence of mostly monopolar isolated vortices from an initially stochastic vorticity distribution. Another beautiful instance is provided by the recent soap film experiments of Couder and Basdevant (1986), in which dipoles are ejected from an unstable wake.

Since the frequency with which these coherent structures appear decreases very rapidly with the increasing number of poles, the first examples of tripoles have only been observed fairly recently. To the best of our knowledge, the earliest reference in the literature is due to Leith (1981), who first conjectured their existence. Larichev and Reznik (1983) observed the formation of a tripole from the collision of two asymmetric dipoles, and the first clear picture was provided by Legras *et al.* (1988) in high-resolution spectral simulations of homogeneous turbulence. Van Heijst and Klosterziel (1988) have been able to produce laboratory examples of tripoles using a two-layer rotating fluid.

Our aim is in this study to investigate the existence and stability of two-dimensional coherent tripolar vortices. Since the complexity of the analytical solution increases drastically with the number of poles (Legras, private communication), we have taken an alternative approach and resorted to contour dynamical techniques to determine the shape of two-dimensional, inviscid, finite-area, constant vorticity, tripoles (*tripolar V-states*). Such techniques have been used successfully to find monopolar, dipolar (Deem and Zabusky, 1978; Pierrehumbert, 1980) and multipolar V-states (Dritschel, 1985), as well as a variety of multi-layer geostrophic V-states (Polvani, 1988). We have investigated the stability of our tripolar V-states via contour surgery.

We have also studied the evolution of tripoles in a continuous and viscous model with the help of a pseudospectral code. Using as initial condition smoothed out versions of the V-states obtained with contour dynamics, continuous long-lived tripolar coherent structures are easily obtained. Furthermore, we show how continuous tripoles can readily be generated from an initial condition composed of three ellipses. The high-resolution spectral computations are found to coincide remarkably well with the results obtained via contour surgery.

2. FINITE-AREA CONSTANT VORTICITY TRIPOLES

We consider the inviscid Euler equations in two-dimensions:

$$(\partial_t + \psi_x \partial_y - \psi_y \partial_x) \quad \omega = 0, \quad \text{and} \quad \nabla^2 \psi = \omega, \tag{1}$$

where ω is the vorticity, and the Eulerian velocities are obtained from the streamfunction ψ by the relations:

$$u = -\psi_y \quad \text{and} \quad v = \psi_x.$$

We seek to determine stationary (i.e. rotating with constant angular velocity Ω without change in shape) solutions to (1), corresponding to the vorticity distribution shown schematically in Figure 1. The vorticity is everywhere zero except in three regions where it has a constant value: without loss of generality we choose $\omega = 1$ in the regions $-1 < x < -a$ and $a < x < 1$ (hereafter referred to as the "satellite" vortices) and $\omega = \gamma < 0$ in the region $-b < x < b$ (the "central" or "core" vortex). The unknowns are the shapes of the three vortices and the angular velocity Ω.

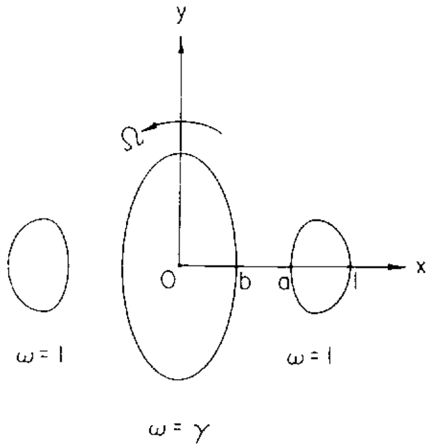

Figure 1 A schematic drawing of a tripolar V-state.

We recall that simpler stationary solutions of (1) are known. Monopole solutions corresponding to a single rotating region of constant vorticity and dipole solutions corresponding to two co-translating regions of equal and opposite vorticity were first found by Deem and Zabusky (1978). By analogy with electromagnetic theory, what we wish to determine here are the shapes of the *quadrupole* distributions of vorticity and thus the term *tripole*, although by now well established in the literature, is somewhat of a misnomer. In this perspective, we point out that the $N=3$ corotating solutions of Dritschel (1985) do not belong to the category of coherent structures that we designate with the term *tripole*, which we apply exclusively to the case $\gamma<0$ to which this study is confined.

2.1. Tripolar V-states

As can be seen from Figure 1 the parameter space of this problem is three dimensional (a, b and γ can be chosen independently) and therefore an exhaustive study would be prohibitively expensive in computational terms. We have decided to present here only the two values $\gamma=-2$ and -4, which are representative of the general properties of such structures. The parameters a and b vary between 0 and 1, and we have attempted to determine the equilibrium shapes for all values of a and b that are multiples of 0.05.

To do this, we have modified the algorithm of Wu *et al.* (1984) to make it applicable to the geometry of our problem; this is a second-order iterative scheme, with overrelaxation, where the unknowns are the radial positions of the nodes discretizing the contour and the angular velocity Ω. We have proceeded as follows: for each of the two values of γ and for each value of a between 0 and 1 in steps of 0.05, we first calculate the state for $b=0.05$ from an initial guess of three circles. Once the scheme has converged to a steady state, we use its expanded shape as the initial guess for the state with $b=0.1$, and the $b=0.1$ state is then used as the initial guess for the $b=0.15$ state, and so on. Eventually, when b becomes

Table 1 Quantitative measures of the V-states for $\gamma = -2$

a	b	Ω	R	d	r_1	r_2	Γ_1	Γ_2
0.9	0.1	−0.01042	0.99912	0.94997	0.10029	0.05103	−0.06320	0.00818
0.9	0.2	−0.04382	0.99943	0.94989	0.20071	0.05391	−0.25312	0.00913
0.9	0.3	−0.09969	0.99891	0.94975	0.30143	0.05798	−0.57091	0.01056
0.9	0.4	−0.17850	0.99773	0.94954	0.40285	0.06277	−1.01968	0.01238
0.9	0.5	−0.28173	0.99501	0.94926	0.50625	0.06795	−1.61032	0.01451
0.9	0.6	−0.42319	0.98187	0.94896	0.62406	0.07284	−2.44699	0.01667
0.8	0.1	−0.00944	0.99845	0.89987	0.10132	0.10216	−0.06450	0.03279
0.8	0.2	−0.04732	0.99767	0.89946	0.20329	0.10874	−0.25966	0.03715
0.8	0.3	−0.11165	0.99438	0.89876	0.30688	0.11799	−0.59174	0.04373
0.8	0.4	−0.20560	0.98735	0.89774	0.41469	0.12874	−1.08048	0.05207
0.8	0.5	−0.34696	0.96564	0.89660	0.54193	0.13931	−1.84533	0.06097
0.7	0.1	−0.00664	0.99612	0.84966	0.10335	0.15320	−0.06711	0.07373
0.7	0.2	−0.05067	0.99300	0.84849	0.20863	0.16481	−0.27349	0.08533
0.7	0.3	−0.12843	0.98217	0.84649	0.31919	0.18100	−0.64016	0.10292
0.7	0.4	−0.25657	0.95530	0.84387	0.44850	0.19897	−1.26390	0.12437
0.6	0.1	−0.00152	0.96952	0.79928	0.10679	0.20399	−0.07166	0.13073
0.6	0.2	−0.05457	0.98137	0.79654	0.21843	0.22300	−0.29978	0.15622
0.6	0.3	−0.15814	0.95070	0.79197	0.34620	0.24917	−0.75306	0.19506
0.5	0.1	0.00644	1.01518	0.74861	0.11238	0.25444	−0.07935	0.20339
0.5	0.2	−0.06166	0.95122	0.74264	0.23689	0.28542	−0.35260	0.25593
0.4	0.1	0.01747	1.01426	0.69724	0.12195	0.30492	−0.09344	0.29209
0.4	0.2	−0.10287	0.85027	0.68516	0.29861	0.35985	−0.56024	0.40681
0.3	0.1	0.02851	1.04346	0.64341	0.14502	0.35967	−0.13214	0.40640

sufficiently large, the algorithm fails to converge and the calculation is then restarted for a different value of γ and a.

Because the arclength of the central and satellite contours can differ substantially, it is necessary to choose an unequal number of points on the two contours. The criterion we have adopted in this study is to discretize the contours in such a way that the distance between two consecutive nodes is approximately the same on the central vortex and the satellites. For the results presented here, the total number of nodes varies between 80 and 500. These were found sufficient to make all the figures shown in Tables 1 and 2 significant.

We have found that, in general, when b becomes close to a we cannot achieve convergence. The type of contour dynamical algorithm we have used is well known to be unable to resolve the very high curvatures that appear as a limiting state is approached; limiting V-states usually possess nondifferentiable points on their contours, as is the case, for instance, with the $m=2,3$ monopolar rotating V-states (Wu *et al.*, 1984). When this happens one must either increase drastically the resolution or resort to higher-order algorithms. Since we are not in this study concerned with the possible existence of limiting V-states, we have contented ourselves with the solutions that can be obtained with the second-order scheme already mentioned. It is also plausible that the algorithm used here may have difficulty converging on unstable V-states, as is suggested by the fact that, as we will show in the next section, the vast majority of the tripolar V-states we have found are also stable.

Table 2 Quantitative measures of the V-states for $\gamma=-4$

a	b	Ω	R	d	r_1	r_2	Γ_1	Γ_2
0.9	0.1	−0.02146	0.99896	0.94994	0.10015	0.05203	−0.12605	0.00851
0.9	0.2	−0.08806	0.99922	0.94978	0.20040	0.05720	−0.50467	0.01028
0.9	0.3	−0.19939	0.99847	0.94948	0.30087	0.06395	−1.13754	0.01285
0.9	0.4	−0.35608	0.99697	0.94901	0.40184	0.07149	−2.02911	0.01606
0.8	0.1	−0.02164	0.99826	0.89973	0.10069	1.10441	−0.12740	0.03425
0.8	0.2	−0.09638	0.99676	0.89891	0.20186	0.11602	−0.51206	0.04229
0.8	0.3	−0.22273	0.99231	0.89743	0.30420	0.13108	−1.16284	0.05398
0.8	0.4	−0.40493	0.98395	0.89509	0.40937	0.14768	−2.10591	0.06852
0.7	0.1	−0.02009	0.99650	0.84929	0.10175	0.15702	−0.13011	0.07746
0.7	0.2	−0.10515	0.99063	0.84699	0.20490	0.17695	−0.52761	0.09837
0.7	0.3	−0.25285	0.97665	0.84272	0.31155	0.20262	−1.21970	0.12898
0.6	0.1	−0.01624	0.99188	0.79848	0.10355	0.20984	−0.13474	0.13833
0.6	0.2	−0.11482	0.97680	0.79316	0.21035	0.24122	−0.55602	0.18279
0.6	0.3	−0.29537	0.94136	0.78309	0.32571	0.28113	−1.33312	0.24830
0.5	0.1	−0.00934	0.97417	0.74704	0.10638	0.26294	−0.14222	0.21721
0.5	0.2	−0.12610	0.94539	0.73557	0.21954	0.31130	−0.60569	0.30445
0.5	0.3	−0.36355	0.86827	0.71510	0.35225	0.36899	−1.55920	0.42773
0.4	0.1	0.00159	1.43490	0.69432	0.11079	0.31676	−0.15426	0.31522
0.4	0.2	−0.13803	0.87006	0.66957	0.23411	0.39238	−0.68874	0.48368
0.3	0.1	0.01826	1.06501	0.63853	0.11784	0.37289	−0.17450	0.43683
0.3	0.2	−0.10359	0.61959	0.57387	0.24232	0.49952	−0.73789	0.78390
0.2	0.1	0.04713	1.08550	0.57337	0.12883	0.43574	−0.20855	0.59648

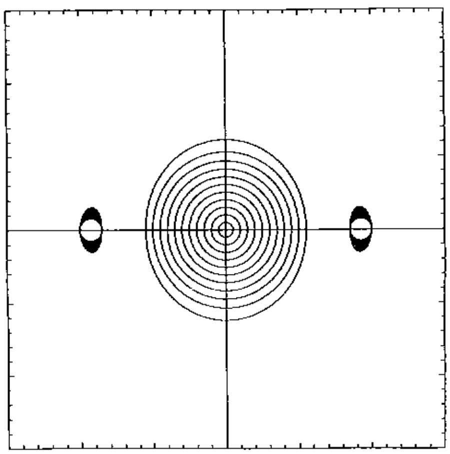

Figure 2 Tripolar V-state solutions for $\gamma=-2$, $a=0.85$ and $b=0.05$ to 0.55. As b increases the satellites become more elongated.

In Figure 2 we show the states for $\gamma=-2$, $a=0.85$ and $b=0.05$ to 0.55. Notice that when the three vortices are sufficiently separated their shapes are not very different from circles (as conjectured by Zabusky, 1981), and in this case an analytical perturbative approach could easily be used to obtain a first approximation to the equilibrium configuration and the angular velocity of the tripole, as was done by Dritschel (1985). For fixed γ and a, as b is increased, the shape of the satellite vortices becomes more elongated since they must resist a greater shear

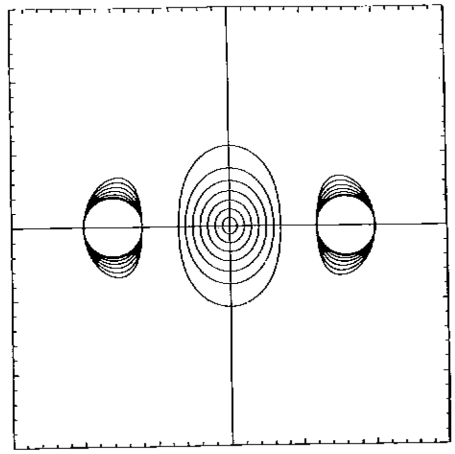

Figure 3 Tripolar V-state solutions for $\gamma=-2$, $a=0.6$ and $b=0.05$ to 0.35. Note the large deformation of the core as the area of the satellites becomes considerable.

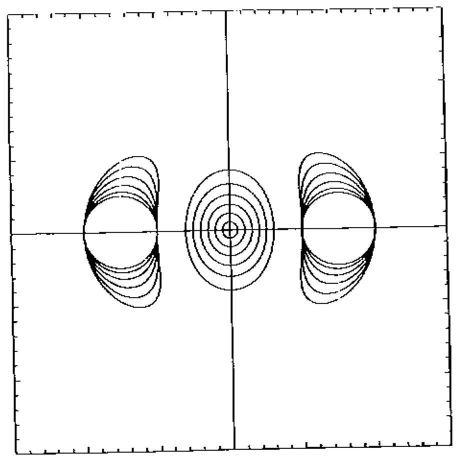

Figure 4 Tripolar V-state solutions for $\gamma=-4$, $a=0.5$ and $b=0.05$ to 0.3. Note the small negative curvature on the satellites of the $b=0.25$ and $b=0.3$ state.

from the central vortex and maintain a equilibrium configuration. As a decreases for fixed γ, the satellites become more considerable in size and the central vortex deforms too, as can be seen from Figure 3 ($\gamma=-2$, $a=0.6$ and $b=0.05$ to 0.35).

As the area of the satellites becomes comparable with the one of the central vortex, the shapes differ significantly from circles, and as b is increased towards a the satellites start to develop a concave region; this occurs, for instance, in the case $\gamma=-4$, $a=0.5$ (see Figure 4). Finally, some extreme shapes appear for small values of a, where the satellites are much greater than the central vortex. In Figure 5, a characteristic example is shown ($\gamma=-4$, $a=0.3$ and $b=0.05$ to 0.2). For tiny cores, the external vortices are nearly circular, while they exhibit very large curvatures as b is increased.

In Tables 1 and 2 we present some quantitative measures for the V-states we have obtained at $\gamma=-2$ and -4 respectively. In particular, we tabulate the angular velocity Ω, the ratio R of Ω to the angular velocity Ω_{pv} of the equivalent

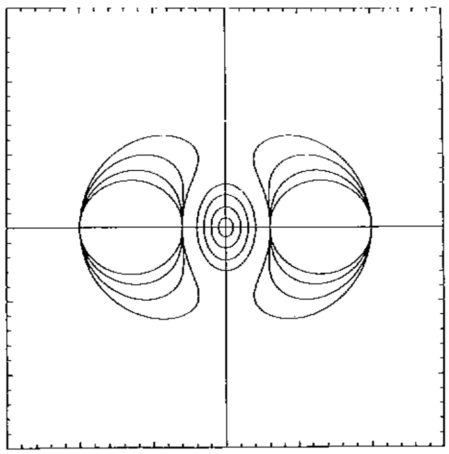

Figure 5 Tripolar V-state solutions for $\gamma=-4$, $a=0.3$ and $b=0.05$ to 0.2.

point-vortex tripole, the distance d of the centroid of the satellite from the origin along the x-axis, the average radii r_1 and r_2, and the circulations Γ_1 and Γ_2 of the core and the satellites respectively.

The value of Ω_{pv} was calculated using the simple formula:

$$\Omega_{pv}=\frac{1}{2\pi d^2}\left(\Gamma_1+\frac{\Gamma_2}{2}\right). \tag{2}$$

For the V-states we have found, the relative difference between Ω and Ω_{pv} is of the order of a few percent, even for the cases when the shapes of the vortices are far from circular. This may be surprising, but a quantitatively similar behavior was found by Dritschel (1985)—large discrepancies between Ω and Ω_{pv} start to appear only when high curvature regions are present on the contours. In general, the value of R is less than one, with the exception of the case when the satellites are greater than the center vortex and nearly circular. In that case, they tend to resemble the $N=2$ V-states of Dritschel (1985), for which R is greater than one.

Finally, it is of interest to point out that since the values of a, b and γ can be chosen in such a way as to obtain tripoles with both positive as well as negative total circulation, there exist stationary states for which $\Omega=0$, i.e. non-rotating tripoles. Of the states that we have numerically determined the one for $\gamma=-2$, $a=0.6$ and $b=0.1$ (for which $\Omega=0.0015$) comes closest to a non-rotating stationary state. It is readily seen from (2) that a point vortex tripole will be non-rotating provided $2\Gamma_1=-\Gamma_2$ irrespective of the distance between the vortices. The numerical value for the case $\gamma=-2$, $a=0.6$ and $b=0.1$ is $-2\Gamma_1/\Gamma_2=1.09$.

Tripoles with zero circulation are of special interest because the velocity fields they generate decrease very rapidly away from them. For a non-rotating point vortex tripole one can show that the velocity field decreases like $1/r^3$ as r becomes large (compared with a $1/r$ dependence for a point vortex monopole, and a $1/r^2$ dependence for a symmetric dipole). This implies that such coherent structures are almost *invisible*, in the sense that their presence is felt only at very short distances.

2.2. Evolution of Perturbed Tripolar V-states

The stability of point-vortex tripoles was analysed by Morikawa and Swenson (1971); they showed that linear instability appears for $\gamma > -1.25$. Our aim in this section is to understand the finite-area problem, and determine the stability of the tripolar V-states obtained in the previous section, whose point-vortex counterparts are linearly stable. One way of doing this is to consider the growth rates of infinitesimal perturbations on the contours. Such an analysis was carried out, for instance, by Dritschel (1985) who performed a very careful study of the linear stability of N corotating states of equal vorticity. This approach, however, does not provide much insight into the large amplitude evolution of the perturbed states, when nonlinear effects are likely to play a major role in the dynamics.

We have thus adopted the alternative approach of following the time evolution of perturbed tripoles with a contour surgery code (Dritschel, 1988); for the runs presented here, we have taken several hundred nodes at $t=0$, and chosen the cut-off scale such that the circulation is typically conserved to better than one part in a thousand (for the worst cases where a lot of surgery has occurred). Two types of initial perturbations have been implemented here: the first one consists of adding a very small random perturbation to the vortex boundary. This method has been used by Dritschel (1989) for the stability of nested multi-contour V-states; it allows the most unstable normal mode of the system—if one exists—to manifest itself. The second type is a finite-amplitude perturbation, in which we displace one satellite from the equilibrium position by some amount; with this method also all the modes are excited.

We have tested all the steady states presented in the previous section, with a perturbation of the first type. Each node (x_e, y_e) on the equilibrium solution was displaced by a random amount (x_r, y_r) to a new position (x_p, y_p) so that:

$$x_p = x_e + \varepsilon x_r \qquad y_p = y_e + \varepsilon y_r,$$

where (x_r, y_r) are taken from a random number generator and are between minus one and plus one, and ε is the amplitude of perturbation; we have tested each V-state for $\varepsilon = 0.01$ and 0.05.

Of the states in Tables 1 and 2, only the $\gamma = -4$, $a = 0.2$ and $b = 0.1$ state was found to be unstable to this type of perturbation. We show its evolution for $\varepsilon = 0.05$ in Figure 6: an asymmetric mode emerges as the central vortex drifts towards one of the satellites and eventually the two satellites merge, yielding a rather dipolar structure. The stable V-states were observed to undergo an oscillation in shape, shedding relatively few filaments, but maintaining a tripolar coherent configuration.

For the majority of the V-states obtained in the previous section, the coherence was found to be so strong that not only we conclude that they are stable, but large amplitude perturbations were necessary to break the tripolar structures apart. To accomplish this, we have used a disturbance of the second form, i.e. the lateral displacement of one of the satellites by a considerable amount—i.e. $O(1)$ with respect to the characteristic length scale of the structure. We report two

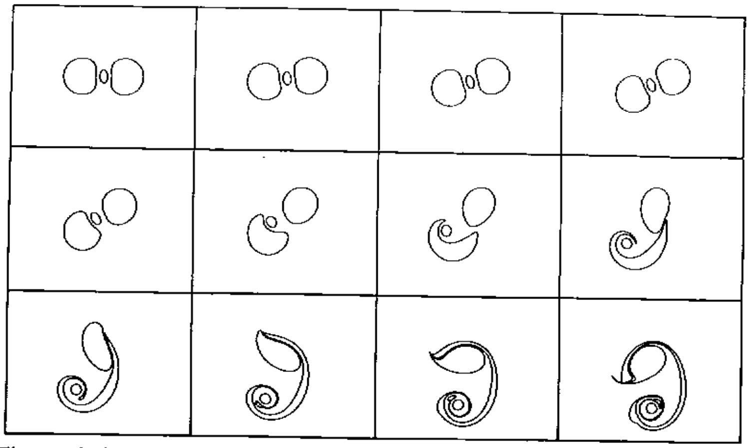

Figure 6 The evolution of a perturbed $\gamma=-4$, $a=0.2$ and $b=0.1$ V-state. Each node of the equilibrium shape was randomly displaced as described in the text; here $\varepsilon=0.05$. Time evolves to the right and downwards; the frames shown are at $t=0,3,6,\ldots$.

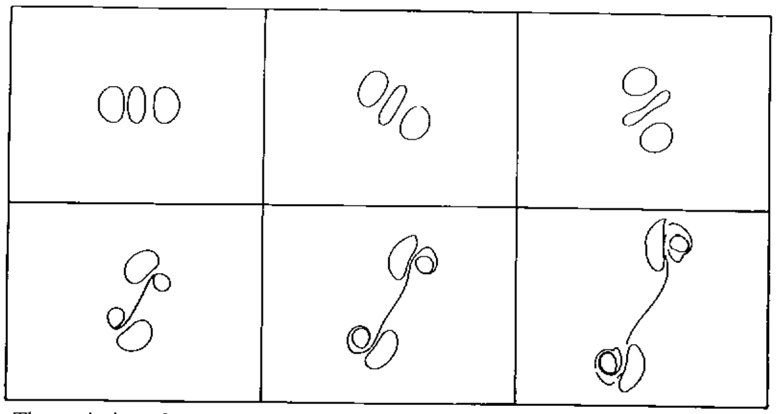

Figure 7 The evolution of a perturbed $\gamma=-2$, $a=0.4$ and $b=0.2$ V-state. At $t=0$ the left satellite was displaced a distance 0.1 to the right. Frames shown at $t=0,6,12,\ldots$.

qualitatively different behaviors under breaking: the formation of two dipoles and axisymmetrization.

An example is shown in Figure 7, where a strongly perturbed $\gamma=-2$, $a=0.4$ and $b=0.2$ tripole is seen to break into two asymmetric dipoles. The mechanism for breaking in this case is the tearing apart of the central vortex by the two satellites, which for the initial condition of Figure 7, occurs in about a quarter of a rotation period. Notice that although the positive and negative vorticities enclosed by the poles of each pair are not equal (actually the negative vorticity is twice the positive one) the area of the negative vortex is about half the area of the positive one in each pair, and thus two rather well balanced dipoles emerge.

The second behavior is observed when tripoles with very large satellites are strongly perturbed, and is illustrated in Figure 8, where a $\gamma=-4$, $a=0.3$ and $b=0.2$ tripole with a large perturbation on the left satellite breaks into a very convoluted shape that eventually closely resembles an axisymmetric shielded

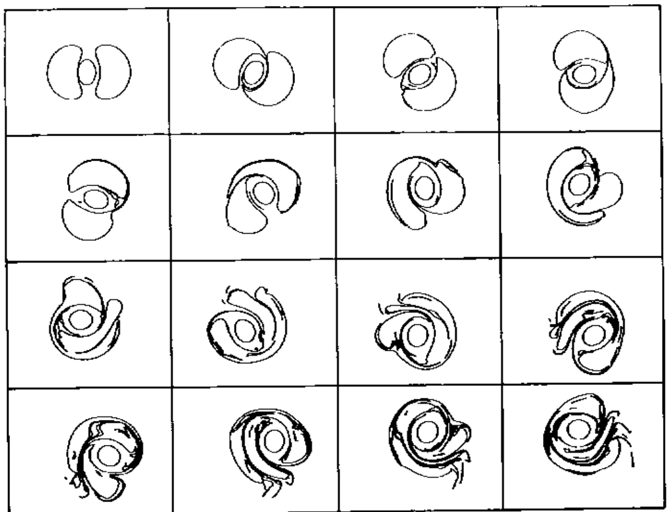

Figure 8 The evolution of a perturbed $\gamma=-4$, $a=0.3$ and $b=0.2$ V-state. At $t=0$ the left satellite was displaced a distance 0.07 to the right. The time frames are at $t=0, 5, 10, \ldots$

vortex. As has been shown by Carton *et al.* (1988), the reverse process can also occur, namely the emergence of a tripole from the evolution of a perturbed unstable axisymmetric shielded vortex.

3. CONTINUOUS TRIPOLES

Since the tripoles that have previously been observed have spontaneously emerged in spectral simulations of two-dimensional turbulent flows, we have proceeded to investigate some of their phenomenology in a continuous model, and more specifically the ways in which they can be generated. The spectral code we have used (Basdevant, 1981) is based on a Galerkin dealiased numerical scheme, with a Heun-type time advection procedure (leapfrog with an Euler correction). As is conventional, some hyperviscosity (νk^{16}) is necessary to keep the enstrophy bounded in the small scales; in spite of this, nonetheless, the results presented here are essentially inviscid, since the energy is typically conserved to better than one part in 10^{-4} by the end of the run. The resolution of our runs is 256 grid points in each direction and, since the time step is bounded by the Courant-Friedrichs-Lewy condition, the simulations are limited to only a few overall rotation times of the coherent structures.

3.1. Generation of Continuous Tripoles from Smoothed V-states

There are several reasons for studying the evolution in a continuous model of the tripolar V-states obtained with contour dynamics. The first one is the simple fact that infinite vorticity gradients are unphysical, and thus never realized in

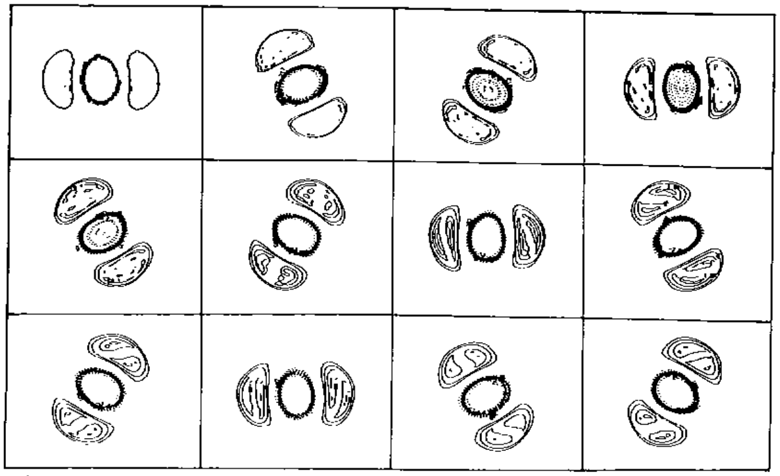

Figure 9 The vorticity field for the spectral evolution of a smoothed $\gamma=-4$, $a=0.5$ and $b=0.3$ V-state. The times shown are $t=0,3,6\ldots$.

geophysical flows and laboratory experiments; therefore a continuous distribution of vorticity is undoubtedly more realistic.

From a different point of view, however, one can interpret the study of the evolution of a continuous system from a V-state initial condition as an investigation of the "robustness" of the V-states themselves. We call a V-state robust if its evolution in a nearly inviscid spectral model does not lead to a drastic change in the topology of the vorticity field. A counter-example is provided by the Kirchhoff ellipse, which is an exact nonlinear V-state solution of the Euler equations. For any value of its aspect ratio, Melander *et al.* (1987) have indicated that it evolves into a nearly axisymmetric vorticity field when taken as an initial condition in a spectral calculation.* It is the robustness of tripolar V-states that we wish to investigate in this section.

In order to use the tripolar V-state solutions obtained with contour dynamics as initial conditions of a spectral calculation, it is necessary to deal with the classical problem constituted by the existence of an infinite vorticity gradient at the boundary of the V-state. Such a gradient is well known to generate strong numerical instabilities in a spectral simulation (Gibbs' phenomenon). This problem is avoided by smoothing the boundaries of the initial vorticity field. To do this we have used a very simple scheme that replaces the vorticity at a grid point by the arithmetic mean of the neighboring values whenever the x- or y-gradient of the vorticity at that grid point is larger than a threshold value.

As a representative example of the several experiments we have conducted, we show, in Figure 9, the evolution obtained by using a smoothed $\gamma=-4$, $a=0.5$ and $b=0.3$ V-state as the initial condition. Note that the vorticity field retains its tripolar character, with the central vortex smaller in size than the satellites, and a

*Incidentally, work in progress by D. G. Dritschel and B. Legras suggests that not all ellipses axisymmetrize (private communication).

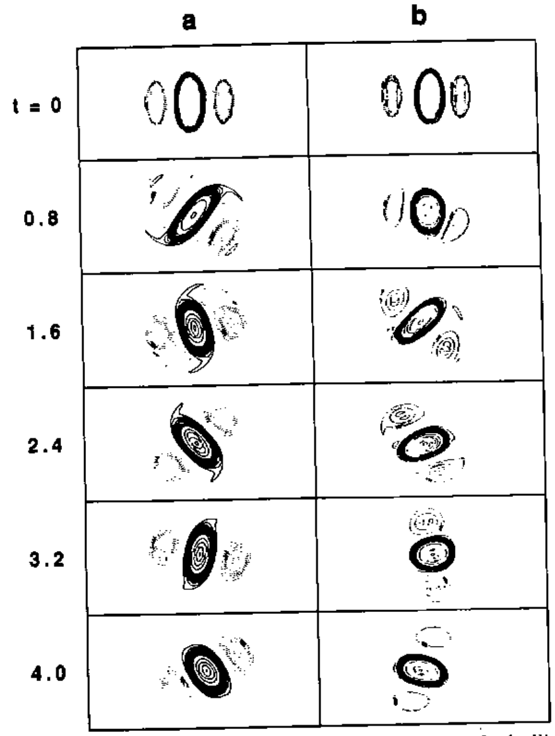

Figure 10 The vorticity field for the spectral evolution of three smoothed ellipses. The central ellipse has vorticity -2, and the satellites $+1$. In case (*a*), the satellites were symmetrically located with respect to the central ellipse, while in (*b*) the core was displaced by 20% towards the right ellipse. The times shown are scaled by the overall rotation period T_R of the coherent structure.

small negative curvature on the satellites persists. [We have chosen to show this particular run, because it exhibits the additional interesting feature of two weak maxima inside each satellite, suggesting the existence of a finite-amplitude equilibrated mode 4 state (a *pentapole*?)]. From a large number of similar runs, we conclude that there exist continuous stationary tripolar solutions to the Euler equations whose geometrical characteristics are similar to the V-state solutions, and that they are stable.

3.2. Generation of Continuous Tripoles from Three Ellipses

One may argue from the results of the preceding section that continuous tripoles were readily obtained solely because the initial conditions were already very close to the steady solutions. We now present an alternative way to generate tripoles. In this section we use three smoothed ellipses as initial conditions. In the run shown in Figure 10a, the major axis is chosen to be of length 1.1 and 0.8 for the central vortex and the satellites respectively, and the minor axes are 0.7 and 0.45 respectively. The central vortex has vorticity $+2$ and the satellites -1.

The total time for this run is longer than for the run of the previous section (nearly 5 turnover times). Some oscillations of the satellites around the core and

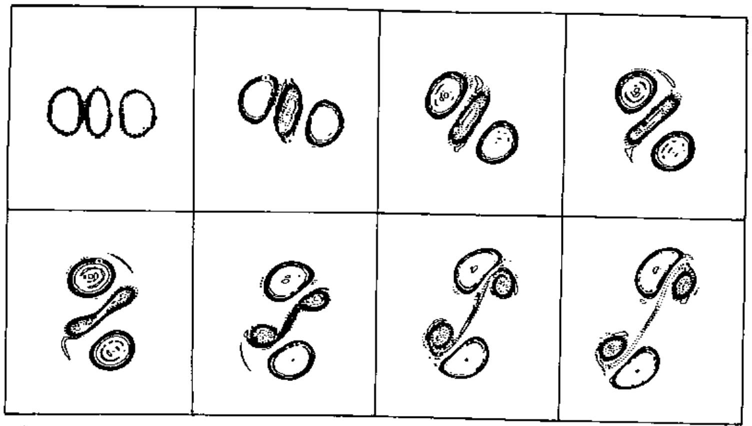

Figure 11 The vorticity field for the spectral evolution of a smoothed perturbed $\gamma=-2$, $a=0.4$ and $b=0.2$ V-state. Both satellites were displaced by 0.1 towards the central vortex at $t=0$. The times shown are $t=0, 3, 6, \ldots$.

some vorticity shedding into filaments occurs, but, after approximately three rotation periods, the tripole is seen to stabilize, and the vortex structure seems strongly stable.

In Figure 10b we show a run whose initial condition is not symmetric about the y-axis; this allows us to excite all the asymmetric modes as well (which were absent in the previous run). The geometrical properties of the ellipses are the same as for Figure 10a at $t=0$, but the vortex core was shifted by 20% to the right. With this initial condition the evolution is more complex and the filamentation processes are stronger but again a coherent tripolar structure can be seen to emerge. From a number of similar experiments we conclude that, in the language of dynamical systems, continuous tripolar coherent vortex structures constitute an important attractor basin.

3.3. *Breakup of Perturbed Continuous Tripoles*

In this last section, we want to show that the qualitative results obtained with contour surgery on the evolution of strongly perturbed tripoles are true manifestations of the underlying nonlinear dynamics of these vortex structures, and are quite independent of the model one uses. In particular we want to confirm the two main processes of breaking and axisymmetrization, that were found with contour surgery, are not a peculiarity of the constant vorticity formulation, but are a true representation of the evolution of certain perturbed tripoles.

In Figure 11, we present the breaking of a smoothed $\gamma=-2$, $a=0.4$ and $b=0.2$ V-state, whose satellites were both displaced by a distance of 0.1 towards the central vortex at $t=0$. Comparison with Figure 7 shows that the same instability mechanism is present as the central vortex is split in two and the two asymmetric dipoles are formed. It is interesting to note that the reverse process, namely the formation of a tripole following the collision of two asymmetric dipoles, has also been observed (Larichev and Reznik, 1983).

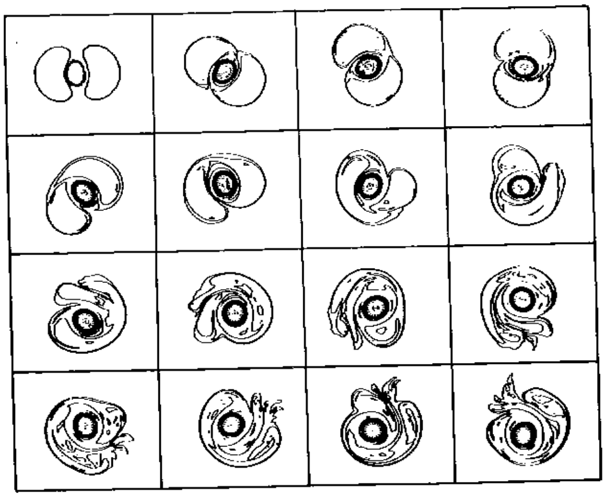

Figure 12 The vorticity field for the spectral evolution of a smoothed perturbed $\gamma=-4$, $a=0.3$ and $b=0.2$ V-state. As in Figure 8, the left satellite was displaced towards the center by an amount 0.07. The time frames are at $t=0, 6, 12, \ldots$.

An example of axisymmetrization is given in Figure 12. As in Figure 8, we perturb a $\gamma=-4$, $a=0.3$ and $b=0.2$ V-state by shifting the left satellite towards the central vortex. Notice how closely many features of the evolution in Figure 8 are also found in Figure 12. Undoubtedly the presence of filaments in the nearly inviscid simulations implies the eventual completion of the axisymmetrization at much longer times than the ones shown in Figure 12, as is the case for the exactly inviscid contour surgery simulations of Figure 8; the presence of Newtonian viscosity (νk^2) would considerably speed up the axisymmetrization.

4. CONCLUSION

Using complementary modelling (contour surgery as well as pseudo-spectral methods) we have investigated the existence and stability of tripolar coherent vortex structures in two-dimensional flows. We have determined the equilibrium contours for rotating finite-area tripolar V-states that are exact nonlinear solutions of the Euler equations in two dimensions. We have examined their stability with the aid of a contour surgical algorithm which has revealed that most of them are stable to small amplitude disturbances but, when strongly perturbed, they either break up into two asymmetric dipoles or axisymmetrize.

We have also shown, by means of a pseudospectral code, how easily tripolar coherent vortex structures can be generated in the case of continuous distributions of vorticity. The nearly inviscid spectral simulations reproduce well the results obtained with contour surgery, and demonstrate the "robustness" of the piecewise-constant V-state solutions. In view of the experimental data, it is not unreasonable to conjecture that similar structures exist in non-homogeneous systems.

Dritschel (1986) remarked that vortex merging and splitting of an ellipse into two vortices are symmetric processes, in the sense that, in some cases, the transition between an elongated ellipse and two like-signed vortices is not very energetic. In the present study we have uncovered two other such "reversible" transitions: the one between a tripole and two dipoles, and the one between a tripole and a shielded axisymmetric vortex. In the light of this and other recent studies, it would seem that the present knowledge of coherent two-dimensional vortex structures is ripe for a synthetic description of their existence, stability and transitions in terms of the nonlinear regimes of a unique dynamical system.

Acknowledgements

The authors wish to thank Drs. G. R. Flierl and B. Legras for their support and criticism, Dr. D. G. Dritschel for providing the contour surgery code, and Dr. S. P. Meacham for his contribution with the early numerical simulations. Discussions with Drs. N. J. Zabusky, F. P. Bretherton, J. C. McWilliams proved timely and valuable. The spectral computations were carried out at the National Center for Atmospheric Research, which is sponsored by the National Science Foundation, while the contour surgery runs were performed at the Naval Research Laboratories. LMP was supported, in part, by the Office of Naval Research under contract N00014-86-K-0325, while XJC was sponsored by the Research Corps of the French Navy.

References

Basdevant, C., "Le modèle de simulation numérique de turbulence bidimensionnelle homogène," *Note Interne L.M.D.* **114** (1981).

Carton, X. J., Flierl, G. R. and Polvani, L. M., "The generation of tripoles from unstable axisymmetric isolated vortex structures," to appear in *Europhys. Lett.* (1989).

Couder, Y. and Basdevant, C., "Experiments and numerical studies of vortex couples in two-dimensional flows," *J. Fluid Mech.* **173**, 225–251 (1986).

Deem, G. S. and Zabusky, N. J., "Stationary V-states: Interactions, recurrence and breaking," *Phys. Rev. Lett.* **40**, 859 (1978).

Deem, G. S. and Zabusky, N. J., "Stationary V-states: Interactions, recurrence and breaking." In: *Solitons in Action* (Eds., K. Lonngren and A. Scott), Academic Press, New York, pp. 277–293 (1978).

Dritschel, D. G., "The stability and energetics of corotating uniform vortices," *J. Fluid Mech.* **157**, 95–134 (1985).

Dritschel, D. G., "The nonlinear evolution of rotating configurations of uniform vorticity," *J. Fluid Mech.* **172**, 157–182 (1986).

Dritschel, D. G., "Contour surgery: A topological reconnection scheme for extended integrations using contour dynamics," *J. Comput. Phys.* **77**, 240–266 (1988).

Dritschel, D. G., "Contour dynamics and contour surgery: Numerical algorithms for extended high-resolution modelling of vortex dynamics in two-dimensional, inviscid, incompressible flows," to appear in *Comp. Phys. Rep.* (1989).

Larichev, V. D. and Reznik, G. M., "On collisions between two-dimensional solitary Rossby waves," *Oceanology* **23**, no. 5, 545 (1983).

Legras, B., Santangelo, P. and Benzi, R., "High resolution numerical experiments for forced two-dimensional turbulence," *Europhys. Lett.* **5**, 37–42 (1988).

Leith, C. E., "Minimum enstrophy vortices," *Phys. of Fluids* **27**, 1388–1395 (1984).

McWilliams, J. C., "The emergence of isolated coherent vortices in turbulent flow," *J. Fluid Mech.* **146**, 21–43 (1984).

Melander, M. V., McWilliams, J. C. and Zabusky, N. J., "Axisymmetrization and vorticity-gradient intensification for an isolated two-dimensional vortex through filamentation," *J. Fluid Mech.* **178,** 137–159 (1987).

Morikawa, G. K. and Swenson, E. V., "Interacting motions of rectilinear geostrophic vortices," *Phys. Fluids* **14,** 1058–1073 (1971).

Pierrehumbert, R. T., "A family of steady translating vortex pairs with distributed vorticity," *J. Fluid Mech.* **99** (1980).

Polvani, L. M., "Geostrophic vortex dynamics," Ph.D. Thesis, Massachusetts Institute of Technology (1988).

Wu, H. M., Overman, E. A. and Zabusky, N. J., "Steady-state solutions of the Euler equations in two dimensions. Rotating and translating V-states with limiting cases. I. Numerical Results," *J. Comput. Phys.* **53,** 42–71 (1984).

Zabusky, N. J., Hughes, M. H. and Roberts, K. V., "Contour dynamics for the Euler equations in two dimensions," *J. Comput. Phys.* **30,** 96–106 (1979).

Zabusky, N. J., "Recent developments in contour dynamics for the Euler equations," *Ann. New York Acad. Sci.* **373,** 160–170 (1981).