Generalized Kirchhoff vortices

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A family of exact solutions of the Euler equations is presented: they are generalizations of the Kirchhoff vortex to N confocal ellipses. Special attention is given to the case $N = 2$, for which the stability is analyzed with a method similar to the one used by Love [Proc. London Math. Soc. **1**, XXV 18 (1893)] for the Kirchhoff vortex. The results are compared with those for the corresponding circular problem.

I. INTRODUCTION

A two-dimensional elliptical patch of homogeneous inviscid fluid of uniform vorticity rotating with constant angular velocity is an exact solution of the Euler equations; it is called a Kirchhoff vortex.[1] Its vorticity Q and angular velocity ω are related by

$$Q = [(a+b)^2/ab]\omega,$$

where a and b are, respectively, the major and minor axes of the ellipse. The system is stable to small perturbations provided $a < 3b$.[2] We present in this work a family of exact solutions which are generalizations of the Kirchhoff vortex to N confocal ellipses, and we investigate the stability for the special case $N = 2$.

In terms of the usual Cartesian coordinates (x,y), the elliptical coordinates (ρ,θ) needed to obtain an analytic solution are implicitly defined by

$$x = c\cosh\rho\cos\theta, \quad y = c\sinh\rho\sin\theta.$$

The lines $\rho = \text{const}$ define confocal ellipses whose foci are located on the x axis at $x = \pm c$.

II. GENERALIZED KIRCHHOFF VORTICES

Consider now the following distribution of vorticity Q (Fig. 1):

$$\begin{aligned} &Q = 0, \quad \text{for } \rho > \rho_1, \\ &Q = Q_j, \quad \text{for } \rho_j > \rho > \rho_{j+1}, \quad j = 1,\ldots, N-1, \\ &Q = Q_n, \quad \text{for } \rho < \rho_N. \end{aligned}$$

The stream function must then satisfy

$$\begin{aligned} &\nabla^2\psi_0 = 0, \quad \text{for } \rho > \rho_1, \\ &\nabla^2\psi_j = Q_j, \quad \text{for } \rho_j > \rho > \rho_{j+1}, \quad j = 1,\ldots, N-1, \\ &\nabla^2\psi_N = Q_N, \quad \text{for } \rho < \rho_N, \end{aligned}$$

which in elliptical coordinates becomes

$$\begin{aligned} &\left(\frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial\theta^2}\right)\psi_0 = 0, \quad \rho > \rho_1, \\ &\left(\frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial\theta^2}\right)\psi_j = \frac{1}{2} Q_j c^2(\cosh 2\rho - \cos 2\theta), \\ &\rho_j > \rho > \rho_{j+1}, \quad j = 1,\ldots, N-1, \\ &\left(\frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial\theta^2}\right)\psi_N = \frac{1}{2} Q_N c^2(\cosh 2\rho - \cos 2\theta), \\ &\qquad \rho < \rho_N. \end{aligned} \tag{1}$$

Since the whole system is in uniform rotation with angular velocity ω, we must impose the condition that the stream function with respect to a rotating reference frame, $\psi - \frac{1}{2}\omega r^2$, must be invariant along the boundaries, i.e.,

$$\frac{\partial}{\partial\theta}\left(\psi - \frac{1}{2}\omega(x^2+y^2)\right) = 0 \quad \text{on } \rho = \rho_j, \quad j = 1,\ldots, N. \tag{2}$$

This condition eliminates all homogeneous solutions except the ones proportional to $\cos 2\theta$. The solution to the inhomogeneous system (1) which satisfies (2) is easily found to be

$$\begin{aligned} &\psi_0 = \tfrac{1}{4}\omega c^2 e^{-2(\rho-\rho_1)}\cos 2\theta + (\Gamma/2\pi)\rho, \quad \text{for } \rho > \rho_1, \\ &\psi_j = \tfrac{1}{8} Q_j c^2(\cosh 2\rho + \cos 2\theta) + B_j \sinh 2\rho \cos 2\theta \\ &\qquad + C_j \cosh 2\rho \cos 2\theta + (\Lambda_j/2\pi)\rho + \psi_j^0, \\ &\qquad \text{for } \rho_j > \rho > \rho_{j+1}, \quad j = 1,\ldots, N-1, \end{aligned} \tag{3}$$

$$\begin{aligned} &\psi_N = \tfrac{1}{8} Q_N c^2(\cosh 2\rho + \cos 2\theta) \\ &\qquad + A\cosh 2\rho\cos 2\theta + \psi_N^0, \quad \text{for } \rho < \rho_N, \end{aligned} \tag{3}$$

where

$$\begin{aligned} &B_j = -\frac{1}{4}c^2\left(\omega - \frac{1}{2}Q_j\right)\frac{\sinh(\rho_j + \rho_{j+1})}{\cosh(\rho_j - \rho_{j+1})}, \\ &C_j = \frac{1}{4}c^2\left(\omega - \frac{1}{2}Q_j\right)\frac{\cosh(\rho_j + \rho_{j+1})}{\cosh(\rho_j - \rho_{j+1})}, \\ &A = \tfrac{1}{4}c^2(\omega - \tfrac{1}{2}Q_N)(\cosh 2\rho_N)^{-1}. \end{aligned}$$

The constants ψ_j^0 can be chosen to make ψ continuous across the boundaries; since our matching conditions are posed in terms of the velocities—i.e., the flow in the rotating frame must be tangent to the boundaries—these constants are im-

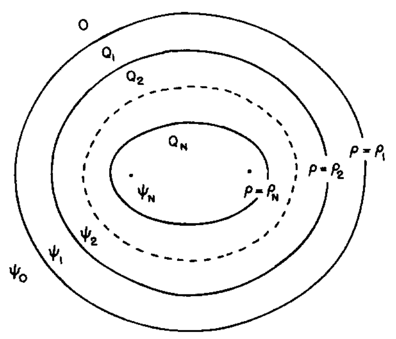

FIG. 1. A schematic drawing of a generalized Kirchhoff vortex.

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material. The Q_j's and Λ_j's are determined by requiring continuity of the tangential velocities:

$$\frac{\partial}{\partial\rho}\psi_{j-1}=\frac{\partial}{\partial\rho}\psi_j \quad \text{on } \rho=\rho_j \quad \text{for } j=1,\ldots,N \qquad (4)$$

which guarantees continuity of pressure. This condition yields

$$\begin{aligned}
&\Gamma=\Lambda_1+\pi a_1 b_1 Q_1,\\
&\Lambda_{j-1}=\Lambda_j+\pi a_j b_j(Q_j-Q_{j-1}),\quad j=2,\ldots,N-1, \qquad (5)\\
&\Lambda_{N-1}=\pi a_N b_N(Q_N-Q_{N-1}),
\end{aligned}$$

and

$$\begin{aligned}
&Q_j=2\omega[1+(-1)^{j+1}\coth(\rho_j-\rho_{j+1})],\\
&\qquad j=1,\ldots,N-1, \qquad (6)\\
&Q_N=2\omega[1+(-1)^{N+1}\coth 2\rho_N],
\end{aligned}$$

where a_j and b_j are, respectively, the major and minor axes of the jth ellipse, and are related to the ρ's by

$$a_j=c\cosh\rho_j \quad \text{and} \quad b_j=c\sinh\rho_j.$$

It is easy to show that the $N=1$ case corresponds to the Kirchhoff vortex. Using (6) one can rewrite (3) as follows:

$$\begin{aligned}
\psi_0&=\tfrac{1}{8}\omega c^2 e^{-2(\rho-\rho_1)}\cos 2\theta+(\Gamma/2\pi)\rho,\\
\psi_j&=\tfrac{1}{8}Q_j c^2(\cosh 2\rho+\cos 2\theta)+(\Lambda_j/2\pi)\rho\\
&\quad+(-1)^{j+1}\frac{1}{4}\omega c^2\frac{\sinh(\rho_j+\rho_{j+1})}{\sinh(\rho_j-\rho_{j+1})}\\
&\quad\times\sinh 2\rho\cos 2\theta \qquad (7)\\
&\quad+(-1)^{j}\frac{1}{4}\omega c^2\frac{\cosh(\rho_j+\rho_{j+1})}{\sinh(\rho_j-\rho_{j+1})}\cosh 2\rho\\
&\quad\times\cos 2\theta+\psi_j^0,\quad j=1,\ldots,N-1,\\
\psi_N&=\tfrac{1}{8}Q_N c^2(\cosh 2\rho+\cos 2\theta)+(-1)^N\tfrac{1}{4}\omega c^2\\
&\quad\times(\sinh 2\rho_N)^{-1}\cosh 2\rho\cos 2\theta+\psi_N^0.
\end{aligned}$$

It should be pointed out that once ω and the ρ_j's $(j=1,\ldots,N)$ have been chosen, the Q_j's follow necessarily from (6) and therefore cannot be set arbitrarily. In particular it is easy to see that Q_j and Q_{j+1} always have opposite signs, and that Q_1 has the sign of ω. A further interesting property of these solutions is that the total circulation Γ is determined uniquely by the angular velocity ω and the size ρ_1 of the outer ellipse. Indeed it can be shown from (5) that

$$\Gamma=\pi\omega(a_1+b_1)^2=\pi\omega c^2 e^{-2\rho_1}$$

for all values of N and ρ_j's $(j>1)$.

III. STABILITY OF THE $N=2$ VORTEX

We now turn our attention to the special case $N=2$, for which the stream function is given by

$$\begin{aligned}
\psi_0&=\tfrac{1}{4}\omega c^2 e^{-2(\rho-\rho_1)}\cos 2\theta+\tfrac{1}{2}\omega c^2 e^{-2\rho_1}\rho,\\
\psi_1&=\tfrac{1}{8}Q_1 c^2(\cosh 2\rho+\cos 2\theta)+\tfrac{1}{2}a_2 b_2(Q_2-Q_1)\rho\\
&\quad+\frac{1}{4}\omega c^2\frac{\sinh(\rho_1+\rho_2)}{\sinh(\rho_1-\rho_2)}\sinh 2\rho\cos 2\theta \qquad (8)\\
&\quad-\frac{1}{4}\omega c^2\frac{\cosh(\rho_1+\rho_2)}{\sinh(\rho_1-\rho_2)}\cosh 2\rho\cos 2\theta+\psi_1^0,\\
\psi_2&=-\tfrac{1}{8}Q_2 c^2(\cosh 2\rho+\cos 2\theta)+\tfrac{1}{4}\omega c^2\\
&\quad\times(\sinh 2\rho_2)^{-1}\cosh 2\rho\cos 2\theta+\psi_2^0,
\end{aligned}$$

with

$$Q_1=2\omega[1+\coth(\rho_1-\rho_2)]$$

and

$$Q_2=2\omega(1-\coth 2\rho_2).$$

Of the four parameters ρ_1, ρ_2, ω, and c only the first two are important in determining the shape, structure, and stability of the vortices.

Contours of the quantity Q_1/Q_2 in the (ρ_1,ρ_2) plane are plotted in Fig. 2. Also shown in that figure are the shapes of the vortices for three typical values of ρ_1 and ρ_2.

In order to investigate the stability of the steady state (8) we use a method similar to the one in Ref. 2. We denote by ϕ_0, ϕ_1, and ϕ_2 the perturbation stream functions. The perturbed boundaries of the ellipses are given by

$$\rho=\rho_1+\rho_1'(\theta) \quad \text{and} \quad \rho=\rho_2+\rho_2'(\theta).$$

Both the ϕ's and the ρ''s are understood to be infinitesimal quantities. The ϕ's must satisfy Laplace's equation together with the following conditions:

$$\begin{aligned}
&\frac{\partial}{\partial\rho}(\psi_0+\phi_0)=\frac{\partial}{\partial\rho}(\psi_1+\phi_1) \quad \text{on } \rho=\rho_1+\rho_1',\\
&\qquad\qquad (9)\\
&\frac{\partial}{\partial\rho}(\psi_1+\phi_1)=\frac{\partial}{\partial\rho}(\psi_2+\phi_2) \quad \text{on } \rho=\rho_2+\rho_2',\\
&\frac{\partial}{\partial\theta}(\psi_0+\phi_0)=\frac{\partial}{\partial\theta}(\psi_1+\phi_1) \quad \text{on } \rho=\rho_1+\rho_1',\\
&\qquad\qquad (10)\\
&\frac{\partial}{\partial\theta}(\psi_1+\phi_1)=\frac{\partial}{\partial\theta}(\psi_2+\phi_2) \quad \text{on } \rho=\rho_2+\rho_2',\\
&\frac{D}{Dt}[\rho-\rho_1-\rho_1'(\theta)]=0 \quad \text{on } \rho=\rho_1+\rho_1',\\
&\qquad\qquad (11)\\
&\frac{D}{Dt}[\rho-\rho_2-\rho_2'(\theta)]=0 \quad \text{on } \rho=\rho_2+\rho_2',
\end{aligned}$$

where

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+J\left(\psi+\phi-\frac{1}{2}\omega r^2\right),$$

$$J(f,g)=\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

The last condition ensures that the deformed ellipses always contain the same fluid. It is possible to show that the above three conditions guarantee continuity of pressure at the boundaries.

Since the ϕ's are solutions of Laplace's equation we can immediately write

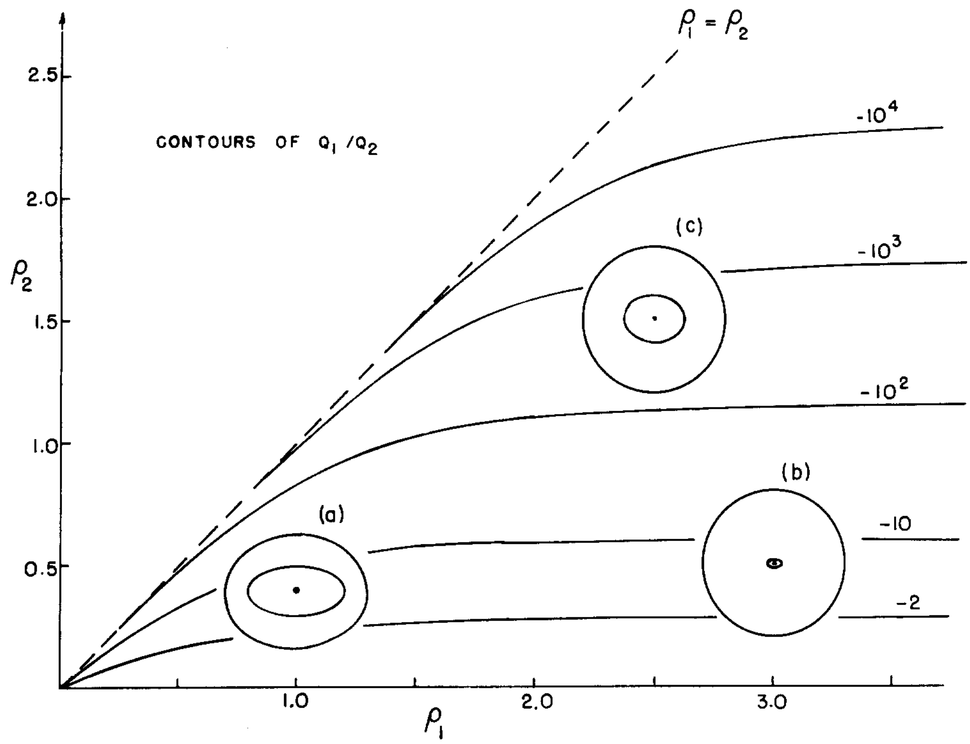

FIG. 2. Contours of Q_1/Q_2 in the (ρ_1, ρ_2) plane. The values of the parameters for the vortices shown are as follows: (a) $\rho_1 = 1.0$, $\rho_2 = 0.5$, $c = 1.94$, $b_1/a_1 = 0.76$, $b_2/a_2 = 0.46$, $a_1/a_2 = 1.37$, $Q_1/Q_2 = -10.1$. (b) $\rho_1 = 3.0$, $\rho_2 = 0.5$, $c = 0.30$, $b_1/a_1 = 0.99$, $b_2/a_2 = 0.46$, $a_1/a_2 = 8.9$, $Q_1/Q_2 = -6.4$. (c) $\rho_1 = 2.5$, $\rho_2 = 1.5$, $c = 0.49$, $b_1/a_1 = 0.98$, $b_2/a_2 = 0.91$, $a_1/a_2 = 2.6$, $Q_1/Q_2 = -465.4$.

$$\begin{aligned}
\phi_0 &= \sum_m (A_m e^{-m\rho} \cos m\theta + B_m e^{-m\rho} \sin m\theta), \\
\phi_1 &= \sum_m (C_m \cosh m\rho \cos m\theta + D_m \sinh m\rho \cos m\theta \\
&\quad + E_m \cosh m\rho \sin m\theta + F_m \\
&\quad \times \sinh m\rho \sin m\theta) + \phi_1^0, \\
\phi_2 &= \sum_m (G_m \cosh m\rho \cos m\theta \\
&\quad + H_m \sinh m\rho \sin m\theta) + \phi_2^0.
\end{aligned} \tag{12}$$

Conditions (9) and (10) can be combined to eliminate four of the eight unknowns. After some algebra they can be shown to simply reduce to

$$\begin{aligned}
&\sum_m m e^{m\rho_1} [(C_m + D_m)\cos m\theta + (E_m + F_m)\sin \theta] \\
&\quad + \frac{1}{2} Q_1 \left(\frac{\rho_1'}{h_1^2}\right) = 0, \\
&\sum_m m \left(\frac{D_m}{\cosh m\rho_2} \cos m\theta - \frac{E_m}{\sinh m\rho_2} \sin m\theta\right) \\
&\quad - \frac{1}{2} (Q_2 - Q_1)\left(\frac{\rho_2'}{h_2^2}\right) = 0,
\end{aligned} \tag{13}$$

where we have defined

$$h_i^{-2} = c^2(\cosh 2\rho_i - \cos 2\theta), \quad i = 1,2 .$$

In a similar way (11) must be Taylor expanded about the unperturbed boundaries. After much algebra we obtain

$$\begin{aligned}
\frac{\partial}{\partial\theta} \phi_1\bigg|_{\rho_1} &= -\frac{\partial}{\partial t}\left(\frac{\rho_1'}{h_1^2}\right) - \frac{1}{2}\omega \frac{\partial}{\partial\theta}\left(\frac{\rho_1'}{h_1^2}\right), \\
\frac{\partial}{\partial\theta} \phi_2\bigg|_{\rho_2} &= -\frac{\partial}{\partial t}\left(\frac{\rho_2'}{h_2^2}\right) + \frac{1}{2}\omega \frac{\partial}{\partial\theta}\left(\frac{\rho_2'}{h_2^2}\right).
\end{aligned} \tag{14}$$

Since ρ_1' and ρ_2' always appear in combination with h_1^{-2} and h_2^{-2} we can expand them together as follows:

$$\begin{aligned}
(\rho_1'/h_1^2)(\theta) &= \sum_m (\alpha_m \cos m\theta + \beta_m \sin m\theta), \\
(\rho_2'/h_2^2)(\theta) &= \sum_m (\gamma_m \cos m\theta + \delta_m \sin m\theta).
\end{aligned} \tag{15}$$

Upon substitutions of (15) into (13) it immediately follows that

$$\begin{aligned}
\alpha_m &= -(2m/Q_1)(C_m + D_m)e^{m\rho_1}, \\
\beta_m &= -(2m/Q_1)(E_m + F_m)e^{m\rho_1},
\end{aligned}$$

and

$$\begin{aligned}
\gamma_m &= \frac{2m}{(Q_2 - Q_1)} \frac{D_m}{\cosh m\rho_2}, \\
\delta_m &= \frac{-2m}{(Q_2 - Q_1)} \frac{E_m}{\sinh m\rho_2}.
\end{aligned}$$

The final step is the substitution of (12) and (15) into (14). Setting the coefficients of $\cos m\theta$ and $\sin m\theta$ to zero yields a system of four homogeneous equations in four unknowns. If the coefficients in (12) are assumed to be proportional to $e^{-i\sigma t}$, the equation for σ is obtained by requiring that a nontrivial solution exist, i.e., that the determinant of the system vanish. This leads to an equation of the form:

$$(\sigma/\omega)^4 - A(m,\rho_1,\rho_2)(\sigma/\omega)^2 - B(m,\rho_1,\rho_2) = 0, \tag{16}$$

where the coefficients A and B are given by

$$\begin{aligned}
A &= \alpha\beta - \zeta\xi + \gamma(2\alpha - 2\beta + \zeta + \xi - \gamma), \\
B &= \alpha\beta(\gamma^2 - \zeta\xi) - \gamma(\alpha^2\zeta + \beta^2\xi),
\end{aligned}$$

and

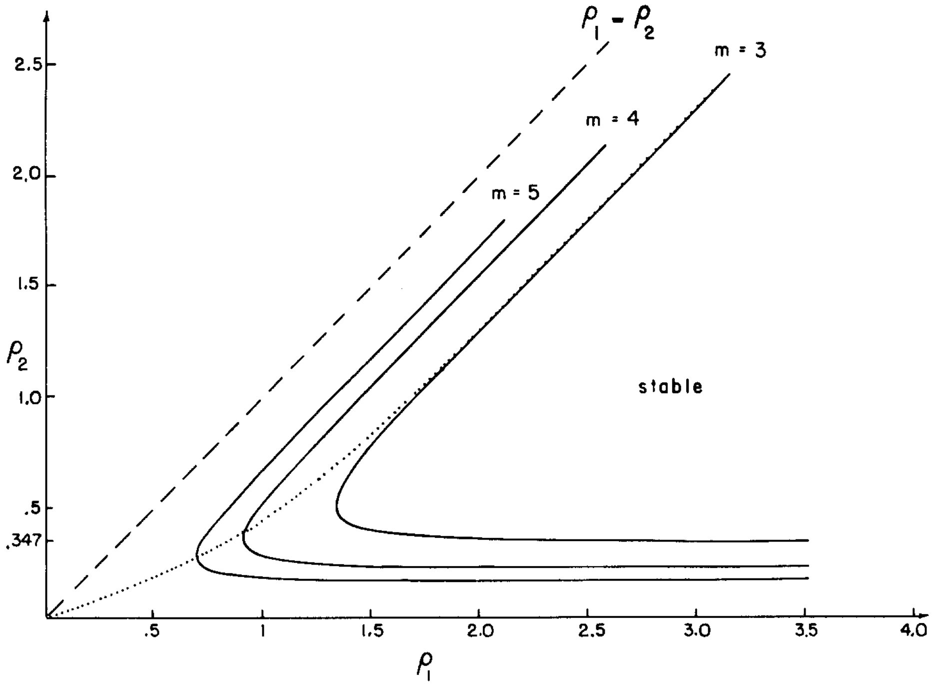

FIG. 3. Stability curves for $N = 2$ for the modes $m = 3,4,5$. The dotted curve is calculated from Flierl's theory for circular vortices for $m = 3$ as described in the text.

$$\alpha = (1/2\omega) Q_1 \cosh m\rho_1 e^{-m\rho_1} - \tfrac{1}{2} m,$$
$$\beta = (1/2\omega) Q_1 \sinh m\rho_1 e^{-m\rho_1} - \tfrac{1}{2} m,$$
$$\gamma = (1/2\omega)(Q_2 - Q_1) \sinh m\rho_2 \cosh m\rho_2,$$
$$\zeta = (1/2\omega)(Q_2 - Q_1) \sinh^2 m\rho_2 - \tfrac{1}{2} m \tanh m\rho_2,$$
$$\xi = (1/2\omega)(Q_2 - Q_1) \cosh^2 m\rho_2 + \tfrac{1}{2} m \coth m\rho_2.$$

From the form of (16) it is clear that ω plays no role in determining the stability of the vortex. Furthermore it is easy to show that for stability the following conditions must be satisfied:

$$A > 0, \quad B < 0, \quad \text{and} \quad A^2 + 4B > 0. \tag{17}$$

Because of the complexity of the coefficients A and B one has to resort to numerical methods in order to determine the shape of the critical curves in the (ρ_1, ρ_2) plane for each mode m; they are shown in Fig. 3. The following results have been established:

(1) The modes $m = 1$ and $m = 2$ are stable (or neutral) for all values of ρ_1 and ρ_2 as was the case for the Kirchhoff vortex ($N = 1$).

(2) In the limit of very large ρ_1 the vortex becomes unstable if $\rho_2 < 0.347$ which corresponds to $b_2/a_2 < \frac{1}{3}$, in agreement with Love's result for the Kirchhoff vortex.

(3) In general, given an inner ellipse of size $\rho_2 > 0.347$ the system will be stable for all m's provided the size of the outer ellipse exceeds some critical value. For very large values of ρ_1 and ρ_2 (i.e., for virtually circular boundaries) the vortex will become unstable for $(\rho_1 - \rho_2) < 0.7$.

Finally, we compare our results with the linear stability theory of Ref. 3 for the circular problem. To each point in the (ρ_1, ρ_2) plane we associate a unique pair of concentric circles of uniform vorticity; if the inner one has radius 1 and vorticity 1, and the outer one radius r and vorticity q, we establish a one-to-one correspondence between (ρ_1, ρ_2) and (r,q) by choosing

$$r^2 = \frac{\sinh 2\rho_1}{\sinh 2\rho_2} \quad \text{and} \quad q = \frac{Q_1}{Q_2},$$

which means that the outer to inner ratios of area and vorticity are the same for the concentric circles and the confocal ellipses.

The stability curve for the $m = 3$ mode is shown in Fig. 3. As expected, it agrees well with the curve for the confocal ellipse for large values of ρ_1 and ρ_2. The circular theory breaks down for $\rho_2 < 1.2$, which corresponds to $b_2/a_2 < 0.83$.

In conclusion, we point out that two quite different kinds of instabilities can be identified from the curves of Fig. 3. The lower branch of these curves (which is asymptotic to $\rho_2 = 0.347$, i.e., $b_2/a_2 = \frac{1}{3}$ for the mode $m = 3$) represents a Love-type instability due to excessive ellipticity of the inner boundary. The upper branch, on the contrary, is really a Rayleigh-type shear instability associated with the fact that the vorticity gradient does not have a unique sign throughout the vortex.

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[3] See National Technical Information Service Document No. AD-A149386/5 (Woods Hole Oceanographic Institution Technical Report, WHOI-84-44). Copies may be ordered from the National Technical Information Service, Springfield, Virginia, 22161. The price is $22.95 plus a $3.00 handling fee. All orders must be prepaid.