The DoF of Two-Way Butterfly Networks

Mehdi Ashraphijuo, Vaneet Aggarwal, and Xiaodong Wang

***Abstract*— This letter studies the two-way butterfly network, a class of two-way four-unicast networks. We first show that bidirectional links do not increase the degrees of freedom for this network, thus giving the first example of a network to the best of our knowledge, where bidirectional links do not increase the degrees of freedom. Furthermore, we show that sufficient caching at the relays or increasing the number of antennas at the relays can double the degrees of freedom for the two-way butterfly network.**

***Index Terms*— Degrees of freedom, four-unicast channels, two-way network, butterfly network, caching, multiple-antenna.**

I. Introduction

Two-unicast channels consist of two sources and two destinations communicating through a general network. Degrees of freedom for one-way $2 \times 2 \times 2$ fully-connected two-unicast channels has been studied in [1]. These results were further generalized to one-way $2 \times 2 \times 2$ non-layered topology in [2]. General one-way two-unicast channel has been considered in [3] and [4] and it was shown in [4] that the DoF for any topology takes one of the values in $\{1, \frac{3}{2}, 2\}$, depending on the topology. Two-way two-unicast channels have been studied for a single relay in [5] and [6]. In [7], three different achievability strategies for two-way MIMO $2 \times 2 \times 2$ fully-connected channel are proposed. A finite-field two-way two-unicast model is also studied in [8] and [9].

In this letter, we study the degrees of freedom for the two-way butterfly network, a class of two-way four-unicast networks. The butterfly network has been widely studied in network coding [10], [11], while not with the wireless links. This network has one-way degrees of freedom of 2 [4]. Here, we show that the two-way degrees of freedom is also 2. This is the first result to the best of our knowledge, where bidirectional links do not improve the degrees of freedom. Indeed there are network configurations where the two-way degrees of freedom double the one-way degrees of freedom [12].

We further consider the case where relays in the two-way butterfly network are equipped with caching memory. Caching is a technique to reduce traffic load by exploiting the high degree of asynchronous content reuse and the fact that storage is cheap and ubiquitous in today's wireless devices [13], [14]. During off-peak periods when network resources are abundant, some content can be stored at the wireless edge (e.g., access points or end user devices), so that demands can be met with reduced access latencies and bandwidth requirements. The caching problem has a long history, dating back to the work by Belady in 1966 [15]. There are various forms of caching, i.e., to store data at user ends, relays, etc. [16]. Both uncoded and coded caching strategies have been developed. The caching process consists of an offline placement phase and an online delivery phase. One important aspect is the design of the placement phase in order to facilitate the delivery phase. There are several recent works that consider communication scenarios where user nodes have pre-cached information from a fixed library of possible files during the offline phase, in order to minimize the transmission from source during the delivery phase [17], [18]. There are only a limited number of works on the degrees of freedom with caching. In particular, [19] study the degrees of freedom for the relay and interference channels with caching, respectively, under some assumptions and provide asymptotic results on the degrees of freedom as the solutions to some optimization problems. In this letter we show that caching increases the degrees of freedom of the two-way butterfly network to 4. This is the first example, to the best of our knowledge, where caching at relays increases the degrees of freedom of two-way networks. Furthermore, we show that another way of achieving the total degrees of freedom for the two-way butterfly network is to employ multiple antennas at the relay, e.g., to equip one of the relay nodes with three antennas or more.

The remainder of this letter is structured as follows. In Section II, the channel model is given. In Section III and Section IV, we present the DoF results for the two-way butterfly network without and with relay caching, respectively. In Section V, a butterfly network where one of the relays is equipped with multiple antennas is studied. Finally, Section VI concludes this letter.

II. Channel Model

As shown in Fig. 1, the two-way butterfly network consists of four transmitters $S_1, \ldots, S_4$, three relays $R_1, \ldots, R_3$, and four receivers $D_1, \ldots, D_4$. Each transmitter S_i has one message that is intended for its designated receiver D_i. (We note that one-way butterfly network does not have transmissions from S_3 and S_4 and destinations D_3 and D_4 do not receive data.) In the first hop, the signals received at relays in time slot m are given as follows.

$$Y_{R_1}[m] = H_{1,R_1} X_1[m] + H_{4,R_1} X_4[m] + Z_{R_1}[m], \quad (1)$$

$$Y_{R_2}[m] = \sum_{i=1}^{4} H_{i,R_2} X_i[m] + Z_{R_2}[m], \quad (2)$$

$$Y_{R_3}[m] = H_{2,R_3} X_2[m] + H_{3,R_3} X_3[m] + Z_{R_3}[m], \quad (3)$$

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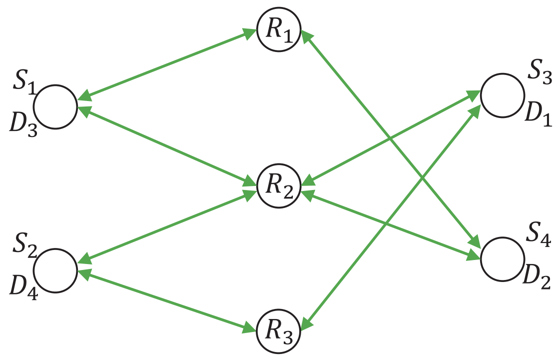


Fig. 1. Two-way butterfly network.

where H_{i,R_k} is the channel coefficient from transmitter S_i to relay R_k, $X_i[m]$ is the signal transmitted from S_i, $Y_{R_k}[m]$ is the signal received at relay R_k and $Z_{R_k}[m]$ is the i.i.d. circularly symmetric complex Gaussian noise with zero mean and unit variance, $i \in \{1, 2, 3, 4\}$, $k \in \{1, 2, 3\}$. In the second hop, the signals received at receivers in time slot m are given by

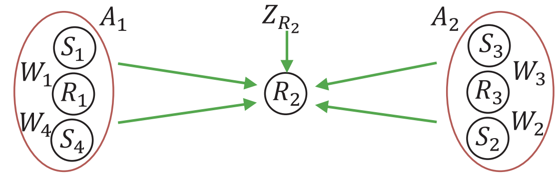
$$Y_i[m] = H_{R_1,i} X_{R_1}[m]$$
$$H_{R_2,i} X_{R_2}[m] + Z_i[m], \quad \text{for } i \in \{2, 3\}, \tag{4}$$
$$Y_i[m] = H_{R_2,i} X_{R_2}[m]$$
$$+ H_{R_3,i} X_{R_3}[m] + Z_i[m], \quad \text{for } i \in \{1, 4\}, \tag{5}$$

where $H_{R_k,i}$ is the channel coefficient from relay R_k to receiver D_i, $X_{R_k}[m]$ is the signal transmitted from R_k, $Y_i[m]$ is the signal received at receiver D_i and $Z_i[m]$ is the i.i.d. circularly symmetric complex Gaussian noise with zero mean and unit variance, $i \in \{1, 2, 3, 4\}$, $k \in \{1, 2, 3\}$. We assume that the channel coefficient values are drawn i.i.d. from a continuous distribution and they are bounded from above and below, i.e., $H_{\min} < |H_{i,R_k}[m]| < H_{\max}$ and $H_{\min} < |H_{R_k,i}[m]| < H_{\max}$ as in [20]. The relays are assumed to be full-duplex and equipped with caches. Furthermore, the relays are assumed to be causal, which means that the signals transmitted from the relays depend only on the signals received in the past and not on the current received signals and can be described as

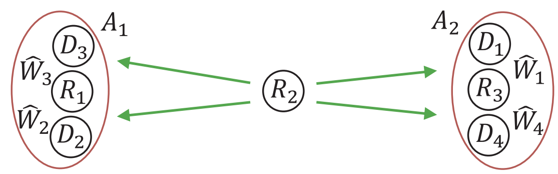
$$X_{R_k}[m] = f(Y_{R_k}^{m-1}, X_{R_k}^{m-1}, C_{R_k}), \tag{6}$$

where $X_{R_k}^{m-1} \triangleq (X_{R_k}[1], \ldots, X_{R_k}[m-1])$, $Y_{R_k}^{m-1} \triangleq (Y_{R_k}[1], \ldots, Y_{R_k}[m-1])$, and C_{R_k} is the cached information in relay R_k. We assume that source S_i, $i \in \{1, 2, 3, 4\}$ only knows channels H_{i,R_k}, $k \in \{1, 2, 3\}$; relay R_k, $k \in \{1, 2, 3\}$ only knows channels H_{i,R_k} and $H_{R_k,i}$, $i \in \{1, 2, 3, 4\}$; and destination D_i, $i \in \{1, 2, 3, 4\}$ only knows channels $H_{R_k,i}$, $k \in \{1, 2, 3\}$.

The source S_i, $i \in \{1, 2, 3, 4\}$ has a message W_i that is intended for destination D_i. $|W_i|$ denotes the size of the message W_i. The rates $\mathcal{R}_i = \frac{\log |W_i|}{n}$, $i \in \{1, 2, 3, 4\}$ are achievable during n channel uses when n is large enough, if the probability of error can be arbitrarily small for all four messages simultaneously. The capacity region $\mathcal{C} = \{(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4)\}$ represents the set of all achievable quadruples. The sum-capacity is the maximum sum-rate that is achievable, *i.e.*, $C_\Sigma(P) = \sum_{i=1}^4 \mathcal{R}_i^c$ where $(\mathcal{R}_1^c, \ldots, \mathcal{R}_4^c) = \arg\max_{(\mathcal{R}_1,\ldots,\mathcal{R}_4)\in\mathcal{C}} \sum_{i=1}^4 \mathcal{R}_i$ and P is the transmit power at each node (source or relay). The degrees of freedom is defined as



(a) The channels from transmitters to the relays.



(b) The channels from relays to the receivers.

Fig. 2. The genie-aided butterfly network.

$$DoF \triangleq \lim_{P\to\infty} \frac{C_\Sigma(P)}{\log P} = \sum_{i=1}^{4} \lim_{P\to\infty} \frac{\mathcal{R}_i^c}{\log P} = \sum_{i=1}^{4} d_i, \tag{7}$$

where $d_i \triangleq \lim_{P\to\infty} \frac{\mathcal{R}_i^c}{\log P}$ is defined as the DoF of source S_i, for $i \in \{1, 2, 3, 4\}$. We note that DoF is the degrees of freedom for almost every channel realization (in other words, with probability 1 over the channel realizations). We denote DoF_C as the degrees of freedom for the case of with relay caching, and DoF_{NC} as the degrees of freedom for the case of no relay caching.

III. Two-Way Butterfly Network Without Relay Caching

The result in this section indicates that the degrees of freedom for the two-way butterfly network in Fig. 1 is 2. The result is surprising since the degrees of freedom for the one-way butterfly network has been shown to be 2 [4, Th. 1, Part B′], thus the total DoF cannot be improved by using the two-way capability of the bidirectional links. To the best of our knowledge, this is the first topology for which the two-way network achieves the same total DoF as the one-way network.

Theorem 1: For the two-way butterfly network in Fig. 1, $DoF_{NC} = 2$.

Proof: We first show the upper bound. Consider S_1, R_1, and S_4 as one group of nodes and S_2, R_3, and S_3 as another group. As genie-aided side information, assume that the nodes in each group have access to all of the messages in the same group. Note that the first group has W_1 and W_4 needed by the second group and the second group has W_2 and W_3 needed by the first group. The genie-aided side information does not give the needed message to any destination, and the two groups can only communicate through R_2. The described channel can be seen in Fig. 2 where nodes A_1 and A_2 both have three antennas. Thus, the cut-set bound gives that sum DoF ≤ 2. The reason is that R_2 is a single antenna node and each of A_1 and A_2 can only decode one DoF of information from it.

The proof for achievability is straightforward. If all the nodes except for S_1, R_2, and S_3 in Fig. 1 are silent, then the channel can be seen as a two-way $1 \times 1 \times 1$ relay network formed by S_1, R_2, and S_3. This channel can achieve two degrees of freedom by simply forwarding the sum of the received signals at relay R_2, which is the sum of the two messages from S_1 and S_3. □

Remark 1: For the one-way butterfly network, the achievability scheme that leads to the total degrees of freedom of 2 is quite involved [4, Th. 1, Part B′]; but the upper bound of 2 is a simple cut-set bound. In contrast, for the two-way butterfly network given in Fig. 1, as seen in the proof of Theorem 1, the upper bound of 2 on the total degrees of freedom is less obvious, but the achievability of 2 is straightforward.

IV. Two-Way Butterfly Network With Relay Caching

We now assume that each relay is equipped with a cache that can store the data from the sources. Our goal is to design strategies for caching and transmission so that the sum rate of all four source-destination pairs is maximized. Our strategy comprises two parts. The first phase is the transmission from sources to the relays, which is performed offline and is known as the placement phase. The second phase is the transmission from relays to the destinations, which is performed online and is known as the delivery phase. We assume that the relays decode W_i, $i = 1, \ldots, 4$ in the offline phase and save $W_1' \triangleq W_1 \oplus W_3$, $W_2' \triangleq W_2 \oplus W_4$ in their caches. The transmitted signals from the relays intend to make W_1' decodable at D_1 and D_3, and W_2' decodable at D_2 and D_4.

The next result shows that the degrees of freedom is 4, thus with the help of relay caching, the bidirectional links can now double the degrees of freedom.

Theorem 2: For the two-way butterfly network with relay caching descried above, $DoF_C = 4$.

Proof: The upper bound follows from the cut-set bound. We now provide an achievability strategy. The relays know the new messages W_1' and W_2', and the encoded signals in all relays for messages W_1' and W_2' at time $m = 1, 2, \ldots, n$ are the same, i.e., $A[m] = f(W_1')$ and $B[m] = f(W_2')$, $m = 1, 2, \ldots, n$. At time m, the relays transmit the following messages

$$\begin{aligned} X_{R_1}[m] &= -\frac{H_{R_2,2}}{H_{R_1,2}}A[m] - \frac{H_{R_2,3}}{H_{R_1,3}}B[m], \\ X_{R_2}[m] &= A[m] + B[m], \\ X_{R_3}[m] &= -\frac{H_{R_2,4}}{H_{R_3,4}}A[m] - \frac{H_{R_2,1}}{H_{R_3,1}}B[m]. \end{aligned}$$

Using this, we see that the received signals at the destinations are as follows

$$\begin{aligned} Y_1[m] &= \left(H_{R_2,1} - \frac{H_{R_3,1}H_{R_2,4}}{H_{R_3,4}}\right)A[m] + Z_1[m], \\ Y_2[m] &= \left(H_{R_2,2} - \frac{H_{R_1,2}H_{R_2,3}}{H_{R_1,3}}\right)B[m] + Z_2[m], \\ Y_3[m] &= \left(H_{R_2,3} - \frac{H_{R_1,3}H_{R_2,2}}{H_{R_1,2}}\right)A[m] + Z_3[m], \\ Y_4[m] &= \left(H_{R_2,4} - \frac{H_{R_3,4}H_{R_2,1}}{H_{R_3,1}}\right)B[m] + Z_4[m]. \end{aligned}$$

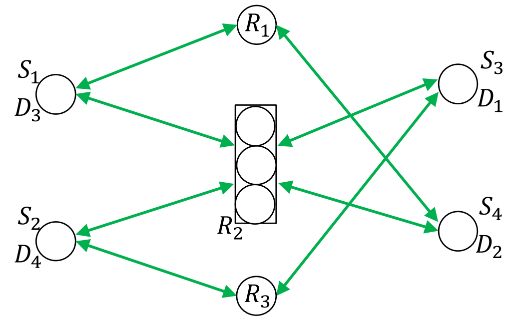


Fig. 3. A two-way butterfly network with 3 antennas at R_2.

Note that the first and the third receivers receive noisy versions of $A[m]$, from which they can decode W_1' and subtract the contribution of their respective own messages to get the respective decoded messages. The argument is similar for the second and the fourth receivers using $B[m]$ and W_2' and thus showing that four degrees of freedom can be achieved. □

In addition, we have the following lower bound on the total DoF for the case of limited relay caching.

Corollary 1: For the two-way butterfly network with each relay caching p portion of each of the messages W_1' and W_2' ($0 \le p \le 1$), the total DoF of $2 + 2p$ is achievable.

Proof: We apply time-sharing to obtain this result. In $(1 - p)$ portion of the time, we do not use the relay caches and DoF of 2 is achievable as in Theorem 1. In the other p portion of the time, we assume that the relays have access to messages W_1' and W_2' available in their caches and apply the same transmission strategy as in Theorem 2. Then the total DoF $= 4$ is achievable in this part of the time-sharing. Hence overall the average sum DoF of $4p + 2(1 - p) = 2 + 2p$ is achievable. □

V. Two-Way Butterfly Network With Multiple-Antenna Relay

In the previous section, we showed that relay caching can increase the degrees of freedom for the two-way butterfly network. In this section we show that increasing the number of antennas at relay R_2 can also increase the degrees of freedom to 4 for the two-way butterfly network.

Fig. 3 represents the two-way butterfly network with 3 antennas at relay R_2. There are some differences in the model compared with the one in previous sections as described in the following. The channels $\mathbf{h}_{i,R_2}$, $\forall i \in \{1, \ldots, 4\}$ are 3×1 vectors, and the channels $\mathbf{h}_{R_2,i}^H$, $\forall i \in \{1, \ldots, 4\}$ are 3×1 vectors. The scalar beamformers v_1 and v_3 are for transmission from relays R_1 and R_3, respectively. Also, $\mathbf{V}_2$ is the 3×3 precoding matrix for transmission from relay R_2. In addition, $\mathbf{y}_{R_2}[m]$ is the 3×1 signal vector received at relay R_2 and $\mathbf{z}_{R_2}[m]$ is the 3×1 i.i.d. circularly symmetric complex Gaussian additive noise vector at relay R_2 with zero mean and unit variance entries.

The following theorem shows that increasing the number of antennas at relays can increase the total degrees of freedom.

Theorem 3: For the two-way butterfly network with 3 antennas at relay R_2 descried above, $DoF_{NC} = 4$.

Proof: The upper bound follows from the cut-set bound. We now provide an achievability strategy, where the relays

R_1 and R_3 are not used ($\upsilon_1 = \upsilon_3 = 0$). The received signals at R_2 is as follows:

$$\mathbf{y}_{R_2}[m] = \sum_{i=1}^{4} \mathbf{h}_{i,R_2} X_i[m] + \mathbf{z}_{R_2}[m], \quad (8)$$

The relay performs amplify-and-forward and the received signal at the destination $D_i, i = 1, \cdots, 4$ is given by:

$$Y_i[m] = \mathbf{h}_{R_2,i} \mathbf{V}_2 \mathbf{y}_{R_2}[m] + Z_i[m], \quad (9)$$

Substituting (8) into (9) we have

$$Y_i[m] = \mathbf{h}_{R_2,i} \mathbf{V}_2 \left(\sum_{j=1}^{4} \mathbf{h}_{j,R_2} X_j[m] + \mathbf{z}_{R_2}[m] \right) + Z_i[m], \quad (10)$$

Thus, we need to find $\mathbf{V}_2$ such that $\mathbf{h}_{R_2,i} \mathbf{V}_2 \mathbf{h}_{j,R_2} = 0$ for $(i, j) = (1, 2), (1, 4), (2, 1), (2, 3), (3, 2), (3, 4), (4, 1)$, and $(4, 3)$. With this, the desired signal can be decoded after canceling the signals known at the receiver. For example, for destination D_1, $X_1[m]$ is the intended signal and it knows $X_3[m]$. $\mathbf{h}_{R_2,i} \mathbf{V}_2 \mathbf{h}_{j,R_2} = 0$ for $(i, j) = (1, 2)$, and $(1, 4)$ makes the multipliers of $X_2[m]$ and $X_4[m]$ zero at destination D_1. Thus, the desired signal is decoded after canceling contribution of $X_3[m]$. We have 9 unknown entries in $\mathbf{V}_2$ and eight equations, which give us infinitely many non-zero solutions thus proving the theorem. For detailed steps of the proof, the reader is referred to [21]. □

Remark 2: We note that relays R_1 and R_3 did not help to achieve optimal degrees of freedom for 1 antenna case at R_2 (Theorem 1), and 3 antennas case at R_2 (Theorem 3). Whether these relays help for two antennas case at R_2 is an open problem.

Remark 3: The achievability strategy in the proof of Theorem 3 works when there are 3 or more antennas at relay R_2. However, with two antennas at R_2, it will not work since in that case there are still 8 equations, but $\mathbf{V}_2$ will be 2×2 and together with υ_1 and υ_3 there are only 6 unknowns.

Remark 4: Caching does not improve the DoF of the two-way butterfly network with 3 antennas at relay R_2, since the DoF with caching is also upper bounded by 4 (cut-set bound).

VI. Conclusions

We have considered the two-way butterfly network, a class of two-way four-unicast networks. We have shown that bidirectional links do not increase the degrees of freedom for this network. We have also shown that enough caching at the relays or increasing the number of antennas at the relays can double the degrees of freedom for this network. Finding the degrees of freedom for the general two-way four-unicast networks, with or without caching, remains an open problem. Moreover, the DoF of two-way butterfly networks with multiple antennas at each node is also open.

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