On the Capacity and Degrees of Freedom Regions of Two-User MIMO Interference Channels With Limited Receiver Cooperation

Mehdi Ashraphijuo, Vaneet Aggarwal, *Member, IEEE*, and Xiaodong Wang, *Fellow, IEEE*

***Abstract*— This paper gives the approximate capacity region of a two-user multiple-input multiple-output (MIMO) interference channel with limited receiver cooperation, where the gap between the inner and outer bounds is in terms of the total number of receive antennas at the two receivers and is independent of the actual channel values. The approximate capacity region is then used to find the degrees of freedom region. For the special case of symmetric interference channels, we also find the amount of receiver cooperation in terms of the backhaul capacity beyond, which the degrees of freedom do not improve. Further, the generalized degrees of freedom is found for MIMO interference channels with equal number of antennas at all nodes. It is shown that the generalized degrees of freedom improves gradually from a W curve to a V curve with increase in cooperation in terms of the backhaul capacity.**

***Index Terms*— MIMO interference channel, limited receiver cooperation, capacity region, generalized degrees of freedom, Han-Kobayashi message splitting.**

I. Introduction

Wireless networks with multiple users are interference-limited rather than noise-limited. Interference channel (IC) is a good starting point for understanding the performance limits of interference-limited communications. In spite of research spanning over three decades, the capacity of the IC has been characterized only for some special cases [1]–[7].

Cooperation between transmitters or receivers in ICs can help mitigate the interference by forming a distributed MIMO system which provides performance gain. In practice, the cellular base stations can be connected via wireless backhaul links [8], or the mobile nodes might have connections to each other via out-of-band device-to-device communication links [9], [10]. In either case, the cooperation between the nodes is limited, and making efficient use of such cooperation link for transmitter and/or receiver cooperation is an important problem [11]–[15]. In this paper, we tackle the fundamental problem of efficient use of limited-capacity backhaul for multiple-input multiple-output (MIMO) uplink ICs (with receiver cooperation). Recently, a number of works have shown that transmitter and receiver cooperation can be employed in ICs to achieve an improvement in data rates [16]–[24]. However, most of the existing works on ICs with cooperation are limited to discrete memoryless channels or to single-input single-output (SISO) channels. This paper analyzes two-user MIMO Gaussian ICs with limited receiver cooperation.

In large wireless networks, having global knowledge of the channel state is infeasible and thus Lozano et al. [25] found a saturation effect in the system capacity. In this paper, we consider the two-user MIMO IC with perfect channel state knowledge at the transmitters and receivers. We assume that all nodes have the channel state information of all links in order to investigate the capacity region of MIMO IC with limited receiver cooperation. While the overhead of gathering the global channel state information must not be neglected, it has been shown (see [26], [27]) that this overhead is manageable in the presence of a reduced number of users. This overhead increases as the number of users increases, and thus some authors have considered the knowledge of channel state in a local neighborhood [28], [29]. With the local network connectivity and channel state information, subnetworks can be scheduled where each subnetwork is operated using an information-theoretic optimal scheme [30], [31]. Thus, even with the knowledge of the local channel state information, understanding of small networks can help to improve throughput of large networks.

A two-user SISO Gaussian IC with limited receiver cooperation is considered in [16] where there are links with fixed capacities between the two receivers and the capacity region of the channel is obtained within two bits. In this paper, we find an outer bound and an inner bound for the capacity region that are within $N_1 + N_2$ bits for a general MIMO IC with limited receiver cooperation, where N_1 and N_2 are the numbers of receive antennas at the two receivers, respectively. A gap of $N_1 + N_2$ is loose when the SNR is low and the number of receive antennas is large. However, this is the worst-case gap, and the actual gap may be smaller in many

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M. Ashraphijuo is with the Department of Electrical Engineering, Columbia University, New York, NY 10027 USA (e-mail: mehdi@ee.columbia.edu).

V. Aggarwal is with AT&T Laboratories-Research, Bedminster, NJ 07921 USA, and also with Columbia University, New York, NY 10027 USA (e-mail: vaneet@alumni.princeton.edu).

X. Wang is with the Department of Electrical Engineering, Columbia University, New York, NY 10027 USA, and also with King Abdulaziz University, Jeddah 22254, Saudi Arabia (e-mail: wangx@ee.columbia.edu).

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cases. More importantly, such a constant gap between the outer and inner bounds of the capacity region makes it possible to obtain the degrees of freedom (DoF) and the generalized degrees of freedom (GDoF) regions of the channel. We use an achievability scheme based on that for the discrete memoryless channel in [16]. In this scheme, receivers do not decode the messages immediately upon receiving the signals from the transmitters. One of the receivers quantizes its received signal at an appropriate distortion, bins the quantization codeword and sends the bin index to the other receiver. For quantizing the received signal, a novel distortion function for MIMO IC is given in this paper. The other receiver decodes its own information based on its own received signal and the received bin index. After decoding, it bin-and-forwards the decoded common messages back to the other receiver and helps it decode. This paper uses the signal distributions and auxiliary variables that are different from those in [16] and in such a way that can be used for a MIMO IC to achieve a constant gap to the capacity region.

We note that the achievability strategy for the SISO IC in [16] is to split the transmit signal into public and private messages using the Han-Kobayashi message splitting where the private message is received at the unintended receiver below the noise floor. For a MIMO IC with limited receiver cooperation, we proposed a counterpart of Han-Kobayashi splitting in [32] where the covariance matrices for the public and the private messages were properly designed. In this paper, we give an achievability scheme based on the splitting scheme in [32]. Further, the authors of [16] proposed different choices of power splits between the public and the private messages for three different regions of SISO IC corresponding to: weak, mixed and strong interferences. In this paper, for MIMO IC, we propose a single choice of covariance matrices for the public and private messages for all regimes rather than considering different regimes separately. For the special case of SISO IC, the achievability scheme used in this paper reduces to a different one from that given in [16]. The achievability scheme uses the convex hull of the regions formed by two strategies corresponding to different decoding orders. The convex hull of the two regions eliminates two constraints in each region resulting in a constant gap between the inner and the outer bounds.

In [33], the capacity region of a Gaussian SISO IC without cooperation is obtained within 1 bit and in [4], the capacity region of a Gaussian MIMO IC without cooperation is obtained within a constant gap to the outer bound. We also note that the proposed Han-Kobayashi split is different from that proposed in [4].

Having characterized the outer and inner bounds within a constant gap, and as a result having the approximate capacity region, we also find the DoF region for the two-user MIMO IC with limited receiver cooperation. We find that the DoF region improves with the increase in cooperation in terms of the backhaul capacity. For the case of symmetric number of antennas in both the transmitters and the receivers, we find that the DoF improves up to a certain point in terms of the backhaul capacity, and beyond which the DoF does not improve anymore.

When each transmitter has M antennas and each receiver has N antennas, the symmetric DoF region is a pentagon with bounds only on individual DoF (d_1, d_2) and sum DoF $(d_1 + d_2)$ for all cases except when $N < M < 2N$. Thus, when $M = N$, the DoF region is a pentagon. However, when $N < M < 2N$, the DoF region also has constraints on $2d_1+d_2$ and d_1+2d_2. These constraints are known to not hold for ICs with no cooperation [5], and for ICs with infinite cooperation which corresponds to a multiple-access channel (MAC) [34]. In this paper, we find that the extra bounds on $2d_1 + d_2$ and d_1+2d_2 are dominant for a finite non-zero limited cooperation (when the backhaul capacity is less than a certain value) for $N < M < 2N$.

Finally, we also characterize the GDoF for a MIMO IC with limited receiver cooperation, when the cooperation links are of the same capacity which is increasing with a base signal-to-noise ratio (SNR) parameter, as $\beta \log \mathsf{SNR}$. Note that even though the DoF region is found in general, we find the GDoF only in a limited setting when the number of antennas at all the nodes are the same (say M). We assume that the direct links have channel strengths SNR while the cross links have channel strengths SNR^{α}. We find that the increase in the cooperation leads to improvement in GDoF. For a given M and α, the GDoF increases till $\beta = M\alpha$ at which point the GDoF with limited cooperation is the same as that with full cooperation. Without any receiver cooperation, the GDoF is a "W" curve in terms of α. We note that the "W" curve changes to a "V" curve and then to an increasing function as the backhaul capacity increases.

The remainder of the paper is organized as follows. Section II introduces the model for a MIMO IC with limited receiver cooperation and the capacity region. Sections III describes our results on the capacity region. Section IV gives our results on the DoF region and the GDoF. Section V concludes the paper. Appendix A summarizes some useful matrix results that are used throughout the paper. The detailed proofs of various results are given in Appendices B–E.

II. Channel Model and Preliminaries

In this section, we describe the channel model that is used in this paper. A two-user MIMO IC consists of two transmitters and two receivers. Transmitter i is labeled as T_i and receiver j is labeled as D_j for $i, j \in \{1, 2\}$. Further, we assume T_i has M_i antennas and D_j has N_j antennas. Henceforth, such a MIMO IC will be referred to as the (M_1, N_1, M_2, N_2) MIMO IC. The channel matrix between transmitter T_i and receiver D_j is denoted by $H_{ij} \in \mathbb{C}^{N_j \times M_i}$. We shall consider a time-invariant channel where the channel matrices remain fixed for the entire duration of the communication. At time t, transmitter T_i transmits a vector $X_i(t) \in \mathbb{C}^{M_i \times 1}$ over the channel with a power constraint $\mathrm{tr}(\mathbb{E}(X_i X_i^{\dagger})) \leq 1$ ($A^{\dagger}$ is the conjugate transpose of the matrix A).

Let $Q_{ij} = \mathbb{E}(X_i X_j^{\dagger})$ for $i, j \in \{1, 2\}$. The covariance matrices Q_{ij} for $i \neq j$ are zero, since the two transmitters do not cooperate with each other. We say $A \preceq B$ if $B - A$ is a positive semi-definite (p.s.d.) matrix and we say $A \succeq B$ if and only if $B \preceq A$. The identity matrix of size $s \times s$ is denoted by I_s. Further, we define $x^+ \triangleq \max\{x, 0\}$. We also note that $0 \preceq Q_{ii} \preceq I$ according to Lemma 3 in Appendix A since $\mathrm{tr}(\mathbb{E}(X_i X_i^\dagger)) \le 1$. By definition of Q_{ii}, we see that $Q_{ii} = Q_{ii}^\dagger$. Moreover, we have $0 \preceq Q_{ii} Q_{ii}^\dagger \preceq I$, where $0 \preceq Q_{ii} Q_{ii}^\dagger$ results from the fact that every matrix in the form of $AA^\dagger$ is p.s.d. and $Q_{ii} Q_{ii}^\dagger \preceq I$ results from $\mathrm{tr}(Q_{ii} {Q_{ii}}^\dagger) = \mathrm{tr}(Q_{ii})\mathrm{tr}(Q_{ii}) \le 1$ which gives $Q_{ii} {Q_{ii}}^\dagger \preceq I$ with a similar argument as that for Q_{ii}.

We also incorporate a non-negative power attenuation factor, denoted as ρ_{ij}, for the signal transmitted from T_i to D_j. The received signal at receiver D_i at time t is denoted as $Y_i(t)$ for $i \in \{1, 2\}$, and can be written as

$$Y_1(t) = \sqrt{\rho_{11}} H_{11} X_1(t) + \sqrt{\rho_{21}} H_{21} X_2(t) + Z_1(t), \quad (1)$$

$$Y_2(t) = \sqrt{\rho_{12}} H_{12} X_1(t) + \sqrt{\rho_{22}} H_{22} X_2(t) + Z_2(t), \quad (2)$$

where $Z_i(t) \in \mathbb{C}^{N_i \times 1} \sim \mathsf{CN}(0, I_{N_i})$ is the i.i.d. complex Gaussian noise, ρ_{ii} is the received SNR at receiver D_i and ρ_{ij} is the received interference-to-noise-ratio at receiver D_j for $i, j \in \{1, 2\}$, $i \ne j$. A MIMO IC with limited receiver cooperation is fully described by four parameters. The first is the set of numbers of antennas at each transmitter and receiver, namely (M_1, N_1, M_2, N_2). The second is the set of channel gains, $\overline{H} = \{H_{11}, H_{12}, H_{21}, H_{22}\}$. The third is the set of average link qualities, $\overline{\rho} = \{\rho_{11}, \rho_{12}, \rho_{21}, \rho_{22}\}$. The fourth parameter is $\overline{C} = \{C_{12}, C_{21}\}$ where C_{ji} is the capacity of the cooperation link from receiver D_j to D_i. We assume that these parameters are known to all transmitters and receivers. Also, the cooperation channels are orthogonal to each other and they are orthogonal to the data channels.

The receiver-cooperation links are noiseless with finite capacities. Encoding is causal in the sense that the signal $\Gamma_{ji}(t)$ transmitted from D_j to D_i at time t, is a function of whatever is received over the data channel and on the cooperation link up to time $t-1$, i.e.,

$$\Gamma_{ij}(t) = g_{it}(\Gamma_{ji}(t-1), \ldots, \Gamma_{ji}(1), Y_i(t-1), \ldots, Y_i(1)) \in \{1, 2, \ldots, 2^{C_{ij}}\}, \quad (3)$$

where g_{it} is the encoding function of D_i for cooperation at time t. Let us assume that T_i transmits information at a rate of R_i to receiver D_i using the codebook $\mathcal{C}_{i,n}$ of length-n codewords with $|\mathcal{C}_{i,n}| = 2^{nR_i}$. Given a message $m_i \in \mathcal{M} = \{1, \ldots, 2^{nR_i}\}$, the corresponding codeword $X_i^n(m_i) \in \mathcal{C}_{i,n}$ satisfies the power constraint mentioned before. In addition, the decoded signal at D_i, $\widehat{m_i}$, is a function of the received signal from the channel, $Y_i(t)$, and the cooperation signal transmitted from D_j to D_i, $\Gamma_{ji}(t)$, for $i \in \{1, 2\}$. Thus, the decoding functions of the two receivers are given as

$$\hat{m_i} = f_i(\Gamma_{ji}(n), \ldots, \Gamma_{ji}(1), Y_i(n), \ldots, Y_i(1)) \in \mathcal{M}, \quad (4)$$

where f_i is the decoding function of D_i. Denote the average probability of decoding error by $e_{i,n} = \Pr(\hat{m_i} \ne m_i)$.

A rate pair (R_1, R_2) is achievable if there exists a family of codebooks $\mathcal{C}_{i,n}$ and decoding functions such that $\max_i\{e_{i,n}\}$ goes to zero as the block length n goes to infinity. The capacity region $\mathbb{C}(\overline{H}, \overline{\rho}, \overline{C})$ of the IC with parameters $\overline{H}$, $\overline{\rho}$ and $\overline{C}$ is defined as the closure of the set of all achievable rate pairs.

Consider a two-dimensional rate region $\mathbb{C}(\overline{H}, \overline{\rho}, \overline{C})$. Then, the region $\mathbb{C}(\overline{H}, \overline{\rho}, \overline{C}) \ominus ([0, a] \times [0, b])$ denotes the region formed by $\{(R_1, R_2) : R_1, R_2 \ge 0, (R_1 + a, R_2 + b) \in \mathbb{C}(\overline{H}, \overline{\rho}, \overline{C})\}$ for some $a, b \ge 0$. Further, we define the notion of an achievable rate region that is within a constant number of bits of the capacity region as follows.

Definition 1: An achievable rate region $\mathbb{A}(\overline{H}, \overline{\rho}, \overline{C})$ is said to be within b bits of the capacity region if $\mathbb{A}(\overline{H}, \overline{\rho}, \overline{C}) \subseteq \mathbb{C}(\overline{H}, \overline{\rho}, \overline{C})$ and $\mathbb{A}(\overline{H}, \overline{\rho}, \overline{C}) \oplus ([0, b] \times [0, b]) \supseteq \mathbb{C}(\overline{H}, \overline{\rho}, \overline{C})$.

III. Inner and Outer Bounds on Capacity Region

In this section, we give the outer and inner bounds on the capacity region of a two-user MIMO IC with limited receiver cooperation. Let $\mathcal{R}_o$ be the region formed by (R_1, R_2) satisfying the following constraints.

$$\begin{aligned} R_1 \le\ & \log\det\left(I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger\right) \\ & + \min\{\log\det\Big(I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger \\ & \quad - \rho_{12}\rho_{11} H_{12} H_{11}^\dagger \left(I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger\right)^{-1} \\ & \quad H_{11} H_{12}^\dagger\Big), C_{21}\}, \end{aligned} \quad (5)$$

$$\begin{aligned} R_2 \le\ & \log\det\left(I_{N_2} + \rho_{22} H_{22} H_{22}^\dagger\right) \\ & + \min\{\log\det\Big(I_{N_1} + \rho_{21} H_{21} H_{21}^\dagger \\ & \quad \rho_{21}\rho_{22} H_{21} H_{22}^\dagger \left(I_{N_2} + \rho_{22} H_{22} H_{22}^\dagger\right)^{-1} \\ & \quad H_{22} H_{21}^\dagger\Big), C_{12}\}, \end{aligned} \quad (6)$$

$$\begin{aligned} R_1 + R_2 \le\ & \log\det\Big(I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger + \rho_{21} H_{21} H_{21}^\dagger \\ & \quad -\rho_{11}\rho_{12} H_{11} H_{12}^\dagger \left(I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger\right)^{-1} \\ & \quad H_{12} H_{11}^\dagger\Big) \\ & + \log\det\Big(I_{N_2} + \rho_{22} H_{22} H_{22}^\dagger + \rho_{12} H_{12} H_{12}^\dagger \\ & \quad - \rho_{22}\rho_{21} H_{22} H_{21}^\dagger \left(I_{N_1} + \rho_{21} H_{21} H_{21}^\dagger\right)^{-1} \\ & \quad H_{21} H_{22}^\dagger\Big) + C_{12} + C_{21}, \end{aligned} \quad (7)$$

$$\begin{aligned} R_1 + R_2 \le\ & \log\det\Big(I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger - \rho_{11}\rho_{12} H_{11} \\ & \quad H_{12}^\dagger \left(I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger\right)^{-1} H_{12} H_{11}^\dagger\Big) \\ & + \log\det\left(I_{N_2} + \rho_{22} H_{22} H_{22}^\dagger + \rho_{12} H_{12} H_{12}^\dagger\right) \\ & + C_{12}, \end{aligned} \quad (8)$$

$$\begin{aligned} R_1 + R_2 \le\ & \log\det\Big(I_{N_2} + \rho_{22} H_{22} H_{22}^\dagger - \rho_{22}\rho_{21} H_{22} \\ & \quad H_{21}^\dagger \left(I_{N_1} + \rho_{21} H_{21} H_{21}^\dagger\right)^{-1} H_{21} H_{22}^\dagger\Big) \\ & + \log\det\left(I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger + \rho_{21} H_{21} H_{21}^\dagger\right) \\ & + C_{21}, \end{aligned} \quad (9)$$

$$\begin{aligned} R_1 + R_2 \le\ & \log\det\Bigg(I_{N_1+N_2} + \begin{bmatrix} \sqrt{\rho_{11}} H_{11} \\ \sqrt{\rho_{12}} H_{12} \end{bmatrix} \\ & \begin{bmatrix} \sqrt{\rho_{11}} H_{11}^\dagger & \sqrt{\rho_{12}} H_{12}^\dagger \end{bmatrix} + \begin{bmatrix} \sqrt{\rho_{21}} H_{21} \\ \sqrt{\rho_{22}} H_{22} \end{bmatrix} \\ & \begin{bmatrix} \sqrt{\rho_{21}} H_{21}^\dagger & \sqrt{\rho_{22}} H_{22}^\dagger \end{bmatrix}\Bigg), \end{aligned} \quad (10)$$

$$\begin{aligned}
2R_1+R_2 \le\ & \log\det\Big(I_{N_1}+\rho_{11}H_{11}H_{11}^{\dagger}-\rho_{11}\rho_{12}H_{11} \\
& \qquad H_{12}^{\dagger}\big(I_{N_2}+\rho_{12}H_{12}H_{12}^{\dagger}\big)^{-1}H_{12}H_{11}^{\dagger}\Big) \\
& + \log\det\Big(I_{N_2}+\rho_{22}H_{22}H_{22}^{\dagger}+\rho_{12}H_{12}H_{12}^{\dagger} \\
& \qquad -\rho_{22}\rho_{21}H_{22}H_{21}^{\dagger} \\
& \qquad \big(I_{N_1}+\rho_{21}H_{21}H_{21}^{\dagger}\big)^{-1}H_{21}H_{22}^{\dagger}\Big) \\
& + \log\det\Big(I_{N_1}+\rho_{11}H_{11}H_{11}^{\dagger}+\rho_{21}H_{21}H_{21}^{\dagger}\Big) \\
& + C_{12}+C_{21}, \qquad (11)
\end{aligned}$$

$$\begin{aligned}
R_1+2R_2 \le\ & \log\det\Big(I_{N_2}+\rho_{22}H_{22}H_{22}^{\dagger}-\rho_{22}\rho_{21}H_{22} \\
& \qquad H_{21}^{\dagger}\big(I_{N_1}+\rho_{21}H_{21}H_{21}^{\dagger}\big)^{-1}H_{21}H_{22}^{\dagger}\Big) \\
& + \log\det\Big(I_{N_1}+\rho_{11}H_{11}H_{11}^{\dagger}+\rho_{21}H_{21}H_{21}^{\dagger} \\
& \qquad -\rho_{11}\rho_{12}H_{11}H_{12}^{\dagger} \\
& \qquad \big(I_{N_2}+\rho_{12}H_{12}H_{12}^{\dagger}\big)^{-1}H_{12}H_{11}^{\dagger}\Big) \\
& + \log\det\Big(I_{N_2}+\rho_{22}H_{22}H_{22}^{\dagger}+\rho_{12}H_{12}H_{12}^{\dagger}\Big) \\
& + C_{21}+C_{12}, \qquad (12)
\end{aligned}$$

$$\begin{aligned}
2R_1+R_2 \le\ & \log\det\Bigg(I_{N_1+N_2}+\begin{bmatrix}\sqrt{\rho_{22}}H_{22}\\ \sqrt{\rho_{21}}H_{21}\end{bmatrix}\big(I_{M_2} \\
& \qquad -\rho_{21}H_{21}^{\dagger}\big(I_{N_1}+\rho_{21}H_{21}H_{21}^{\dagger}\big)^{-1}H_{21}\big) \\
& \qquad \Big[\sqrt{\rho_{22}}H_{22}^{\dagger}\ \sqrt{\rho_{21}}H_{21}^{\dagger}\Big]+\begin{bmatrix}\sqrt{\rho_{12}}H_{12}\\ \sqrt{\rho_{11}}H_{11}\end{bmatrix} \\
& \qquad \Big[\sqrt{\rho_{12}}H_{12}^{\dagger}\ \sqrt{\rho_{11}}H_{11}^{\dagger}\Big]\Bigg) \\
& + \log\det\Big(I_{N_1}+\rho_{11}H_{11}H_{11}^{\dagger}+\rho_{21}H_{21}H_{12}^{\dagger}\Big) \\
& + C_{21}, \qquad (13)
\end{aligned}$$

$$\begin{aligned}
R_1+2R_2 \le\ & \log\det\Bigg(I_{N_1+N_2}+\begin{bmatrix}\sqrt{\rho_{11}}H_{11}\\ \sqrt{\rho_{12}}H_{12}\end{bmatrix}\big(I_{M_1} \\
& \qquad -\rho_{12}H_{12}^{\dagger}\big(I_{N_2}+\rho_{12}H_{12}H_{12}^{\dagger}\big)^{-1}H_{12}\big) \\
& \qquad \Big[\sqrt{\rho_{11}}H_{11}^{\dagger}\ \sqrt{\rho_{12}}H_{12}^{\dagger}\Big]+\begin{bmatrix}\sqrt{\rho_{21}}H_{21}\\ \sqrt{\rho_{22}}H_{22}\end{bmatrix} \\
& \qquad \Big[\sqrt{\rho_{21}}H_{21}^{\dagger}\ \sqrt{\rho_{22}}H_{22}^{\dagger}\Big]\Bigg) \\
& + \log\det\Big(I_{N_2}+\rho_{22}H_{22}H_{22}^{\dagger}+\rho_{12}H_{12}H_{12}^{\dagger}\Big) \\
& + C_{12}. \qquad (14)
\end{aligned}$$

The following theorem shows that the capacity region of a two-user MIMO IC with limited receiver cooperation is within N_1+N_2 bits of $\mathcal{R}_o$.

Theorem 1: The capacity region $\mathbb{C}_{RC}$ *for an* (M_1, N_1, M_2, N_2) *two-user MIMO IC with limited receiver cooperation is bounded from outside and inside as*

$$\mathcal{R}_o \ominus ([0, N_1+N_2]\times[0, N_1+N_2]) \subseteq \mathbb{C}_{RC} \subseteq \mathcal{R}_o. \quad (15)$$

Thus, the inner and outer bounds are within N_1+N_2 *bits.*

Note that the gap of N_1+N_2 bits is the worst-case. It is loose when the SNR is low and only informative at high SNR.

Outer Bound: The complete proof that $\mathcal{R}_o$ is an outer bound for the capacity region of the two-user MIMO IC with limited receiver cooperation is given in Appendix B.

Note that (5), (6) and (10) are cut-set based upper bounds. The other bounds are obtained based on genie-aided strategies making use of Fano's inequality, the data processing inequality, and the fact that Gaussian distribution maximizes the entropy. The detailed derivations are given in Appendix B. For evaluating and outer-bounding the joint entropies, the following result on the monotonicity of a function, is used extensively in the proof.

Lemma 1: Let $L(K, S)$ *be defined as*

$$L(K, S) \triangleq K - KS(I_N + S^{\dagger}KS)^{-1}S^{\dagger}K, \quad (16)$$

for some $M\times M$ *p.s.d. Hermitian matrix* K *and some* $M\times N$ *matrix* S. *Then if* $0 \preceq K_1 \preceq K_2$ *for some Hermitian matrices* K_1 *and* K_2*, we have*

$$L(K_1, S) \preceq L(K_2, S). \quad (17)$$

Proof: The proof uses the matrix differential and the eigenvalue continuity properties (e.g., Wielandt-Hoffman Theorem), and can be seen in the proof of [32, Lemma 11]. ■

Inner Bound: Here, we will give a brief description of the achievability strategy. The complete proof can be found in Appendix C.

The achievability scheme is based on a two-round strategy, similar to that used in [16] for SISO interference channels. It consists of two parts: 1) the transmission scheme and 2) the cooperation protocol.

1) Transmission Scheme:

Each transmitter T_i splits its own message into private and common sub-messages and sends

$$X_i = X_{ip} + X_{ic}, \quad (18)$$

where $X_{ip} \sim \mathsf{CN}(0, Q_{ip})$ denotes the private message, and $X_{ic} \sim \mathsf{CN}(0, Q_{ic})$ denotes the common message. We assume that X_{ip} and X_{ic} are independent with

$$Q_{ip} = I_{M_i} - \sqrt{\rho_{ij}}H_{ij}^{\dagger}(I_{N_j} + \rho_{ij}H_{ij}H_{ij}^{\dagger})^{-1}\sqrt{\rho_{ij}}H_{ij}, \quad (19)$$

and

$$Q_{ic} = I_{M_i} - Q_{ip}, \quad (20)$$

for $i \in \{1, 2\}$.

It is shown in Corollary 6 in Appendix A that $Q_{ip} \succeq 0$ and $Q_{ic} \succeq 0$. Further, this message split is such that a private signal is received at the other receiver with constant power. More specifically, as discussed in Corollary 5 in Appendix A, received signal at receiver D_j corresponding to the private signal from transmitter T_i is below the noise floor.

Remark 1: Note that the power allocation in (19)-(20) *is different from that given in [16] even for a SISO channel. In [16] different achievability schemes were given for weak, mixed, and strong interference regimes. Here what we propose is a single choice of parameters for all interference regimes. For the special case of SISO IC, the above scheme constitutes an alternative choice of variances to those proposed in [16].*

Remark 2: Note that the covariance matrices for the public and private messages proposed in this paper are different from those in [4]. With the expressions in [4], we were not *able to get an inner bound within constant gap from the outer bound. The covariance of the public message in our work is* $\rho_{ij} H_{ij}^{\dagger}(I_{N_j} + \rho_{ij} H_{ij} H_{ij}^{\dagger})^{-1} H_{ij}$*, while in [4], it is* $\frac{1}{M_i}(I_{M_i} + \rho_{ij} H_{ij}^{\dagger} H_{ij})^{-1}$*. For example, for a SISO channel the variance of the public message in our work is* $\frac{\rho_{ij}|H_{ij}|^2}{1+\rho_{ij}|H_{ij}|^2}$*, while in [4], it is* $\frac{1}{1+\rho_{ij}|H_{ij}|^2}$*. Using a covariance matrix of* $I_{M_i} - \rho_{ij} H_{ij}^{\dagger}(I_{N_j} + \rho_{ij} H_{ij} H_{ij}^{\dagger})^{-1} H_{ij}$ *for the private message, we were able to make* $h(Y_i|X_i, U_j)$ *outer-bounded by a constant, which helped in the analysis.*

2) Cooperation Protocol:

We use a two-round cooperation protocol similar to that in [16]. In the first round, D_j quantizes its received signal and sends out the bin index. And then in the second round, D_i $i \neq j$ $(i, j) \in \{1, 2\}$ receives this side information and decodes its desired messages (its own message plus the other's public message). After decoding, D_i randomly bins the decoded public messages, and sends the bin indices to D_j. Finally, D_j decodes its message. In this two-round strategy, $STG_{j\to i\to j}$, the processing order is: D_j quantize-and-bins, D_i decode-and-bins and finally D_j decodes. Its achievable rate region is denoted by $\mathcal{R}_{j\to i\to j}$. By time-sharing, the rate region $\mathcal{R} \triangleq conv\{\mathcal{R}_{2\to 1\to 2} \cup \mathcal{R}_{1\to 2\to 1}\}$, i.e. the convex hull of the union of the two rate regions is achievable.

For simplicity, we consider strategy $\mathcal{R}_{2\to 1\to 2}$. D_2 does not decode messages immediately upon receiving its signal. It first quantizes its signal by a pre-generated Gaussian quantization codebook with certain distortion and then sends out a bin index determined by a pre-generated binning function. It sets the distortion level equal to the aggregate power level of the noise and T_2's private signal. D_1 decodes the two common messages and its own private message, by searching in transmitters' codebooks for a codeword triplet that is jointly typical with its received signal and some quantization point (codeword) in the given bin after retrieving the receiver-cooperative side information (the bin index). After D_1 decodes, it uses two pre-generated binning functions to bin the two common messages and sends out these two bin indices to D_2. After receiving these two bin indices, D_2 decodes the two common messages and its own private message, by searching in the corresponding bins and T_2's private codebook for a codeword triplet that is jointly typical with its received signal.

Although the cooperation protocol is similar to that in [16], the distortion function used for the quantization of the received signal needs to be extended to the case of multiple antennas. We here describe the distortion function for $STG_{2\to 1\to 2}$. For the quantization, we use the quantization codebook satisfying

$$\hat{Y}_2 \triangleq Y_2 + \hat{Z}_2, \tag{21}$$

where the distortion $\hat{Z}_2 \sim \mathsf{CN}(0, \Delta)$ with

$$\Delta = I_{N_2} + \rho_{22} H_{22} Q_{2p} H_{22}^{\dagger}. \tag{22}$$

D_2 then sends the bin index to D_1. The rate loss due to this quantization, ξ, is given as

$$\begin{aligned}
\xi &\triangleq I(\hat{Y}_2; Y_2 | X_{1c}, X_1, X_{2c}, Y_1)\\
&= h\left(\sqrt{\rho_{22}} H_{22} X_{2p} + Z_2 + \hat{Z}_2 | \sqrt{\rho_{21}} H_{21} X_{2p} + Z_1\right) - h\left(\hat{Z}_2\right)\\
&= h\left(\sqrt{\rho_{22}} H_{22} X_{2p} + Z_2 + \hat{Z}_2, \sqrt{\rho_{21}} H_{21} X_{2p} + Z_1\right)\\
&\quad - h\left(\sqrt{\rho_{21}} H_{21} X_{2p} + Z_1\right) - h\left(\hat{Z}_2\right)\\
&= \log\det\\
&\quad \begin{bmatrix} I_{N_2} + \Delta + \rho_{22} H_{22} Q_{2p} H_{22}^{\dagger} & \sqrt{\rho_{21}\rho_{22}} H_{22} Q_{2p} H_{21}^{\dagger} \\ [0.3pc] \sqrt{\rho_{21}\rho_{22}} H_{21} Q_{2p} H_{22}^{\dagger} & I_{N_1} + \rho_{21} H_{21} Q_{2p} H_{21}^{\dagger} \end{bmatrix}\\
&\quad - \log\det\left(I_{N_1} + \rho_{21} H_{21} Q_{2p} H_{21}^{\dagger}\right) - \log\det(\Delta)\\
&= \log\det\Big(I_{N_2} + \Delta + \rho_{22} H_{22} Q_{2p} H_{22}^{\dagger}\\
&\qquad - \sqrt{\rho_{21}\rho_{22}} H_{22} Q_{2p} H_{21}^{\dagger} \left(I_{N_1} + \rho_{21} H_{21} Q_{2p} H_{21}^{\dagger}\right)^{-1}\\
&\qquad \sqrt{\rho_{21}\rho_{22}} H_{21} Q_{2p} H_{22}^{\dagger}\Big) - \log\det(\Delta)\\
&= \log\det\Big(I_{N_2} + \Delta + \rho_{22} H_{22}\Big(Q_{2p} - \rho_{21} Q_{2p} H_{21}^{\dagger}\\
&\qquad \left(I_{N_1} + \rho_{21} H_{21} Q_{2p} H_{21}^{\dagger}\right)^{-1} H_{21} Q_{2p}\Big) H_{22}^{\dagger}\Big)\\
&\quad - \log\det(\Delta)\\
&\overset{(a)}{\leq} \log\det\left(I_{N_2} + \Delta + \rho_{22} H_{22} Q_{2p} H_{22}^{\dagger}\right) - \log\det(\Delta)\\
&= \log\det(2\Delta) - \log\det(\Delta)\\
&= N_2.
\end{aligned} \tag{23}$$

where (a) follows from Lemma 2 and $Q_{2p} \succeq Q_{2p} - \rho_{21} Q_{2p} H_{21}^{\dagger}\left(I_{N_1} + \rho_{21} H_{21} Q_{2p} H_{21}^{\dagger}\right)^{-1} H_{21} Q_{2p} \succeq 0$ by Lemma 1. Thus, we see that the rate loss ξ is upper bounded by the constant N_2. That is, replacing $\hat{Y}_2$ by Y_2 incurs at most N_2 bits.

Remark 3: The distortion specified in (22) *may not be optimal. The achievable rates can be further improved if we optimize over all possible distortions. For instance, if the cooperative link capacity is relatively large, we could lower the distortion level to achieve a better description of the received signals. With the expression of* Δ *in* (22)*, however, we can show that the achievable rate region is within a constant number of bits to the capacity region for any channel parameters.*

Considering the convex hull of the union of the achievable rate regions by the strategies $STG_{2\to 1\to 2}$ and $STG_{1\to 2\to 1}$ for MIMO IC, we show in Appendix C that we can get the achievable rate region for the general MIMO IC. Moreover, we will show in Appendix C that two of the bounds in each region will not play a role in the convex hull. This is because if any of these bounds is active, the bound on $R_1 + R_2$ is active and thus following the arguments in [16] we get that these bounds will not be active when we take the convex hull of the two regions. This is illustrated in Figure 1, where it is seen that two of the bounds in each region are not dominant when a convex hull of the regions is taken.

Having considered the inner and outer bounds for the capacity region of the two-user IC with limited receiver cooperation, we have shown that the inner bound and the outer bound are within $N_1 + N_2$ bits, thus finding the capacity region of the two-user IC with limited receiver cooperation, approximately.

The authors of [16] found the capacity region for the SISO IC with limited receiver cooperation within 2 bits. Theorem 1 generalizes the result to find the capacity region of MIMO IC with limited receiver cooperation within $N_1 + N_2$ bits.

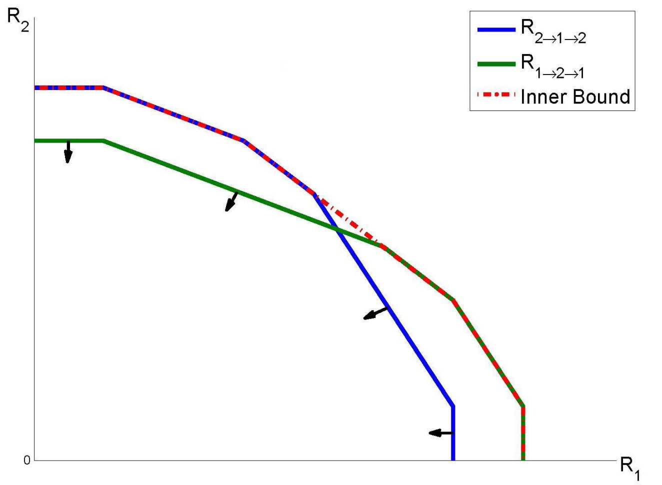


Fig. 1. Time sharing of two regions $\mathcal{R}_{2\to1\to2}$ and $\mathcal{R}_{1\to2\to1}$. The four lines with arrow marks indicate that the corresponding bounds are not active when the convex hull of the two regions is taken.

In Figure 2, we compare the inner bound given in [16, Sec. V] for a SISO IC to that obtained in Lemma 9 through some numerical examples. Since the outer bounds are the same for SISO, we only plot one outer bound. Let $\mathsf{SNR}_i = \rho_{ii}|H_{ii}|^2$ and $\mathsf{INR}_i = \rho_{ji}|H_{ji}|^2$ for $j \neq i$. In Figure 2(a), we consider a weak interference regime ($\mathsf{SNR}_1 \geq \mathsf{INR}_2$ and $\mathsf{SNR}_2 \geq \mathsf{INR}_1$) with $C_{21} = 1.1$, $C_{12} = 1.1$, $\mathsf{SNR}_1 = 5$, $\mathsf{SNR}_2 = 5$, $\mathsf{INR}_1 = 2$ and $\mathsf{INR}_2 = 2$. In Figure 2(b), we consider strong interference regime ($\mathsf{SNR}_1 \leq \mathsf{INR}_2$ and $\mathsf{SNR}_2 \leq \mathsf{INR}_1$) with $C_{21} = 6$, $C_{12} = 11$, $\mathsf{SNR}_1 = 1000$, $\mathsf{SNR}_2 = 1500$, $\mathsf{INR}_1 = 4000$ and $\mathsf{INR}_2 = 10000$. In Figure 2(c), we consider a mixed interference regime ($\mathsf{SNR}_1 \geq \mathsf{INR}_2$ and $\mathsf{SNR}_2 \leq \mathsf{INR}_1$) with $C_{21} = 6$, $C_{12} = 11$, $\mathsf{SNR}_1 = 9000$, $\mathsf{SNR}_2 = 1500$, $\mathsf{INR}_1 = 5000$ and $\mathsf{INR}_2 = 1000$. We see from Figure 2 that the inner bounds are comparable. In the above example for weak interference channel, the strategy in this paper gives better achievable region than that in [16, Sec. V-C].

In Figure 3, we see the improvement in the capacity region (outer bound) for a MIMO IC with limited receiver cooperation. The parameters chosen for limited cooperation are $M_1 = N_2 = 3$, $M_2 = N_1 = 4$, $\rho_{11} = \rho_{22} = \rho_{12} = \rho_{21} = 10^8$, $C_{21} = 21$, $C_{12} = 15$,

$$H_{11} = \begin{bmatrix} 0.3096 & 0.1974 & 0.1080 \\ 0.3066 & 0.4470 & 0.3885 \\ 0.3595 & 0.6582 & 0.9854 \\ 0.4595 & 0.6582 & 0.4566 \end{bmatrix},$$

$$H_{22} = \begin{bmatrix} 0.9070 & 0.6690 & 0.6854 & 0.6565 \\ 0.6067 & 0.9480 & 0.6585 & 0.6645 \\ 0.4465 & 0.6167 & 0.6845 & 0.3685 \end{bmatrix},$$

$$H_{21} = \begin{bmatrix} 0.8660 & 0.9767 & 0.4595 & 0.6582 \\ 0.8603 & 0.5850 & 0.6582 & 0.9854 \\ 0.3066 & 0.4470 & 0.6585 & 0.3885 \\ 0.3066 & 0.6167 & 0.4470 & 0.3885 \end{bmatrix},$$

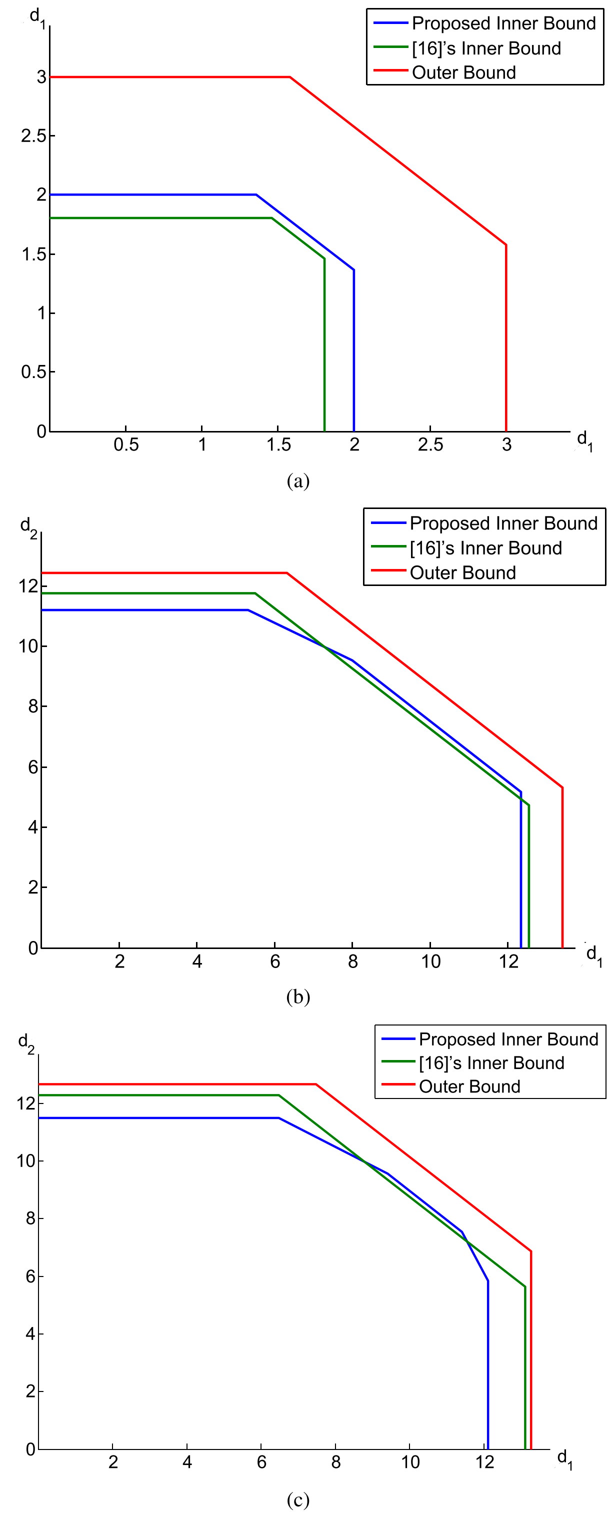


Fig. 2. Comparison of the inner bound in this paper with that in [16] for SISO interference channels. (a) Weak Interference. (b) Strong Interference. (c) Mixed Interference.

and

$$H_{12} = \begin{bmatrix} 0.1890 & 0.7650 & 0.3864 \\ 0.6678 & 0.2880 & 0.3867 \\ 0.4886 & 0.7904 & 0.2684 \end{bmatrix}. \quad (24)$$

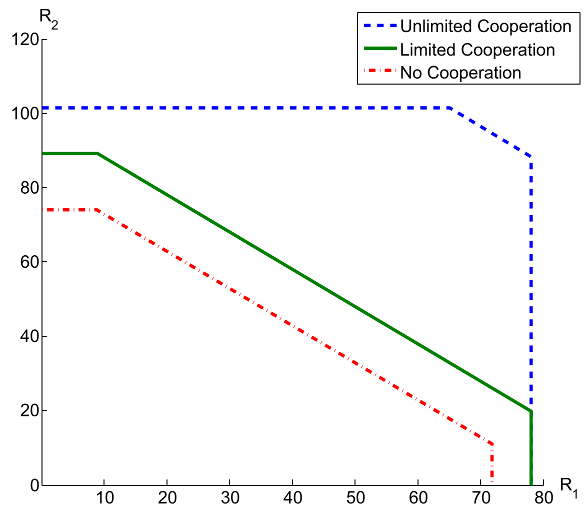


Fig. 3. The outer bounds for a MIMO IC with no cooperation, limited cooperation and unlimited cooperation.

IV. DoF and GDoF Regions

In this section, we will use the DoF and GDoF regions to characterize the capacity region of the MIMO IC with limited receiver cooperation in the limit of high SNR. We first describe our results on the DoF region of the two-user MIMO IC with limited receiver cooperation, and then proceed to the results on GDoF.

A. DoF Region

The DoF characterizes the simultaneously accessible fractions of spatial and signal-level dimensions (per channel use) by the two users when all the average channel parameters are an exponent of a nominal SNR parameter. Thus, we assume that

$$\lim_{\log \mathsf{SNR}\to\infty} \frac{\log C_{ij}}{\log \mathsf{SNR}} = \beta_{ij}, \text{ and}$$
$$\lim_{\log \mathsf{SNR}\to\infty} \frac{\log \rho_{ij}}{\log \mathsf{SNR}} = 1, \tag{25}$$

where $\beta_{12}, \beta_{21} \in \mathbb{R}^+$.

The DoF region is defined as the region formed by the set of all (d_1, d_2) such that $(d_1 \log \mathsf{SNR} - o(\log \mathsf{SNR}), d_2 \log \mathsf{SNR} - o(\log \mathsf{SNR}))$[1] is inside the capacity region. Further, the DoF is the maximum d such that (d, d) is in the DoF region.

The elements of the channel matrix are chosen from a random and continuous space and consequently, the channel matrices are of full rank with probability 1. As a result we will have the DoF and GDoF (next subsection) regions with probability 1 over the randomness of channel matrices.

It has been shown that for basic networks (such as the MAC, BC and the IC) with a reasonably small number of users, the DoF analysis offers good insight on the performance at moderate SNR [35]. However, for cellular networks with many nodes, it may be necessary to consider the saturation effect on the spectral efficiency at high SNR [25], since it is infeasible to obtain precise global channel state information.

[1] $a = o(\log \mathsf{SNR})$ indicates that $\lim_{\mathsf{SNR}\to\infty} \frac{a}{\log \mathsf{SNR}} = 0$.

In this subsection, we find the DoF region for the two-user MIMO IC with limited receiver cooperation. We use the approximate capacity region characterization in Theorem 1 to get the DoF region for the two-user MIMO IC as follows.

Theorem 2: The DoF region for a general MIMO IC with limited receiver cooperation is given as follows:

$$d_1 \le \min(M_1, N_1) + \min\{\min\{N_2, (M_1 - N_1)^+\}, \beta_{21}\}, \tag{26}$$
$$d_2 \le \min(M_2, N_2) + \min\{\min\{N_1, (M_2 - N_2)^+\}, \beta_{12}\}, \tag{27}$$
$$d_1 + d_2 \le \min\{N_1, (M_1 - N_2)^+ + M_2\} + \min\{N_2, (M_2 - N_1)^+ + M_1\} + \beta_{12} + \beta_{21}, \tag{28}$$
$$d_1 + d_2 \le \min\{N_1, (M_1 - N_2)^+\} + \min\{N_2, M_1 + M_2\} + \beta_{12}, \tag{29}$$
$$d_1 + d_2 \le \min\{N_2, (M_2 - N_1)^+\} + \min\{N_1, M_1 + M_2\} + \beta_{21}, \tag{30}$$
$$d_1 + d_2 \le \min\{N_1 + N_2, M_1 + M_2\}, \tag{31}$$
$$2d_1 + d_2 \le \min\{N_2, (M_2 - N_1)^+ + M_1\} + \min\{N_1, (M_1 - N_2)^+\} + \min\{N_1, M_1 + M_2\} + \beta_{12} + \beta_{21}, \tag{32}$$
$$d_1 + 2d_2 \le \min\{N_1, (M_1 - N_2)^+ + M_2\} + \min\{N_2, (M_2 - N_1)^+\} + \min\{N_2, M_2 + M_1\} + \beta_{12} + \beta_{21}, \tag{33}$$
$$2d_1 + d_2 \le \min\{N_1 + N_2, M_1\} + \min\{N_1, M_1 + M_2\} + \beta_{21}, \tag{34}$$
$$d_1 + 2d_2 \le \min\{N_1 + N_2, M_2\} + \min\{N_2, M_1 + M_2\} + \beta_{12}. \tag{35}$$

Proof: The proof can be found in Appendix C. ■

Corollary 1: The symmetric DoF region where $\beta_{12} = \beta_{21} = \beta$, $N_1 = N_2 = N$, and $M_1 = M_2 = M$, is given as follows:

For $M \le N$:

$$d_1 \le M,$$
$$d_2 \le M,$$
$$d_1 + d_2 \le N + \beta; \tag{36}$$

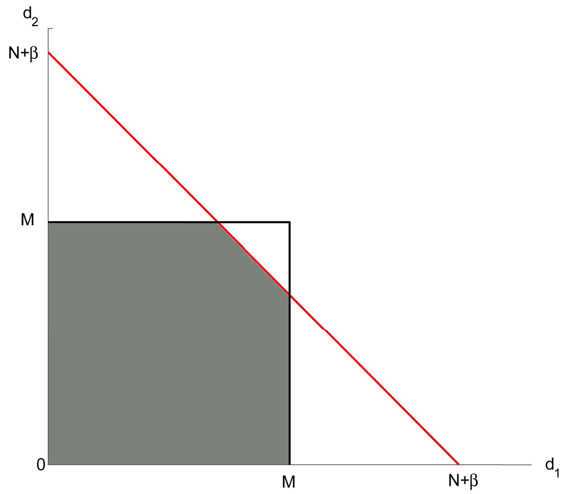
For $2N \le M$:

$$d_1 \le N + \beta,$$
$$d_2 \le N + \beta,$$
$$d_1 + d_2 \le 2N; \tag{37}$$

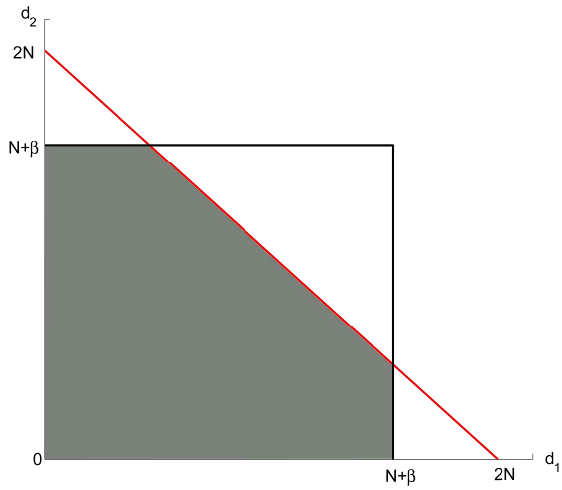
For $N \le M \le 2N$:

$$d_1 \le \min\{M, N + \beta\},$$
$$d_2 \le \min\{M, N + \beta\},$$
$$d_1 + d_2 \le \min\{M + \beta, 2N\},$$
$$2d_1 + d_2 \le N + M + \beta,$$
$$d_1 + 2d_2 \le N + M + \beta. \tag{38}$$

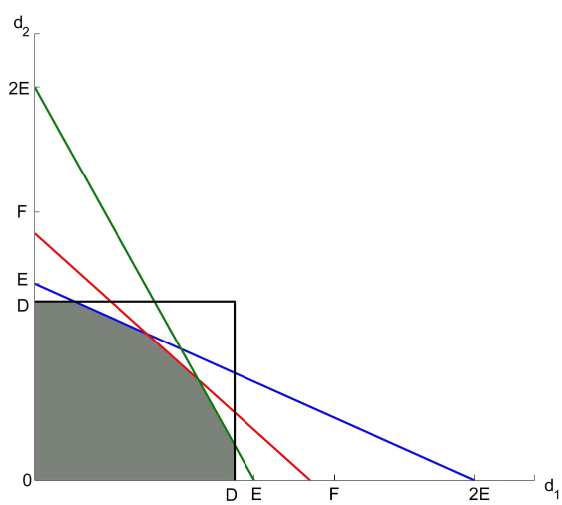
These three cases are illustrated in Figure 4.



(a)



(b)



(c)

Fig. 4. The DoF region for symmetric MIMO IC with limited receiver cooperation (grey areas). (a) $M \le N$. (b) $2N \le M$. (c) $N \le M \le 2N$, where $D = \min(M, N+\beta)$, $E = \frac{M+N+\beta}{2}$, and $F = \min(M+\beta, 2N)$.

Corollary 2: For the symmetric DoF region where $\beta_{12} = \beta_{21} = \beta$, $N_1 = N_2 = N$, and $M_1 = M_2 = M$, cooperation improve the DoF region for $\beta \le \min\{N, (2M-N)^+\}$.

Proof: For $M \le N$ it can be seen from (36) that the cooperation improves the DoF region for $\beta \le (2M-N)^+ = \min\{N, (2M-N)^+\}$.

Also, for $2N \le M$ it can be seen from (37) that the cooperation improves the DoF region for $\beta \le N = \min\{N, (2M-N)^+\}$.

For $N \le M \le 2N$, we consider the following four cases.

Case 1 - $\beta \le M-N$, $\beta \le 2N-M$: In this case, the symmetric DoF region reduces to

$$\begin{aligned} d_1 &\le N+\beta,\\ d_2 &\le N+\beta,\\ d_1+d_2 &\le \beta,\\ 2d_1+d_2 &\le N+M+\beta,\\ d_1+2d_2 &\le N+M+\beta. \end{aligned} \tag{39}$$

In this region, β is always less than $\min\{N, (2M-N)^+\}$ because $\beta \le M-N \le N = \min\{N, (2M-N)^+\}$. Hence increasing β always enlarges the region.

Case 2 - $\beta \ge M-N$, $\beta \le 2N-M$: In this case, the symmetric DoF region reduces to

$$\begin{aligned} d_1 &\le M,\\ d_2 &\le M,\\ d_1+d_2 &\le M+\beta,\\ 2d_1+d_2 &\le N+M+\beta,\\ d_1+2d_2 &\le N+M+\beta. \end{aligned} \tag{40}$$

In this region, β is always less than $\min\{N, (2M-N)^+\}$ because $\beta \le 2N-M \le N = \min\{N, (2M-N)^+\}$. In this case, increasing β always enlarges the region. According to Figure 4(c), while $\beta \le 2N-M$, we get $2E \le 3N$ and $F \le 2N$ which indicates none of the red, green and blue lines could include the point $(d_1, d_2) = (M, M)$ below them. Also, increasing β leads to the increase of E and F in Figure 4(c) and as a result, enlarges the symmetric DoF region.

Case 3 - $\beta \le M-N$, $\beta \ge 2N-M$: In this case, the symmetric DoF region reduces to

$$\begin{aligned} d_1 &\le N+\beta,\\ d_2 &\le N+\beta,\\ d_1+d_2 &\le 2N,\\ 2d_1+d_2 &\le N+M+\beta,\\ d_1+2d_2 &\le N+M+\beta. \end{aligned} \tag{41}$$

In this region, β is always less than $\min\{N, (2M-N)^+\}$ because $\beta \le M-N \le N = \min\{N, (2M-N)^+\}$. In this case, increasing β always enlarges the region. According to Figure 4(c), when $\beta \le M-N$, we get $D, E \le M \le 2N = F$ and also, increasing β leads to the increase of D and E in Figure 4(c) and as a result, enlarges the symmetric DoF region.

Case 4 - $\beta \ge M-N$, $\beta \ge 2N-M$: In this case, the symmetric DoF region reduces to

$$\begin{aligned} d_1 &\le M,\\ d_2 &\le M,\\ d_1+d_2 &\le 2N,\\ 2d_1+d_2 &\le N+M+\beta,\\ d_1+2d_2 &\le N+M+\beta. \end{aligned} \tag{42}$$

In this region, changing β only changes E in Figure 4(c). Also, we can easily see that the black line and red line intersects at $(d_1, d_2) = (M, 2N - M)$. The green line includes this intersection point when $\beta \geq N$ and will be below this point when $\beta \leq N$ which means increasing β improves the DoF region until $\beta \leq N = \min\{N, (2M - N)^+\}$. ■

B. GDoF Region

The notion of GDoF generalizes the DoF metric by additionally emphasizing the signal level as a signaling dimension. It characterizes the simultaneously accessible fractions of spatial and signal-level dimensions (per channel use) by the two users when all the average channel parameters vary as exponents of a nominal SNR parameter as follows

$$\lim_{\log \mathsf{SNR} \to \infty} \frac{\log C_{ij}}{\log \mathsf{SNR}} = \beta_{ij},$$
$$\lim_{\log \mathsf{SNR} \to \infty} \frac{\log \rho_{ij}}{\log \mathsf{SNR}} = \begin{cases} 1, & \text{if } i = j \\ \alpha, & \text{if } i \neq j \end{cases}, \tag{43}$$

where $\alpha, \beta_{12}, \beta_{21} \in \mathbb{R}^+$.

The GDoF region is defined as the region formed by the set of all (d_1, d_2) such that $(d_1 \log \mathsf{SNR} - o(\log \mathsf{SNR}), d_2 \log \mathsf{SNR} - o(\log \mathsf{SNR}))$ is inside the capacity region. Further, the GDoF is the maximum d such that (d, d) is in the GDoF region. Thus, both the GDoF region and GDoF are functions of link quality scaling exponent α.

Next we present our results on the GDoF region for the two-user MIMO IC with limited receiver cooperation. For the general case, the computation of GDoF region is hard and thus we will only consider the case that $M_1 = M_2 = N_1 = N_2 = M$. We also assume that $\beta_{21} = \beta_{12} = \beta$. With these assumptions, the GDoF region for the two user MIMO IC with limited receiver cooperation is given in the following Theorem.

Theorem 3: The GDoF region for a two-user symmetric MIMO IC with limited receiver cooperation is equivalent to the convex hull of the:

$$d_1 \leq M + \min\{(\alpha - 1)^+ M, \beta\}, \tag{44}$$
$$d_2 \leq M + \min\{(\alpha - 1)^+ M, \beta\}, \tag{45}$$
$$d_1 + d_2 \leq 2M \max\{(1 - \alpha)^+, \alpha\} + 2\beta, \tag{46}$$
$$d_1 + d_2 \leq (1 - \alpha)^+ M + M \max\{1, \alpha\} + \beta, \tag{47}$$
$$d_1 + d_2 \leq 2M \max\{1, \alpha\}, \tag{48}$$
$$d_1 + 2d_2 \leq M \max\{(1 - \alpha)^+, \alpha\} + (1 - \alpha)^+ M + M \max\{1, \alpha\} + 2\beta, \tag{49}$$
$$2d_1 + d_2 \leq M \max\{(1 - \alpha)^+, \alpha\} + (1 - \alpha)^+ M + M \max\{1, \alpha\} + 2\beta, \tag{50}$$
$$d_1 + 2d_2 \leq M \max\{(2 - \alpha)^+, \alpha\} + M \max\{1, \alpha\} + \beta, \tag{51}$$
$$2d_1 + d_2 \leq M \max\{(2 - \alpha)^+, \alpha\} + M \max\{1, \alpha\} + \beta. \tag{52}$$

Proof: The proof can be found in Appendix D. ■

Corollary 3: The GDoF for a two-user MIMO IC with limited receiver cooperation, when $M_1 = M_2 = N_1 = N_2 = M$ and $\beta_{21} = \beta_{12} = \beta$ is given as

$$\begin{aligned} \text{GDOF}_{RC} &= \min\{M + \min\{(\alpha - 1)^+ M, \beta\}, M \max\{(1 - \alpha)^+, \alpha\} \\ &+\beta, \frac{1}{2}(1 - \alpha)^+ M + \frac{1}{2} M \max\{1, \alpha\} + \frac{1}{2}\beta, M \\ &\max\{1, \alpha\}, \frac{1}{3} M \max\{(1 - \alpha)^+, \alpha\} + \frac{1}{3}(1 - \alpha)^+ M \\ &+\frac{1}{3} M \max\{1, \alpha\} + \frac{2}{3}\beta, \frac{1}{3} M \max\{(2 - \alpha)^+, \alpha\} \\ &+\frac{1}{3} M \max\{1, \alpha\} + \frac{1}{3}\beta\}. \end{aligned} \tag{53}$$

Since the GDoF in Corollary 3 is the minimum of many terms, we evaluate the minimum in (53) to reduce the expression of GDoF as follows.

For $0 \leq \beta \leq \frac{M}{2}$:

$$\text{GDoF}_{RC} = \begin{cases} M, & \text{if } 0 \leq \alpha \leq \frac{\beta}{M}, \\ M(1 - \alpha)^+ + \beta, & \text{if } \frac{\beta}{M} \leq \alpha \leq \frac{1}{2}, \\ M\alpha + \beta, & \text{if } \frac{1}{2} \leq \alpha \leq \frac{2}{3} - \frac{\beta}{3M}, \\ \frac{1}{2}(M(2 - \alpha)^+ + \beta), & \text{if } \frac{2}{3} - \frac{\beta}{3M} \leq \alpha \leq 1, \\ \frac{1}{2}(M\alpha + \beta), & \text{if } 1 \leq \alpha \leq 2 + \frac{\beta}{M}, \\ M + \beta, & \text{if } 2 + \frac{\beta}{M} \leq \alpha. \end{cases} \tag{54}$$

For $\frac{M}{2} \leq \beta \leq M$:

$$\text{GDoF}_{RC} = \begin{cases} M, & \text{if } 0 \leq \alpha \leq \frac{\beta}{M}, \\ \frac{1}{2}(M(2 - \alpha)^+ + \beta), & \text{if } \frac{\beta}{M} \leq \alpha \leq 1, \\ \frac{1}{2}(M\alpha + \beta), & \text{if } 1 \leq \alpha \leq 2 + \frac{\beta}{M}, \\ M + \beta, & \text{if } 2 + \frac{\beta}{M} \leq \alpha. \end{cases} \tag{55}$$

For $M \leq \beta$:

$$\text{GDoF}_{RC} = \begin{cases} M, & \text{if } 0 \leq \alpha \leq 1, \\ M\alpha, & \text{if } 1 \leq \alpha \leq \frac{\beta}{M}, \\ \frac{1}{2}(M\alpha + \beta), & \text{if } \frac{\beta}{M} \leq \alpha \leq 2 + \frac{\beta}{M}, \\ M + \beta, & \text{if } 2 + \frac{\beta}{M} \leq \alpha. \end{cases} \tag{56}$$

The authors of [1] found the GDoF for the two-user symmetric MIMO IC without cooperation as follows

$$\text{GDoF}_{NRC} = \begin{cases} M(1 - \alpha)^+, & \text{if } 0 \leq \alpha \leq \frac{1}{2}, \\ M\alpha, & \text{if } \frac{1}{2} \leq \alpha \leq \frac{2}{3}, \\ \frac{1}{2}(M(2 - \alpha)^+), & \text{if } \frac{2}{3} \leq \alpha \leq 1, \\ \frac{1}{2}M\alpha, & \text{if } 1 \leq \alpha \leq 2, \\ M, & \text{if } 2 \leq \alpha. \end{cases} \tag{57}$$

Figure 5 compares the GDoF for the two-user symmetric MIMO IC with and without receiver cooperation. In Figure 5(a), the "W"-curve obtained without cooperation delineates the very weak ($0 \leq \alpha \leq \frac{1}{2}$), weak ($\frac{1}{2} \leq \alpha \leq \frac{2}{3}$), moderate ($\frac{2}{3} \leq \alpha \leq 1$), strong ($1 \leq \alpha \leq 2$) and very strong ($\alpha \geq 2$) interference regimes. In the presence of weak collaboration ($0 \leq \beta \leq \frac{M}{2}$), the "W"-curve improves to another "W"-curve which delineates to extremely weak ($0 \leq \alpha \leq \frac{\beta}{M}$),

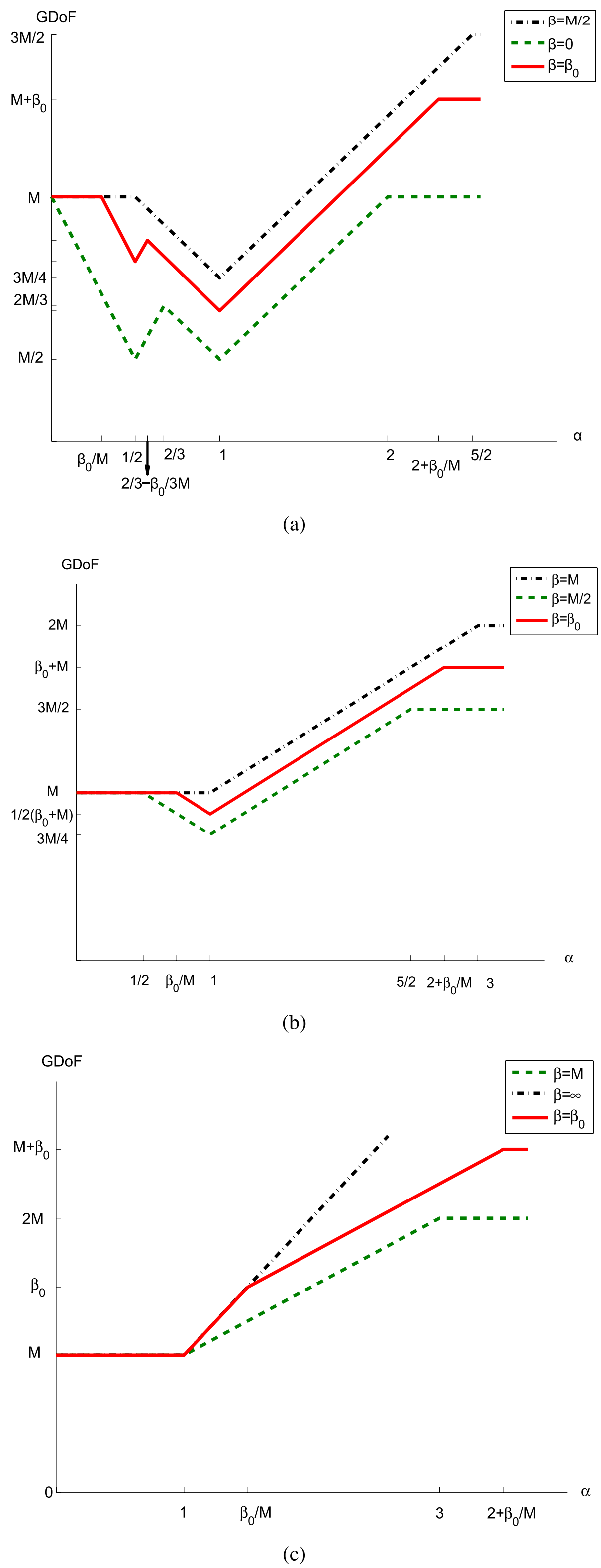


Fig. 5. GDoF for MIMO IC with limited receiver cooperation when all nodes have the same number of antennas M. (a) $0 \le \beta_0 \le \frac{M}{2}$. (b) $\frac{M}{2} \le \beta_0 \le M$. (c) $\beta_0 \ge M$.

very weak ($\frac{\beta}{M} \le \alpha \le \frac{1}{2}$), weak ($\frac{1}{2} \le \alpha \le \frac{2}{3} - \frac{\beta}{3M}$), moderate ($\frac{2}{3} - \frac{\beta}{3M} \le \alpha \le 1$), strong ($1 \le \alpha \le 2 + \frac{\beta}{M}$) and very strong ($2 + \frac{\beta}{M} \le \alpha$) interference regimes. In the presence of weak collaboration ($0 \le \beta \le \frac{M}{2}$), we see that the GDoF is strictly greater than that without collaboration for every $\alpha > 0$. The GDoF improvement indicates an unbounded gap in the corresponding capacity regions as the SNR goes to infinity.

For moderate collaboration ($\frac{M}{2} \le \beta \le M$), the "W"-curve improves to a "V"-curve which delineates to the very weak ($0 \le \alpha \le \frac{\beta}{M}$), weak ($\frac{\beta}{M} \le \alpha \le 1$), strong ($1 \le \alpha \le 2 + \frac{\beta}{M}$) and very strong ($2 + \frac{\beta}{M} \le \alpha$) interference regimes, and we see that the GDoF with collaboration is strictly greater than that without collaboration for $\alpha > 0$ similar to the weak collaboration.

For strong collaboration ($\beta \ge M$), the "W"-curve improves to an increasing curve which delineates to the very weak ($0 \le \alpha \le 1$), weak ($1 \le \alpha \le \frac{\beta}{M}$), strong ($\frac{\beta}{M} \le \alpha \le 2 + \frac{\beta}{M}$) and very strong ($2 + \frac{\beta}{M} \le \alpha$) interference regimes. The slopes of increase of GDoF with α changes at the border of these regimes.

We note that for a given M and α, increasing β improves the GDoF till $\beta = M\alpha$, after which there is no improvement in the GDoF since the GDoF at this point is the same as that with full cooperation. This can be seen also in the following corollary.

Corollary 4: The symmetric GDoF for a two-user MIMO IC with limited receiver cooperation, when $M_1 = M_2 = N_1 = N_2 = M$ and $\beta_{21} = \beta_{12} = \beta = M\alpha$ is equal to $M\max(1, \alpha)$ which is the same as that with full cooperation.

Proof: We only need to compare $M\max(1, \alpha)$ with all the bounds of the Corollary 3 and see that it is smaller or equal to all of them in Corollary 3, or

$$\begin{aligned}
M\max\{1, \alpha\} &\le M + \min\{(\alpha - 1)^+ M, \beta\},\\
M\max\{1, \alpha\} &\le M\max\{(1-\alpha)^+, \alpha\} + \beta,\\
M\max\{1, \alpha\} &\le \frac{1}{2}(1-\alpha)^+ M + \frac{1}{2}M\max\{1, \alpha\} + \frac{1}{2}\beta,\\
M\max\{1, \alpha\} &\le M\max\{1, \alpha\},\\
M\max\{1, \alpha\} &\le \frac{1}{3}M\max\{(1-\alpha)^+, \alpha\} + \frac{1}{3}(1-\alpha)^+ M\\
&\quad + \frac{1}{3}M\max\{1, \alpha\} + \frac{2}{3}\beta,\\
M\max\{1, \alpha\} &\le \frac{1}{3}M\max\{(2-\alpha)^+, \alpha\}\\
&\quad + \frac{1}{3}M\max\{1, \alpha\} + \frac{1}{3}\beta. \qquad (58)
\end{aligned}$$

Since all these expressions can be shown to hold, $M\max(1, \alpha)$ is achievable. Further, since $M\max(1, \alpha)$ is also an outer bound, the Corollary 4 holds. ■

V. Conclusion

This paper characterizes the approximate capacity region of the two-user MIMO ICs with limited receiver cooperation within $N_1 + N_2$ bits. This approximate capacity region is used to find the DoF region for the two user MIMO ICs with limited receiver cooperation. We also find the maximum amount of cooperation needed to achieve the outer bound of unlimited receiver cooperation. Further, the GDoF region is found for a two-user MIMO IC with equal antennas at all the nodes. With the GDoF region, we find that the

"W" curve without cooperation changes gradually to "V" curve with full cooperation. The cooperation improves the GDoF till the capacity of the cooperation link is of the order of $\alpha M \log \mathsf{SNR}$ when the GDoF reaches the GDoF with full cooperation.

This paper gives a specific strategy for Han-Kobayashi message splitting for two-user MIMO IC where the covariance matrices for the public and the private messages are properly designed. This scheme has been shown to achieve a region that is within a constant gap of the capacity region for MIMO IC with feedback in [32], and MIMO IC with limited receiver cooperation in this paper. We believe this strategy can help achieve general results for a variety of other scenarios where Han-Kobayashi message splitting is used. Investigation of these scenarios (such as transmitter cooperation, limited feedback, etc.) is an interesting future problem.

Appendix A
Useful Matrix Results

In this section, we will describe some results related to matrices that are extensively used in this paper.

A Hermitian matrix $M \in \mathbf{R}^{n\times n}$ is said to be positive semi-definite (p.s.d.) if z^*Mz is real and non-negative for all complex vectors z. A positive semi-definite matrix M is denoted as $M \succeq 0$. If in addition, z^*Mz is non-zero for $z \neq 0$, M is positive definite. Further, $M \succeq N$ for $n \times n$ Hermitian matrices M and N if $M - N \succeq 0$. We first show the monotone property for det(.) function.

Lemma 2: For two Hermitian positive definite matrices A and B of size $n \times n$, if $A \succeq B$, $\det(A) \geq \det(B)$.

Proof: The proof follows from [36, Corollary 7.7.4.b]. ∎

Also, the following lemma is useful in bounding the covariance matrix of the input signal from the point of view of positive definiteness.

Lemma 3: For Hermitian matrix A of size $n \times n$, $A \preceq I_n$ if and only if the largest eigenvalue is less than or equal to 1.

Proof: The proof follows from Theorem 7.7.3 of [36]. ∎

The following lemma will be useful to outer bound some entropy relations in the paper.

Lemma 4 ([32]): The following holds for any $M_i \times N_j$ matrix S

$$\det(I_{N_j} + S^\dagger S - S^\dagger S(I_{N_j} + S^\dagger S)^{-1} S^\dagger S) \leq 2^{N_j}. \quad (59)$$

As a corollary, we have the following result.

Corollary 5: If $Q_{ip} = I_{M_i} - \sqrt{\rho_{ij}} H_{ij}^\dagger (I_{N_j} + \rho_{ij} H_{ij} H_{ij}^\dagger)^{-1} \sqrt{\rho_{ij}} H_{ij}$, we can conclude from Lemma 4 that

$$\log\det(I_{N_i} + \rho_{ji} H_{ji} Q_{jp} H_{ji}^\dagger) \overset{(a)}{\leq} \log\det\left(2 I_{N_i}\right) = N_i, \quad (60)$$

where (a) follows from Lemma 4 by substituting $\sqrt{\rho_{ji}} H_{ji}^\dagger$ in S.

The following lemma will be used in this paper to prove the positive semi-definite property for some covariance matrices.

Lemma 5 ([32]): The following holds for any $M_i \times N_j$ matrix S

$$S(I_{N_j} + S^\dagger S)^{-1} S^\dagger \succeq 0. \quad (61)$$

As a corollary, we have the following result.

Corollary 6: Let $Q_{ip} = I_{M_i} - \sqrt{\rho_{ij}} H_{ij}^\dagger (I_{N_j} + \rho_{ij} H_{ij} H_{ij}^\dagger)^{-1} \sqrt{\rho_{ij}} H_{ij}$, and $Q_{ic} = I_{M_i} - Q_{ip}$. Then, $Q_{ip} \succeq 0$ and $Q_{ic} \succeq 0$.

Proof: $Q_{ip} \succeq 0$ follows from Lemma 1 by substituting $K_2 = I_{M_i}$ and $K_1 = 0_{M_i}$. Also, $Q_{ic} = I_{M_i} - Q_{ip} \succeq 0$ follows from Lemma 5 by substituting $\sqrt{\rho_{ij}} H_{ij}^\dagger$ into S. ∎

In addition, the next result gives the determinant of a block matrix, which will be used extensively in this sequel.

Lemma 6 ([37]): For block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with submatrices A, B, C, and D, we have:

$$\det M = \begin{cases} \det A \det(D - CA^{-1}B), & \text{if } A \text{ is invertible,} \\ \det D \det(A - BD^{-1}C), & \text{if } D \text{ is invertible.} \end{cases} \quad (62)$$

Appendix B
Proof of Outer Bound for Theorem 1

In this Appendix, we will show that $\mathbb{C}_{RC} \subseteq \mathcal{R}_o$. The set of upper bounds to the capacity region will be derived in two steps. First, the capacity region is outer-bounded by a region defined in terms of the differential entropy of the random variables associated with the signals. These outer-bounds use genie-aided information at the receivers. Second, we outer-bound this region to prove the outer-bound as described in the statement of Theorem 1.

The following result outer-bounds the capacity region of a two-user MIMO IC with limited receiver cooperation.

Lemma 7: Let S_i and $\tilde{S}_i$ be defined as $S_i \triangleq \sqrt{\rho_{ij}} H_{ij} X_i + Z_j$ and $\tilde{S}_i \triangleq \sqrt{\rho_{ij}} H_{ij} X_i + \tilde{Z}_j$, respectively, where $\tilde{Z}_i \sim \mathcal{CN}(0, I_{M_i})$ is independent of everything else. Then, the capacity region of a two-user MIMO IC with limited receiver cooperation is outerbounded by the region formed by (R_1, R_2) satisfying

$$\begin{aligned} R_1 \leq\ & h(H_{11}X_1 + Z_1) - h(Z_1) \\ & + \min\{h(H_{12}X_1 + Z_2 | H_{11}X_1 + Z_1) \\ & \quad - h(Z_2), C_{21}\}, && (63) \\ R_2 \leq\ & h(H_{22}X_2 + Z_2) - h(Z_2) \\ & + \min\{h(H_{21}X_2 + Z_1 | H_{22}X_2 + Z_2) \\ & \quad - h(Z_1), C_{12}\}, && (64) \\ R_1 + R_2 \leq\ & h(Y_1|\tilde{S}_1) + h(Y_2|\tilde{S}_2) - h(\tilde{Z}_1) - h(\tilde{Z}_1) \\ & + C_{21} + C_{12}, && (65) \\ R_1 + R_2 \leq\ & h(H_{11}X_1 + Z_1|S_1) + h(Y_2) - h(Z_1, Z_2) \\ & + C_{12}, && (66) \\ R_1 + R_2 \leq\ & h(H_{22}X_2 + Z_2|S_2) + h(Y_1) - h(Z_1, Z_2) \\ & + C_{21}, && (67) \\ R_1 + R_2 \leq\ & h(Y_1, Y_2) - h(Z_1, Z_2), && (68) \\ 2R_1 + R_2 \leq\ & h(H_{11}X_1 + Z_1|S_1) + h(Y_1) + h(Y_2|S_2) \\ & - h(Z_1, Z_2) - h(Z_1) + C_{21} + C_{12}, && (69) \\ R_1 + 2R_2 \leq\ & h(H_{22}X_2 + Z_2|S_2) + h(Y_2) + h(Y_1|S_1) \\ & - h(Z_1, Z_2) - h(Z_2) + C_{21} + C_{12}, && (70) \\ 2R_1 + R_2 \leq\ & h(Y_1, Y_2|\tilde{S}_2) + h(Y_1) - h(Z_1, Z_2) - h(Z_1) \\ & + C_{21}, && (71) \\ R_1 + 2R_2 \leq\ & h(Y_1, Y_2|\tilde{S}_1) + h(Y_2) - h(Z_1, Z_2) - h(Z_2) \\ & + C_{12}. && (72) \end{aligned}$$

Proof: The proof follows the same lines as the proof of [16, Lemma 5.1], replacing SISO channel gains by MIMO channel matrices and is thus omitted here. ■

The rest of the section outer-bounds this region to get the outer bound in Theorem 1. For this, we will introduce some useful Lemmas.

The next result outer-bounds the entropies and the conditional entropies of two random variables by their corresponding Gaussian random variables.

Lemma 8 ([38]): Let X and Y be two random vectors, and let X^G and Y^G be Gaussian vectors with covariance matrices satisfying

$$Cov\begin{bmatrix} X \\ Y \end{bmatrix} = Cov\begin{bmatrix} X^G \\ Y^G \end{bmatrix}, \tag{73}$$

Then, we have

$$h(Y) \leq h(Y^G), \tag{74}$$

$$h\left(Y \mid X\right) \leq h\left(Y^G \,\middle|\, X^G\right). \tag{75}$$

Define X_1^G and X_2^G as having a Gaussian distribution with the covariance matrix

$$Cov\begin{bmatrix} X_1^G \\ X_2^G \end{bmatrix} = Cov\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}. \tag{76}$$

Define $S_i^G \triangleq \sqrt{\rho_{ij}} H_{ij} X_i^G + Z_j$, $\tilde{S}_i^G \triangleq \sqrt{\rho_{ij}} H_{ij} X_i^G + \tilde{Z}_j$ and $Y_i^G \triangleq \sqrt{\rho_{ii}} H_{ii} X_i^G + \sqrt{\rho_{ji}} H_{ji} X_j^G + Z_i$. Also define $H_1 \triangleq I_{N_1} + \rho_{11} H_{11} Q_{11} H_{11}^{\dagger} + \rho_{21} H_{21} Q_{22} H_{21}^{\dagger}$ and $H_2 \triangleq I_{N_2} + \rho_{22} H_{22} Q_{22} H_{22}^{\dagger} + \rho_{12} H_{12} Q_{11} H_{12}^{\dagger}$.

The rest of the section considers the 10 terms in Lemma 7 and outer-bounds each of them to get the terms in the outer-bound of Theorem 1.

(63)→(5): We can split the bound in (63) into two upper bounds. The first bound is

$$\begin{aligned}
R_1 &\leq h(H_{11}X_1 + Z_1) - h(Z_1) \\
&\quad + h(H_{12}X_1 + Z_2 | H_{11}X_1 + Z_1) - h(Z_2) \\
&= h(H_{12}X_1 + Z_2, H_{11}X_1 + Z_1) - h(Z_1) - h(Z_2) \\
&\overset{(a)}{\leq} \log\det \begin{bmatrix} I_{N_2} + \rho_{12} H_{12} Q_{11} H_{12}^{\dagger} & \sqrt{\rho_{12}\rho_{11}} H_{12} Q_{11} H_{11}^{\dagger} \\ \sqrt{\rho_{12}\rho_{11}} H_{11} Q_{11} H_{12}^{\dagger} & I_{N_1} + \rho_{11} H_{11} Q_{11} H_{11}^{\dagger} \end{bmatrix} \\
&\overset{(b)}{=} \log\det(I_{N_1} + \rho_{11} H_{11} Q_{11} H_{11}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{12} H_{12} Q_{11} H_{12}^{\dagger} - \rho_{12}\rho_{11} H_{12} Q_{11} H_{11}^{\dagger} \\
&\qquad (I_{N_1} + \rho_{11} H_{11} Q_{11} H_{11}^{\dagger})^{-1} H_{11} Q_{11} H_{12}^{\dagger}) \\
&\overset{(c)}{\leq} \log\det(I_{N_1} + \rho_{11} H_{11} H_{11}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{12} H_{12} Q_{11} H_{12}^{\dagger} - \rho_{12}\rho_{11} H_{12} Q_{11} H_{11}^{\dagger} \\
&\qquad (I_{N_1} + \rho_{11} H_{11} Q_{11} H_{11}^{\dagger})^{-1} H_{11} Q_{11} H_{12}^{\dagger}) \\
s &= \log\det(I_{N_1} + \rho_{11} H_{11} H_{11}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{12} H_{12}(Q_{11} - \rho_{11} Q_{11} H_{11}^{\dagger} \\
&\qquad (I_{N_1} + \rho_{11} H_{11} Q_{11} H_{11}^{\dagger})^{-1} H_{11} Q_{11}) H_{12}^{\dagger}) \\
&\overset{(d)}{\leq} \log\det(I_{N_1} + \rho_{11} H_{11} H_{11}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{12} H_{12} H_{12}^{\dagger} - \rho_{12}\rho_{11} H_{12} H_{11}^{\dagger} \\
&\qquad (I_{N_1} + \rho_{11} H_{11} H_{11}^{\dagger})^{-1} H_{11} H_{12}^{\dagger}),
\end{aligned} \tag{77}$$

where (a) follows from Lemma 8 and from the fact that $h(Z_i) = \log\det\left(2\pi e I_{N_i}\right)$, (b) follows from Lemma 6, (c) follows from Lemma 2 and $Q_{ii} \preceq I_{M_i}$ for $i \in \{1, 2\}$, and (d) follows from Lemma 1 where $K_1 = Q_{11}$, $K_2 = I_{M_1}$ and $S = \sqrt{\rho_{11}} H_{11}^{\dagger}$. It gives the first part of the bound (5).

The second bound is

$$\begin{aligned}
R_1 &\leq h(H_{11}X_1 + Z_1) - h(Z_1) + C_{21} \\
&\overset{(a)}{=} \log\det(I_{N_1} + \rho_{11} H_{11} Q_{11} H_{11}^{\dagger}) + C_{21} \\
&\overset{(b)}{\leq} \log\det(I_{N_1} + \rho_{11} H_{11} H_{11}^{\dagger}) + C_{21},
\end{aligned} \tag{78}$$

where (a) follows from Lemma 8 and from the fact that $h(Z_i) = \log\det\left(2\pi e I_{N_i}\right)$, (b) follows from Lemma 2 and $Q_{ii} \preceq I_{M_i}$ for $i \in \{1, 2\}$. It gives the second part of the bound (5).

(64)→(6): This is obtained similarly to the last bound by exchanging 1 and 2 in the indices.

(65)→(7): For the bound (65) in Lemma 7,

$$\begin{aligned}
R_1 + R_2 &\leq h(Y_1|\tilde{S}_1) + h(Y_2|\tilde{S}_2) - h(\tilde{Z}_1) - h(\tilde{Z}_1) \\
&\quad + C_{21} + C_{12} \\
&= h(\sqrt{\rho_{11}} H_{11} X_1 + \sqrt{\rho_{21}} H_{21} X_2 + Z_1 | \\
&\quad \sqrt{\rho_{12}} H_{12} X_1 + \tilde{Z}_2) + h(\sqrt{\rho_{12}} H_{12} X_1 \\
&\quad + \sqrt{\rho_{22}} H_{22} X_2 + Z_2 | \sqrt{\rho_{21}} H_{21} X_2 + \tilde{Z}_1) \\
&\quad - h(\tilde{Z}_1) - h(\tilde{Z}_1) + C_{21} + C_{12} \\
&\overset{(a)}{\leq} h(\sqrt{\rho_{11}} H_{11} X_1^G + \sqrt{\rho_{21}} H_{21} X_2^G + Z_1 | \\
&\quad \sqrt{\rho_{12}} H_{12} X_1^G + \tilde{Z}_2) + h(\sqrt{\rho_{12}} H_{12} X_1^G \\
&\quad + \sqrt{\rho_{22}} H_{22} X_2^G + Z_2 | \sqrt{\rho_{21}} H_{21} X_2^G + \tilde{Z}_1) \\
&\quad - h(\tilde{Z}_1) - h(\tilde{Z}_1) + C_{21} + C_{12} \\
&= h(\sqrt{\rho_{11}} H_{11} X_1^G + \sqrt{\rho_{21}} H_{21} X_2^G + Z_1, \\
&\quad \sqrt{\rho_{12}} H_{12} X_1^G + \tilde{Z}_2) \\
&\quad - h(\sqrt{\rho_{12}} H_{12} X_1^G + \tilde{Z}_2) \\
&\quad + h(\sqrt{\rho_{12}} H_{12} X_1^G + \sqrt{\rho_{22}} H_{22} X_2^G + Z_2, \\
&\quad \sqrt{\rho_{21}} H_{21} X_2^G + \tilde{Z}_1) \\
&\quad - h(\sqrt{\rho_{21}} H_{21} X_2^G + \tilde{Z}_1) \\
&\quad - h(\tilde{Z}_1) - h(\tilde{Z}_1) + C_{21} + C_{12} \\
&\overset{(b)}{=} \log\det \begin{bmatrix} H_1 & \sqrt{\rho_{12}\rho_{11}} H_{11} Q_{11} H_{12}^{\dagger} \\ \sqrt{\rho_{12}\rho_{11}} H_{12} Q_{11} H_{11}^{\dagger} & I_{N_2} + \rho_{12} H_{12} Q_{11} H_{12}^{\dagger} \end{bmatrix} \\
&\quad + \log\det \begin{bmatrix} H_2 & \sqrt{\rho_{21}\rho_{22}} H_{22} Q_{22} H_{21}^{\dagger} \\ \sqrt{\rho_{21}\rho_{22}} H_{21} Q_{22} H_{22}^{\dagger} & I_{N_1} + \rho_{21} H_{21} Q_{22} H_{21}^{\dagger} \end{bmatrix} \\
&\quad - \log\det(I_{N_2} + \rho_{12} H_{12} Q_{11} H_{12}^{\dagger}) \\
&\quad - \log\det(I_{N_1} + \rho_{21} H_{21} Q_{22} H_{21}^{\dagger}) + C_{21} + C_{12} \\
&\overset{(c)}{=} \log\det(I_{N_1} + \rho_{11} H_{11} Q_{11} H_{11}^{\dagger} \\
&\qquad + \rho_{21} H_{21} Q_{22} H_{21}^{\dagger} - \rho_{11}\rho_{12} H_{11} Q_{11} H_{12}^{\dagger} \\
&\qquad (I_{N_2} + \rho_{12} H_{12} Q_{11} H_{12}^{\dagger})^{-1} H_{12} Q_{11} H_{11}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{22} H_{22} Q_{22} H_{22}^{\dagger} \\
&\qquad + \rho_{12} H_{12} Q_{11} H_{12}^{\dagger} - \rho_{22}\rho_{21} H_{22} Q_{22} H_{21}^{\dagger} \\
&\qquad (I_{N_1} + \rho_{21} H_{21} Q_{22} H_{21}^{\dagger})^{-1} H_{21} Q_{22} H_{22}^{\dagger}) \\
&\quad + C_{12} + C_{21}
\end{aligned}$$

$$\begin{aligned}
&\overset{(d)}{\leq} \log\det(I_{N_1} + \rho_{11} H_{11} H_{11}^{\dagger} + \rho_{21} H_{21} H_{21}^{\dagger} \\
&\qquad - \rho_{11}\rho_{12} H_{11} H_{12}^{\dagger} (I_{N_2} + \rho_{12} H_{12} H_{12}^{\dagger})^{-1} \\
&\qquad H_{12} H_{11}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{22} H_{22} H_{22}^{\dagger} \\
&\qquad + \rho_{12} H_{12} H_{12}^{\dagger} - \rho_{22}\rho_{21} H_{22} H_{21}^{\dagger} \\
&\qquad (I_{N_1} + \rho_{21} H_{21} H_{21}^{\dagger})^{-1} H_{21} H_{22}^{\dagger}) \\
&\quad + C_{12} + C_{21},
\end{aligned} \tag{79}$$

where (a) follows from Lemma 8, (b) follows from the fact that $h(Z_i) = \log\det\left(2\pi e I_{N_i}\right)$, (c) follows from Lemma 6, and (d) follows from Lemma 2 and $Q_{ii} \preceq I_{M_i}$ for $i \in \{1, 2\}$, and Lemma 1 where for the first term $K_1 = Q_{11}$, $K_2 = I_{M_1}$ and $S = \sqrt{\rho_{12}} H_{12}^{\dagger}$ and for the second term where $K_1 = Q_{22}$, $K_2 = I_{M_2}$ and $S = \sqrt{\rho_{21}} H_{21}^{\dagger}$. It gives the bound (7).

(66)→(8): For the bound (66) in Lemma 7,

$$\begin{aligned}
R_1 + R_2 &\leq h(H_{11}X_1 + Z_1|S_1) + h(Y_2) - h(Z_1, Z_2) + C_{12} \\
&= h(H_{11}X_1 + Z_1|H_{12}X_1 + Z_2) + h(Y_2) \\
&\quad - h(Z_1, Z_2) + C_{12} \\
&\leq h(H_{11}X_1^G + Z_1|H_{12}X_1^G + Z_2) + h(Y_2^G) \\
&\quad - h(Z_1, Z_2) + C_{12} \\
&= h(H_{11}X_1^G + Z_1, H_{12}X_1^G + Z_2) \\
&\quad - h(H_{12}X_1^G + Z_2) + h(Y_2^G) - h(Z_1, Z_2) + C_{12} \\
&\overset{(a)}{\leq} \log\det \\
&\quad \begin{bmatrix} I_{N_1} + \rho_{11} H_{11} Q_{11} H_{11}^{\dagger} & \sqrt{\rho_{12}\rho_{11}} H_{11} Q_{11} H_{12}^{\dagger} \\ \sqrt{\rho_{12}\rho_{11}} H_{12} Q_{11} H_{11}^{\dagger} & I_{N_2} + \rho_{12} H_{12} Q_{11} H_{12}^{\dagger} \end{bmatrix} \\
&\quad + \log\det(I_{N_2} + \rho_{12} H_{12} Q_{11} H_{12}^{\dagger} \\
&\qquad + \rho_{22} H_{22} Q_{22} H_{22}^{\dagger}) \\
&\quad - \log\det(I_{N_2} + \rho_{12} H_{12} Q_{11} H_{12}^{\dagger}) + C_{12} \\
&\overset{(b)}{=} \log\det(I_{N_1} + \rho_{11} H_{11} Q_{11} H_{11}^{\dagger} \\
&\qquad - \rho_{11}\rho_{12} H_{11} Q_{11} H_{12}^{\dagger} \\
&\qquad (I_{N_2} + \rho_{12} H_{12} Q_{11} H_{12}^{\dagger})^{-1} H_{12} Q_{11} H_{11}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{12} H_{12} Q_{11} H_{12}^{\dagger} \\
&\qquad + \rho_{22} H_{22} Q_{22} H_{22}^{\dagger}) + C_{12} \\
&\overset{(c)}{\leq} \log\det(I_{N_1} + \rho_{11} H_{11} H_{11}^{\dagger} - \rho_{11}\rho_{12} H_{11} H_{12}^{\dagger} \\
&\qquad (I_{N_2} + \rho_{12} H_{12} H_{12}^{\dagger})^{-1} H_{12} H_{11}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{12} H_{12} Q_{11} H_{12}^{\dagger} \\
&\qquad + \rho_{22} H_{22} Q_{22} H_{22}^{\dagger}) + C_{12} \\
&\overset{(d)}{\leq} \log\det(I_{N_1} + \rho_{11} H_{11} H_{11}^{\dagger} - \rho_{11}\rho_{12} H_{11} H_{12}^{\dagger} \\
&\qquad (I_{N_2} + \rho_{12} H_{12} H_{12}^{\dagger})^{-1} H_{12} H_{11}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{22} H_{22} H_{22}^{\dagger} + \rho_{12} H_{12} H_{12}^{\dagger}) \\
&\quad + C_{12},
\end{aligned} \tag{80}$$

where (a) follows from the fact that $h(Z_i) = \log\det\left(2\pi e I_{N_i}\right)$, and (b) follows from Lemma 6, and (c) follows from Lemma 1 where $K_1 = Q_{11}$, $K_2 = I_{M_1}$ and $S = \sqrt{\rho_{12}} H_{12}^{\dagger}$, and (d) follows from Lemma 2 and $Q_{ii} \preceq I_{M_i}$ for $i \in \{1, 2\}$. It gives the bound (8).

(67)→(9): This is obtained similarly to the last bound by exchanging 1 and 2 in the indices.

(68)→(10): For the bound (68) in Lemma 7, assume infinite capacity between the receivers, i.e., consider a single receiver. We get

$$\begin{aligned}
R_1 + R_2 &\leq h(Y_1, Y_2) - h(Z_1, Z_2) \\
&\overset{(a)}{\leq} \log\det\Bigg(I_{N_1+N_2} + \begin{bmatrix} \sqrt{\rho_{11}} H_{11} \\ \sqrt{\rho_{12}} H_{12} \end{bmatrix} Q_{11} \\
&\qquad \left[\sqrt{\rho_{11}} H_{11}^{\dagger} \ \sqrt{\rho_{12}} H_{12}^{\dagger}\right] + \begin{bmatrix} \sqrt{\rho_{21}} H_{21} \\ \sqrt{\rho_{22}} H_{22} \end{bmatrix} Q_{22} \\
&\qquad \left[\sqrt{\rho_{21}} H_{21}^{\dagger} \ \sqrt{\rho_{22}} H_{22}^{\dagger}\right]\Bigg) \\
&\overset{(b)}{\leq} \log\det\Bigg(I_{N_1+N_2} + \begin{bmatrix} \sqrt{\rho_{11}} H_{11} \\ \sqrt{\rho_{12}} H_{12} \end{bmatrix} \\
&\qquad \left[\sqrt{\rho_{11}} H_{11}^{\dagger} \ \sqrt{\rho_{12}} H_{12}^{\dagger}\right] + \begin{bmatrix} \sqrt{\rho_{21}} H_{21} \\ \sqrt{\rho_{22}} H_{22} \end{bmatrix} \\
&\qquad \left[\sqrt{\rho_{21}} H_{21}^{\dagger} \ \sqrt{\rho_{22}} H_{22}^{\dagger}\right]\Bigg),
\end{aligned} \tag{81}$$

where (a) follows from Lemma 8 and from the fact that $h(Z_i) = \log\det\left(2\pi e I_{N_i}\right)$, and (b) follows from Lemma 2 and $Q_{ii} \preceq I_{M_i}$ for $i \in \{1, 2\}$. It gives the bound (10).

(69)→(11): For the bound (69) in Lemma 7,

$$\begin{aligned}
2R_1 + R_2 &\leq h(\sqrt{\rho_{11}} H_{11} X_1 + Z_1|S_1) + h(Y_1) \\
&\quad + h(Y_2|S_2) - h(Z_1, Z_2) - h(Z_1) + C_{21} + C_{12} \\
&\leq h(\sqrt{\rho_{11}} H_{11} X_1^G + Z_1|S_1^G) + h(Y_1^G) \\
&\quad + h(Y_2^G|S_2^G) - h(Z_1, Z_2) - h(Z_1) + C_{21} + C_{12} \\
&= h(\sqrt{\rho_{11}} H_{11} X_1^G + Z_1, S_1^G) - h(S_1^G) \\
&\quad + h(Y_1^G) + h(Y_2^G, S_2^G) - h(S_2^G) - h(Z_1, Z_2) \\
&\quad - h(Z_1) + C_{21} + C_{12} \\
&= h(\sqrt{\rho_{11}} H_{11} X_1^G + Z_1, \sqrt{\rho_{12}} H_{12} X_1^G + Z_2) \\
&\quad + sh(\sqrt{\rho_{12}} H_{12} X_1^G + \sqrt{\rho_{22}} H_{22} X_2^G + Z_2, \\
&\qquad \sqrt{\rho_{21}} H_{21} X_2^G + Z_1) \\
&\quad + h(\sqrt{\rho_{11}} H_{11} X_1^G + \sqrt{\rho_{21}} H_{21} X_2^G + Z_1) \\
&\quad - h(\sqrt{\rho_{12}} H_{12} X_1^G + Z_2) \\
&\quad - h(\sqrt{\rho_{21}} H_{21} X_2^G + Z_1^G) - h(Z_1, Z_2) \\
&\quad - h(Z_1) + C_{21} + C_{12} \\
&\overset{(a)}{\leq} \log\det \\
&\quad \begin{bmatrix} H_2 & \sqrt{\rho_{22}\rho_{21}} H_{22} Q_{22} H_{21}^{\dagger} \\ \sqrt{\rho_{22}\rho_{21}} H_{21} Q_{22} H_{22}^{\dagger} & I_{N_1} + \rho_{21} H_{21} Q_{22} H_{21}^{\dagger} \end{bmatrix} \\
&\quad + \log\det \\
&\quad \begin{bmatrix} I_{N_1} + \rho_{11} H_{11} Q_{11} H_{11}^{\dagger} & \sqrt{\rho_{12}\rho_{11}} H_{11} Q_{11} H_{12}^{\dagger} \\ \sqrt{\rho_{12}\rho_{11}} H_{12} Q_{11} H_{11}^{\dagger} & I_{N_2} + \rho_{12} H_{12} Q_{11} H_{12}^{\dagger} \end{bmatrix} \\
&\quad + \log\det(I_{N_1} + \rho_{11} H_{11} Q_{11} H_{11}^{\dagger} \\
&\qquad + \rho_{21} H_{21} Q_{22} H_{21}^{\dagger}) \\
&\quad - \log\det(I_{N_2} + \rho_{12} H_{12} Q_{11} H_{12}^{\dagger}) \\
&\quad - \log\det(I_{N_1} + \rho_{21} H_{21} Q_{22} H_{21}^{\dagger}) + C_{12} + C_{21}
\end{aligned}$$

$$
\begin{aligned}
&\overset{(b)}{\leq} \log\det(I_{N_1}+\rho_{11}H_{11}Q_{11}H_{11}^{\dagger}-\rho_{11}\rho_{12} \\
&\qquad H_{11}Q_{11}H_{12}^{\dagger}(I_{N_2}+\rho_{12}H_{12}Q_{11}H_{12}^{\dagger})^{-1} \\
&\qquad H_{12}Q_{11}H_{11}^{\dagger}) \\
&\quad+\log\det(I_{N_2}+\rho_{22}H_{22}Q_{22}H_{22}^{\dagger} \\
&\qquad+\rho_{12}H_{12}Q_{11}H_{12}^{\dagger}-\rho_{22}\rho_{21}H_{22}Q_{22}H_{21}^{\dagger} \\
&\qquad(I_{N_1}+\rho_{21}H_{21}Q_{22}H_{21}^{\dagger})^{-1}H_{21}Q_{22}H_{22}^{\dagger}) \\
&\quad+\log\det(I_{N_1}+\rho_{11}H_{11} \\
&\qquad Q_{11}H_{11}^{\dagger}+\rho_{21}H_{21}Q_{22}H_{21}^{\dagger}) \\
&\quad+C_{12}+C_{21} \\
&\overset{(c)}{\leq} \log\det(I_{N_1}+\rho_{11}H_{11}H_{11}^{\dagger}-\rho_{11}\rho_{12}H_{11}H_{12}^{\dagger} \\
&\qquad(I_{N_2}+\rho_{12}H_{12}H_{12}^{\dagger})^{-1}H_{12}H_{11}^{\dagger}) \\
&\quad+\log\det(I_{N_2}+\rho_{22}H_{22}H_{22}^{\dagger}+\rho_{12}H_{12}H_{12}^{\dagger} \\
&\qquad-\rho_{22}\rho_{21}H_{22}H_{21}^{\dagger}(I_{N_1}+\rho_{21}H_{21}H_{21}^{\dagger})^{-1} \\
&\qquad H_{21}H_{22}^{\dagger}) \\
&\quad+\log\det(I_{N_1}+\rho_{11}H_{11}H_{11}^{\dagger}+\rho_{21}H_{21}H_{21}^{\dagger}) \\
&\quad+C_{12}+C_{21}, \qquad (82)
\end{aligned}
$$

where (a) follows from Lemma 8 and from the fact that $h(Z_i)=\log\det\left(2\pi e I_{N_i}\right)$, (b) follows from Lemma 6, and (c) follows from Lemma 1 and Lemma 2 and $Q_{ii}\preceq I_{M_i}$ for $i\in\{1,2\}$. It gives the bound (11).

(70)→(12)**:** This is obtained similarly to the last bound by exchanging 1 and 2 in the indices.

(71)→(13)**:** Define K as in (83), as shown at the bottom of the page. For the bound (71) in Lemma 7,

$$
\begin{aligned}
&2R_1+R_2 \\
&\leq h(Y_1,Y_2|\tilde{S}_2)+h(Y_1)-h(Z_1,Z_2)-h(Z_1)+C_{21} \\
&\overset{(a)}{\leq} h(Y_1^G,Y_2^G|\tilde{S}_2{}^G)+h(Y_1^G)-h(Z_1,Z_2)-h(Z_1) \\
&\quad+C_{21} \\
&= h(Y_1^G,Y_2^G,H_{21}X_2^G+\hat{Z}_1)-h(H_{21}X_2^G+\hat{Z}_1) \\
&\quad+h(Y_1^G)-h(Z_1,Z_2)-h(Z_1)+C_{21} \\
&\overset{(b)}{\leq} h(Y_1,Y_2,H_{21}X_2+\hat{Z}_1)-h(H_{21}X_2+\hat{Z}_1) \\
&\quad-h(Z_1,Z_2)+\log\det(I_{N_1}+\rho_{11}H_{11}H_{11}^{\dagger} \\
&\quad+\rho_{21}H_{21}H_{21}^{\dagger})+C_{21} \\
&\overset{(c)}{\leq} \log\det K \\
&\quad-\log\det(I_{N_1}+\rho_{21}H_{21}Q_{22}H_{21}^{\dagger})+\log\det(I_{N_1} \\
&\quad+\rho_{11}H_{11}H_{11}^{\dagger}+\rho_{21}H_{21}H_{21}^{\dagger})+C_{21} \\
&\overset{(d)}{=} \log\det\Bigg(I_{N_1+N_2}+\begin{bmatrix} s\sqrt{\rho_{22}}H_{22} \\ \sqrt{\rho_{21}}H_{21}\end{bmatrix} \\
&\qquad(Q_{22}-Q_{22}H_{21}^{\dagger}(I_{N_1}+\rho_{21}H_{21}Q_{22}H_{21}^{\dagger})^{-1}H_{21}Q_{22}) \\
&\qquad[\sqrt{\rho_{22}}H_{22}^{\dagger}\ \sqrt{\rho_{21}}H_{21}^{\dagger}]+\begin{bmatrix}\sqrt{\rho_{12}}H_{12} \\ \sqrt{\rho_{11}}H_{11}\end{bmatrix}Q_{11} \\
&\qquad[\sqrt{\rho_{12}}H_{12}^{\dagger}\ \sqrt{\rho_{11}}H_{11}^{\dagger}]\Bigg) \\
&\quad+\log\det(I_{N_1}+\rho_{11}H_{11}H_{11}^{\dagger}+\rho_{21}H_{21}H_{12}^{\dagger})+C_{21} \\
&\overset{(e)}{\leq} \log\det\Bigg(I_{N_1+N_2}+\begin{bmatrix}\sqrt{\rho_{22}}H_{22} \\ \sqrt{\rho_{21}}H_{21}\end{bmatrix} \\
&\qquad(I_{M_2}-H_{21}^{\dagger}(I_{N_1}+\rho_{21}H_{21}H_{21}^{\dagger})^{-1}H_{21}) \\
&\qquad[\sqrt{\rho_{22}}H_{22}^{\dagger}\ \sqrt{\rho_{21}}H_{21}^{\dagger}]+\begin{bmatrix}\sqrt{\rho_{12}}H_{12} \\ \sqrt{\rho_{11}}H_{11}\end{bmatrix} \\
&\qquad[\sqrt{\rho_{12}}H_{12}^{\dagger}\ \sqrt{\rho_{11}}H_{11}^{\dagger}]\Bigg) \\
&\quad+\log\det(I_{N_1}+\rho_{11}H_{11}H_{11}^{\dagger}+\rho_{21}H_{21}H_{12}^{\dagger})+C_{21}, \qquad (84)
\end{aligned}
$$

where (a) and (b) follow from Lemma 8 and from the fact that $h(Z_i)=\log\det\left(2\pi e I_{N_i}\right)$ and Lemma 2 and $Q_{ii}\preceq I_{M_i}$ for $i\in\{1,2\}$, (c) follows from Lemma 8, (d) follows from Lemma 6, and (e) follows from Lemma 1 and Lemma 2 and $Q_{ii}\preceq I_{M_i}$ for $i\in\{1,2\}$. It gives the bound (13).

(72)→(14)**:** This is obtained similarly to the last bound by exchanging 1 and 2 in the indices.

APPENDIX C
PROOF OF ACHIEVABILITY FOR THEOREM 1

In this section, we prove the achievability for Theorem 1. Denote the RHS of the 10 terms in (5)-(14) as I_1 to I_{10}, respectively. We will show a constant gap achiavability result for the two-user MIMO Gaussian IC with limited receiver cooperation in the following Lemma.

Lemma 9: The capacity region for the two-user MIMO IC with receiver cooperation contains the region formed by (R_1,R_2) *such that*

$$
\begin{aligned}
R_1 &\leq I_1-N_1-N_2, \\
R_2 &\leq I_2-N_1-N_2, \\
R_1+R_2 &\leq \min\{I_3,I_4,I_5,I_6\}-N_1-N_2 \\
&\quad-\max(N_1,N_2), \\
2R_1+R_2 &\leq \min\{I_7,I_9\}-2N_1-2N_2, \\
R_1+2R_2 &\leq \min\{I_8,I_{10}\}-2N_1-3N_2. \qquad (85)
\end{aligned}
$$

The rest of this section proves this Lemma. This region is within N_1+N_2 bits of the outer bound giver by $\mathcal{R}_o$ and thus proves the achievability for Theorem 1. In the following, we will consider the rate regions for $STG_{2\to1\to2}$ and then take the convex hull of $STG_{2\to1\to2}$ and $STG_{1\to2\to1}$ to get this result.

Lemma 10: If we consider $STG_{2\to1\to2}$*, the capacity region of the two-user MIMO Gaussian IC with limited receiver cooperation includes the set of* (R_1,R_2) *such that*

$$
\begin{aligned}
R_1 &\leq I(X_1;Y_1|X_{2c}), \qquad (86)\\
R_1 &\leq I(X_1;Y_1|X_{1c},X_{2c})+I(X_{1c},X_2;Y_2|X_{2c}) \\
&\quad+C_{12}, \qquad (87)\\
R_2 &\leq I(X_2;Y_2|X_{1c})+C_{12}, \qquad (88)\\
R_2 &\leq I(X_{2c};Y_1|X_1)+I(X_2;Y_2|X_{1c},X_{2c}), \qquad (89)
\end{aligned}
$$

$$
K \triangleq \begin{bmatrix}
I_{N_2}+\rho_{22}H_{22}Q_{22}H_{22}^{\dagger}+\rho_{12}H_{12}Q_{11}H_{12}^{\dagger} & \sqrt{\rho_{21}\rho_{22}}H_{22}Q_{22}H_{21}^{\dagger}+\sqrt{\rho_{11}\rho_{12}}H_{12}Q_{11}H_{11}^{\dagger} & \sqrt{\rho_{22}\rho_{21}}H_{22}Q_{22}H_{21}^{\dagger} \\
\sqrt{\rho_{21}\rho_{22}}H_{21}Q_{22}H_{22}^{\dagger}+\sqrt{\rho_{11}\rho_{12}}H_{11}Q_{11}H_{12}^{\dagger} & I_{N_1}+\rho_{11}H_{11}Q_{11}H_{11}^{\dagger}+\rho_{21}H_{21}Q_{22}H_{21}^{\dagger} & \rho_{21}H_{21}Q_{22}H_{21}^{\dagger} \\
\sqrt{\rho_{22}\rho_{21}}H_{21}Q_{22}H_{22}^{\dagger} & \rho_{21}H_{21}Q_{22}H_{21}^{\dagger} & I_{N_1}+\rho_{21}H_{21}Q_{22}H_{21}^{\dagger}
\end{bmatrix} \qquad (83)
$$

$$R_1+R_2 \le I(X_{2c}, X_1; Y_1)+I(X_2; Y_2|X_{1c}, X_{2c}) + (C_{21}-\xi)^+, \quad (90)$$
$$R_1+R_2 \le I(X_{2c}, X_1; Y_1, \hat{Y}_2) + I(X_2; Y_2|X_{1c}, X_{2c}), \quad (91)$$
$$R_1+R_2 \le I(X_{2c}, X_1; Y_1|X_{1c})+I(X_{1c}, X_2; Y_2|X_{2c}) + C_{12}+(C_{21}-\xi)^+, \quad (92)$$
$$R_1+R_2 \le I(X_{2c}, X_1; Y_1, \hat{Y}_2|X_{1c}) + I(X_{1c}, X_2; Y_2|X_{2c})+C_{12}, \quad (93)$$
$$R_1+R_2 \le I(X_1; Y_1|X_{1c}, X_{2c})+I(X_{1c}, X_2; Y_2) + C_{12}, \quad (94)$$
$$R_1+R_2 \le I(X_1; Y_1|X_{1c}, X_{2c})+I(X_{2c}; Y_1|X_1) + I(X_{1c}, X_2; Y_2|X_{2c})+C_{12}, \quad (95)$$
$$2R_1+R_2 \le I(X_1, X_{2c}; Y_1)+I(X_1; Y_1|X_{1c}, X_{2c}) + I(X_{1c}, X_2; Y_2|X_{2c})+C_{12}+(C_{21}-\xi)^+, \quad (96)$$
$$2R_1+R_2 \le I(X_1, X_{2c}; Y_1, \hat{Y}_2)+I(X_1; Y_1|X_{1c}, X_{2c}) + I(X_{1c}, X_2; Y_2|X_{2c})+C_{12}, \quad (97)$$
$$R_1+2R_2 \le I(X_1, X_{2c}; Y_1|X_{1c})+I(X_{1c}, X_2; Y_2) + I(X_2; Y_2|X_{1c}, X_{2c})+C_{12}+(C_{21}-\xi)^+, \quad (98)$$
$$R_1+2R_2 \le I(X_1, X_{2c}; Y_1|X_{1c})+I(X_{2c}; Y_1|X_1) + I(X_{1c}, X_2; Y_2|X_{2c})s+I(X_2; Y_2|X_{1c}, X_{2c}) + C_{12}+(C_{21}-\xi)^+, \quad (99)$$
$$R_1+2R_2 \le I(X_1, X_{2c}; Y_1, \hat{Y}_2|X_{1c})+I(X_{1c}, X_2; Y_2) + I(X_2; Y_2|X_{1c}, X_{2c})+C_{12}, \quad (100)$$
$$R_1+2R_2 \le I(X_1, X_{2c}; Y_1, \hat{Y}_2|X_{1c})+I(X_{2c}; Y_1|X_1) + I(X_{1c}, X_2; Y_2|X_{2c})+I(X_2; Y_2|X_{1c}, X_{2c}) + C_{12}. \quad (101)$$

where $\hat{Y}_i$ is defined in (21).

Proof: The proof follows similarly to that in [16, Sec. V-C], replacing scalars in the SISO channel by vectors for the MIMO channel. ■

The rest of the section inner bounds the convex hull of union of this region and the one achieved from $STG_{1\to 2\to 1}$ to get the inner bound in Theorem 1.

The achievability scheme is a 2-round protocol as described in Section III and the transmission scheme is based on (18), (19) and (20).

We will first evaluate some entropies that will be used in inner bounds of the achievable rate region.

$$h(Y_i) = \log\det(I_{N_i} + \rho_{ii} H_{ii} H_{ii}^\dagger + \rho_{ji} H_{ji} H_{ji}^\dagger) + N_i \log(2\pi e), \quad (102)$$
$$h(Y_i|X_i) = \log\det\left(I_{N_i} + \rho_{ji} H_{ji} H_{ji}^\dagger\right) + N_i \log(2\pi e). \quad (103)$$

In addition, we have

$$\begin{aligned} h(Y_i|X_{ic}, X_{jc}) &\ge h(Y_i|X_{ic}, X_{jc}, X_j) \\ &= \log\det(I_{N_i} + \rho_{ii} H_{ii} Q_{ip} H_{ii}^\dagger) + N_i \log(2\pi e) \\ &= \log\det(I_{N_i} + \rho_{ii} H_{ii} H_{ii}^\dagger - \sqrt{\rho_{ii}\rho_{ij}} H_{ii} H_{ij}^\dagger (I_{N_j} + \rho_{ij} H_{ij} H_{ij}^\dagger)^{-1} \\ &\quad \sqrt{\rho_{ii}\rho_{ij}} H_{ij} H_{ii}^\dagger) + N_i \log(2\pi e). \end{aligned} \quad (104)$$

Moreover, we have

$$\begin{aligned} h(Y_i|X_{jc}, X_i) &\le \log\det(I_{N_i} + \rho_{ji} H_{ji} Q_{jp} H_{ji}^\dagger) + N_i \log(2\pi e) \\ &\overset{(a)}{\le} \log\det\left(2I_{N_i}\right) + N_i \log(2\pi e) \\ &= N_i + N_i \log(2\pi e), \end{aligned} \quad (105)$$

where (a) follows from Corollary 5. This shows that $h(Y_i|X_{jc}, X_i)$ is upper-bounded by N_i.

The rest of the section evaluates some terms in Lemma 10. We will not evaluate the bounds (86) and (97) for now and show that the rest of the bounds contain a region within $N_1 + N_2$ bits of the outer bounds.

(87)**:** For this bound in Lemma 10, we have

$$\begin{aligned} &I(X_1; Y_1|X_{1c}, X_{2c}) + I(X_{1c}, X_2; Y_2|X_{2c}) + C_{12} \\ &= I(X_1; Y_1|X_{1c}, X_{2c}) + I(X_{1c}, X_{2p}; \sqrt{\rho_{12}} H_{12} X_1 + \sqrt{\rho_{22}} H_{22} X_{2p} + Z_2) + C_{12} \\ &= I(X_1; Y_1|X_{1c}, X_{2c}) + h(\sqrt{\rho_{12}} H_{12} X_1 + \sqrt{\rho_{22}} H_{22} X_{2p} + Z_2) - h(\sqrt{\rho_{12}} H_{12} X_{1p} + Z_2) + C_{12} \\ &\ge I(X_1; Y_1|X_{1c}, X_{2c}) + h(\sqrt{\rho_{12}} H_{12} X_1 + Z_2) - h(\sqrt{\rho_{12}} H_{12} X_{1p} + Z_2) \\ &= I(X_1; Y_1|X_{1c}, X_{2c}) + I(X_{1c}; \sqrt{\rho_{12}} H_{12} X_1 + Z_2) \\ &= I(X_1; Y_1|X_{2c}, X_{1c}) + I(X_{1c}; Y_2|X_2) \\ &= h(Y_2|X_2) - h(Y_2|X_2, X_{1c}) + h(Y_1|X_{2c}, X_{1c}) - h(Y_1|X_{2c}, X_{1c}, X_1) \\ &\overset{(a)}{\ge} h(Y_2|X_2) + h(Y_1|X_{2c}, X_{1c}) - N_1 - N_2 - (N_1 + N_2)\log(2\pi e) \\ &\overset{(b)}{=} \log\det(I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger) + \log\det(I_{N_1} + \rho_{11} H_{11} Q_{1p} H_{11}^\dagger + \rho_{21} H_{21} Q_{2p} H_{21}^\dagger) - N_1 - N_2 \\ &\overset{(c)}{\ge} \log\det(I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger) + \log\det(I_{N_1} + \rho_{11} H_{11} Q_{1p} H_{11}^\dagger) - N_1 - N_2 \\ &\overset{(d)}{=} \log\det(I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger) + \log\det(I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger - \rho_{12}\rho_{11} H_{12} H_{11}^\dagger \\ &\quad (I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger)^{-1} H_{11} H_{12}^\dagger) - N_1 - N_2, \end{aligned} \quad (106)$$

where (a) follows from (105), (b) follows from the assumed Gaussian distributions, (c) follows from Lemma 2, and (d) follows from the fact that using Lemma 6,

$$\begin{aligned} &\log\det(I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger) + \log\det(I_{N_1} + \rho_{11} H_{11} Q_{1p} H_{11}^\dagger) \\ &= \log\det\begin{bmatrix} I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger & \sqrt{\rho_{11}\rho_{12}} H_{12} H_{11}^\dagger \\ \sqrt{\rho_{11}\rho_{12}} H_{11} H_{12}^\dagger & I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger \end{bmatrix} \\ &= \log\det(I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger - \rho_{12}\rho_{11} H_{12} H_{11}^\dagger \\ &\quad (I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger)^{-1} H_{11} H_{12}^\dagger) + \log\det(I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger). \end{aligned} \quad (107)$$

Thus, we see that this R_1 bound is within $N_1 + N_2$ bits to the outer bound in (5).

(88)**:** For this term in Lemma 10, we have

$$\begin{aligned}
I(X_2; Y_2|X_{1c}) + C_{12} &= h(Y_2|X_{1c}) - h(Y_2|X_{1c}, X_2) + C_{12} \\
&\overset{(a)}{\geq} h(Y_2|X_{1c}) + C_{12} - N_2 - N_2 \log(2\pi e) \\
&= \log\det(I_{N_2} + \rho_{22} H_{22} H_{22}^{\dagger} + \rho_{12} H_{12} Q_{1p} H_{12}^{\dagger}) \\
&\quad - N_2 + C_{12} \\
&\overset{(b)}{\geq} \log\det(I_{N_2} + \rho_{22} H_{22} H_{22}^{\dagger}) + C_{12} - N_2, \qquad (108)
\end{aligned}$$

where (a) follows from (105) and (b) follows from Lemma 2.

Thus, we see that this R_2 bound is within N_2 bits of the outer bound in (6).

(89)**:** For this term in Lemma 10, we have

$$\begin{aligned}
I(X_{2c}; Y_1|X_1) &+ I(X_2; Y_2|X_{1c}, X_{2c}) \\
&= h(Y_1|X_1) - h(Y_1|X_1, X_{2c}) + h(Y_2|X_{1c}, X_{2c}) \\
&\quad - h(Y_2|X_{1c}, X_{2c}, X_2) \\
&\overset{(a)}{\geq} h(Y_1|X_1) + h(Y_2|X_{1c}, X_{2c}) - N_1 - N_2 \\
&\quad - (N_1 + N_2) \log(2\pi e) \\
&\overset{(b)}{=} \log\det(I_{N_1} + \rho_{21} H_{21} H_{21}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{22} H_{22} Q_{2p} H_{22}^{\dagger} + \rho_{12} H_{12} Q_{1p} H_{12}^{\dagger}) \\
&\quad - N_1 - N_2 \\
&\overset{(c)}{\geq} \log\det(I_{N_1} + \rho_{21} H_{21} H_{21}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{22} H_{22} Q_{2p} H_{22}^{\dagger}) - N_1 - N_2 \\
&\overset{(d)}{=} \log\det(I_{N_2} + \rho_{22} H_{22} H_{22}^{\dagger}) \\
&\quad + \log\det(I_{N_1} + \rho_{21} H_{21} H_{21}^{\dagger} - \rho_{21}\rho_{22} H_{21} H_{22}^{\dagger} \\
&\quad (I_{N_2} + \rho_{22} H_{22} H_{22}^{\dagger})^{-1} H_{22} H_{21}^{\dagger}) - N_1 - N_2, \qquad (109)
\end{aligned}$$

where (a) follows from (105), (b) follows from the assumed Gaussian distributions, and (c) follows from Lemma 2 and (d) follows from Lemma 6. Using Lemma 6 it is easy to see that

$$\begin{aligned}
\log\det(I_{N_1} &+ \rho_{21} H_{21} H_{21}^{\dagger}) \\
&+ \log\det(I_{N_2} + \rho_{22} H_{22} Q_{2p} H_{22}^{\dagger}) \\
&= \log\det \begin{bmatrix} I_{N_1} + \rho_{21} H_{21} H_{21}^{\dagger} & \sqrt{\rho_{21}\rho_{21}} H_{21} H_{22}^{\dagger} \\ \sqrt{\rho_{22}\rho_{21}} H_{22} H_{21}^{\dagger} & I_{N_2} + \rho_{22} H_{22} H_{22}^{\dagger} \end{bmatrix} \\
&= \log\det(I_{N_1} + \rho_{21} H_{21} H_{21}^{\dagger} - \rho_{21}\rho_{22} H_{21} H_{22}^{\dagger} \\
&\qquad (I_{N_2} + \rho_{22} H_{22} H_{22}^{\dagger})^{-1} H_{22} H_{21}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{22} H_{22} H_{22}^{\dagger}). \qquad (110)
\end{aligned}$$

Thus, we see that this R_2 bound is within $N_1 + N_2$ bits of the outer bound in (6).

(90)**:** For this bound in Lemma 10, we have

$$\begin{aligned}
I(X_{2c}, X_1; Y_1) &+ I(X_2; Y_2|X_{1c}, X_{2c}) + (C_{21} - \xi)^+ \\
&= h(Y_1) - h(Y_1|X_{2c}, X_1) + h(Y_2|X_{1c}, X_{2c}) \\
&\quad - h(Y_2|X_{1c}, X_2) + (C_{21} - \xi)^+ \\
&\overset{(a)}{\geq} h(Y_1) + h(Y_2|X_{1c}, X_{2c}) + C_{21} - N_1 - 2N_2 \\
&\quad - (N_1 + N_2) \log(2\pi e) \\
&= \log\det(I_{N_1} + \rho_{21} H_{21} H_{21}^{\dagger} + \rho_{11} H_{11} H_{11}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{22} H_{22} Q_{2p} H_{22}^{\dagger} + \rho_{12} H_{12} Q_{1p} H_{12}^{\dagger}) \\
&\quad + C_{21} - N_1 - 2N_2 \\
&\overset{(b)}{\geq} \log\det(I_{N_1} + \rho_{21} H_{21} H_{21}^{\dagger} + \rho_{11} H_{11} H_{11}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{22} H_{22} Q_{2p} H_{22}^{\dagger}) \\
&\quad + C_{21} - N_1 - 2N_2 \\
&= \log\det(I_{N_2} + \rho_{22} H_{22} H_{22}^{\dagger} - \rho_{22}\rho_{21} H_{22} H_{21}^{\dagger} \\
&\qquad (I_{N_1} + \rho_{21} H_{21} H_{21}^{\dagger})^{-1} H_{21} H_{22}^{\dagger}) \\
&\quad + \log\det(I_{N_1} + \rho_{11} H_{11} H_{11}^{\dagger} + \rho_{21} H_{21} H_{21}^{\dagger}) \\
&\quad + C_{21} - N_1 - 2N_2, \qquad (111)
\end{aligned}$$

where (a) follows from (105) and (23), and (b) follows from Lemma 2.

Thus, we see that this $R_1 + R_2$ bound is within $N_1 + 2N_2$ bits of the outer bound in (9).

(91)**:** Define L_1 as in (112), as shown at the bottom of the page. For this bound in Lemma 10, we have

$$\begin{aligned}
I(X_{2c}, X_1; Y_1, \hat{Y}_2) &+ I(X_2; Y_2|X_{1c}, X_{2c}) \\
&= h(Y_1, \hat{Y}_2) - h(Y_1, \hat{Y}_2|X_{2c}, X_1) + h(Y_2|X_{1c}, X_{2c}) \\
&\quad - h(Y_2|X_{1c}, X_2) \\
&\overset{(a)}{\geq} h(Y_1, \hat{Y}_2) - h(Y_1, \hat{Y}_2|X_{2c}, X_1) + h(Y_2|X_{1c}, X_{2c}) \\
&\quad - N_2 - N_2 \log(2\pi e) \\
&= h(Y_1, \hat{Y}_2) - h(Y_1, \hat{Y}_2|X_{2c}, X_1) \\
&\quad + \log\det(I_{N_2} + \rho_{22} H_{22} Q_{2p} H_{22}^{\dagger} + \rho_{12} H_{12} Q_{1p} H_{12}^{\dagger}) - N_2 \\
&\overset{(b)}{\geq} h(Y_1, \hat{Y}_2) - h(Y_1, \hat{Y}_2|X_{2c}, X_1) \\
&\quad + \log\det(I_{N_2} + \rho_{22} H_{22} Q_{2p} H_{22}^{\dagger}) - N_2 \\
&= h(Y_1, \hat{Y}_2) - h(\sqrt{\rho_{21}} H_{21} X_{2p} + Z_1, \sqrt{\rho_{22}} H_{22} X_{2p} \\
&\quad + Z_2 + \hat{Z}_2) + \log\det(I_{N_2} + \rho_{22} H_{22} Q_{2p} H_{22}^{\dagger}) - N_2 \\
&\overset{(c)}{=} h(Y_1, \hat{Y}_2) - \log\det(\Delta + I_{N_2} + H_{22} Q_{2p} H_{22}^{\dagger}) \\
&\quad - \log\det(I_{N_1} + H_{12} Q_{2p} H_{12}^{\dagger} - H_{12} Q_{2p} H_{22}^{\dagger} \\
&\qquad (\Delta + I_{N_2} + H_{22} Q_{2p} H_{22}^{\dagger})^{-1} H_{22} Q_{2p} H_{12}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{22} H_{22} Q_{2p} H_{22}^{\dagger}) \\
&\quad - N_2 - (N_1 + N_2) \log(2\pi e) \\
&\overset{(d)}{=} h(Y_1, \hat{Y}_2) - \log\det(I_{N_2} + H_{22} Q_{2p} H_{22}^{\dagger}) \\
&\quad - \log\det(I_{N_1} + H_{12} Q_{2p} H_{12}^{\dagger} - H_{12} Q_{2p} H_{22}^{\dagger} \\
&\qquad (\Delta + I_{N_2} + H_{22} Q_{2p} H_{22}^{\dagger})^{-1} H_{22} Q_{2p} H_{12}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{22} H_{22} Q_{2p} H_{22}^{\dagger}) \\
&\quad - 2N_2 - (N_1 + N_2) \log(2\pi e)
\end{aligned}$$

$$L_1 \triangleq \begin{bmatrix} \rho_{11} H_{11} H_{11}^{\dagger} + \rho_{21} H_{21} H_{21}^{\dagger} & \sqrt{\rho_{11}\rho_{12}} H_{11} H_{12}^{\dagger} + \sqrt{\rho_{21}\rho_{22}} H_{21} H_{22}^{\dagger} \\ \sqrt{\rho_{11}\rho_{12}} H_{12} H_{11}^{\dagger} + \sqrt{\rho_{21}\rho_{22}} H_{22} H_{21}^{\dagger} & \rho_{22} H_{22} H_{22}^{\dagger} + \rho_{12} H_{12} H_{12}^{\dagger} \end{bmatrix} \qquad (112)$$

$$
\begin{aligned}
&= h(Y_1,\hat{Y}_2) - \log\det(I_{N_1} + H_{12}Q_{2p}H_{12}^{\dagger} - H_{12}Q_{2p}H_{22}^{\dagger} \\
&\qquad (\Delta + I_{N_2} + H_{22}Q_{2p}H_{22}^{\dagger})^{-1} H_{22}Q_{2p}H_{12}^{\dagger}) \\
&\quad -2N_2 - (N_1+N_2)\log(2\pi e) \\
&\overset{(e)}{\geq} h(Y_1,\hat{Y}_2) - \log\det(I_{N_1} + H_{12}Q_{2p}H_{12}^{\dagger}) - 2N_2 \\
&\quad - (N_1+N_2)\log(2\pi e) \\
&\overset{(f)}{\geq} h(Y_1,\hat{Y}_2) - N_1 - 2N_2 - (N_1+N_2)\log(2\pi e) \\
&= \log\det\left(L_1 + \begin{bmatrix} I_{N_1} & 0 \\ 0 & \Delta + I_{N_2} \end{bmatrix}\right) \\
&\quad - N_1 - 2N_2 - (N_1+N_2)\log(2\pi e) \\
&= \log\det(\Delta + I_{N_2} + \rho_{22}H_{22}H_{22}^{\dagger} + \rho_{12}H_{12}H_{12}^{\dagger}) \\
&\quad + \log\det(I_{N_1} + \rho_{11}H_{11}H_{11}^{\dagger} + \rho_{21}H_{21}H_{21}^{\dagger} \\
&\qquad - \sqrt{\rho_{11}\rho_{12}}H_{11}H_{12}^{\dagger} + \sqrt{\rho_{21}\rho_{22}}H_{21}H_{22}^{\dagger} \\
&\qquad (\Delta + I_{N_2} + \rho_{22}H_{22}H_{22}^{\dagger} + \rho_{12}H_{12}H_{12}^{\dagger})^{-1} \\
&\qquad \sqrt{\rho_{11}\rho_{12}}H_{12}H_{11}^{\dagger} + \sqrt{\rho_{21}\rho_{22}}H_{22}H_{21}^{\dagger}) \\
&\quad - N_1 - 2N_2 \\
&\geq \log\det(I_{N_2} + \rho_{22}H_{22}H_{22}^{\dagger} + \rho_{12}H_{12}H_{12}^{\dagger}) \\
&\quad + \log\det(I_{N_1} + \rho_{11}H_{11}H_{11}^{\dagger} + \rho_{21}H_{21}H_{21}^{\dagger} \\
&\qquad - \sqrt{\rho_{11}\rho_{12}}H_{11}H_{12}^{\dagger} + \sqrt{\rho_{21}\rho_{22}}H_{21}H_{22}^{\dagger} \\
&\qquad (I_{N_2} + \rho_{22}H_{22}H_{22}^{\dagger} + \rho_{12}H_{12}H_{12}^{\dagger})^{-1} \\
&\qquad \sqrt{\rho_{11}\rho_{12}}H_{12}H_{11}^{\dagger} + \sqrt{\rho_{21}\rho_{22}}H_{22}H_{21}^{\dagger}) \\
&\quad - N_1 - 2N_2 \\
&= \log\det\left(L_1 + \begin{bmatrix} I_{N_1} & 0 \\ 0 & I_{N_2} \end{bmatrix}\right) \\
&\quad - N_1 - 2N_2 \\
&= \log\det\left(I_{N_1+N_2} + \begin{bmatrix} \sqrt{\rho_{11}}H_{11} \\ \sqrt{\rho_{12}}H_{12} \end{bmatrix} \right. \\
&\qquad [\sqrt{\rho_{11}}H_{11}^{\dagger} \ \sqrt{\rho_{12}}H_{12}^{\dagger}] + \begin{bmatrix} \sqrt{\rho_{21}}H_{21} \\ \sqrt{\rho_{22}}H_{22} \end{bmatrix} \\
&\qquad \left. [\sqrt{\rho_{21}}H_{21}^{\dagger} \ \sqrt{\rho_{22}}H_{22}^{\dagger}] \right) - N_1 - 2N_2 \\
&= h(Y_1, Y_2) - N_1 - 2N_2 \\
&= \log\det\left(I_{N_1+N_2} + \begin{bmatrix} \sqrt{\rho_{11}}H_{11} \\ \sqrt{\rho_{12}}H_{12} \end{bmatrix} \right. \\
&\qquad [\sqrt{\rho_{11}}H_{11}^{\dagger} \ \sqrt{\rho_{12}}H_{12}^{\dagger}] + \begin{bmatrix} \sqrt{\rho_{21}}H_{21} \\ \sqrt{\rho_{22}}H_{22} \end{bmatrix} \\
&\qquad \left. [\sqrt{\rho_{21}}H_{21}^{\dagger} \ \sqrt{\rho_{22}}H_{22}^{\dagger}] \right) - N_1 - 2N_2,
\end{aligned} \tag{113}
$$

where (a) and (f) follow from (105), and (b) and (e) follow from Lemma 2, (c) follows from the fact that

$$
\begin{aligned}
&h(\sqrt{\rho_{21}}H_{21}X_{2p} + Z_1, \sqrt{\rho_{22}}H_{22}X_{2p} + Z_2 + \hat{Z}_2) \\
&\quad = \log\det \\
&\quad \begin{bmatrix} I_{N_1} + \rho_{21}H_{21}Q_{2p}H_{21}^{\dagger} & \sqrt{\rho_{21}\rho_{22}}H_{21}Q_{2p}H_{22}^{\dagger} \\ \sqrt{\rho_{21}\rho_{22}}H_{22}Q_{2p}H_{21}^{\dagger} & \Delta + I_{N_2} + \rho_{22}H_{22}Q_{2p}H_{22}^{\dagger} \end{bmatrix} \\
&\quad + (N_1+N_2)\log(2\pi e) \\
&\quad = \log\det(\Delta + I_{N_2} + H_{22}Q_{2p}H_{22}^{\dagger}) \\
&\quad + \log\det(I_{N_1} + H_{12}Q_{2p}H_{12}^{\dagger} - H_{12}Q_{2p}H_{22}^{\dagger}(\Delta + I_{N_2} \\
&\quad + H_{22}Q_{2p}H_{22}^{\dagger})^{-1}H_{22}Q_{2p}H_{12}^{\dagger}) + (N_1+N_2)\log(2\pi e),
\end{aligned} \tag{114}
$$

and (d) follows from the fact that $\Delta = I_{N_2} + H_{22}Q_{2p}H_{22}^{\dagger}$ and hence:

$$
\begin{aligned}
&\log\det(\Delta + I_{N_2} + H_{22}Q_{2p}H_{22}^{\dagger}) \\
&\qquad = \log\det 2(I_{N_2} + H_{22}Q_{2p}H_{22}^{\dagger}) \\
&\qquad = \log\det(I_{N_2} + H_{22}Q_{2p}H_{22}^{\dagger}) + N_2.
\end{aligned} \tag{115}
$$

Thus, we see that this $R_1 + R_2$ bound is within $N_1 + 2N_2$ bits of the outer bound in (10).

(92): For this bound in Lemma 10, we have

$$
\begin{aligned}
&I(X_{2c}, X_1; Y_1|X_{1c}) + I(X_{1c}, X_2; Y_2|X_{2c}) + C_{12} \\
&\quad + (C_{21} - \xi)^+ \\
&\quad = h(Y_1|X_{1c}) - h(Y_1|X_{1c}, X_{2c}, X_1) + h(Y_2|X_{2c}) \\
&\quad - h(Y_2|X_{2c}, X_{1c}, X_2) + C_{12} + (C_{21} - \xi)^+ \\
&\quad \overset{(a)}{\geq} h(\sqrt{\rho_{11}}H_{11}X_{1p} + \sqrt{\rho_{21}}H_{21}X_2 + Z_1) \\
&\quad + h(\sqrt{\rho_{12}}H_{12}X_1 + \sqrt{\rho_{22}}H_{22}X_{2p} + Z_2) \\
&\quad + C_{12} + C_{21} - N_1 - 2N_2 - (N_1+N_2)\log(2\pi e) \\
&\quad = \log\det(I_{N_1} + \rho_{11}H_{11}Q_{1p}H_{11}^{\dagger} + \rho_{21}H_{21}H_{21}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{22}H_{22}Q_{2p}H_{22}^{\dagger} + \rho_{12}H_{12}H_{12}^{\dagger}) \\
&\quad + C_{12} + C_{21} - N_1 - 2N_2 \\
&\quad = \log\det(I_{N_1} + \rho_{11}H_{11}H_{11}^{\dagger} + \rho_{21}H_{21}H_{21}^{\dagger} \\
&\quad - \rho_{11}\rho_{12}H_{11}H_{12}^{\dagger}(I_{N_2} + \rho_{12}H_{12}H_{12}^{\dagger})^{-1}H_{12}H_{11}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{22}H_{22}H_{22}^{\dagger} + \rho_{12}H_{12}H_{12}^{\dagger} \\
&\quad - \rho_{22}\rho_{21}H_{22}H_{21}^{\dagger}(I_{N_1} + \rho_{21}H_{21}H_{21}^{\dagger})^{-1}H_{21}H_{22}^{\dagger}) \\
&\quad + C_{12} + C_{21} - N_1 - 2N_2.
\end{aligned} \tag{116}
$$

where (a) follows from (105) and (23).

Thus, we see that this $R_1 + R_2$ bound is within $N_1 + 2N_2$ bits of the outer bound in (7).

(93): For this bound in Lemma 10, we have

$$
\begin{aligned}
&I(X_{2c}, X_1; Y_1, \hat{Y}_2|X_{1c}) + I(X_{1c}, X_2; Y_2|X_{2c}) + C_{12} \\
&\quad = I(X_{2c}; Y_1, \hat{Y}_2|X_{1c}) + I(X_1; Y_1, \hat{Y}_2|X_{1c}, X_{2c}) \\
&\quad + I(X_{1c}, X_2; Y_2|X_{2c}) + C_{12} \\
&\quad \geq I(X_{2c}; \hat{Y}_2|X_{1c}) + I(X_1; Y_1|X_{1c}, X_{2c}) \\
&\quad + I(X_{1c}, X_2; Y_2|X_{2c}) + C_{12} \\
&\quad \overset{(a)}{\geq} I(X_{2c}; Y_2|X_{1c}) - N_2 + I(X_1; Y_1|X_{1c}, X_{2c}) \\
&\quad + I(X_{1c}, X_2; Y_2|X_{2c}) + C_{12} \\
&\quad \overset{(b)}{\geq} I(X_1; Y_1|X_{1c}, X_{2c}) + I(X_{1c}, X_2; Y_2) + C_{12} - N_2 \\
&\quad = h(Y_1|X_{1c}, X_{2c}) - h(Y_1|X_1, X_{1c}, X_{2c}) + h(Y_2) \\
&\quad - h(Y_2|X_{1c}, X_2) + C_{12} - N_2 \\
&\quad \overset{(c)}{\geq} h(Y_1|X_{1c}, X_{2c}) + h(Y_2) + C_{12} - N_1 - 2N_2 \\
&\quad - (N_1+N_2)\log(2\pi e) \\
&\quad = \log\det(I_{N_1} + \rho_{11}H_{11}Q_{1p}H_{11}^{\dagger} + \rho_{21}H_{21}Q_{2p}H_{21}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{22}H_{22}H_{22}^{\dagger} + \rho_{12}H_{12}H_{12}^{\dagger}) + C_{12} \\
&\quad - N_1 - 2N_2 \\
&\quad \overset{(d)}{\geq} \log\det(I_{N_1} + \rho_{11}H_{11}Q_{1p}H_{11}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{22}H_{22}H_{22}^{\dagger} + \rho_{12}H_{12}H_{12}^{\dagger}) \\
&\qquad + C_{12} - N_1 - 2N_2
\end{aligned}
$$

$$= \log\det(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger - \rho_{11}\rho_{12}H_{11}H_{12}^\dagger (I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger)^{-1}H_{12}H_{11}^\dagger) + \log\det(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}H_{12}H_{12}^\dagger) + C_{12} - N_1 - 2N_2, \quad (117)$$

where (a) follows from

$$I(X_{2c}; \hat{Y}_2|X_{1c}) \geq I(X_{2c}; Y_2|X_{1c}) - N_2, \quad (118)$$

which is true since

$$\begin{aligned} &I(X_{2c}; \hat{Y}_2|X_{1c}) - I(X_{2c}; Y_2|X_{1c}) + N_2 \\ &= h(\hat{Y}_2|X_{1c}) - h(\hat{Y}_2|X_{1c}, X_{2c}) - h(Y_2|X_{1c}) + h(Y_2|X_{1c}, X_{2c}) + N_2 \\ &= \log\det(\Delta + I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}Q_{1p}H_{12}H_{12}^\dagger) - \log\det(\Delta + I_{N_2} + \rho_{22}H_{22}Q_{2p}H_{22}^\dagger + \rho_{12}H_{12}Q_{1p}H_{12}^\dagger) \\ &\quad - \log\det(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}H_{12}Q_{1p}H_{12}^\dagger) + \log\det(I_{N_2} + \rho_{22}H_{22}Q_{2p}H_{22}^\dagger + \rho_{12}H_{12}Q_{1p}H_{12}^\dagger) + N_2 \\ &= \log\det(\Delta + I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}Q_{1p}H_{12}H_{12}^\dagger) - \log\det(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}H_{12}Q_{1p}H_{12}^\dagger) \\ &\quad - \log\det(2\Delta + \rho_{12}H_{12}Q_{1p}H_{12}^\dagger) + \log\det(\Delta + \rho_{12}H_{12}Q_{1p}H_{12}^\dagger) + N_2 \geq 0, \end{aligned} \quad (119)$$

(b) follows from the fact that

$$\begin{aligned} &I(X_{2c}; Y_2|X_{1c}) + I(X_{1c}, X_2; Y_2|X_{2c}) \\ &= I(X_{2c}; Y_2, X_{1c}) + I(X_{1c}, X_2; Y_2|X_{2c}) \\ &\geq I(X_{2c}; Y_2) + I(X_{1c}, X_2; Y_2|X_{2c}) \\ &= I(X_{1c}, X_2, X_{2c}; Y_2) + I(X_{1c}, X_2; Y_2), \end{aligned} \quad (120)$$

(c) follows from (105) and (d) follows from Lemma 2.

Thus, we see that this $R_1 + R_2$ bound is within $N_1 + 2N_2$ bits of the outer bound in (8).

(94)**:** For this bound in Lemma 10, similar to the last term we have

$$\begin{aligned} &I(X_1; Y_1|X_{1c}, X_{2c}) + I(X_{1c}, X_2; Y_2) + C_{12} \\ &\geq \log\det(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger - \rho_{11}\rho_{12}H_{11}H_{12}^\dagger (I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger)^{-1}H_{12}H_{11}^\dagger) \\ &\quad + \log\det(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}H_{12}H_{12}^\dagger) + C_{12} - N_1 - N_2, \end{aligned} \quad (121)$$

which results from the proof of the last bound.

Thus, we see that this $R_1 + R_2$ bound is within $N_1 + N_2$ bits of the outer bound in (8).

(95)**:** For this bound in Lemma 10 we have

$$\begin{aligned} &I(X_1; Y_1|X_{1c}, X_{2c}) + I(X_{2c}; Y_1|X_1) + I(X_{1c}, X_2; Y_2|X_{2c}) + C_{12} \\ &= h(Y_1|X_{1c}, X_{2c}) - h(Y_1|X_{1c}, X_{2c}, X_1) + h(Y_1|X_1) - h(Y_1|X_1, X_{2c}) + h(Y_2|X_{2c}) - h(Y_2|X_{1c}, X_2, X_{2c}) + C_{12} \\ &\overset{(a)}{\geq} h(Y_1|X_{1c}, X_{2c}) + h(Y_1|X_1) + h(Y_2|X_{2c}) + C_{12} - 2N_1 - N_2 - (2N_1 + N_2)\log(2\pi e) \\ &= h(Y_1|X_{1c}, X_{2c}) + h(\sqrt{\rho_{21}}H_{21}X_2 + Z_1) + h(Y_2, X_{2c}) - h(X_{2c}) + C_{12} - 2N_1 - N_2 - (2N_1 + N_2)\log(2\pi e) \\ &= \log\det\begin{bmatrix} \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}H_{12}H_{12}^\dagger & \sqrt{\rho_{22}}H_{22}Q_{2c} \\ \sqrt{\rho_{22}}Q_{2c}H_{22}^\dagger & Q_{2c} \end{bmatrix} - \log\det(Q_{2c}) + h(Y_1|X_{1c}, X_{2c}) \\ &\quad + h(\sqrt{\rho_{21}}H_{21}X_2 + Z_1) + C_{12} - 2N_1 - N_2 - (2N_1)\log(2\pi e) \\ &= \log\det\left(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}H_{12}H_{12}^\dagger\right) + \log\det\left(Q_{2c} - \rho_{22}Q_{2c}H_{22}^\dagger\,(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}H_{12}H_{12}^\dagger)^{-1}\,H_{22}Q_{2c}\right) \\ &\quad - \log\det(Q_{2c}) + h(Y_1|X_{1c}, X_{2c}) + h(\sqrt{\rho_{21}}H_{21}X_2 + Z_1) + C_{12} - 2N_1 - N_2 - (2N_1)\log(2\pi e) \\ &= \log\det\left(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}H_{12}H_{12}^\dagger\right) + \log\det(I_{N_1} + \rho_{11}H_{11}Q_{1p}H_{11}^\dagger + \rho_{21}H_{21}Q_{2p}H_{21}^\dagger) \\ &\quad + \log\det\left(Q_{2c} - \rho_{22}Q_{2c}H_{22}^\dagger\,(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}H_{12}H_{12}^\dagger)^{-1}\,H_{22}Q_{2c}\right) - \log\det(Q_{2c}) \\ &\quad + \log\det\left(I_{N_1} + \rho_{21}H_{21}H_{21}^\dagger\right) + C_{12} - 2N_1 - N_2 \\ &\geq \log\det\left(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}H_{12}H_{12}^\dagger\right) + \log\det(I_{N_1} + \rho_{11}H_{11}Q_{1p}H_{11}^\dagger) \\ &\quad + \log\det\left(Q_{2c} - \rho_{22}Q_{2c}H_{22}^\dagger(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger)^{-1}H_{22}Q_{2c}\right) - \log\det(Q_{2c}) + \log\det\left(I_{N_1} + \rho_{21}H_{21}H_{21}^\dagger\right) + C_{12} - 2N_1 - N_2 \\ &\geq \log\det\left(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}H_{12}H_{12}^\dagger\right) + \log\det(I_{N_1} + \rho_{11}H_{11}Q_{1p}H_{11}^\dagger) + \log\det\left(Q_{2c} - Q_{2c}^2\right) \\ &\quad - \log\det(Q_{2c}) + \log\det\left(I_{N_1} + \rho_{21}H_{21}H_{21}^\dagger\right) + C_{12} - 2N_1 - N_2 \\ &= \log\det\left(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}H_{12}H_{12}^\dagger\right) + \log\det(I_{N_1} + \rho_{11}H_{11}Q_{1p}H_{11}^\dagger) + \log\det\left(Q_{2p}\right) \\ &\quad + \log\det\left(I_{N_1} + \rho_{21}H_{21}H_{21}^\dagger\right) + C_{12} - 2N_1 - N_2 \\ &= \log\det\left(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}H_{12}H_{12}^\dagger\right) + \log\det(I_{N_1} + \rho_{11}H_{11}Q_{1p}H_{11}^\dagger) + \log\det\left(Q_{2p}\right) \\ &\quad + \log\det\left(I_{M_2} + \rho_{21}H_{21}^\dagger H_{21}\right) + C_{12} - 2N_1 - N_2 \\ &= \log\det\left(I_{N_2} + \rho_{22}H_{22}H_{22}^\dagger + \rho_{12}H_{12}H_{12}^\dagger\right) + \log\det(I_{N_1} + \rho_{11}H_{11}Q_{1p}H_{11}^\dagger) \\ &\quad + \log\det\left(I_{M_2} - \rho_{21}H_{21}^\dagger(I_{N_1} + \rho_{21}H_{21}H_{21}^\dagger)^{-1}H_{21}\right) + \log\det\left(I_{M_2} + \rho_{21}H_{21}^\dagger H_{21}\right) + C_{12} - 2N_1 - N_2 \end{aligned}$$

$$= \log\det\left(I_{N_2}+\rho_{22}H_{22}H_{22}^\dagger+\rho_{12}H_{12}H_{12}^\dagger\right) + \log\det(I_{N_1}+\rho_{11}H_{11}Q_{1p}H_{11}^\dagger) + \log\det\Big(I_{M_2}-\rho_{21}H_{21}^\dagger(I_{N_1}+\rho_{21}H_{21}H_{21}^\dagger)^{-1}H_{21} +\rho_{21}H_{21}^\dagger H_{21}-\rho_{21}H_{21}^\dagger H_{21}\rho_{21}H_{21}^\dagger (I_{N_1}+\rho_{21}H_{21}H_{21}^\dagger)^{-1}H_{21}\Big) + C_{12}-2N_1-N_2$$

$$= \log\det\left(I_{N_2}+\rho_{22}H_{22}H_{22}^\dagger+\rho_{12}H_{12}H_{12}^\dagger\right) + \log\det(I_{N_1}+\rho_{11}H_{11}Q_{1p}H_{11}^\dagger) + \log\det\Big(I_{M_2}+\rho_{21}H_{21}^\dagger\Big(I_{M_2}-H_{21}\rho_{21}H_{21}^\dagger (I_{N_1}+\rho_{21}H_{21}H_{21}^\dagger)^{-1}-(I_{N_1}+\rho_{21}H_{21}H_{21}^\dagger)^{-1}\Big)H_{21}\Big) + C_{12}-2N_1-N_2$$

$$\stackrel{(c)}{=} \log\det(I_{N_1}+\rho_{11}H_{11}Q_{1p}H_{11}^\dagger) + \log\det(I_{N_2}+\rho_{22}H_{22}H_{22}^\dagger+\rho_{12}H_{12}H_{12}^\dagger) + C_{12}-2N_1-N_2$$

$$= \log\det(I_{N_1}+\rho_{11}H_{11}H_{11}^\dagger-\rho_{11}\rho_{12}H_{11}H_{12}^\dagger (I_{N_2}+\rho_{12}H_{12}H_{12}^\dagger)^{-1}H_{12}H_{11}^\dagger) + \log\det(I_{N_2}+ \rho_{22}H_{22}H_{22}^\dagger+\rho_{12}H_{12}H_{12}^\dagger) + C_{12}-2N_1-N_2, \tag{122}$$

where (a) follows from (105), (b) follows from Lemma 6 and (c) follows from Lemma 2.

Thus, we see that this $R_1 + R_2$ bound is within $2N_1 + N_2$ bits of the outer bound in (8).

(96)**:** For this bound in Lemma 10 we have

$$I(X_1, X_{2c}; Y_1) + I(X_1; Y_1|X_{1c}, X_{2c}) + I(X_{1c}, X_2; Y_2|X_{2c}) + C_{12} + (C_{21}-\xi)^+$$
$$= h(Y_1) - h(Y_1|X_1, X_{2c}) + h(Y_1|X_{1c}, X_{2c}) - h(Y_1|X_{1c}, X_{2c}, X_1) + h(Y_2|X_{2c}) - h(Y_2|X_{1c}, X_2, X_{2c}) + C_{12} + (C_{21}-\xi)^+$$
$$\stackrel{(a)}{\geq} h(Y_1) + h(Y_1|X_{1c}, X_{2c}) + h(Y_2|X_{2c}) + C_{12} + C_{21} - 2N_1 - 2N_2 - (2N_1+N_2)\log(2\pi e)$$
$$= \log\det(I_{N_1}+\rho_{11}H_{11}H_{11}^\dagger+\rho_{21}H_{21}H_{21}^\dagger) + \log\det(I_{N_1}+\rho_{11}H_{11}Q_{1p}H_{11}^\dagger+\rho_{21}H_{21}Q_{2p}H_{21}^\dagger) + \log\det(I_{N_2}+\rho_{22}H_{22}Q_{2p}H_{22}^\dagger+\rho_{12}H_{12}H_{12}^\dagger) + C_{12}+C_{21}-2N_1-2N_2$$
$$\stackrel{(b)}{\geq} \log\det(I_{N_1}+\rho_{11}H_{11}H_{11}^\dagger+\rho_{21}H_{21}H_{21}^\dagger) + \log\det(I_{N_1}+\rho_{11}H_{11}Q_{1p}H_{11}^\dagger) + \log\det(I_{N_2}+\rho_{22}H_{22}Q_{2p}H_{22}^\dagger+\rho_{12}H_{12}H_{12}^\dagger) + C_{12}+C_{21}-2N_1-2N_2$$
$$= \log\det(I_{N_1}+\rho_{11}H_{11}H_{11}^\dagger-\rho_{11}\rho_{12}H_{11}H_{12}^\dagger(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger)^{-1}H_{12}H_{11}^\dagger) + \log\det(I_{N_2}+\rho_{22}H_{22}H_{22}^\dagger+\rho_{12}H_{12}H_{12}^\dagger - \rho_{22}\rho_{21}H_{22}H_{21}^\dagger(I_{N_1}+\rho_{21}H_{21}H_{21}^\dagger)^{-1}H_{21}H_{22}^\dagger) + \log\det(I_{N_1}+\rho_{11}H_{11}H_{11}^\dagger+\rho_{21}H_{21}H_{21}^\dagger) + C_{12}+C_{21}-2N_1-2N_2, \tag{123}$$

where (a) follows from (105) and (23), and (b) follows from Lemma 2.

Thus, we see that this $2R_1+R_2$ bound is within $2N_1+2N_2$ bits of the outer bound in (11).

(98)**:** For this bound in Lemma 10 we have

$$I(X_1, X_{2c}; Y_1|X_{1c}) + I(X_{1c}, X_2; Y_2) + I(X_2; Y_2|X_{1c}, X_{2c}) + C_{12} + (C_{21}-\xi)^+$$
$$= h(Y_1|X_{1c}) - h(Y_1|X_{1c}, X_1, X_{2c}) + h(Y_2) - h(Y_2|X_{1c}, X_2) + h(Y_2|X_{1c}, X_{2c}) - h(Y_2|X_{1c}, X_{2c}, X_2) + C_{12} + (C_{21}-\xi)^+$$
$$\stackrel{(a)}{\geq} h(Y_1|X_{1c}) + h(Y_2) + h(Y_2|X_{1c}, X_{2c}) + C_{12} + C_{21} - 2N_1 - 2N_2 - (N_1+2N_2)\log(2\pi e)$$
$$= \log\det(I_{N_1}+\rho_{11}H_{11}Q_{1p}H_{11}^\dagger+\rho_{21}H_{21}H_{21}^\dagger) + \log\det(I_{N_2}+\rho_{22}H_{22}H_{22}^\dagger+\rho_{12}H_{12}H_{12}^\dagger) + \log\det(I_{N_2}+\rho_{22}H_{22}Q_{2p}H_{22}^\dagger+\rho_{12}H_{12}Q_{1p}H_{12}^\dagger) + C_{12}+C_{21}-2N_1-2N_2$$
$$\stackrel{(b)}{\geq} \log\det(I_{N_1}+\rho_{11}H_{11}Q_{1p}H_{11}^\dagger+\rho_{21}H_{21}H_{21}^\dagger) + \log\det(I_{N_2}+\rho_{22}H_{22}H_{22}^\dagger+\rho_{12}H_{12}H_{12}^\dagger) + \log\det(I_{N_2}+\rho_{22}H_{22}Q_{2p}H_{22}^\dagger) + C_{12}+C_{21} - 2N_1-2N_2$$
$$= \log\det(I_{N_2}+\rho_{22}H_{22}H_{22}^\dagger-\rho_{22}\rho_{21}H_{22}H_{21}^\dagger (I_{N_1}+\rho_{21}H_{21}H_{21}^\dagger)^{-1}H_{21}H_{22}^\dagger) + \log\det(I_{N_1}+\rho_{11}H_{11}H_{11}^\dagger+\rho_{21}H_{21}H_{21}^\dagger - \rho_{11}\rho_{12}H_{11}H_{12}^\dagger(I_{N_2}+\rho_{12}H_{12}H_{12}^\dagger)^{-1} H_{12}H_{11}^\dagger) + \log\det(I_{N_2}+ \rho_{22}H_{22}H_{22}^\dagger+\rho_{12}H_{12}H_{12}^\dagger) +C_{21}+C_{12}- 2N_1-2N_2, \tag{124}$$

where (a) follows from (105) and (23), and (b) follows from Lemma 2.

Thus, we see that this R_1+2R_2 bound is within $2N_1+2N_2$ bits of the outer bound in (12).

(99)**:** For this bound in Lemma 10 we have

$$I(X_1, X_{2c}; Y_1|X_{1c}) + I(X_{2c}; Y_1|X_1) + I(X_{1c}, X_2; Y_2|X_{2c}) + I(X_2; Y_2|X_{1c}, X_{2c}) + C_{12} + (C_{21}-\xi)^+$$
$$= h(Y_1|X_{1c}) - h(Y_1|X_1, X_{2c}, X_{1c}) + h(Y_1|X_1) - h(Y_1|X_1, X_{2c}) + h(Y_2|X_{2c}) - h(Y_2|X_{1c}, X_2, X_{2c}) + h(Y_2|X_{1c}, X_{2c}) - h(Y_2|X_{1c}, X_{2c}, X_2) + C_{12} + (C_{21}-\xi)^+$$
$$\stackrel{(a)}{\geq} h(Y_1|X_{1c}) + h(Y_1|X_1) + h(Y_2|X_{2c}) + h(Y_2|X_{1c}, X_{2c}) + C_{12} + C_{21} - 2N_1 - 3N_2 - 2(N_1+N_2)\log(2\pi e)$$
$$= \log\det(I_{N_2}+\rho_{22}H_{22}Q_{2p}H_{22}^\dagger+\rho_{12}H_{12}Q_{1p}H_{12}^\dagger) + \log\det(I_{N_1}+\rho_{11}H_{11}Q_{1p}H_{11}^\dagger+\rho_{21}H_{21}H_{21}^\dagger) + \log\det(I_{N_1}+\rho_{21}H_{21}H_{21}^\dagger) + \log\det(I_{N_2}+ \rho_{22}H_{22}Q_{2p}H_{22}^\dagger+\rho_{12}H_{12}H_{12}^\dagger) +C_{21}+C_{12}-2N_1-3N_2$$

$$\begin{aligned}
&\stackrel{(b)}{\geq} \log\det(I_{N_2} + \rho_{22}H_{22}Q_{2p}H_{22}^{\dagger}) \\
&\quad + \log\det(I_{N_1} + \rho_{11}H_{11}Q_{1p}H_{11}^{\dagger} + \rho_{21}H_{21}H_{21}^{\dagger}) \\
&\quad + \log\det(I_{N_1} + \rho_{21}H_{21}H_{21}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{22}H_{22}Q_{2p}H_{22}^{\dagger} + \rho_{12}H_{12}H_{12}^{\dagger}) \\
&\quad + C_{21} + C_{12} - 2N_1 - 3N_2 \\
&\stackrel{(c)}{\geq} \log\det(I_{N_2} + \rho_{22}H_{22}Q_{2p}H_{22}^{\dagger}) \\
&\quad + \log\det(I_{N_1} + \rho_{11}H_{11}Q_{1p}H_{11}^{\dagger} + \rho_{21}H_{21}H_{21}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{22}H_{22}H_{22}^{\dagger} + \rho_{12}H_{12}H_{12}^{\dagger}) \\
&\quad + C_{21} + C_{12} - 2N_1 - 3N_2 \\
&= \log\det(I_{N_2} + \rho_{22}H_{22}H_{22}^{\dagger} - \rho_{22}\rho_{21}H_{22}H_{21}^{\dagger} \\
&\qquad (I_{N_1} + \rho_{21}H_{21}H_{21}^{\dagger})^{-1}H_{21}H_{22}^{\dagger}) \\
&\quad + \log\det(I_{N_1} + \rho_{11}H_{11}H_{11}^{\dagger} + \rho_{21}H_{21}H_{21}^{\dagger} \\
&\qquad - \rho_{11}\rho_{12}H_{11}H_{12}^{\dagger}(I_{N_2} + \rho_{12}H_{12}H_{12}^{\dagger})^{-1} \\
&\qquad H_{12}H_{11}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{22}H_{22}H_{22}^{\dagger} + \rho_{12}H_{12}H_{12}^{\dagger}) \\
&\quad + C_{21} + C_{12} - 2N_1 - 3N_2, \qquad (125)
\end{aligned}$$

where (a) follows from (105) and (23), and (b) follows from Lemma 2 and (c) follows from

$$\begin{aligned}
&\log\det(I_{N_1} + \rho_{21}H_{21}H_{21}^{\dagger}) \\
&\quad = h(Y_1|X_1) \\
&\quad \stackrel{(d)}{\geq} h(Y_2) - h(Y_2|X_{2c}) \\
&\quad = \log\det(I_{N_2} + \rho_{22}H_{22}H_{22}^{\dagger} + \rho_{12}H_{12}H_{12}^{\dagger}) \\
&\qquad - \log\det(I_{N_2} + \rho_{22}H_{22}Q_{2p}H_{22}^{\dagger} + \rho_{12}H_{12}H_{12}^{\dagger}), \qquad (126)
\end{aligned}$$

where (d) follows from (122).

Thus, we see that this $R_1 + 2R_2$ bound is within $2N_1 + 3N_2$ bits of the outer bound in (12).

(100)**:** For this bound in Lemma 10 we have

$$\begin{aligned}
&I(X_1, X_{2c}; Y_1, \hat{Y}_2|X_{1c}) + I(X_{1c}, X_2; Y_2) \\
&\quad + I(X_2; Y_2|X_{1c}, X_{2c}) + C_{12} \\
&= h(Y_1, \hat{Y}_2|X_{1c}) - h(Y_1, \hat{Y}_2|X_1, X_{2c}, X_{1c}) + h(Y_2) \\
&\quad - h(Y_2|X_{1c}, X_2) + h(Y_2|X_{1c}, X_{2c}) \\
&\quad - h(Y_2|X_{1c}, X_{2c}, X_2) + C_{12} \\
&\stackrel{(a)}{\geq} h(Y_1, \hat{Y}_2|X_{1c}) - h(Y_1, \hat{Y}_2|X_1, X_{2c}, X_{1c}) + h(Y_2) \\
&\quad + h(Y_2|X_{1c}, X_{2c}) + C_{12} - 2N_2 - 2N_2\log(2\pi e) \\
&= h(Y_1, \hat{Y}_2|X_{1c}) - h(\sqrt{\rho_{21}}H_{21}X_{2p} + Z_1, \\
&\quad \sqrt{\rho_{22}}H_{22}X_{2p} + Z_2 + \hat{Z}_2) + h(Y_2|X_{1c}, X_{2c}) + h(Y_2) \\
&\quad + C_{12} - 2N_2 - 2N_2\log(2\pi e) \\
&\stackrel{(b)}{=} h(Y_1, \hat{Y}_2|X_{1c}) - \log\det(\Delta + I_{N_2} + \rho_{22}H_{22}Q_{2p}H_{22}^{\dagger}) \\
&\quad - \log\det(I_{N_1} + \rho_{12}H_{12}Q_{2p}H_{12}^{\dagger} \\
&\quad - \rho_{12}\rho_{22}H_{12}Q_{2p}H_{22}^{\dagger}(\Delta + I_{N_2} + \rho_{22}H_{22}Q_{2p}H_{22}^{\dagger})^{-1} \\
&\quad H_{22}Q_{2p}H_{12}^{\dagger}) + h(Y_2|X_{1c}, X_{2c}) + h(Y_2) \\
&\quad + C_{12} - 2N_2 - (N_1 + 3N_2)\log(2\pi e) \\
&\stackrel{(c)}{\geq} h(Y_1, \hat{Y}_2|X_{1c}) - \log\det(\Delta + I_{N_2} + \rho_{22}H_{22}Q_{2p}H_{22}^{\dagger}) \\
&\quad - \log\det(I_{N_1} + \rho_{12}H_{12}Q_{2p}H_{12}^{\dagger}) + h(Y_2|X_{1c}, X_{2c}) \\
&\quad + h(Y_2) + C_{12} - 2N_2 - (N_1 + 3N_2)\log(2\pi e) \\
&= h(Y_1, \hat{Y}_2|X_{1c}) - \log\det(\Delta + I_{N_2} + \rho_{22}H_{22}Q_{2p}H_{22}^{\dagger}) \\
&\quad - \log\det(I_{N_1} + \rho_{12}H_{12}Q_{2p}H_{12}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{22}H_{22}Q_{2p}H_{22}^{\dagger} + \rho_{12}H_{12}Q_{1p}H_{12}^{\dagger}) \\
&\quad + h(Y_2) + C_{12} - 2N_2 - (N_1 + 2N_2)\log(2\pi e) \\
&\stackrel{(d)}{=} h(Y_1, \hat{Y}_2|X_{1c}) - \log\det 2(I_{N_2} + \rho_{22}H_{22}Q_{2p}H_{22}^{\dagger}) \\
&\quad - \log\det(I_{N_1} + \rho_{12}H_{12}Q_{2p}H_{12}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{22}H_{22}Q_{2p}H_{22}^{\dagger} + \rho_{12}H_{12}Q_{1p}H_{12}^{\dagger}) \\
&\quad + h(Y_2) + C_{12} - 2N_2 - (N_1 + 2N_2)\log(2\pi e) \\
&= h(Y_1, \hat{Y}_2|X_{1c}) - \log\det(I_{N_2} + \rho_{22}H_{22}Q_{2p}H_{22}^{\dagger}) \\
&\quad - \log\det(I_{N_1} + \rho_{12}H_{12}Q_{2p}H_{12}^{\dagger}) \\
&\quad + \log\det(I_{N_2} + \rho_{22}H_{22}Q_{2p}H_{22}^{\dagger} + \rho_{12}H_{12}Q_{1p}H_{12}^{\dagger}) \\
&\quad + h(Y_2) + C_{12} - 3N_2 - (N_1 + 2N_2)\log(2\pi e) \\
&\geq h(Y_1, \hat{Y}_2|X_{1c}) - \log\det(I_{N_1} + \rho_{12}H_{12}Q_{2p}H_{12}^{\dagger}) \\
&\quad + h(Y_2) + C_{12} - 3N_2 - (N_1 + 2N_2)\log(2\pi e) \\
&\geq h(Y_1, \hat{Y}_2|X_{1c}) + h(Y_2) + C_{12} - N_1 - 3N_2 \\
&\quad - (N_1 + 2N_2)\log(2\pi e) \\
&\stackrel{(e)}{\geq} \log\det\Bigg(I_{N_1+N_2} + \begin{bmatrix}\sqrt{\rho_{11}}H_{11} \\ \sqrt{\rho_{12}}H_{12}\end{bmatrix}(I_{M_1} - H_{12}^{\dagger}(I_{N_2} \\
&\qquad + \rho_{12}H_{12}H_{12}^{\dagger})^{-1}H_{12})\,[\sqrt{\rho_{11}}H_{11}^{\dagger}\ \sqrt{\rho_{12}}H_{12}^{\dagger}] \\
&\qquad + \begin{bmatrix}\sqrt{\rho_{21}}H_{21} \\ \sqrt{\rho_{22}}H_{22}\end{bmatrix}[\sqrt{\rho_{21}}H_{21}^{\dagger}\ \sqrt{\rho_{22}}H_{22}^{\dagger}]\Bigg) + h(Y_2) \\
&\quad + C_{12} - N_1 - 3N_2 - N_2\log(2\pi e) \\
&= \log\det\Bigg(I_{N_1+N_2} + \begin{bmatrix}\sqrt{\rho_{11}}H_{11} \\ \sqrt{\rho_{12}}H_{12}\end{bmatrix}(I_{M_1} - H_{12}^{\dagger}(I_{N_2} \\
&\qquad + \rho_{12}H_{12}H_{12}^{\dagger})^{-1}H_{12})\,[\sqrt{\rho_{11}}H_{11}^{\dagger}\ \sqrt{\rho_{12}}H_{12}^{\dagger}] \\
&\qquad + \begin{bmatrix}\sqrt{\rho_{21}}H_{21} \\ \sqrt{\rho_{22}}H_{22}\end{bmatrix}[\sqrt{\rho_{21}}H_{21}^{\dagger}\ \sqrt{\rho_{22}}H_{22}^{\dagger}]\Bigg) \\
&\quad + \log\det(I_{N_2} + \rho_{22}H_{22}H_{22}^{\dagger} + \rho_{12}H_{12}H_{12}^{\dagger}) + C_{12} \\
&\quad - N_1 - 3N_2, \qquad (127)
\end{aligned}$$

where (a) follows from (105), (b) is achieved similar to (114), and (c) follows from Lemma 2, (d) follows from (22), and (e) is due to

$$\begin{aligned}
&h(Y_1, \hat{Y}_2|X_{1c}) \\
&\quad = h(Y_1, \hat{Y}_2, X_{1c}) - h(X_{1c}) \\
&\quad = \log\det(L_2) - h(X_{1c}) + (M_1 + N_1 + N_2)\log(2\pi e) \\
&\quad \stackrel{(f)}{=} \log\det\Bigg(\Bigg(L_1 + \begin{bmatrix} I_{N_1} & 0 \\ 0 & \Delta + I_{N_2}\end{bmatrix}\Bigg) \\
&\qquad - \begin{bmatrix}H_{11} \\ H_{12}\end{bmatrix} Q_{1c}({Q_{1c}}^{-1})Q_{1c}[H_{11}^{\dagger}H_{12}^{\dagger}]\Bigg) \\
&\qquad + h(X_{1c}) - h(X_{1c}) + (N_1 + N_2)\log(2\pi e) \\
&\quad = \log\det\Bigg(\Bigg(L_1 + \begin{bmatrix} I_{N_1} & 0 \\ 0 & \Delta + I_{N_2}\end{bmatrix}\Bigg) \\
&\qquad - \begin{bmatrix}H_{11} \\ H_{12}\end{bmatrix}(I_{M_1} - Q_{1p})[H_{11}^{\dagger}H_{12}^{\dagger}]\Bigg) \\
&\qquad + (N_1 + N_2)\log(2\pi e) \\
&\quad \stackrel{(g)}{\geq} \log\det\Bigg(\Bigg(L_1 + \begin{bmatrix} I_{N_1} & 0 \\ 0 & I_{N_2}\end{bmatrix}\Bigg) \\
&\qquad - \begin{bmatrix}H_{11} \\ H_{12}\end{bmatrix}(I_{M_1} - Q_{1p})[H_{11}^{\dagger}H_{12}^{\dagger}]\Bigg) \\
&\quad + (N_1 + N_2)\log(2\pi e)
\end{aligned}$$

$$= \log\det\Bigg(I_{N_1+N_2} + \begin{bmatrix}\sqrt{\rho_{11}}H_{11}\\ \sqrt{\rho_{12}}H_{12}\end{bmatrix}(I_{M_1} - H_{12}^{\dagger}(I_{N_2}+\rho_{12}H_{12}H_{12}^{\dagger})^{-1}H_{12})\,[\sqrt{\rho_{11}}H_{11}^{\dagger}\ \sqrt{\rho_{12}}H_{12}^{\dagger}] + \begin{bmatrix}\sqrt{\rho_{21}}H_{21}\\ \sqrt{\rho_{22}}H_{22}\end{bmatrix}[\sqrt{\rho_{21}}H_{21}^{\dagger}\ \sqrt{\rho_{22}}H_{22}^{\dagger}]\Bigg) + (N_1+N_2)\log(2\pi e), \quad (128)$$

where L_2 is defined as in (129), as shown at the bottom of the page, L_1 is defined as in (112), (f) is due to Lemma 6 and (g) results from Lemma 6 and Lemma 2 and also the fact that Δ is a positive definite matrix.

Thus, we see that this $R_1 + 2R_2$ bound is within $N_1 + 3N_2$ bits of the outer bound in (14).

(101)**:** For this bound in Lemma 10 we have

$$\begin{aligned}
&I(X_1, X_{2c}; Y_1, \hat{Y}_2|X_{1c}) + I(X_{2c}; Y_1|X_1) + I(X_{1c}, X_2; Y_2|X_{2c}) + I(X_2; Y_2|X_{1c}, X_{2c}) + C_{12}\\
&= h(Y_1, \hat{Y}_2|X_{1c}) - h(Y_1, \hat{Y}_2|X_{1c}, X_1, X_{2c}) + h(Y_1|X_1) - h(Y_1|X_1, X_{2c}) + h(Y_2|X_{2c}) - h(Y_2|X_{2c}, X_{1c}, X_2)\\
&\quad + h(Y_2|X_{1c}, X_{2c}) - h(Y_2|X_{1c}, X_{2c}, X_2) + C_{12}\\
&\overset{(a)}{\geq} h(Y_1, \hat{Y}_2|X_{1c}) - h(Y_1, \hat{Y}_2|X_{1c}, X_1, X_{2c}) + h(Y_1|X_1) + h(Y_2|X_{2c}) + h(Y_2|X_{1c}, X_{2c}) + C_{12} - N_1 - 2N_2\\
&\quad - (N_1 + 2N_2)\log(2\pi e)\\
&\overset{(b)}{\geq} h(Y_1, \hat{Y}_2|X_{1c}) + h(Y_1|X_1) + h(Y_2|X_{2c}) + C_{12} - 2N_1 - 3N_2 - 2(N_1+N_2)\log(2\pi e)\\
&= h(Y_1, \hat{Y}_2|X_{1c}) + \log\det(I_{N_1} + \rho_{21}H_{21}H_{21}^{\dagger}) + \log\det(I_{N_2} + \rho_{22}H_{22}Q_{2p}H_{22}^{\dagger} + \rho_{12}H_{12}H_{12}^{\dagger})\\
&\quad + C_{12} - 2N_1 - 3N_2 - (N_1+N_2)\log(2\pi e)\\
&\overset{(c)}{\geq} h(Y_1, \hat{Y}_2|X_{1c}) + \log\det(I_{N_2} + \rho_{22}H_{22}H_{22}^{\dagger} + \rho_{12}H_{12}H_{12}^{\dagger}) + C_{12} - 2N_1 - 3N_2 - (N_1+N_2)\log(2\pi e)\\
&\overset{(d)}{\geq} \log\det\Bigg(I_{N_1+N_2} + \begin{bmatrix}\sqrt{\rho_{11}}H_{11}\\ \sqrt{\rho_{12}}H_{12}\end{bmatrix}(I_{M_1} - H_{12}^{\dagger}(I_{N_2}+\rho_{12}H_{12}H_{12}^{\dagger})^{-1}H_{12})\,[\sqrt{\rho_{11}}H_{11}^{\dagger}\ \sqrt{\rho_{12}}H_{12}^{\dagger}] + \begin{bmatrix}\sqrt{\rho_{21}}H_{21}\\ \sqrt{\rho_{22}}H_{22}\end{bmatrix}[\sqrt{\rho_{21}}H_{21}^{\dagger}\ \sqrt{\rho_{22}}H_{22}^{\dagger}]\Bigg)\\
&\quad + \log\det(I_{N_2} + \rho_{22}H_{22}H_{22}^{\dagger} + \rho_{12}H_{12}H_{12}^{\dagger}) + C_{12} - 2N_1 - 3N_2,
\end{aligned} \quad (130)$$

where (a) follows from (105), (c) follows from (122), and (b) and (d) can be seen similar to the proof of the last bound.

Thus, we see that this $R_1 + 2R_2$ bound is within $2N_1 + 3N_2$ bits of the outer bound in (14).

We define the region $\mathcal{R}^p$ including all the achievability bounds in (86)-(101) except for (86) and (97). Up to now, we have analyzed all the bounds of $\mathcal{R}^p$. We proved in $\mathcal{R}^p$ that:

$$\begin{aligned}
R_1 &\leq I_1 - N_1 - N_2,\\
R_2 &\leq I_2 - N_1 - N_2,\\
R_1 + R_2 &\leq \min\{I_3, I_4, I_5, I_6\} - N_1 - N_2 - \max(N_1, N_2),\\
2R_1 + R_2 &\leq \min\{I_7, I_9\} - 2N_1 - 2N_2,\\
R_1 + 2R_2 &\leq \min\{I_8, I_{10}\} - 2N_1 - 3N_2.
\end{aligned} \quad (131)$$

Thus, $\mathcal{R}^p$ contains the region which is within $N_1 + N_2$ bits to the outer bound $\mathcal{R}_0$.

Now, add constraints (97) and (86) to $\mathcal{R}^p$. [16] proved that whenever (97) is active, at least one of the $R_1 + R_2$ bounds is active, which can be extended to the MIMO case because [16, Claim 5.6] is true in general independent of the number of antennas. We will now present similar reasoning for bound (86) to show that whenever bound (86) is active, at least one of the $R_1 + R_2$ bounds is active.

The value of $R_1 + R_2$ at the intersection of (86) and (98) is greater than the average value of $R_1 + R_2$ in (90) and (94):

$$\begin{aligned}
&\text{RHS of (86)} + \text{RHS of (98)}\\
&= I(X_1; Y_1|X_{2c}) + I(X_1, X_{2c}; Y_1|X_{1c}) + I(X_{1c}, X_2; Y_2) + I(X_2; Y_2|X_{1c}, X_{2c}) + C_{12} + (C_{21} - \xi)^+\\
&\overset{(a)}{\geq} I(X_{2c}, X_1; Y_1) + I(X_2; Y_2|X_{1c}, X_{2c}) + (C_{21} - \xi)^+ + I(X_1; Y_1|X_{1c}, X_{2c}) + I(X_{1c}, X_2; Y_2) + C_{12}\\
&= \text{RHS of (90)} + \text{RHS of (94)},
\end{aligned} \quad (132)$$

where (a) follows from the fact that

$$\begin{aligned}
&I(X_1; Y_1|X_{2c}) + I(X_1, X_{2c}; Y_1|X_{1c}) - I(X_{2c}, X_1; Y_1) - I(X_1; Y_1|X_{1c}, X_{2c})\\
&= h(Y_1|X_{2c}) + h(Y_1|X_{1c}) - h(Y_1) - h(Y_1|X_{1c}, X_{2c})\\
&\overset{(b)}{\geq} 0,
\end{aligned} \quad (133)$$

where (b) results from the following fact that if A, B, C and D are invertible positive semi-definite $M \times M$ matrices then

$$\det(A+B).\det(A+C) \geq \det(A+B+C).\det(A), \quad (134)$$

because it is equivalent to

$$\det(A+B).\det(A^{-1}).\det(A+C) \geq \det(A+B+C), \quad (135)$$

or

$$\det(A+B+C+BA^{-1}C) \geq \det(A+B+C), \quad (136)$$

which is trivial.

It shows that when both the bounds (86) and (98) are active, at least one of the bounds (90) or (94) will be active also.

$$L_2 \triangleq \begin{bmatrix} I_{N_1} + \rho_{11}H_{11}H_{11}^{\dagger} + \rho_{21}H_{21}H_{21}^{\dagger} & \sqrt{\rho_{12}\rho_{11}}H_{11}H_{12}^{\dagger} + \sqrt{\rho_{22}\rho_{21}}H_{21}H_{22}^{\dagger} & \sqrt{\rho_{11}}H_{11}Q_{1c}\\ \sqrt{\rho_{12}\rho_{11}}H_{12}H_{11}^{\dagger} + \sqrt{\rho_{22}\rho_{21}}H_{22}H_{21}^{\dagger} & \Delta + I_{N_2} + \rho_{22}H_{22}H_{22}^{\dagger} + \rho_{12}H_{12}H_{12}^{\dagger} & \sqrt{\rho_{12}}H_{12}Q_{1c}\\ \sqrt{\rho_{11}}Q_{1c}H_{11}^{\dagger} & \sqrt{\rho_{12}}Q_{1c}H_{12}^{\dagger} & Q_{1c} \end{bmatrix} \quad (129)$$

The value of $R_1 + R_2$ at the intersection of (86) and (99) is greater than the average value of $R_1 + R_2$ in (90) and (95):

$$\begin{aligned}&\text{RHS of } (86) + \text{RHS of } (99)\\ &= I(X_1; Y_1|X_{2c}) + I(X_1, X_{2c}; Y_1|X_{1c})\\ &\quad + I(X_{2c}; Y_1|X_1) + I(X_{1c}, X_2; Y_2|X_{2c})\\ &\quad + I(X_2; Y_2|X_{1c}, X_{2c}) + C_{12} + (C_{21} - \xi)^+\\ &\overset{(a)}{\geq} I(X_{2c}, X_1; Y_1) + I(X_2; Y_2|X_{1c}, X_{2c}) + (C_{21} - \xi)^+\\ &\quad + I(X_1; Y_1|X_{1c}, X_{2c}) + I(X_{2c}; Y_1|X_1)\\ &\quad + I(X_{1c}, X_2; Y_2|X_{2c}) + C_{12}\\ &= \text{RHS of } (90) + \text{RHS of } (95), \end{aligned} \quad (137)$$

where (a) follows from (133).

It shows that when both the bounds (86) and (99) are active, at least one of the bounds (90) or (95) will be active also.

The value of $R_1 + R_2$ at the intersection of (86) and (100) is greater than the average value of $R_1 + R_2$ in (91) and (94):

$$\begin{aligned}&\text{RHS of } (86) + \text{RHS of } (100)\\ &= I(X_1; Y_1|X_{2c}) + I(X_1, X_{2c}; Y_1, \hat{Y}_2|X_{1c})\\ &\quad + I(X_{1c}, X_2; Y_2) + I(X_2; Y_2|X_{1c}, X_{2c}) + C_{12}\\ &\overset{(a)}{\geq} I(X_1, X_{2c}; Y_1, \hat{Y}_2|X_{1c}) + I(X_2; Y_2|X_{1c}, X_{2c})\\ &\quad + I(X_1; Y_1|X_{1c}, X_{2c}) + I(X_{1c}, X_2; Y_2) + C_{12}\\ &= \text{RHS of } (91) + \text{RHS of } (94), \end{aligned} \quad (138)$$

where (a) follows from the fact that

$$\begin{aligned}&I(X_1; Y_1|X_{2c}) - I(X_1; Y_1|X_{1c}, X_{2c})\\ &= I(Y_1|X_{2c}) - I(Y_1|X_{1c}, X_{2c})\\ &= \log\det(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger + \rho_{21}H_{21}Q_{2p}H_{21}^\dagger)\\ &\quad - \log\det(I_{N_1} + \rho_{11}H_{11}Q_{1p}H_{11}^\dagger + \rho_{21}H_{21}Q_{2p}H_{21}^\dagger)\\ &\geq 0. \end{aligned} \quad (139)$$

It shows that when both the bounds (86) and (100) are active, at least one of the bounds (91) or (94) will be active also.

The value of $R_1 + R_2$ at the intersection of (86) and (101) is greater than the average value of $R_1 + R_2$ in (91) and (95):

$$\begin{aligned}&\text{RHS of } (86) + \text{RHS of } (101)\\ &= I(X_1; Y_1|X_{2c}) + I(X_1, X_{2c}; Y_1, \hat{Y}_2|X_{1c})\\ &\quad + I(X_{2c}; Y_1|X_1) + I(X_{1c}, X_2; Y_2|X_{2c})\\ &\quad + I(X_2; Y_2|X_{1c}, X_{2c}) + C_{12}\\ &\overset{(a)}{\geq} I(X_1, X_{2c}; Y_1, \hat{Y}_2|X_{1c}) + I(X_2; Y_2|X_{1c}, X_{2c})\\ &\quad + I(X_1; Y_1|X_{1c}, X_{2c}) + I(X_{2c}; Y_1|X_1)\\ &\quad + I(X_{1c}, X_2; Y_2|X_{2c}) + C_{12}\\ &= \text{RHS of } (91) + \text{RHS of } (95), \end{aligned} \quad (140)$$

where (a) follows from (139).

It shows that when both the bounds (86) and (101) are active, at least one of the bounds (91) or (95) will be active also.

So, when (86) is active, we can see that at least one of the R_1+R_2 bounds in (90)-(95) is active in $\mathcal{R}_{2\to1\to2}$. Hence, with a strategy similar to the one in [16, Claim 5.6] for (97) we can see that the bound (86) does not show up in $conv\{\mathcal{R}_{2\to1\to2} \cup \mathcal{R}_{1\to2\to1}\}$.

Therefore, the R_1 bound (86) and the $2R_1 + R_2$ bound (97) do not show up in $\mathcal{R} = conv\{\mathcal{R}_{2\to1\to2} \cup \mathcal{R}_{1\to2\to1}\}$ and $\mathcal{R}$ is within N_1+N_2 bits per user to the outer bounds in Theorem 1.

APPENDIX D
PROOF OF THEOREM 2

In this section, we will find the limit of $\mathcal{R}_o/\log \mathsf{SNR}$ as $\mathsf{SNR} \to \infty$ to get the result stated in Theorem 2 when $C_{ij} \sim \mathsf{SNR}^{\beta_{ij}}$ and $\rho_{ij} \sim \mathsf{SNR}$ where $\beta_{12}, \beta_{21} \in \mathbb{R}^+$.

This follows from Theorem 1 since the capacity region is inner and outer- bounded by $\mathcal{R}_o$ with constant gaps which would vanish for the DoF. Before going over each of the above terms and finding their high SNR limit, we first give some lemmas that will be used in the proof.

Lemma 11 ([4]): Let $H_1 \in \mathbb{C}^{N\times M_1}$, $H_2 \in \mathbb{C}^{N\times M_2}$,..., and $H_k \in \mathbb{C}^{N\times M_k}$ be k full rank and independent channel matrices. Then, the following holds

$$\begin{aligned}&\log\det(I_N + \rho H_1H_1^\dagger + \rho H_2H_2^\dagger + \cdots + \rho H_kH_k^\dagger)\\ &= \log\det(I_N + \rho[H_1 \ldots H_k][H_1 \ldots H_k]^\dagger)\\ &= \min\{N, M_1 + M_2 + \cdots + M_k\}\log\rho + o(\log\rho). \end{aligned} \quad (141)$$

Lemma 12 ([32]): Let $H_{ii} \in \mathbb{C}^{N_i\times M_i}$ and $H_{ij} \in \mathbb{C}^{N_i\times M_j}$ be two channel matrices with each entry independently chosen from $\mathsf{CN}(0, 1)$. Then, the following holds with probability 1 (over the randomness of channel matrices).

$$\begin{aligned}&\log\det(I_{N_i} + \rho H_{ii}H_{ii}^\dagger - \rho H_{ii}H_{ij}^\dagger(I_{N_j} + \rho H_{ij}H_{ij}^\dagger)^{-1}\rho H_{ij}H_{ii}^\dagger)\\ &= \min\{N_i, (M_i - N_j)^+\}\log\rho + o(\log\rho). \end{aligned} \quad (142)$$

Lemma 13: Let $H_{ii} \in \mathbb{C}^{N_i\times M_i}$ and $H_{ij} \in \mathbb{C}^{N_j\times M_i}$ be two channel matrices with each entry independently chosen from $\mathsf{CN}(0, 1)$. Then, the following holds with probability 1 (over the randomness of channel matrices).

$$\begin{aligned}&\log\det(I_{N_j} + \rho H_{ij}H_{ij}^\dagger - \rho H_{ij}H_{ii}^\dagger(I_{N_i} + \rho H_{ii}H_{ii}^\dagger)^{-1}\rho H_{ii}H_{ij}^\dagger)\\ &= \min\{N_j, (M_i - N_i)^+\}\log\rho + o(\log\rho). \end{aligned} \quad (143)$$

Proof: The proof is similar to that of Lemma 12. ∎

Now we find the high SNR limits of the bounds in (5)-(14) leading to Theorem 2.

(5)→(26)**:** Consider bound (5) in $\mathcal{R}_o$, we have

$$\begin{aligned}&\log\det\left(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger\right)\\ &\quad + \min\{\log\det\Big(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger - \rho_{12}\rho_{11}H_{12}H_{11}^\dagger\\ &\qquad \left(I_{N_1} + \rho_{11}H_{11}H_{11}^\dagger\right)^{-1}H_{11}H_{12}^\dagger\Big), C_{21}\}\\ &= \log\det(I_{N_1} + \rho H_{11}H_{11}^\dagger)\\ &\quad + \min\{\log\det(I_{N_2} + \rho^\alpha H_{12}H_{12}^\dagger - \rho^2 H_{12}H_{11}^\dagger\\ &\qquad (I_{N_1} + \rho H_{11}H_{11}^\dagger)^{-1}H_{11}H_{12}^\dagger), C_{21}\}\\ &\overset{(a)}{=} (\min\{M_1, N_1\} + \min\{\min\{N_2, (M_1 - N_1)^+\}, \beta_{21}\})\\ &\quad \log\mathsf{SNR} + o(\log\mathsf{SNR}), \end{aligned} \quad (144)$$

where (a) follows from Lemma 11 and Lemma 13. Now, dividing both sides by $\log \mathsf{SNR}$, we obtain (26).

(6)→(27)**:** This is obtained similarly to the last bound by exchanging 1 and 2 in the indices.

(7)→(28)**:** Consider bound (7) in $\mathcal{R}_o$, we have

$$\begin{aligned}
&\log\det\Big(I_{N_1}+\rho_{11}H_{11}H_{11}^\dagger+\rho_{21}H_{21}H_{21}^\dagger \\
&\qquad -\rho_{11}\rho_{12}H_{11}H_{12}^\dagger\Big(I_{N_2}+\rho_{12}H_{12}H_{12}^\dagger\Big)^{-1}H_{12}H_{11}^\dagger\Big) \\
&\quad +\log\det\Big(I_{N_2}+\rho_{22}H_{22}H_{22}^\dagger+\rho_{12}H_{12}H_{12}^\dagger- \\
&\qquad \rho_{22}\rho_{21}H_{22}H_{21}^\dagger\Big(I_{N_1}+\rho_{21}H_{21}H_{21}^\dagger\Big)^{-1}H_{21}H_{22}^\dagger\Big) \\
&\quad +C_{12}+C_{21} \\
&=\log\det(I_{N_1}+\rho H_{11}H_{11}^\dagger+\rho H_{21}H_{21}^\dagger-\rho\rho H_{11}H_{12}^\dagger \\
&\qquad (I_{N_2}+\rho H_{12}H_{12}^\dagger)^{-1}H_{12}H_{11}^\dagger) \\
&\quad +\log\det(I_{N_2}+\rho H_{22}H_{22}^\dagger+\rho H_{12}H_{12}^\dagger-\rho\rho H_{22}H_{21}^\dagger \\
&\qquad (I_{N_1}+\rho H_{21}H_{21}^\dagger)^{-1}H_{21}H_{22}^\dagger)+C_{12}+C_{21} \\
&\overset{(a)}{=}(\min\{N_1,(M_1-N_2)^+ +M_2\} \\
&\quad +\min\{N_2,(M_2-N_1)^+ +M_1\} \\
&\quad +\beta_{12}+\beta_{21})\log\mathsf{SNR}+o(\log\mathsf{SNR})), \qquad (145)
\end{aligned}$$

where (a) follows from Lemma 11 and Lemma 12. Now, dividing both sides by $\log \mathsf{SNR}$, we obtain (28).

(8)→(29)**:** Consider bound (8) in $\mathcal{R}_o$, we have

$$\begin{aligned}
&\log\det\Big(I_{N_1}+\rho_{11}H_{11}H_{11}^\dagger-\rho_{11}\rho_{12}H_{11}H_{12}^\dagger \\
&\qquad \Big(I_{N_2}+\rho_{12}H_{12}H_{12}^\dagger\Big)^{-1}H_{12}H_{11}^\dagger\Big) \\
&\quad +\log\det\Big(I_{N_2}+\rho_{22}H_{22}H_{22}^\dagger+\rho_{12}H_{12}H_{12}^\dagger\Big)+C_{12} \\
&=\log\det(I_{N_1}+\rho H_{11}H_{11}^\dagger-\rho\rho H_{11}H_{12}^\dagger \\
&\qquad (I_{N_2}+\rho^{\alpha}H_{12}H_{12}^\dagger)^{-1}H_{12}H_{11}^\dagger) \\
&\quad +\log\det(I_{N_2}+\rho H_{22}H_{22}^\dagger+\rho H_{12}H_{12}^\dagger)+C_{12} \\
&\overset{(a)}{=}(\min\{N_1,(M_1-N_2)^+\}+\min\{N_2,M_1+M_2\}+\beta_{12}) \\
&\quad \log\mathsf{SNR}+o(\log\mathsf{SNR}), \qquad (146)
\end{aligned}$$

where (a) follows from Lemma 11 and Lemma 13. Now, dividing both sides by $\log \mathsf{SNR}$, we obtain (29).

(9)→(30)**:** This is obtained similarly to the previous bound by exchanging 1 and 2 in the indices.

(10)→(31)**:** Consider bound (10) in $\mathcal{R}_o$, using Lemma 11 we have

$$\begin{aligned}
&\log\det\left(I_{N_1+N_2}+\begin{bmatrix}\sqrt{\rho_{11}}H_{11}\\ \sqrt{\rho_{12}}H_{12}\end{bmatrix}[\sqrt{\rho_{11}}H_{11}^\dagger\ \sqrt{\rho_{12}}H_{12}^\dagger]\right. \\
&\qquad \left.+\begin{bmatrix}\sqrt{\rho_{21}}H_{21}\\ \sqrt{\rho_{22}}H_{22}\end{bmatrix}[\sqrt{\rho_{21}}H_{21}^\dagger\ \sqrt{\rho_{22}}H_{22}^\dagger]\right) \\
&=\log\det\left(I_{N_1+N_2}+\rho\begin{bmatrix}H_{11}\\ H_{12}\end{bmatrix}[H_{11}^\dagger\ H_{12}^\dagger]+\rho\begin{bmatrix}H_{21}\\ H_{22}\end{bmatrix}\right. \\
&\qquad \left.[H_{21}^\dagger\ H_{22}^\dagger]\right) \\
&=\min\{N_1+N_2,M_1+M_2\}\log\mathsf{SNR}\ +o(\log\mathsf{SNR}\ ). \qquad (147)
\end{aligned}$$

(11)→(32)**:** Consider bound bound (11) in $\mathcal{R}_o$, we have

$$\begin{aligned}
&\log\det\Big(I_{N_1}+\rho_{11}H_{11}H_{11}^\dagger-\rho_{11}\rho_{12}H_{11}H_{12}^\dagger \\
&\qquad \Big(I_{N_2}+\rho_{12}H_{12}H_{12}^\dagger\Big)^{-1}H_{12}H_{11}^\dagger\Big) \\
&\quad +\log\det\Big(I_{N_2}+\rho_{22}H_{22}H_{22}^\dagger+\rho_{12}H_{12}H_{12}^\dagger \\
&\qquad -\rho_{22}\rho_{21}H_{22}H_{21}^\dagger\Big(I_{N_1}+\rho_{21}H_{21}H_{21}^\dagger\Big)^{-1}H_{21}H_{22}^\dagger\Big) \\
&\quad +\log\det\Big(I_{N_1}+\rho_{11}H_{11}H_{11}^\dagger+\rho_{21}H_{21}H_{21}^\dagger\Big) \\
&\quad +C_{12}+C_{21} \\
&=\log\det\log\det(I_{N_1}+\rho H_{11}H_{11}^\dagger-\rho H_{11}H_{12}^\dagger \\
&\qquad (I_{N_2}+\rho H_{12}H_{12}^\dagger)^{-1}\rho H_{12}H_{11}^\dagger) \\
&\quad +\log\det(I_{N_2}+\rho H_{22}H_{22}^\dagger+\rho H_{12}H_{12}^\dagger-\rho H_{22}H_{21}^\dagger \\
&\qquad (I_{N_1}+\rho H_{21}H_{21}^\dagger)^{-1}\rho H_{21}H_{22}^\dagger) \\
&\quad +\log\det(I_{N_1}+\rho H_{11}H_{11}^\dagger+\rho H_{21}H_{21}^\dagger)+\beta_{12}+\beta_{21} \\
&\overset{(a)}{=}\min\{N_2,(M_2-N_1)^+ +M_1\} \\
&\quad +\min\{N_1,(M_1-N_2)^+\}+\min\{N_1,M_1+M_2\} \\
&\quad +\beta_{12}+\beta_{21}, \qquad (148)
\end{aligned}$$

where (a) is obtained from Lemma 11 and Lemma 12. Now, dividing both sides by $\log \mathsf{SNR}$, we obtain (32).

(12)→(33)**:** This is obtained similarly to the previous bound by exchanging 1 and 2 in the indices.

(13)→(34)**:** Consider bound (13) in $\mathcal{R}_o$, we have

$$\begin{aligned}
&\log\det\left(I_{N_1+N_2}+\begin{bmatrix}\sqrt{\rho_{22}}H_{22}\\ \sqrt{\rho_{21}}H_{21}\end{bmatrix}\right. \\
&\qquad (I_{M_2}-\rho_{12}H_{21}^\dagger(I_{N_1}+\rho_{21}H_{21}H_{21}^\dagger)^{-1}H_{21}) \\
&\qquad [\sqrt{\rho_{22}}H_{22}^\dagger\ \sqrt{\rho_{21}}H_{21}^\dagger]+\begin{bmatrix}\sqrt{\rho_{12}}H_{12}\\ \sqrt{\rho_{11}}H_{11}\end{bmatrix} \\
&\qquad \left.[\sqrt{\rho_{12}}H_{12}^\dagger\ \sqrt{\rho_{11}}H_{11}^\dagger]\right) \\
&\quad +\log\det\Big(I_{N_1}+\rho_{11}H_{11}H_{11}^\dagger+\rho_{21}H_{21}H_{12}^\dagger\Big)+C_{21} \\
&=\log\det\left(I_{N_1+N_2}+\rho\begin{bmatrix}H_{22}\\ H_{21}\end{bmatrix}\right. \\
&\qquad (I_{M_2}-\rho H_{21}^\dagger(I_{N_1}+\rho H_{21}H_{21}^\dagger)^{-1}H_{21})[H_{22}^\dagger\ H_{21}^\dagger] \\
&\qquad \left.+\rho\begin{bmatrix}H_{12}\\ H_{11}\end{bmatrix}[H_{12}^\dagger\ H_{11}^\dagger]\right) \\
&\quad +\log\det\Big(I_{N_1}+\rho H_{11}H_{11}^\dagger+\rho H_{21}H_{12}^\dagger\Big)+C_{21} \\
&=(\min\{N_1+N_2,M_1\}+\min\{N_1,M_1+M_2\}+\beta_{21}) \\
&\quad \log\mathsf{SNR}+o(\log\mathsf{SNR}). \qquad (149)
\end{aligned}$$

(14)→(35)**:** This is obtained similarly to the previous bound by exchanging 1 and 2 in the indices.

Combining the above results we obtain Theorem 2 results.

Appendix E

Proof of Theorem 3

In this section, we will find the limit of $\mathcal{R}_o/\log\mathsf{SNR}$ as $\mathsf{SNR}\to\infty$ to get the result stated in Theorem 3 when

$C_{ij} \sim \mathsf{SNR}^{\beta_{ij}}$, $\rho_{ij} \sim \mathsf{SNR}$ for $i = j$ and $\rho_{ij} \sim \mathsf{SNR}^{\alpha}$ for $i \neq j$ where $\beta_{12}, \beta_{21} \in \mathbb{R}^{+}$.

This follows from Theorem 1 since the capacity region is inner and outer- bounded by $\mathcal{R}_o$ with constant gaps which would vanish for the DoF. Before going over each of the above terms and finding their high SNR limit. We first give some lemmas that will be used for the proof.

Lemma 14 ([4]): Let $H_1 \in \mathbb{C}^{M\times M}$, $H_2 \in \mathbb{C}^{M\times M}$, ..., and $H_k \in \mathbb{C}^{M\times M}$ be k full rank channel matrices. Then, the following holds

$$\log\det(I_M + \rho^{\alpha_1} H_1 H_1^{\dagger} + \rho^{\alpha_2} H_2 H_2^{\dagger} + \cdots + \rho^{\alpha_k} H_k H_k^{\dagger}) \\ = \max\{\alpha_1, \alpha_2, \ldots, \alpha_k\} M \log\rho + o(\log\rho). \quad (150)$$

Lemma 15 ([32]): Let $H_{ii} \in \mathbb{C}^{M\times M}$ and $H_{ij} \in \mathbb{C}^{M\times M}$ be two channel matrices with each entry independently chosen from $\mathsf{CN}(0,1)$. Then, the following holds with probability 1 (over the randomness of channel matrices).

$$\log\det(I_M + \rho H_{ii} H_{ii}^{\dagger} - \sqrt{\rho\rho^{\alpha}} H_{ii} H_{ij}^{\dagger} \\ (I_M + \rho^{\alpha} H_{ij} H_{ij}^{\dagger})^{-1} \sqrt{\rho\rho^{\alpha}} H_{ij} H_{ii}^{\dagger}) \\ = (1-\alpha)^{+} M \log\rho + o(\log\rho). \quad (151)$$

Lemma 16: Let $H_{ii} \in \mathbb{C}^{M\times M}$ and $H_{ij} \in \mathbb{C}^{M\times M}$ be two channel matrices with each entry independently chosen from $\mathsf{CN}(0,1)$. Then, the following holds with probability 1 (over the randomness of channel matrices).

$$\log\det(I_{N_j} + \rho^{\alpha} H_{ij} H_{ij}^{\dagger} - \sqrt{\rho\rho^{\alpha}} H_{ij} H_{ii}^{\dagger} \\ (I_{N_i} + \rho H_{ii} H_{ii}^{\dagger})^{-1} \sqrt{\rho\rho^{\alpha}} H_{ii} H_{ij}^{\dagger}) \\ = (\alpha - 1)^{+} M \log\rho + o(\log\rho). \quad (152)$$

Proof: The proof is similar to that of given in [32]. ∎

Lemma 17: Let $H \in \mathbb{C}^{M\times M}$ be a full rank channel matrix. Then, the following holds

$$I_M - \rho H^{\dagger}(I_M + \rho H H^{\dagger})^{-1} H = (I_M + \rho H^{\dagger} H)^{-1} \quad (153)$$

Proof: Let $B \triangleq I_M + \rho H^{\dagger} H$. Thus,

$$I_M - \rho H^{\dagger}(B^{\dagger})^{-1} H = (B)^{-1} \quad (154)$$

Since B is invertible, it is enough to show that

$$B - \rho H^{\dagger}(B^{\dagger})^{-1} H B = I_M, \quad (155)$$

which is equivalent to showing

$$\rho H^{\dagger}(B^{\dagger})^{-1} H B = \rho H^{\dagger} H \quad (156)$$

So it is enough to prove $(B^{\dagger})^{-1} H B = H$. Or, $HB = B^{\dagger} H$, which holds since $B = I_M + \rho H^{\dagger} H$. ∎

Now we find the high SNR limits of the bounds in (5)-(14) leading to Theorem 3.

(5)→(44)**:** Consider bound (5) in $\mathcal{R}_o$, we have

$$\log\det(I_M + \rho H_{11} H_{11}^{\dagger}) \\ + \min\{\log\det(I_M + \rho^{\alpha} H_{12} H_{12}^{\dagger} - \rho^{\alpha+1} H_{12} H_{11}^{\dagger} \\ (I_M + \rho H_{11} H_{11}^{\dagger})^{-1} H_{11} H_{12}^{\dagger}), C_{21}\} \\ \overset{(a)}{=} (M + \min\{(\alpha-1)^{+} M, \beta\}) \log\mathsf{SNR} \\ + o(\log\mathsf{SNR})), \quad (157)$$

where (a) follows from Lemma 14 and Lemma 16. Now, dividing both sides by $\log\mathsf{SNR}$, we obtain (44).

(6)→(45)**:** This is obtained similarly to the last bound by exchanging 1 and 2 in the indices.

(7)→(46)**:** Consider bound (7) in $\mathcal{R}_o$, we have

$$\log\det(I_M + \rho H_{11} H_{11}^{\dagger} + \rho^{\alpha} H_{21} H_{21}^{\dagger} - \rho^{\alpha+1} H_{11} H_{12}^{\dagger} \\ (I_{N_2} + \rho^{\alpha} H_{12} H_{12}^{\dagger})^{-1} H_{12} H_{11}^{\dagger}) \\ + \log\det(I_M + \rho H_{22} H_{22}^{\dagger} + \rho^{\alpha} H_{12} H_{12}^{\dagger} - \rho^{\alpha+1} H_{22} H_{21}^{\dagger} \\ (I_{N_1} + \rho^{\alpha} H_{21} H_{21}^{\dagger})^{-1} H_{21} H_{22}^{\dagger}) + C_{12} + C_{21} \\ \overset{(a)}{=} (2M \max\{(1-\alpha)^{+}, \alpha\} + 2\beta) \log\mathsf{SNR} + o(\log\mathsf{SNR})), \quad (158)$$

where (a) follows from Lemma 14 and Lemma 15. Now, dividing both sides by $\log\mathsf{SNR}$, we obtain (46).

(8)→(47)**:** Consider bound (8) in $\mathcal{R}_o$, we have

$$\log\det(I_M + \rho H_{11} H_{11}^{\dagger} - \rho^{\alpha+1} H_{11} H_{12}^{\dagger} \\ (I_M + \rho^{\alpha} H_{12} H_{12}^{\dagger})^{-1} H_{12} H_{11}^{\dagger}) \\ + \log\det(I_M + \rho H_{22} H_{22}^{\dagger} + \rho^{\alpha} H_{12} H_{12}^{\dagger}) + C_{12} \\ \overset{(a)}{=} ((1-\alpha)^{+} M + M \max\{1, \alpha\} + \beta) \log\mathsf{SNR} \\ + o(\log\mathsf{SNR})), \quad (159)$$

where (a) follows from Lemma 14 and Lemma 16. Now, dividing both sides by $\log\mathsf{SNR}$, we obtain (47).

(9)→(47)**:** This is obtained similarly to the last bound by exchanging 1 and 2 in the indices which gives the same bound as the last one.

(10)→(48)**:** Consider bound (10) in $\mathcal{R}_o$, we have

$$\log\det\left(I_{2M} + \begin{bmatrix} \sqrt{\rho} H_{11} \\ \sqrt{\rho^{\alpha}} H_{12} \end{bmatrix} \begin{bmatrix} \sqrt{\rho} H_{11}^{\dagger} & \sqrt{\rho^{\alpha}} H_{12}^{\dagger} \end{bmatrix} \right. \\ \left. + \begin{bmatrix} \sqrt{\rho^{\alpha}} H_{21} \\ \sqrt{\rho} H_{22} \end{bmatrix} \begin{bmatrix} \sqrt{\rho^{\alpha}} H_{21}^{\dagger} & \sqrt{\rho} H_{22}^{\dagger} \end{bmatrix} \right) \\ = \log\det\left(I_{2M} \right. \\ \left. + \begin{bmatrix} \rho H_{11} H_{11}^{\dagger} + \rho^{\alpha} H_{21} H_{21}^{\dagger} & \rho^{\frac{\alpha+1}{2}} (H_{11} H_{12}^{\dagger} + H_{21} H_{22}^{\dagger}) \\ \rho^{\frac{\alpha+1}{2}} (H_{12} H_{11}^{\dagger} + H_{22} H_{21}^{\dagger}) & \rho H_{22} H_{22}^{\dagger} + \rho^{\alpha} H_{12} H_{12}^{\dagger} \end{bmatrix} \right) \\ \overset{(a)}{=} \log\det(I_M + \rho H_{11} H_{11}^{\dagger} + \rho^{\alpha} H_{21} H_{21}^{\dagger}) \\ + \log\det(I_M + \rho H_{22} H_{22}^{\dagger} + \rho^{\alpha} H_{12} H_{12}^{\dagger} \\ - (\rho^{\frac{\alpha+1}{2}} H_{12} H_{11}^{\dagger} + \rho^{\frac{\alpha+1}{2}} H_{22} H_{21}^{\dagger}) \\ (I_M + \rho H_{11} H_{11}^{\dagger} + \rho^{\alpha} H_{21} H_{21}^{\dagger})^{-1} \\ (\rho^{\frac{\alpha+1}{2}} H_{11} H_{12}^{\dagger} + \rho^{\frac{\alpha+1}{2}} H_{21} H_{22}^{\dagger})) \\ = \log\det(I_M + \rho H_{11} H_{11}^{\dagger} + \rho^{\alpha} H_{21} H_{21}^{\dagger}) \\ + \log\det(I_M + \rho H_{22} H_{22}^{\dagger} + \rho^{\alpha} H_{12} H_{12}^{\dagger} - \rho^{\alpha+1} \\ (H_{12} H_{11}^{\dagger} + H_{22} H_{21}^{\dagger})(I_M + \rho H_{11} H_{11}^{\dagger} + \rho^{\alpha} H_{21} H_{21}^{\dagger})^{-1} \\ (H_{11} H_{12}^{\dagger} + H_{21} H_{22}^{\dagger})) \\ = (2M \max\{1, \alpha\}) \log\mathsf{SNR} + o(\log\mathsf{SNR}), \quad (160)$$

where (a) is obtained from Lemma 6. Now, dividing both sides by $\log\mathsf{SNR}$, we obtain (48).

(11)→(49): Consider bound (11) in $\mathcal{R}_o$, we have

$$\begin{aligned}
&\log\det\log\det(I_M+\rho H_{11}H_{11}^\dagger-\rho^{\alpha+1}H_{11}H_{12}^\dagger\\
&\qquad (I_{N_2}+\rho^\alpha H_{12}H_{12}^\dagger)^{-1}H_{12}H_{11}^\dagger)\\
&\quad+\log\det(I_M+\rho H_{22}H_{22}^\dagger+\rho^\alpha H_{12}H_{12}^\dagger\\
&\qquad -\rho^{\alpha+1}H_{22}H_{21}^\dagger(I_{N_1}+\rho^\alpha H_{21}H_{21}^\dagger)^{-1}H_{21}H_{22}^\dagger)\\
&\quad+\log\det(I_M+\rho H_{11}H_{11}^\dagger+\rho^\alpha H_{21}H_{21}^\dagger)+C_{12}+C_{21}\\
&\stackrel{(a)}{=}(M\max\{(1-\alpha)^+,\alpha\}+(1-\alpha)^+M+M\max\{1,\alpha\}+2\beta)\\
&\quad\log\mathsf{SNR}+o(\log\mathsf{SNR}),
\end{aligned}\tag{161}$$

where (a) is obtained from Lemma 14, Lemma 14 and Lemma 15. Now, dividing both sides by $\log\mathsf{SNR}$, we obtain (49).

(12)→(50): This is obtained similarly to the last bound by exchanging 1 and 2 in the indices.

(13)→(51): Define P as in (162), as shown at the bottom of the page. Consider bound (13) in $\mathcal{R}_o$, we have

$$\begin{aligned}
&\log\det\Bigg(I_{2M}+s\begin{bmatrix}\sqrt{\rho}H_{22}\\ \sqrt{\rho^\alpha}H_{21}\end{bmatrix}\Big(I_M-\rho^\alpha H_{21}^\dagger\\
&\qquad\Big(I_{N_1}+\rho^\alpha H_{21}H_{21}^\dagger\Big)^{-1}H_{21}\Big)\Big[\sqrt{\rho}H_{22}^\dagger\ \sqrt{\rho^\alpha}H_{21}^\dagger\Big]\\
&\qquad+\begin{bmatrix}\sqrt{\rho^\alpha}H_{12}\\ \sqrt{\rho}H_{11}\end{bmatrix}\Big[\sqrt{\rho^\alpha}H_{12}^\dagger\ \sqrt{\rho}H_{11}^\dagger\Big]\Bigg)\\
&\quad+\log\det\Big(I_M+\rho H_{11}H_{11}^\dagger+\rho^\alpha H_{21}H_{12}^\dagger\Big)+C_{21}\\
&\stackrel{(a)}{=}\log\det\Bigg(I_{2M}+\begin{bmatrix}\sqrt{\rho}H_{22}\\ \sqrt{\rho^\alpha}H_{21}\end{bmatrix}\Big(I_M+\rho^\alpha H_{21}^\dagger H_{21}\Big)^{-1}\\
&\qquad\Big[\sqrt{\rho}H_{22}^\dagger\ \sqrt{\rho^\alpha}H_{21}^\dagger\Big]+\begin{bmatrix}\sqrt{\rho^\alpha}H_{12}\\ \sqrt{\rho}H_{11}\end{bmatrix}\\
&\qquad\Big[\sqrt{\rho^\alpha}H_{12}^\dagger\ \sqrt{\rho}H_{11}^\dagger\Big]\Bigg)\\
&\quad+\log\det\Big(I_M+\rho H_{11}H_{11}^\dagger+\rho^\alpha H_{21}H_{12}^\dagger\Big)+C_{21}\\
&\stackrel{(b)}{=}(M\max\{1,\alpha\}+\beta)\log\mathsf{SNR}+o(\log\mathsf{SNR})+\log\det(P)\\
&\stackrel{(c)}{=}(M\max\{1,\alpha\}+\beta)\log\mathsf{SNR}+o(\log\mathsf{SNR})\\
&\quad+\log\det(I_M+\rho H_{11}H_{11}^\dagger+\rho^\alpha H_{21}\\
&\quad(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{21}^\dagger)\\
&\quad+\log\det(I_M+\rho H_{22}(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{22}^\dagger+\rho^\alpha H_{12}H_{12}^\dagger\\
&\quad-(\rho^{\frac{\alpha+1}{2}}(H_{22}(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{21}^\dagger+H_{12}H_{11}^\dagger))\\
&\quad(I_M+\rho H_{11}H_{11}^\dagger+\rho^\alpha H_{21}(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{21}^\dagger)^{-1}\\
&\quad(\rho^{\frac{\alpha+1}{2}}(H_{21}(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{22}^\dagger+H_{11}H_{12}^\dagger)))\\
&=(M+M\max\{1,\alpha\}+\beta)\log\mathsf{SNR}\\
&\quad+\log\det(I_M+\rho H_{22}(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{22}^\dagger+\rho^\alpha H_{12}H_{12}^\dagger\\
&\quad-\rho^{\alpha+1}(H_{22}(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{21}^\dagger+H_{12}H_{11}^\dagger)\\
&\quad(I_M+\rho H_{11}H_{11}^\dagger+\rho^\alpha H_{21}(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{21}^\dagger)^{-1}\\
&\quad(H_{21}(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{22}^\dagger+H_{11}H_{12}^\dagger))+o(\log\mathsf{SNR})\\
&=(M+M\max\{1,\alpha\}+\beta)\log\mathsf{SNR}\\
&\quad+\log\det(I_M+\rho H_{22}(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{22}^\dagger+\rho^\alpha H_{12}H_{12}^\dagger\\
&\quad-\rho^{\alpha+1}H_{12}H_{11}^\dagger(I_M+\rho H_{11}H_{11}^\dagger+\rho^\alpha H_{21}\\
&\quad(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{21}^\dagger)^{-1}H_{11}H_{12}^\dagger-\rho^{\alpha+1}H_{22}\\
&\quad(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{21}^\dagger(I_M+\rho H_{11}H_{11}^\dagger+\rho^\alpha H_{21}\\
&\quad(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{21}^\dagger)^{-1}H_{11}H_{12}^\dagger-\rho^{\alpha+1}H_{12}H_{11}^\dagger\\
&\quad(I_M+\rho H_{11}H_{11}^\dagger+\rho^\alpha H_{21}(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{21}^\dagger)^{-1}H_{21}\\
&\quad(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{22}^\dagger-\rho^{\alpha+1}H_{22}\\
&\quad(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{21}^\dagger(I_M+\rho H_{11}H_{11}^\dagger+\rho^\alpha H_{21}\\
&\quad(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{21}^\dagger)^{-1}H_{21}(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{22}^\dagger)\\
&\quad+o(\log\mathsf{SNR})\\
&\stackrel{(d)}{=}(M+M\max\{1,\alpha\}+\beta)\log\mathsf{SNR}\\
&\quad+\log\det(I_M+\rho H_{22}(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{22}^\dagger\\
&\quad+\rho^\alpha H_{12}H_{12}^\dagger-\rho^{\alpha+1}H_{12}H_{11}^\dagger(I_M+\rho H_{11}H_{11}^\dagger+\rho^\alpha H_{21}\\
&\quad(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{21}^\dagger)^{-1}H_{11}H_{12}^\dagger)+o(\log\mathsf{SNR})\\
&=(M+M\max\{1,\alpha\}+\beta)\log\mathsf{SNR}\\
&\quad+\log\det(I_M+\rho H_{22}(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{22}^\dagger\\
&\quad+\rho^\alpha H_{12}H_{12}^\dagger-\rho^{\alpha+1}H_{12}H_{11}^\dagger(I_M+\rho H_{11}H_{11}^\dagger\\
&\quad+(\rho^{-\alpha}H_{21}^{\dagger\,-1}(I_M+\rho^\alpha H_{21}^\dagger H_{21})H_{21}^{-1})^{-1})^{-1}H_{11}H_{12}^\dagger)\\
&\quad+o(\log\mathsf{SNR})\\
&=(M+M\max\{1,\alpha\}+\beta)\log\mathsf{SNR}\\
&\quad+\log\det(I_M+\rho H_{22}(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{22}^\dagger\\
&\quad+\rho^\alpha H_{12}H_{12}^\dagger-\rho^{\alpha+1}H_{12}H_{11}^\dagger(I_M+\rho H_{11}H_{11}^\dagger\\
&\quad+(\rho^{-\alpha}H_{21}^{\dagger\,-1}H_{21}^{-1}+I_M)^{-1})^{-1}H_{11}H_{12}^\dagger)+o(\log\mathsf{SNR})\\
&=(M+M\max\{1,\alpha\}+\beta)\log\mathsf{SNR}\\
&\quad+\log\det(I_M+\rho H_{22}(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{22}^\dagger\\
&\quad+\rho^\alpha H_{12}(I_M-\rho H_{11}^\dagger(I_M+\rho H_{11}H_{11}^\dagger\\
&\quad+(\rho^{-\alpha}H_{21}^{\dagger\,-1}H_{21}^{-1}+I_M)^{-1})^{-1}H_{11})H_{12}^\dagger)\\
&\quad+o(\log\mathsf{SNR})\\
&=(M+M\max\{1,\alpha\}+\beta)\log\mathsf{SNR}\\
&\quad+\log\det(I_M+\rho H_{22}(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{22}^\dagger\\
&\quad+\rho^\alpha H_{12}(I_M-(\rho^{-1}H_{11}^{-1}(I_M+\rho H_{11}H_{11}^\dagger\\
&\quad+(\rho^{-\alpha}H_{21}^{\dagger\,-1}H_{21}^{-1}+I_M)^{-1})H_{11}^{\dagger\,-1})^{-1})H_{12}^\dagger)\\
&\quad+o(\log\mathsf{SNR})\\
&\stackrel{(e)}{=}(M+M\max\{1,\alpha\}+\beta)\log\mathsf{SNR}\\
&\quad+\log\det(I_M+\rho H_{22}(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{22}^\dagger\\
&\quad+\rho^\alpha H_{12}(\rho^{-1}H_{11}^{-1}(I_M+\rho H_{11}H_{11}^\dagger\\
&\quad+(\rho^{-\alpha}H_{21}^{\dagger\,-1}H_{21}^{-1}+I_M)^{-1})H_{11}^{\dagger\,-1}-I_M)\\
&\quad(\rho^{-1}H_{11}^{-1}(I_M+\rho H_{11}H_{11}^\dagger+(\rho^{-\alpha}H_{21}^{\dagger\,-1}H_{21}^{-1}\\
&\quad+I_M)^{-1})H_{11}^{\dagger\,-1})^{-1}H_{12}^\dagger)+o(\log\mathsf{SNR})
\end{aligned}$$

$$P\triangleq\begin{bmatrix} I_M+\rho H_{22}(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{22}^\dagger+\rho^\alpha H_{12}H_{12}^\dagger & \rho^{\frac{\alpha+1}{2}}(H_{22}(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{21}^\dagger+H_{12}H_{11}^\dagger)\\ \rho^{\frac{\alpha+1}{2}}(H_{21}(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{22}^\dagger+H_{11}H_{12}^\dagger) & I_M+\rho H_{11}H_{11}^\dagger+\rho^\alpha H_{21}(I_M+\rho^\alpha H_{21}^\dagger H_{21})^{-1}H_{21}^\dagger\end{bmatrix}\tag{162}$$

$$
\begin{aligned}
&= (M+M\max\{1,\alpha\}+\beta)\log\mathsf{SNR} \\
&\quad +\log\det(I_M+\rho H_{22}(I_M+\rho^{\alpha}H_{21}^{\dagger}H_{21})^{-1}H_{22}^{\dagger} \\
&\quad +\rho^{\alpha}H_{12}(\rho^{-1}H_{11}^{-1}(I_M+\rho H_{11}H_{11}^{\dagger} \\
&\quad +(\rho^{-\alpha}H_{21}^{\dagger\,-1}H_{21}^{-1}+I_M)^{-1})H_{11}^{\dagger\,-1} \\
&\quad -\rho^{-1}H_{11}^{-1}(\rho H_{11}H_{11}^{\dagger})H_{11}^{\dagger\,-1})(\rho^{-1}H_{11}^{-1} \\
&\quad (I_M+\rho H_{11}H_{11}^{\dagger}+(\rho^{-\alpha}H_{21}^{\dagger\,-1}H_{21}^{-1} \\
&\quad +I_M)^{-1})H_{11}^{\dagger\,-1})^{-1}H_{12}^{\dagger})+o(\log\mathsf{SNR}) \\
&= (M+M\max\{1,\alpha\}+\beta)\log\mathsf{SNR} \\
&\quad +\log\det(I_M+\rho H_{22}(I_M+\rho^{\alpha}H_{21}^{\dagger}H_{21})^{-1}H_{22}^{\dagger}+\rho^{\alpha}H_{12} \\
&\quad (\rho^{-1}H_{11}^{-1}(I_M+(\rho^{-\alpha}H_{21}^{\dagger\,-1}H_{21}^{-1}+I_M)^{-1})H_{11}^{\dagger\,-1}) \\
&\quad (\rho^{-1}H_{11}^{-1}(I_M+\rho H_{11}H_{11}^{\dagger}+(\rho^{-\alpha}H_{21}^{\dagger\,-1}H_{21}^{-1} \\
&\quad +I_M)^{-1})H_{11}^{\dagger\,-1})^{-1}H_{12}^{\dagger})+o(\log\mathsf{SNR}) \\
&= (M+M\max\{1,\alpha\}+\beta)\log\mathsf{SNR} \\
&\quad +M\max\{(1-\alpha)^{+},\alpha-1\}\log\mathsf{SNR}\ +o(\log\mathsf{SNR}) \\
&= (M\max\{(2-\alpha)^{+},\alpha\}+M\max\{1,\alpha\}+\beta)\log\mathsf{SNR} \\
&\quad +o(\log\mathsf{SNR}), \qquad (163)
\end{aligned}
$$

where (a) is obtained from Lemma 17, (b) is obtained from Lemma 14, (c) is obtained from Lemma 6, (d) is because the three eliminated sentences have a constant upper bounds and (e) follows from the fact that $I_M - X^{-1} = (X - I_M)X^{-1}$ where $X = \rho^{-1}H_{11}^{-1}(I_M + \rho H_{11}H_{11}^{\dagger} + (\rho^{-\alpha}H_{21}^{\dagger\,-1}H_{21}^{-1}+I_M)^{-1})H_{11}^{\dagger\,-1}$. Now, dividing both sides by $\log\mathsf{SNR}$, we obtain (51).

(14)→(52)**:** This is obtained similarly to the last bound by exchanging 1 and 2 in the indices.

Combining the above results we obtain Theorem 3 results.

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Mehdi Ashraphijuo received the B.Sc. and M.Sc. degrees in electrical engineering from Sharif University of Technology, Tehran, Iran, in 2010 and 2012, respectively. He is currently working towards the Ph.D. degree in the Department of Electrical Engineering, Columbia University, New York, NY. He was the winner of 2014 Qualcomm Innovation Fellowship. His research interests lie in the general areas of network information theory and statistical signal processing with applications in wireless communication.

He has worked in the areas of digital communications, underwater sensor networks, and applications of the queuing theory in wireless sensor networks. His current research focuses primarily on developing fundamental principles for communication network design, with emphasis on developing interference management and understanding the role of feedback and cooperation.

Vaneet Aggarwal (S'08–M'11) received the B.Tech. degree in 2005 from the Indian Institute of Technology, Kanpur, India, and the M.A. and Ph.D. degrees in 2007 and 2010, respectively from Princeton University, Princeton, NJ, USA, all in Electrical Engineering.

He is currently a Senior Member of Technical Staff at AT&T Labs-Research, Florham Park, NJ, and an Adjunct Assistant Professor at Columbia University, New York, NY. His research interests are in applications of information and coding theory to wireless systems and distributed storage systems. Dr. Aggarwal was the recipient of Princeton University's Porter Ogden Jacobus Honorific Fellowship in 2009.

Xiaodong Wang (S'98–M'98–SM'04–F'08) received the Ph.D degree in Electrical Engineering from Princeton University. He is a Professor of Electrical Engineering at Columbia University in New York. Dr. Wang's research interests fall in the general areas of computing, signal processing and communications, and has published extensively in these areas. Among his publications is a book entitled *Wireless Communication Systems: Advanced Techniques for Signal Reception* (Prentice Hall, 2003). His current research interests include wireless communications, statistical signal processing, and genomic signal processing. Dr. Wang received the 1999 NSF CAREER Award, the 2001 IEEE Communications Society and Information Theory Society Joint Paper Award, and the 2011 IEEE Communication Society Award for Outstanding Paper on New Communication Topics. He has served as an Associate Editor for the IEEE TRANSACTIONS ON COMMUNICATIONS, the IEEE TRANSACTIONS ON WIRELESS COMMUNICATIONS, the IEEE TRANSACTIONS ON SIGNAL PROCESSING, and the IEEE TRANSACTIONS ON INFORMATION THEORY. He is listed as an ISI Highly-cited Author.