On the DoF of Two-Way $2 \times 2 \times 2$ MIMO Relay Networks

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***Abstract*—This paper studies the degrees of freedom (DoF) of two-way $2 \times 2 \times 2$ MIMO interference networks, a class of two-way four-unicast networks. We first provide upper and lower bounds on the sum DoF of general two-way $2 \times 2 \times 2$ MIMO interference networks with any number of antennas in each node. We then investigate the special case where all user nodes have M antennas and relay nodes have N antennas, and provide the simplified bounds on the sum DoF. We show that our proposed achievable rate for this special case is higher than the achievable rates recently proposed in [1]. Moreover, for this special case, we obtain the exact DoF $= 4M$ when $N \geq 2M$, and DoF $= 4N$ when $M \leq 2N$.**

***Index Terms*—Degrees of freedom, four-unicast channels, two-way relay, $2 \times 2 \times 2$ network, MIMO interference network.**

I. Introduction

TWO-WAY relay is an effective approach to improving spectral efficiency in wireless networks [2], [3], where a pair of user nodes exchange information via the intermediate relay node in two phases: the multiple-access phase and broadcast phase. Specifically, in the multiple-access phase, the two user nodes transmit their messages simultaneously to the relay node; while in the broadcast phase the relay node broadcasts the superimposed signal to the two user nodes. Each user node extracts its intended information through self-interference cancellation. Multi-pair two-way relay has also been considered where a common relay node is used to facilitate simultaneous information exchanges of multiple user pairs. To mitigate the inter-pair interference, either some orthogonal multiple-access scheme is employed to assign different user pairs to different orthogonal resources [4], [5], or the MIMO technique such as beamforming [6], [7] or interference alignment [8] is applied. In this paper, our objective is to investigate the fundamental performance limit of a two-way MIMO relay network with two pairs of user nodes and two relays, known as a $2 \times 2 \times 2$ network, in terms of its degrees of freedom.

Even though the one-way $2 \times 2 \times 2$ SISO interference network has two degrees of freedom (DoF), reccently in [9] we showed that the DoF of the two-way $2 \times 2 \times 2$ SISO interference network is no larger than $8/3$ indicating that the bidirectional links cannot double the DoF for this network. Finite-field two-way $2 \times 2 \times 2$ SISO models are also studied in [10], [11], which is however not applicable to Gaussian models. The two-way $2 \times 2 \times 2$ interference network is a class of two-way four-unicast networks, also known as the two-way layered interference channel.

In this paper, we study the DoF for the two-way $2 \times 2 \times 2$ MIMO interference network. It is shown in [12] that for the one-way $2 \times 2 \times 2$ interference network with M antennas at all terminals, the DoF is $2M$. And in [13] the DoF of the general one-way $2 \times 2 \times 2$ MIMO interference network is obtained. Also, the DoF of a symmetric one-way $2 \times 2 \times 2$ MIMO interference network with a non-symmetric delayed feedback is investigated in [14]. Moreover, recently in [1], three different achievability strategies are proposed for the two-way $2 \times 2 \times 2$ MIMO interference network where all user nodes have M antennas and relay nodes have N antennas.

The main contribution of this paper is to provide upper and lower bounds on the DoF for the general two-way $2 \times 2 \times 2$ MIMO interference network with arbitrary number of antennas at each node. Specifically, a new achievability scheme is proposed that performs an interference neutralization scheme, which exploits side-information inherent to two-way communications so as to obtain both interference neutralization gain and network coding gain. For the case where all user nodes have M antennas and relay nodes have N antennas, we show that our proposed achievability strategy outperforms all achievability strategies in [1]. For this special case, for some cases that N or M is the bottleneck for the transmission, we found the exact DoF, i.e., if $M \geq 2N$, DoF $= 4N$ and if $N > 2M$, DoF $= 4M$.

The remainder of this paper is as follows. In Section II, the two-way $2 \times 2 \times 2$ interference channel model is given. In Sections III and IV, we present upper and lower bounds on the DoF of the two-way $2 \times 2 \times 2$ MIMO interference network, respectively. We specialize these bounds to the case where the user nodes have M nodes and the relay nodes have N nodes in Section V. Finally, Section VI concludes the paper.

II. Channel Model

As shown in Fig. 1, the two-way $2 \times 2 \times 2$ MIMO interference network consists of four transceiver nodes and two relays R_1, R_2. Transceiver node i is equipped with M_i antennas and

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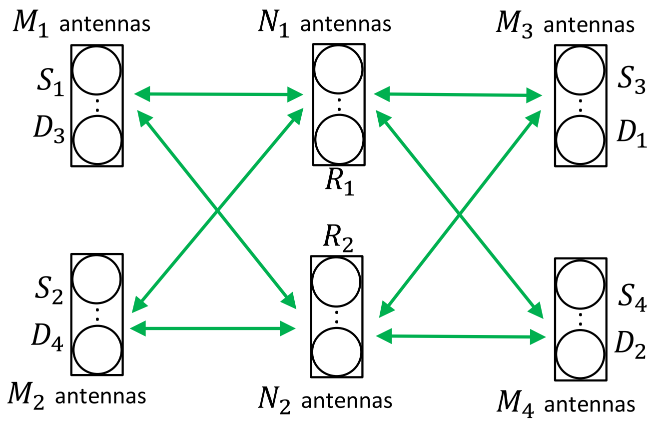
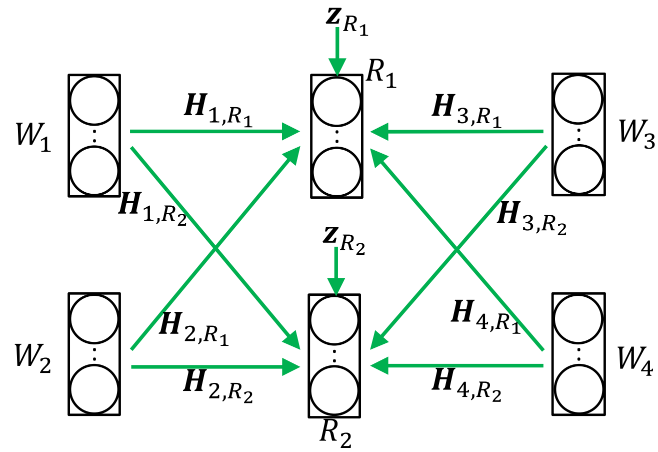
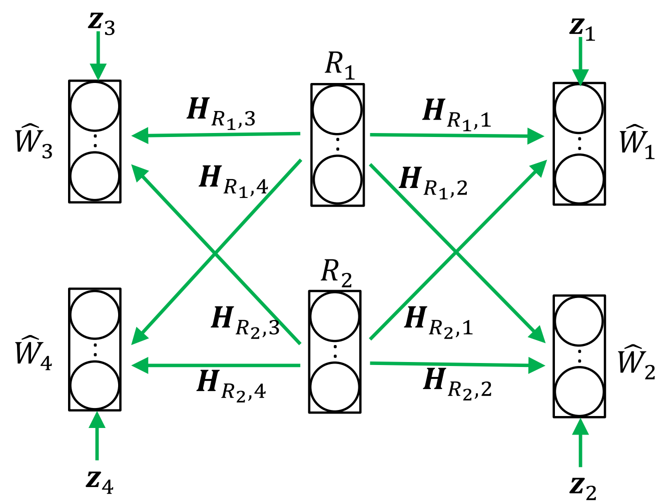


Fig. 1. A two-way 2 × 2 × 2 MIMO interference network.



(a) The channels from transmitters to the relays.



(b) The channels from relays to the receivers.

Fig. 2. The channels from and to relays in a two-way 2 × 2 × 2 MIMO interference network.

consists of transmitter (source) S_i and receiver (destination) $D_{q(i)}$, where $q_i = i + 2$ for $i = 1, 2$ and $q_i = i - 2$ for $i = 3, 4$. Each transmitter S_i has one message that is intended for its designated receiver D_i, $i \in \{1, \dots, 4\}$. The relay R_k comprises of N_k antennas, $k \in \{1, 2\}$. Fig. 2 shows the two hops of this system separately. In the first hop (Fig. 2(a)), the signal received at relay R_k, $k \in \{1, 2\}$, in time slot m is expressed as

$$\mathbf{y}_{R_k}[m] = \sum_{i=1}^{4} \mathbf{H}_{i,R_k} \mathbf{x}_i[m] + \mathbf{z}_{R_k}[m], \tag{1}$$

where $\mathbf{H}_{i,R_k}$ is the $N_k \times M_i$ complex channel matrix from transmitter S_i to relay R_k, $\mathbf{x}_i[m]$ is the $M_i \times 1$ signal vector transmitted from S_i, $\mathbf{y}_{R_k}[m]$ is the $N_k \times 1$ signal vector received at relay R_k and $\mathbf{z}_{R_k}[m]$ is the $N_k \times 1$ circularly symmetric complex Gaussian noise vector with i.i.d. zero mean and unit variance entries, $i \in \{1, 2, 3, 4\}$, $k \in \{1, 2\}$. In the second hop (Fig. 2(b)), the signal received at receiver D_i in time slot m is given by

$$\mathbf{y}_i[m] = \mathbf{H}_{R_1,i}\mathbf{x}_{R_1}[m] + \mathbf{H}_{R_2,i}\mathbf{x}_{R_2}[m] + \mathbf{z}_i[m], \tag{2}$$

for $i \in \{1, \dots, 4\}$, where $\mathbf{H}_{R_k,i}$ is the $M_{q(i)} \times N_k$ complex channel matrix from relay R_k to receiver D_i, $\mathbf{x}_{R_k}[m]$ is the $N_k \times 1$ signal vector transmitted from R_k, $\mathbf{y}_i[m]$ is the $M_{q(i)} \times 1$ signal received at receiver D_i and $\mathbf{z}_i[m]$ is the $M_{q(i)} \times 1$ circularly symmetric complex Gaussian noise vector with i.i.d. zero mean and unit variance entries, $i \in \{1, 2, 3, 4\}$, $k \in \{1, 2\}$. We assume that the channel coefficient values are drawn i.i.d. from a continuous distribution and their magnitudes are bounded from below and above by $H_{\min}$ and $H_{\max}$ respectively as in [15]. Furthermore, the relays are assumed to be causal, which means that the signals transmitted from the relays depend only on the signals received in the past and not on the current received signals and can be described as

$$\mathbf{x}_{R_k}[m] = f(\mathbf{Y}_{R_k}^{m-1}, \mathbf{X}_{R_k}^{m-1}), \tag{3}$$

where $\mathbf{X}_{R_k}^{m-1} \triangleq (\mathbf{x}_{R_k}[1], \dots, \mathbf{x}_{R_k}[m-1])$, $\mathbf{Y}_{R_k}^{m-1} \triangleq (\mathbf{y}_{R_k}[1], \dots, \mathbf{y}_{R_k}[m-1])$. We assume that each source S_i knows only channels $\mathbf{H}_{i,R_k}$, $k \in \{1, 2\}$; each relay knows all the channels; and each destination D_i knows only channels $\mathbf{H}_{R_k,i}$, $k \in \{1, 2\}$.

The source S_i has a message W_i that is intended for destination D_i. $|W_i|$ denotes the size of the message W_i. The rates $\mathcal{R}_i = \frac{\log |W_i|}{n}$, $i \in \{1, 2, 3, 4\}$ are achievable during n channel uses when n is large enough, if the probability of error can be arbitrarily small for all four messages simultaneously. The capacity region $\mathcal{C} = \{(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4)\}$ represents the set of all achievable quadruples. The sum-capacity is the maximum sum-rate that is achievable, i.e., $\mathcal{C}_\Sigma(P) = \sum_{i=1}^4 \mathcal{R}_i^c$ where $(\mathcal{R}_1^c, \mathcal{R}_2^c, \mathcal{R}_3^c, \mathcal{R}_4^c) = \arg\max_{(\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3,\mathcal{R}_4)\in\mathcal{C}} \sum_{i=1}^4 \mathcal{R}_i$ and P is the transmit power at each node (source or relay). The degrees of freedom (DoF) is defined as

$$DoF \triangleq \lim_{P\to\infty} \frac{\mathcal{C}_\Sigma(P)}{\log P} = \sum_{i=1}^{4} \lim_{P\to\infty} \frac{\mathcal{R}_i^c}{\log P} = \sum_{i=1}^{4} d_i, \tag{4}$$

where $d_i \triangleq \lim_{P\to\infty} \frac{\mathcal{R}_i^c}{\log P}$ is the DoF of source S_i, for $i \in \{1, 2, 3, 4\}$.

III. DoF Upper Bounds

In [9], an upper bound on the DoF of two-way $2 \times 2 \times 2$ SISO interference networks is given. Next we generalize that bound to the MIMO case with any number of antennas in each node.

Theorem 1: For the general two-way $2 \times 2 \times 2$ MIMO interference network, DoF $\leq \frac{2}{3} \max\{M_1 + M_3, N_1 + N_2\} + \frac{2}{3} \max\{M_2 + M_4, N_1 + N_2\}$.

Proof: For the upper bound, we assume that the relays have access to each other's message as side information. Consider n time slots of the channel use and assume that nR_i represents the maximum rate achievable for transmitter i in the total n time slots. Define $\mathbf{Y}_i^n \triangleq (\mathbf{y}_i[1], \ldots, \mathbf{y}_i[n])$ and $\mathbf{X}_i^n \triangleq (\mathbf{x}_i[1], \ldots, \mathbf{x}_i[n])$. We also define $\mathbf{H}_{j,R} \triangleq [\mathbf{H}_{j,R_1} \;\; \mathbf{H}_{j,R_2}]$, $\mathbf{Y}_R^n \triangleq [\mathbf{Y}_{R_1}^n \;\; \mathbf{Y}_{R_2}^n]$, and $\mathbf{Z}_R^n \triangleq [\mathbf{Z}_{R_1}^n \;\; \mathbf{Z}_{R_2}^n]$, where $\mathbf{Y}_{R_k}^n \triangleq (\mathbf{y}_{R_k}[1], \ldots, \mathbf{y}_{R_k}[n])$ and $\mathbf{Z}_{R_k}^n \triangleq (\mathbf{z}_{R_k}[1], \ldots, \mathbf{z}_{R_k}[n])$. Then, we have:

$$
\begin{aligned}
&nR_3 \\
&\overset{(a)}{\leq} I(W_3; \mathbf{Y}_3^n) + n\epsilon_n \\
&\overset{(b)}{\leq} I(W_3; \mathbf{Y}_3^n | W_1) + n\epsilon_n \\
&\overset{(c)}{\leq} I(W_3; \mathbf{Y}_R^n | W_1) + n\epsilon_n \\
&= h(\mathbf{Y}_R^n | W_1) - h(\mathbf{Y}_R^n | W_1, W_3) + n\epsilon_n \\
&= h(\mathbf{Y}_R^n | W_1) - I(\mathbf{Y}_R^n; W_2, W_4 | W_1, W_3) \\
&\quad - h(\mathbf{Y}_R^n | W_1, W_3, W_2, W_4) + n\epsilon_n \\
&\overset{(d)}{=} h(\mathbf{Y}_R^n | W_1) - I(\mathbf{Y}_R^n; W_2, W_4 | W_1, W_3) - h(\mathbf{Z}_R^n) \\
&\quad + n\epsilon_n \\
&= h(\mathbf{Y}_R^n | W_1) - H(W_2, W_4 | W_1, W_3) \\
&\quad + H(W_2, W_4 | W_1, W_3, \mathbf{Y}_R^n) - 2n(N_1 + N_2) \log(2\pi e) \\
&\quad + n\epsilon_n \\
&\overset{(e)}{\leq} h(\mathbf{Y}_R^n | W_1) - H(W_2, W_4) \\
&\quad + H(W_2, W_4 | \mathbf{Y}_R^n - \mathbf{H}_{1,R}\mathbf{X}_1^n - \mathbf{H}_{3,R}\mathbf{X}_3^n) \\
&\quad - 2n(N_1 + N_2) \log(2\pi e) + n\epsilon_n \\
&= h(\mathbf{Y}_R^n | W_1) - H(W_2, W_4) \\
&\quad + H(W_2, W_4 | \mathbf{H}_{2,R}\mathbf{X}_2^n + \mathbf{H}_{4,R}\mathbf{X}_4^n + \mathbf{Z}_R^n) \\
&\quad - 2n(N_1 + N_2) \log(2\pi e) + n\epsilon_n \\
&\overset{(f)}{\leq} h(\mathbf{Y}_R^n | W_1) - H(W_2, W_4) - 2n(N_1 + N_2) \log(2\pi e) \\
&\quad + (M_2 + M_4 - N_1 - N_2)^+ \left(\log \left(2\pi e (2H_{\max}^2 P) \right)^n \right) \\
&\quad + n\epsilon_n^{'} + n\epsilon_n \\
&\overset{(g)}{\leq} h(\mathbf{Y}_R^n) - H(W_2, W_4) - 2n(N_1 + N_2) \log(2\pi e) \\
&\quad + (M_2 + M_4 - N_1 - N_2)^+ \left(\log \left(2\pi e (2H_{\max}^2 P) \right)^n \right) \\
&\quad + n\epsilon_n^{''} \\
&\overset{(h)}{\leq} h(\mathbf{Y}_{R_1}^n) + h(\mathbf{Y}_{R_2}^n) - H(W_2, W_4) \\
&\quad - 2n(N_1 + N_2) \log(2\pi e) \\
&\quad + (M_2 + M_4 - N_1 - N_2)^+ \left(\log \left(2\pi e (2H_{\max}^2 P) \right)^n \right) \\
&\quad + n\epsilon_n^{''} \\
&\overset{(i)}{\leq} (N_1 + N_2) \left(\log \left(2\pi e (4H_{\max}^2 P + 1) \right)^n \right) \\
&\quad - H(W_2, W_4) - 2n(N_1 + N_2) \log(2\pi e) \\
&\quad + (M_2 + M_4 - N_1 - N_2)^+ \left(\log \left(2\pi e (2H_{\max}^2 P) \right)^n \right) \\
&\quad + n\epsilon_n^{''},
\end{aligned} \tag{5}
$$

where (a) follows since the transmission rate is less than or equal to the mutual information between the message and the received signal, and ϵ_n can be arbitrarily small by increasing n; (b) follows since $I(W_3; \mathbf{Y}_3^n | W_1) - I(W_3; \mathbf{Y}_3^n) = I(W_3; \mathbf{Y}_3^n; W_1) \geq -\min\{I(W_3; \mathbf{Y}_3^n), I(W_1; \mathbf{Y}_3^n), I(W_3; W_1)\} = 0$ (as $I(W_3; W_1) = 0$); (c) holds since $W_3 \rightarrow \mathbf{Y}_R^n \rightarrow \mathbf{Y}_3^n$; (d) follows since by subtracting the contributions of $\mathbf{X}_i^n$, $i = 1, \ldots, 4$ from $\mathbf{Y}_R^n$, we will only have Gaussian noise at the relays; (e) follows from Lemma 1 below; (f) follows from Lemma 2 below; (g) holds because conditioning decreases the entropy; (h) holds since $h(X, Y) \leq h(X) + h(Y)$; and (i) holds since $\mathbf{y}_{R_i}$ is in the form of (1), with $|H_{i,R_k}[m]| \leq H_{\max}$, and $X_i \sim \mathcal{CN}(0, P)$.

Lemma 1: The following inequality holds:

$$
\begin{aligned}
&H(W_2, W_4 | W_1, W_3, \mathbf{Y}_R^n) \\
&\leq H(W_2, W_4 | \mathbf{Y}_R^n - \mathbf{H}_{1,R}\mathbf{X}_1^n - \mathbf{H}_{3,R}\mathbf{X}_3^n).
\end{aligned} \tag{6}
$$

Proof: For any X and Y, the following relations hold:

$$
H(X|Y) = H(X|Y, f(Y)) \leq H(X|f(Y)), \tag{7}
$$

where the equality follows from the fact that $f(Y)$ is only a function of Y and the inequality follows from the fact that conditioning reduces entropy. Therefore, since $\mathbf{X}_1^n$ and $\mathbf{X}_3^n$ are functions of W_1, W_3, and $\mathbf{Y}_R^n$, it can be seen that (6) follows from (7). ■

Lemma 2: The following inequality holds:

$$
\begin{aligned}
&H(W_2, W_4 | \mathbf{H}_{2,R}\mathbf{X}_2^n + \mathbf{H}_{4,R}\mathbf{X}_4^n + \mathbf{Z}_R^n) \\
&\leq (M_2 + M_4 - N_1 - N_2)^+ \left(\log \left(2\pi e (2H_{\max}^2 P) \right)^n \right) + n\epsilon_n^{'}.
\end{aligned} \tag{8}
$$

Proof: The signal $\mathbf{H}_{2,R}\mathbf{X}_2^n + \mathbf{H}_{4,R}\mathbf{X}_4^n + \mathbf{Z}_R^n$ includes $N_1 + N_2$ equations, but $[\mathbf{X}_2^n \; \mathbf{X}_4^n]$ includes streams from $M_2 + M_4$ antennas. If we consider another $(M_2 + M_4 - N_1 - N_2)^+$ sets of equations as $\mathbf{H}_{2,R}^{'}\mathbf{X}_2^n + \mathbf{H}_{4,R}^{'}\mathbf{X}_4^n$ such that all of the entries of $\mathbf{H}_{i,R}^{'}$ are chosen independently from the same continuous distribution as $\mathbf{H}_{i,R}$, then having $M_2 + M_4$ streams of signals $\mathbf{H}_{2,R}\mathbf{X}_2^n + \mathbf{H}_{4,R}\mathbf{X}_4^n$ together with $\mathbf{H}_{2,R}^{'}\mathbf{X}_2^n + \mathbf{H}_{4,R}^{'}\mathbf{X}_4^n$ allows us to decode the transmitted $M_2 + M_4$ streams, i.e.,

$[\mathbf{X}_2^n\ \mathbf{X}_4^n]$, with probability one. Therefore we can write:

$$
\begin{aligned}
&H(W_2, W_4|\mathbf{H}_{2,R}\mathbf{X}_2^n + \mathbf{H}_{4,R}\mathbf{X}_4^n + \mathbf{Z}_R^n)\\
&= I(\mathbf{H}'_{2,R}\mathbf{X}_2^n + \mathbf{H}'_{4,R}\mathbf{X}_4^n; W_2, W_4|\mathbf{H}_{2,R}\mathbf{X}_2^n + \mathbf{H}_{4,R}\mathbf{X}_4^n + \mathbf{Z}_R^n)\\
&\quad + H(W_2, W_4|\mathbf{H}_{2,R}\mathbf{X}_2^n + \mathbf{H}_{4,R}\mathbf{X}_4^n + \mathbf{z}_R^n, \mathbf{H}'_{2,R}\mathbf{X}_2^n + \mathbf{H}'_{4,R}\mathbf{X}_4^n)\\
&\overset{(a)}{\leq} I(\mathbf{H}'_{2,R}\mathbf{X}_2^n + \mathbf{H}'_{4,R}\mathbf{X}_4^n; W_2, W_4|\mathbf{H}_{2,R}\mathbf{X}_2^n + \mathbf{H}_{4,R}\mathbf{X}_4^n + \mathbf{Z}_R^n)\\
&\quad + n\epsilon'_n\\
&\overset{(b)}{\leq} h(\mathbf{H}'_{2,R}\mathbf{X}_2^n + \mathbf{H}'_{4,R}\mathbf{X}_4^n) + n\epsilon'_n\\
&\overset{(c)}{\leq} (M_2 + M_4 - N_1 - N_2)^+ \left(\log\left(2\pi e(2H_{\max}^2 P)\right)^n\right)\\
&\quad + n\epsilon'_n, \qquad (9)
\end{aligned}
$$

where (a) follows from Fano's inequality and the fact that probability of error in decoding W_2 and W_4 given $\mathbf{X}_2^n$ and $\mathbf{X}_4^n$ with a limited noise goes to zero based on the the discussion at the beginning of the proof for high SNR; (b) follows from the fact that the mutual information between two signals is less than or equal to the entropy of each individual signal; and (c) holds since each of the $M_2 + M_4 - N_1 - N_2$ streams in $\mathbf{H}'_{2,R}\mathbf{X}_2^n + \mathbf{H}'_{4,R}\mathbf{X}_4^n$ is with channel entries upper bounded by $H_{\max}$, and the transmitted signals from each antenna has distribution $\mathcal{CN}(0, P)$. ■

Dividing both sides of (5) by $n \log P$, and using $n(R_2 + R_4 - \epsilon'''_n) \leq I(W_2; \mathbf{Y}_2) + I(W_4; \mathbf{Y}_4) = H(W_2) - H(W_2|\mathbf{Y}_2) + H(W_4) - H(W_4|\mathbf{Y}_4) \leq H(W_2) + H(W_4) = H(W_2, W_4)$, results in:

$$
\begin{aligned}
&\frac{R_3}{\log P}\\
&\leq \frac{(N_1 + N_2)\log\left(2\pi e(4H_{\max}^2 P + 1)\right)^n}{n\log P}\\
&\quad - \frac{(R_2 + R_4 - \epsilon'''_n)}{\log P} - \frac{2n(N_1 + N_2)\log(2\pi e)}{n\log P}\\
&\quad + \frac{(M_2 + M_4 - N_1 - N_2)^+ \log\left(2\pi e(2H_{\max}^2 P + 1)\right)^n}{n\log P}\\
&\quad + \frac{\epsilon''_n}{\log P}, \qquad (10)
\end{aligned}
$$

and with $n \to \infty$ and $P \to \infty$, we obtain the following bound:

$$
\begin{aligned}
d_2 + d_3 + d_4 &\leq (N_1 + N_2) + (M_2 + M_4 - N_1 - N_2)^+\\
&= \max\{M_2 + M_4, N_1 + N_2\}. \qquad (11)
\end{aligned}
$$

Similarly, we also have

$$d_1 + d_2 + d_3 \leq \max\{M_1 + M_3, N_1 + N_2\}, \qquad (12)$$

$$d_1 + d_2 + d_4 \leq \max\{M_2 + M_4, N_1 + N_2\}, \qquad (13)$$

$$d_1 + d_3 + d_4 \leq \max\{M_1 + M_3, N_1 + N_2\}. \qquad (14)$$

Summing up (11)–(14) we get $3(d_1 + d_2 + d_3 + d_4) \leq 2\max\{M_1 + M_3, N_1 + N_2\} + 2\max\{M_2 + M_4, N_1 + N_2\}$ which completes the proof of Theorem 1. ■

Remark 1: When $M_i = 1$, $i = 1, 2, 3, 4$, $N_k = 1$, $k = 1, 2$, we obtain $DoF = 8/3$ which is the upper bound given in [9].

The following two theorems are simple cut-set bounds:

Theorem 2: For the general two-way MIMO 2 × 2 × 2 relay network, $DoF \leq 2(N_1 + N_2)$.

Proof: The proof follows from the fact that the DoF in each direction is upper bounded by the number of relays $N_1 + N_2$. ■

Theorem 3: For the general two-way MIMO 2 × 2 × 2 relay network, $DoF \leq M_1 + M_2 + M_3 + M_4$.

Proof: The proof follows from the fact that the DoF is bounded by the total number of transmit antennas. ■

IV. DoF Lower Bounds

We first give the following two lower bounds on the DoF of two-way 2 × 2 × 2 MIMO interference networks.

Theorem 4: For the general two-way 2 × 2 × 2 MIMO interference network we have $\text{DoF} \geq 2\min\{N_1 + N_2, \max\{\min\{M_1, M_3\}, \min\{M_2, M_4\}\}\}$.

Proof: If nodes S_2 and S_4 in Fig. 1 are silent, then the channel can be seen as a two-way 1 × 1 × 1 MIMO interference network formed by S_1, a super relay node consisting of R_1 and R_2 together, and S_3. This channel can achieve the DoF of $\min\{N_1 + N_2, M_1, M_3\}$ in each direction by simply forwarding the sum of the received signals at the super relay node, which is the sum of the two messages from S_1 and S_3 (with a total DoF of $2\min\{N_1 + N_2, M_1, M_3\}$). By using S_2 and S_4 instead of S_1 and S_3, the DoF of $2\min\{N_1 + N_2, M_2, M_4\}$ is achievable. Therefore, the maximum of these two bounds, i.e., $2\min\{N_1 + N_2, \max\{\min\{M_1, M_3\}, \min\{M_2, M_4\}\}\}$ is achievable, as well. ■

Theorem 5: For the general two-way 2 × 2 × 2 MIMO interference network, we have $\text{DoF} \geq \min\{\max\{N_1, N_2\}, 2\min\{M_1, M_3\} + 2\min\{M_2, M_4\}\}$.

Proof: Without loss of generality, assume $N_1 \leq N_2$. We divide the proof into two parts:

1) If $N_2 \geq 2\min\{M_1, M_3\} + 2\min\{M_2, M_4\}$, i.e., the total number of transmit antennas is no more than N_2, assume nodes S_1 and S_3 transmit only from their top $\min\{M_1, M_3\}$ antennas and nodes S_2 and S_4 transmit only from their top $\min\{M_2, M_4\}$ antennas. The relay R_2 is able to decode all messages by solving a set of linear equations since the number of received signals is no less than the number of messages. Then, R_2 broadcasts the decoded messages to the destinations, and achieves the DoF of $2\min\{M_1, M_3\} + 2\min\{M_2, M_4\}$, as it is no more than $\min\{N_2, \sum_{i=1}^4 M_i\}$ suggested by Lemma 3 below.

2) If $N_2 < 2\min\{M_1, M_3\} + 2\min\{M_2, M_4\}$, then only a total of N_2 antennas in sources transmit and using the same argument as above, it can be seen that the signals will be decoded in relay R_2 and then in their corresponding destinations, as the DoF of N_2 is suggested by Lemma 3 below (since $N_2 < 2\min\{M_1, M_3\} + 2\min\{M_2, M_4\}$, we get $\min\{N_2, \sum_{i=1}^4 M_i\} = N_2$). ■

Lemma 3: [16] The DoF of a broadcast channel with N antennas at the transmitter and M_i antennas at receiver i, $i = 1, \ldots, K$, is $\min\{N, \sum_{i=1}^K M_i\}$.

The following theorem provides a lower bound on the DoF of the $2 \times 2 \times 2$ MIMO interference network based on the DoF for each source-destination pair.

Theorem 6: If d_1, d_2, d_3, d_4 are non-negative integers that satisfy the following conditions:

- $d_1, d_3 \leq \min\{M_1, M_3, N_1 + N_2\}$,
- $d_2, d_4 \leq \min\{M_2, M_4, N_1 + N_2\}$,
- $d_1 + d_2, d_3 + d_4 \leq N_1 + N_2$,
- $2(d_1 d_4 + d_1 d_2 + d_3 d_4 + d_2 d_3) \leq N_1^2 + N_2^2 - 1$,

then the DoF of $\sum_{i=1}^{4} d_i$ is achievable.

Proof: We show that if all the conditions in the theorem statement hold, each source-destination pair $(S_i, D_i), i \in \{1, \ldots, 4\}$ can achieve the DoF of d_i. The first two conditions in the theorem statement ensure that the DoF for each link is no more than the number of transmit antennas, the number of receive antennas, and also the number of antennas in the relay between them. The third condition ensures that the DoF in each direction is no more than the number of relay antennas. In the following, we show that by adding the fourth condition, the DoF of (d_1, d_2, d_3, d_4) is achievable.

The received signals at relays are given in (1). Then, each relay R_i, $i \in \{1, 2\}$, performs amplify-and-forward by transmitting $\mathbf{V}_i \mathbf{y}_{R_i}[m]$ using an $N_i \times N_i$ precoding matrix $\mathbf{V}_i$, and the received signals at the destinations are given by:

$$\mathbf{y}_i[m] = \mathbf{H}_{R_1,i}\mathbf{V}_1\mathbf{y}_{R_1}[m] + \mathbf{H}_{R_2,i}\mathbf{V}_2\mathbf{y}_{R_2}[m] + \mathbf{z}_i[m], \quad (15)$$

for $i \in \{1, \ldots, 4\}$. Substituting (1) into (15) results in:

$$\begin{aligned}\mathbf{y}_i[m] = \mathbf{H}_{R_1,i}\mathbf{V}_1 &\left(\sum_{j=1}^{4} \mathbf{H}_{j,R_1}\mathbf{x}_j[m] + \mathbf{z}_{R_1}[m]\right) \\ &+ \mathbf{H}_{R_2,i}\mathbf{V}_2 \left(\sum_{j=1}^{4} \mathbf{H}_{j,R_2}\mathbf{x}_j[m] + \mathbf{z}_{R_2}[m]\right) + \mathbf{z}_i[m],\end{aligned} \quad (16)$$

for $i \in \{1, \ldots, 4\}$. We assume that each transmitter S_i transmits signals from the top d_i antennas and nothing from the rest of the antennas. We will show the existence of $\mathbf{V}_1$ and $\mathbf{V}_2$ such that each receiver D_i can decode the d_i information streams from its corresponding transmitter S_i, and then the proof of achievability will be complete.

Now, we analyze the interfering signals that should be nulled. For destination D_1, the signal $\mathbf{x}_1[m]$ is the intended signal and the receiver knows $\mathbf{x}_3[m]$ as it is transmitter S_3 as well. Therefore, the interference from the signals $\mathbf{x}_2[m]$ and $\mathbf{x}_4[m]$ should be nulled at destination D_1.

1) The interfering signal from $\mathbf{x}_4[m]$ to D_1:

$$\begin{aligned}\mathbf{q}_{4\to1}[m] = &\underbrace{\left(\mathbf{H}_{R_1,1}\mathbf{V}_1\mathbf{H}_{4,R_1} + \mathbf{H}_{R_2,1}\mathbf{V}_2\mathbf{H}_{4,R_2}\right)}_{\triangleq \mathbf{G}_{4\to1}} \\ &\times \left[x_4^{(1)}[m], \ldots, x_4^{(d_4)}[m], 0, \ldots, 0\right]^T;\end{aligned} \quad (17)$$

where $x_i^{(j)}[m]$ represents the j^{th} entry of vector $\mathbf{x}_i[m]$.

2) The interfering signal from $\mathbf{x}_2[m]$ to D_1:

$$\begin{aligned}\mathbf{q}_{2\to1}[m] = &\underbrace{\left(\mathbf{H}_{R_1,1}\mathbf{V}_1\mathbf{H}_{2,R_1} + \mathbf{H}_{R_2,1}\mathbf{V}_2\mathbf{H}_{2,R_2}\right)}_{\triangleq \mathbf{G}_{2\to1}} \\ &\times \left[x_2^{(1)}[m], \ldots, x_2^{(d_2)}[m], 0, \ldots, 0\right]^T;\end{aligned} \quad (18)$$

We will choose $\mathbf{V}_1$ and $\mathbf{V}_2$ such that the top d_1 antennas at D_1 contain the d_1 intended data streams, by enforcing that first d_1 elements of both $\mathbf{q}_{4\to1}[m]$ and $\mathbf{q}_{2\to1}[m]$ does not contain elements of $\mathbf{x}_4[m]$ and $\mathbf{x}_2[m]$, respectively. That is, we force the corresponding submatrices in (17) and (18) to be zero, i.e.,

$$\begin{aligned}&\mathbf{G}_{4\to1}[1:d_1, 1:d_4] = \mathbf{0}, \\ &\mathbf{G}_{2\to1}[1:d_1, 1:d_2] = \mathbf{0}.\end{aligned} \quad (19)$$

(19) consists of $d_1(d_2 + d_4)$ linear equations of elements of $\mathbf{V}_1$ and $\mathbf{V}_2$.

Also, the interference from the signals $\mathbf{x}_1[m]$ and $\mathbf{x}_3[m]$ should be nulled at destination D_2, which are defined as

$$\begin{aligned}\mathbf{q}_{1\to2}[m] = &\underbrace{\left(\mathbf{H}_{R_1,2}\mathbf{V}_1\mathbf{H}_{1,R_1} + \mathbf{H}_{R_2,2}\mathbf{V}_2\mathbf{H}_{1,R_2}\right)}_{\triangleq \mathbf{G}_{1\to2}} \\ &\times \left[x_1^{(1)}[m], \ldots, x_1^{(d_1)}[m], 0, \ldots, 0\right]^T,\end{aligned} \quad (20)$$

$$\begin{aligned}\mathbf{q}_{3\to2}[m] = &\underbrace{\left(\mathbf{H}_{R_1,2}\mathbf{V}_1\mathbf{H}_{3,R_1} + \mathbf{H}_{R_2,2}\mathbf{V}_2\mathbf{H}_{3,R_2}\right)}_{\triangleq \mathbf{G}_{3\to2}} \\ &\times \left[x_3^{(1)}[m], \ldots, x_3^{(d_3)}[m], 0, \ldots, 0\right]^T.\end{aligned} \quad (21)$$

Therefore, the followings should hold:

$$\begin{aligned}&\mathbf{G}_{1\to2}[1:d_2, 1:d_1] = \mathbf{0}, \\ &\mathbf{G}_{3\to2}[1:d_2, 1:d_3] = \mathbf{0}.\end{aligned} \quad (22)$$

(22) consists of $d_2(d_1 + d_3)$ linear equations of elements of $\mathbf{V}_1$ and $\mathbf{V}_2$.

Similarly, $\mathbf{x}_2[m]$ and $\mathbf{x}_4[m]$ should be nulled at destination D_3, which are defined as

$$\begin{aligned}\mathbf{q}_{2\to3}[m] = &\underbrace{\left(\mathbf{H}_{R_1,3}\mathbf{V}_1\mathbf{H}_{2,R_1} + \mathbf{H}_{R_2,3}\mathbf{V}_2\mathbf{H}_{2,R_2}\right)}_{\triangleq \mathbf{G}_{2\to3}} \\ &\times \left[x_2^{(1)}[m], \ldots, x_2^{(d_2)}[m], 0, \ldots, 0\right]^T,\end{aligned} \quad (23)$$

$$\begin{aligned}\mathbf{q}_{4\to3}[m] = &\underbrace{\left(\mathbf{H}_{R_1,3}\mathbf{V}_1\mathbf{H}_{4,R_1} + \mathbf{H}_{R_2,3}\mathbf{V}_2\mathbf{H}_{4,R_2}\right)}_{\triangleq \mathbf{G}_{4\to3}} \\ &\times \left[x_4^{(1)}[m], \ldots, x_4^{(d_4)}[m], 0, \ldots, 0\right]^T.\end{aligned} \quad (24)$$

Therefore, the followings should hold:

$$\begin{aligned}&\mathbf{G}_{2\to3}[1:d_3, 1:d_2] = \mathbf{0}, \\ &\mathbf{G}_{4\to3}[1:d_3, 1:d_4] = \mathbf{0}.\end{aligned} \quad (25)$$

(25) consists of $d_3(d_2 + d_4)$ linear equations of elements of $\mathbf{V}_1$ and $\mathbf{V}_2$.

Finally, $\mathbf{x}_1[m]$ and $\mathbf{x}_3[m]$ should be nulled at destination D_4, which are defined as

$$\mathbf{q}_{1\to 4}[m] = \underbrace{(\mathbf{H}_{R_1,4}\mathbf{V}_1\mathbf{H}_{1,R_1} + \mathbf{H}_{R_2,4}\mathbf{V}_2\mathbf{H}_{1,R_2})}_{\triangleq \mathbf{G}_{1\to 4}} \times \left[x_1^{(1)}[m], \dots, x_1^{(d_1)}[m], 0, \dots, 0\right]^T, \quad (26)$$

$$\mathbf{q}_{3\to 4}[m] = \underbrace{(\mathbf{H}_{R_1,4}\mathbf{V}_1\mathbf{H}_{3,R_1} + \mathbf{H}_{R_2,4}\mathbf{V}_2\mathbf{H}_{3,R_2})}_{\triangleq \mathbf{G}_{3\to 4}} \times \left[x_3^{(1)}[m], \dots, x_3^{(d_3)}[m], 0, \dots, 0\right]^T. \quad (27)$$

Therefore, the followings should hold:

$$\mathbf{G}_{1\to 4}[1:d_4, 1:d_1] = \mathbf{0},$$
$$\mathbf{G}_{3\to 4}[1:d_4, 1:d_3] = \mathbf{0}. \quad (28)$$

(28) consists of $d_4(d_1+d_3)$ linear equations of elements of $\mathbf{V}_1$ and $\mathbf{V}_2$.

Combining (19), (22), (25) and (28) we have in total $2(d_1+d_3)(d_2+d_4)$ linear equations of the form $\mathbf{Gv} = \mathbf{0}$ of the $N_1^2 + N_2^2$ elements of $\mathbf{V}_1$ and $\mathbf{V}_2$. When the fourth condition in the theorem holds, then with probability 1 there exists a non-zero solution of $\mathbf{V}_1$ and $\mathbf{V}_2$. ■

V. SPECIAL CASE

We now apply the upper and lower bounds in the previous sections to a special case of $2 \times 2 \times 2$ MIMO interference networks where $M_i = M$, $i \in \{1, \dots, 4\}$, and $N_k = N$, $k \in \{1, 2\}$. The following theorem provides the bounds on the DoF for the such networks.

Theorem 7: For the $2 \times 2 \times 2$ MIMO interference network where each relay node has N antennas and each user has M nodes, we have $2\min\{2N, M\} + \min\{\lfloor \frac{N^2-1}{2\min\{2N,M\}} \rfloor, 4N - 2\min\{2N, M\}, 2M\} \le$ DoF $\le \min\{\frac{8}{3}\max\{N, M\}, 4M, 4N\}$.

Proof: The upper bound of $\frac{8}{3}\max\{N, M\}$ follows from Theorem 1, the upper bound of $4N$ follows from Theorem 2, and the upper bound of $4M$ follows from Theorem 3.

The lower bound is obtained by finding the highest one among the lower bounds given by Theorem 6.

In particular, we want to maximize $\sum_{i=1}^4 d_i$ with d_i being non-negative integers subject to

$$d_i \le M, \qquad i \in \{1, \dots, 4\}, \quad (29)$$
$$d_1 + d_2 \le 2N, \quad (30)$$
$$d_3 + d_4 \le 2N, \quad (31)$$
$$(d_1+d_3)(d_2+d_4) \le (N^2-1). \quad (32)$$

Solving (29)–(31) for obtaining the largest achievable bound $\sum_{i=1}^4 d_i$ with preference on maximizing $d_1 + d_3$ can be performed using Lemma 4 below. It results $d_1' = \min\{2N, M\}$, $d_3' = \min\{2N, M\}$, $d_2' = \min\{2N - d_1', M\}$, and $d_4' = \min\{2N - d_3', M\}$. It is easy to see that these values lead to the largest $\sum_{i=1}^4 d_i$ based on (29)–(31), without considering (32). To solve for (d_1, d_2, d_3, d_4), given that $d_1' = d_3' \ge d_2', d_4'$, we choose $d_1 = d_1' = \min\{2N, M\}$, $d_3 = d_3' = \min\{2N, M\}$ and solve

$$d_2 \le \min\{2N - \min\{2N, M\}, M\},$$
$$d_4 \le \min\{2N - \min\{2N, M\}, M\},$$
$$d_2 + d_4 \le \left\lfloor \frac{(N^2-1)}{2\min\{2N, M\}} \right\rfloor \le \frac{(N^2-1)}{(d_1+d_3)},$$

using Lemma 4 with $a = \min\{2N - \min\{2N, M\}, M\}$, $b = \lfloor \frac{(N^2-1)}{2\min\{2N,M\}} \rfloor \le \frac{(N^2-1)}{2\min\{2N,M\}}$, $x = d_2$, $y = d_4$ to maximize $d_2 + d_4$. Therefore, we obtain

$$d_2 = \min\left\{ \left\lfloor \frac{(N^2-1)}{2\min\{2N, M\}} \right\rfloor, 2N - \min\{2N, M\}, M \right\},$$
$$d_4 = \min\left\{ \left\lfloor \frac{(N^2-1)}{2\min\{2N, M\}} \right\rfloor - d_2, 2N - \min\{2N, M\}, M \right\}. \quad (33)$$

Moreover, we also have

$$d_1 = d_3 = \min\{2N, M\}. \quad (34)$$

Finally, Lemma 5 below completes the proof of Theorem 7. ■

Lemma 4: Assume that for constants a and b, the inequalities $x, y \le a$ and $x + y \le b$ hold. Then, we have $\max\{x+y\} = \min\{2a, b\}$ and the pair $x_0 = \min\{a, b\}$, $y_0 = \min\{a, b - x_0\}$ satisfies this.

Proof: It is easy to see that $\max\{x+y\}$ cannot be more than $\min\{2a, b\}$ due to the constraints. Therefore, if we can obtain this value, it is optimal.

On the other hand, $x_0, y_0 \le a$ hold. So it is enough to show that $x_0 + y_0 = \min\{2a, b\}$. We have $x_0 + y_0 = \min\{a, b\} + \min\{a, b - \min\{a, b\}\} = \min\{2a, a + b, a + b - \min\{a, b\}, 2b - \min\{a, b\}\} = \min\{2a, a + b, \max\{a, b\}, 2b - \min\{a, b\}\}$. Let divide the proof into 3 cases as below.

- $2a \ge b \ge a$: In this case, $x_0 + y_0 = \min\{2a, b, 2b - a\} = \min\{2a, b\}$.
- $a \ge b$: In this case, $x_0 + y_0 = b = \min\{2a, b\}$.
- $b \ge 2a$: In this case, $x_0 + y_0 = 2a = \min\{2a, b\}$. ■

Lemma 5: (d_1, d_2, d_3, d_4) in (33)–(34) maximizes $\sum_{i=1}^4 d_i$ with d_i being non-negative integers subject to (29)–(32).

Proof: The proof is given in Appendix A. ■

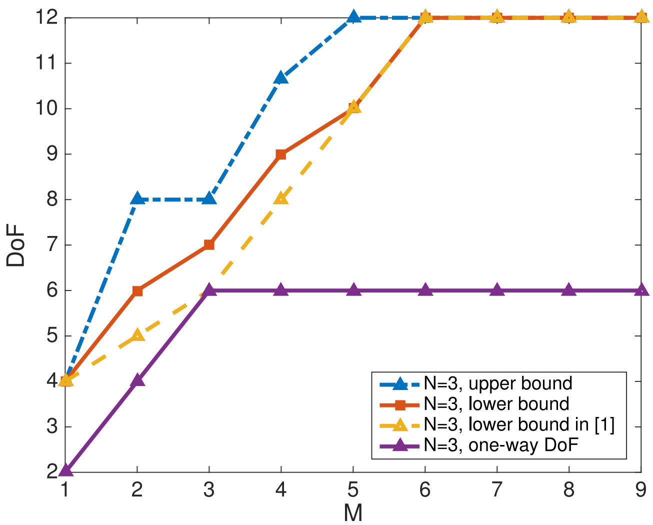
The achievable DoF given in [1, Theorem 1] is $\max\{\min\{4N, 2M\}, \min\{2N, 2\lfloor \frac{4}{3}M \rfloor\}, \min\{2N-1, 4M\}\}$. The following corollary shows that our lower bound in Theorem 7 is higher than the one given in [1]:

Corollary 1: The lower bound in Theorem 7 is better than the one proposed in [1, Theorem 1].

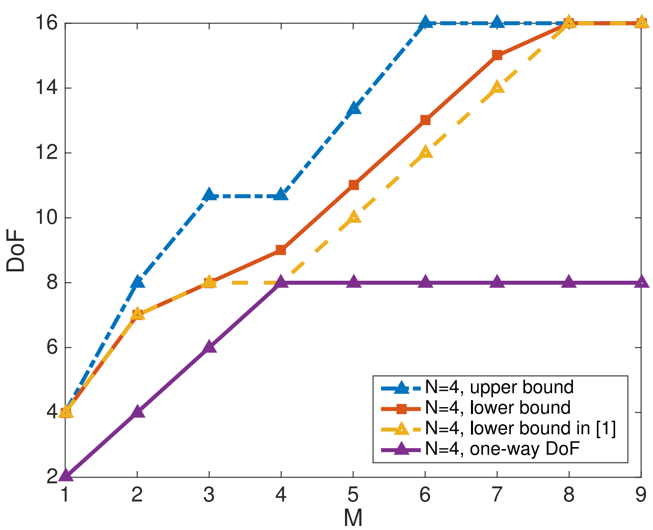
Proof: The proof is given in Appendix B. ■

Theorem 7 also results in the following corollary.

Corollary 2: For the two-way $2 \times 2 \times 2$ MIMO interference network where each relay node has N antennas and each user node has M antennas, if $M \ge 2N$ holds, then $DoF = 4N$. And if $N > 2M$, then $DoF = 4M$.

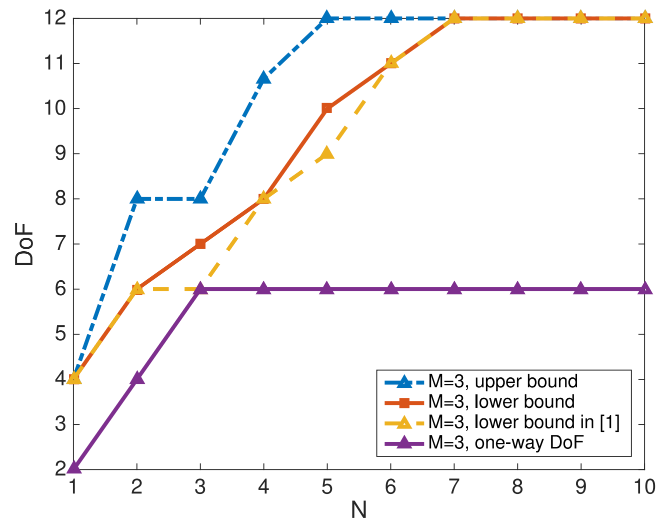


(a) $N = 3$.

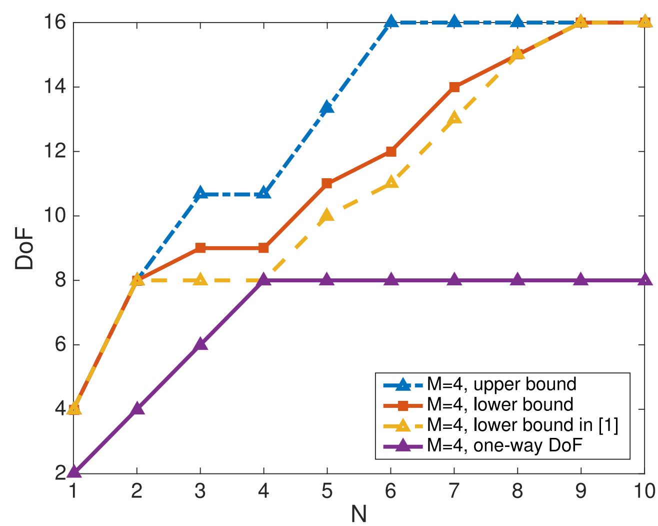


(b) $N = 4$.

Fig. 3. Comparison of the bounds given in Theorem 7 and the lower bound in [1].



(a) $M = 3$.



(b) $M = 4$.

Fig. 4. Comparison of the bounds given in Theorem 7 and the lower bound in [1].

Proof: If $M \geq 2N$, the lower bound in Theorem 7 can be written as

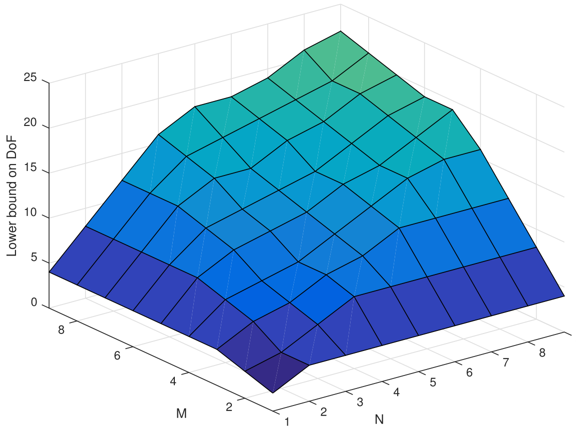
$$\begin{aligned} &2\min\{2N, M\} + \min\left\{\left\{\left\lfloor \frac{N^2-1}{2\min\{2N, M\}} \right\rfloor\right\}, 4N \right.\\ &\qquad\qquad \left. - 2\min\{2N, M\}, 2M\right\} \\ &= 4N + \min\left\{\left\{\left\lfloor \frac{N^2-1}{2\min\{2N, M\}} \right\rfloor\right\}, 4N \right.\\ &\qquad\qquad \left. - 2\min\{2N, M\}, 2M\right\} \\ &= 4N + \min\left\{\left\{\left\lfloor \frac{N^2-1}{4N} \right\rfloor\right\}, 4N - 4N, 2M\right\} = 4N. \end{aligned} \tag{35}$$

And the upper bound can be written as

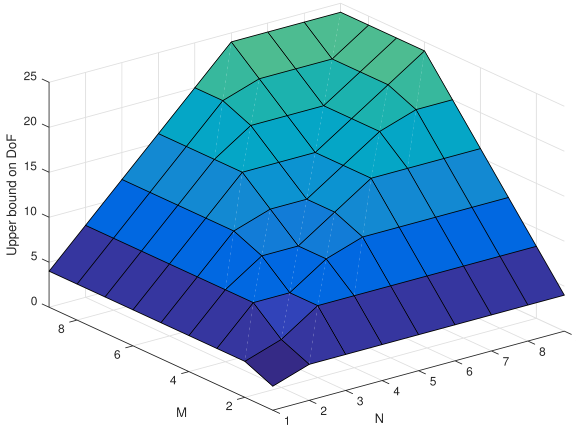
$$\begin{aligned} &\min\left\{\frac{8}{3}\max\{N, M\}, 4M, 4N\right\} \\ &= \min\left\{\frac{8}{3}M, 4M, 4N\right\} = 4N. \end{aligned} \tag{36}$$

Similarly, if $N > 2M$, both the lower and upper bounds become $4M$. ∎

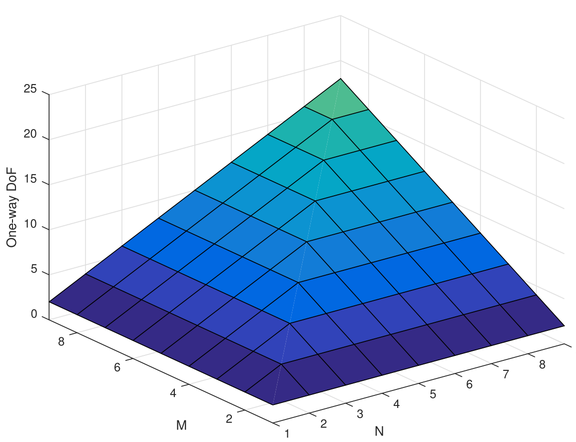
The above corollary states that when $M \geq 2N$, then the bottleneck on the DoF is the number of relay antennas and since there are a total of $2N$ antennas in relays, in each direction the DoF is $2N$ (with a total of $4N$ in two directions). Similarly, when $N > 2M$, then the bottleneck on the DoF is the number of transmitter antennas and since there are a total of $4M$ antennas in transmitters, the DoF is $4M$.



(a) The lower bound on DoF.



(b) The upper bound on DoF.



(c) The one-way DoF.

Fig. 5. The upper and lower bounds on DoF given in Theorem 7, in comparison with the one-way DoF.

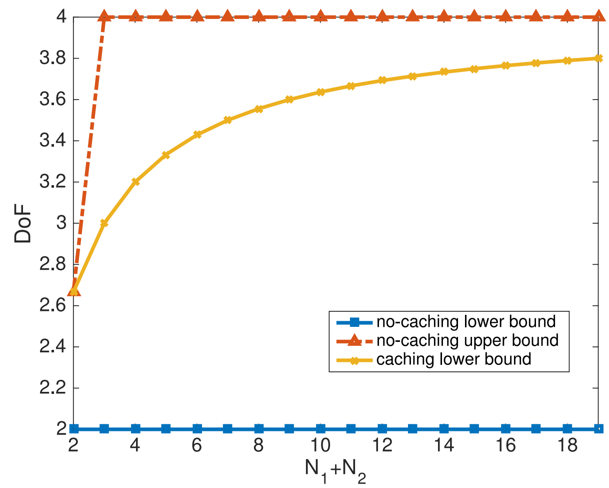


Fig. 6. A comparison of the DoF bounds and the achievable DoF with caching for two-way $2 \times 2 \times 2$ networks with single-antenna user nodes and multiple-antenna relays.

Now, we will briefly compare our results with the one-way model. It is shown in [13] that the DoF of the one-way $2 \times 2 \times 2$ MIMO interference network where each relay node has N antennas and each user node has M antennas is $2\min\{M, N\}$.

Comparing this result with Theorem 7 we see that the two-way communication capability in general can increase the DoF, with the actual gain depending on the values of M and N. In particular for the cases of $M \geq 2N$ and $N > 2M$, Corollary 2 indicates that two-way transmission doubles the DoF.

The bounds given in Theorem 7 are illustrated in Figs. 3 and 4 for different values of M, N. In Fig. 3(a) the tight results for $M = 1$ and $M \geq 6$ follow from Corollary 2. Similarly the tight results in Figs. 3(b)–4(b) also follow from Corollary 2. We observe that for all values of M and N in Figs. 3 and 4 there is an improvement in the lower bound of DoF in comparison with the one-way DoF. Moreover, our lower bound is better than that given in [1]. In Fig. 5, we plot the upper and lower bounds given in Theorem 7 as well as the one-way DoF as a function of M and N.

Finally, we consider the case of multi-antenna relays and single-antenna source/destination nodes, i.e., $M_i = 1$, $i = 1, \ldots, 4$. It is shown in [9] that if each relay is equipped with cache, then $\text{DoF}_c \geq \frac{4(N_1+N_2)}{N_1+N_2+1}$ which is plotted in Fig. 6. On the other hand, by applying the upper and lower bounds given by Theorems 1–6 to the special case of $M_i = 1$, $i = 1, 2, 3, 4$, we obtain $2 \leq \text{DoF} \leq \max\{\frac{4}{3}(N_1 + N_2), 4\}$ which is also plotted in Fig. 6. It is seen that with caching the DoF approaches the upper bound of 4 as $N_1 + N_2 \to \infty$, indicating that caching could potentially increase the DoF.

VI. Conclusions

We have considered the two-way $2 \times 2 \times 2$ MIMO interference network, a class of two-way four-unicast MIMO interference networks. We have obtained upper and lower bounds on the sum DoF of the two-way $2 \times 2 \times 2$ MIMO interference network with any number of antennas in each node. We have also considered the special case where there are M antennas at each user node and N antennas at each relay node and obtained a better achievable DoF than that in the literature.

This work is also a first step towards the study of general two-way relay-assisted networks. Another future work direction is studying the impact of practical considerations such as channel estimation error.

APPENDIX A
PROOF OF LEMMA 5

We divide the proof into five cases:

1) $N < M < 2N$ and $\lfloor \frac{(N^2-1)}{2M} \rfloor > 4N - 2M$.
2) $N < M < 2N$ and $\lfloor \frac{(N^2-1)}{2M} \rfloor \le 4N - 2M$.
3) $M \le N$ and $\lfloor \frac{(N^2-1)}{2M} \rfloor \ge 2M$.
4) $M \le N$ and $\lfloor \frac{(N^2-1)}{2M} \rfloor < 2M$.
5) $M \ge 2N$.

The achieved DoF set (d_1, d_2, d_3, d_4) obtained in Theorem 7 is such that $d_2 + d_4 = \min\{\lfloor \frac{(N^2-1)}{2\min\{2N,M\}} \rfloor, 4N - 2\min\{2N, M\}, 2M\}$ and $d_1 + d_3 = 2\min\{2N, M\}$.

For cases 1, 3 and 5 it is easy to show that $\sum_{i=1}^4 d_i$ cannot be increased:

- $N < M < 2N$ and $\lfloor \frac{(N^2-1)}{2M} \rfloor > 4N - 2M$: We get $d_1 + d_3 = 2\min\{2N, M\} = 2M$ and $d_2 + d_4 = \min\{\lfloor \frac{(N^2-1)}{2\min\{2N,M\}} \rfloor, 4N - 2\min\{2N, M\}, 2M\} = \min\{\lfloor \frac{(N^2-1)}{2M} \rfloor, 4N - 2M\} = \min\{\lfloor \frac{(N^2-1)}{2M} \rfloor, 4N - 2M\} = 4N - 2M$. Hence $\sum_{i=1}^4 d_i = 4N$ which is optimal given (30)–(31).
- $M \le N$ and $\lfloor \frac{(N^2-1)}{2M} \rfloor \ge 2M$: We get $d_1 + d_3 = 2\min\{2N, M\} = 2M$ and $d_2 + d_4 = \min\{\lfloor \frac{(N^2-1)}{2\min\{2N,M\}} \rfloor, 4N - 2\min\{2N, M\}, 2M\} = \min\{\lfloor \frac{(N^2-1)}{2M} \rfloor, 2M\} = 2M$. Hence $\sum_{i=1}^4 d_i = 4M$ which is optimal given (29).
- $M \ge 2N$: We get $d_1 = d_3 = 2N$ and $d_2 = d_4 = 0$. Hence gets $\sum_{i=1}^4 d_i = 4N$ which is optimal given (30)–(31).

For cases 2 and 4, we show the optimality by contradiction. Assume that the DoF set $(d_1', d_2', d_3', d_4') = (d_1 - p, d_2 + n, d_3 - q, d_4 + m)$ with integers m, n, p, q is achievable such that $m + n > p + q$ and thus $\sum d_i' > \sum d_i$. Define $a \triangleq m + n$ and $b \triangleq p + q$ (and therefore $a > b$). Note that $p, q \ge 0$ (and therefore $b \ge 0$) since d_1' and d_3' cannot be more than $\min\{2N, M\}$ (see (29)–(31)). We show that this does not satisfy the bound (32). Let first simplify $(d_1' + d_3')(d_2' + d_4')$ for these two cases as below:

- $N < M < 2N$ and $\lfloor \frac{(N^2-1)}{2M} \rfloor \le 4N - 2M$:

$$
\begin{aligned}
&(d_2' + d_4')(d_1' + d_3') \\
&= \left(\min\left\{\left\lfloor \frac{(N^2-1)}{2\min\{2N,M\}} \right\rfloor, 4N - 2\min\{2N, M\}, 2M\right\} + a\right)(2\min\{2N, M\} - b) \\
&= \left(\min\left\{\left\lfloor \frac{(N^2-1)}{2M} \right\rfloor, 4N - 2M, 2M\right\} + a\right) \times (2M - b) \\
&= \left(\min\left\{\left\lfloor \frac{(N^2-1)}{2M} \right\rfloor, 4N - 2M\right\} + a\right) \times (2M - b) \\
&= \left(\left\lfloor \frac{(N^2-1)}{2M} \right\rfloor + a\right)(2M - b).
\end{aligned} \tag{37}
$$

- $M \le N$ and $\lfloor \frac{(N^2-1)}{2M} \rfloor < 2M$:

$$
\begin{aligned}
&(d_2' + d_4')(d_1' + d_3') \\
&= \left(\min\left\{\left\lfloor \frac{(N^2-1)}{2\min\{2N,M\}} \right\rfloor, 4N - 2\min\{2N, M\}, 2M\right\} + a\right)(2\min\{2N, M\} - b) \\
&= \left(\min\left\{\left\lfloor \frac{(N^2-1)}{2M} \right\rfloor, 2M\right\} + a\right)(2M - b) \\
&= \left(\left\lfloor \frac{(N^2-1)}{2M} \right\rfloor + a\right)(2M - b).
\end{aligned} \tag{38}
$$

Therefore, for these two cases we have $(d_2' + d_4') = (\lfloor \frac{(N^2-1)}{2M} \rfloor + a)$. Given (29)–(31), we get $(d_2' + d_4') \le 2\min\{2N, M\} = 2M$ which results $(\lfloor \frac{(N^2-1)}{2M} \rfloor + a) \le 2M$. Then, for these two cases we get:

$$
\begin{aligned}
&(d_2' + d_4')(d_1' + d_3') \\
&= \left(\left\lfloor \frac{(N^2-1)}{2M} \right\rfloor + a\right)(2M - b) \\
&= \left(2M \left\lfloor \frac{(N^2-1)}{2M} \right\rfloor + 2aM - b \left\lfloor \frac{(N^2-1)}{2M} \right\rfloor - ab\right) \\
&\overset{(a)}{\ge} \left(2M \left\lfloor \frac{(N^2-1)}{2M} \right\rfloor + 2aM - b(2M - a) - ab\right) \\
&= 2M \left(\left\lfloor \frac{(N^2-1)}{2M} \right\rfloor + \underbrace{(a - b)}_{\ge 1}\right) \\
&> (N^2 - 1),
\end{aligned} \tag{39}
$$

where (a) follows from $\left(\left\lfloor \frac{(N^2-1)}{2M} \right\rfloor + a\right) \le 2M$ as mentioned above.

APPENDIX B
PROOF OF COROLLARY 1

We only need to show that all three terms in the lower bound in [1, Theorem 1] have a solution (d_1, d_2, d_3, d_4) that satisfy (29)–(32).

- The first term, $\min\{4N, 2M\}$, is achievable with $d_1 = d_3 = \min\{2N, M\}$ and $d_2 = d_4 = 0$ which satisfies (29)–(32).
- The second term, $\min\{2N, 2\lfloor \frac{4}{3}M \rfloor\}$, is achievable as follows.
  - If $N \le \lfloor \frac{4}{3}M \rfloor$ choose $d_1 = \min\{M, N\} - 1$, $d_2 = \min\{M, N\}$, $d_3 = N - \min\{M, N\}$, and $d_4 = N + 1 - \min\{M, N\}$.
    - (29) holds since $N + 1 - \min\{M, N\} \le M$ is true as:
      - If $\min\{M, N\} = M$: It is enough to show $N \le 2M - 1$ which holds as $\lfloor \frac{4}{3}M \rfloor \le 2M - 1$.

      - If $\min\{M, N\} = N$: It is easy to verify in this case as $1 \leq M$.
    - It is easy to see that (30)–(31) hold.
    - (32) holds since $(d_1 + d_3)(d_2 + d_4) = N^2 - 1$.
  - If $N > \lfloor \frac{4}{3}M \rfloor$, we choose $d_1 = \min\{M, 2N\}$, $d_2 = \lfloor \frac{4}{3}M \rfloor - \min\{M, 2N\}$, $d_3 = \lfloor \frac{4}{3}M \rfloor - \min\{M, 2N\}$, and $d_4 = \min\{M, 2N\}$.
    - (29) holds since $\lfloor \frac{4}{3}M \rfloor - \min\{M, 2N\} \leq M$ is true, as $\frac{1}{2}\lfloor \frac{M}{3} \rfloor \leq \lfloor \frac{4M}{3} \rfloor \leq N \Rightarrow \lfloor \frac{M}{3} \rfloor \leq 2N \Rightarrow \lfloor \frac{M}{3} \rfloor \leq \min\{2N, M\} \Rightarrow \lfloor \frac{4M}{3} \rfloor \leq M + \min\{2N, M\}$.
    - (30)–(31) hold since $\lfloor \frac{4}{3}M \rfloor < 2N$.
    - (32) holds as $\lfloor \frac{4}{3}M \rfloor < N$ results $\lfloor \frac{4}{3}M \rfloor^2 \leq N^2 - 1$.
- The third term, $\min\{2N - 1, 4M\}$, is achievable as follows.
  - If $2N - 1 < 4M$, we choose $d_1 = \min\{M, N\}$, $d_2 = \min\{M, N\} - 1$, $d_3 = N - \min\{M, N\}$, and $d_4 = N - \min\{M, N\}$.
    - (29) holds since $N - \min\{M, N\} \leq M$ is true, which is due to the following:
      - If $\min\{M, N\} = M$: It is enough to show $N - M \leq M$. We have $2N - 1 < 4M$, and since M and N are integers we have $2N \leq 4M$.
      - If $\min\{M, N\} = N$: It holds since $0 \leq M$.
    - (30)–(31) hold since $2\min\{M, N\} - 1 \leq 2N$ and $2N - 2\min\{M, N\} \leq 2N$.
    - (32) holds since $N(N - 1) < N^2 - 1$.
  - If $2N - 1 > 4M$, we choose $d_1 = d_2 = d_3 = d_4 = M$.
    - It is trivial that (29) holds.
    - (30)–(31) hold since $M < \frac{2N-1}{4} < N$.
    - (32) holds since $M < \frac{2N-1}{4}$ results $4M^2 < \left(\frac{2N-1}{2}\right)^2 = N^2 - N + \frac{1}{4}$. Since M and N are integers, then $4M^2 \leq N^2 - N < N^2 - 1$.

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