On the DoF of Two-Way MIMO Butterfly Networks

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***Abstract*—This paper studies the degrees of freedom (DoF) of two-way multiple-input multiple-output (MIMO) butterfly networks, a class of two-way four-unicast networks, that consists of four source/destination nodes and three relay nodes. We first give upper and lower bounds on the sum DoF of such network with any number of antennas in each node. We also study the DoF of the two-way MIMO butterfly network with caching at the relays. For the special case that each source/destination node has M antennas and each relay node has N antennas, we obtain the exact DoF for some special cases. Specifically, if $N \leq M$, then without caching, $\mathrm{DoF} = 2N$, which is equal to the DoF of a one-way MIMO butterfly network, and with relay caching, $\mathrm{DoF} = 4N$. Moreover, if $N > 2M$, then with or without relay caching, $\mathrm{DoF} = 4M$ which doubles the DoF of a one-way MIMO butterfly network. Hence, for this MIMO butterfly network, when the number of relay antennas is large compared with the number of source/destination antennas, bidirectional transmission improves the DoF; and when the number of relay antennas is small, relay caching improves the DoF.**

***Index Terms*—Degrees of freedom, four-unicast channels, two-way channels, butterfly network, MIMO interference network, caching.**

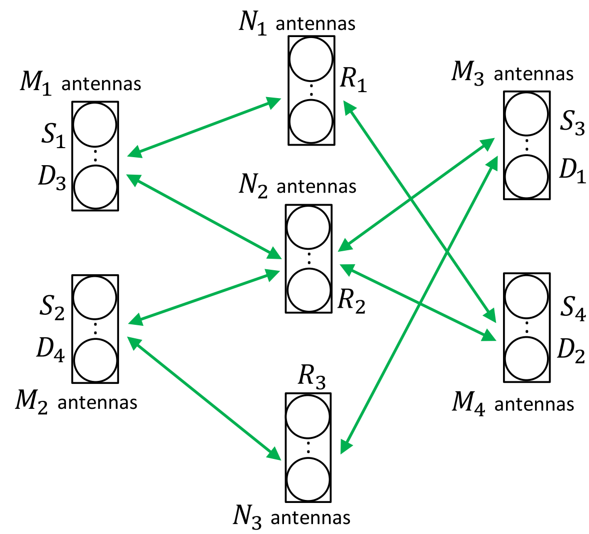

Fig. 1. A two-way MIMO butterfly interference network.

I. Introduction

The two-way butterfly network shown in Fig. 1 is a class of two-way four-unicast networks, that consists of four source/destination nodes and three relay nodes, with bidirectional communication links between relay and source/destination nodes. The butterfly network is motivated by the well-known network coding example [1]. In [2] the degrees of freedom (DoF) of a one-way butterfly network with single-antenna nodes is shown to be 2. In [3], we showed that the DoF of a two-way butterfly network with single-antenna nodes is also 2, which to the best of my knowledge, is the first result where bidirectional links do not improve the DoF. Indeed there are relay network configurations where the two-way DoF doubles the one-way DoF [4]. This paper generalizes [3] to multiple-input multiple-output (MIMO) butterfly networks, where each node is equipped with multiple antennas.

We further consider the case where each relay is equipped with cache that can offline store the transmitted messages. Content caching is a technique to reduce traffic load by exploiting the high degree of asynchronous content reuse and the fact that storage is cheap and ubiquitous in today's wireless devices [5], [6]. During off-peak periods when network resources are abundant, some content can be stored at the wireless edge (e.g., access points or end user devices), so that demands can be met with reduced access latencies and bandwidth requirements. The caching problem has a long history, dating back to the early work by Belady [7]. There are various forms of caching, i.e., to store data at user ends, relays, etc. [8], and both uncoded and coded caching strategies have been developed [9]. The caching process consists of an offline placement phase and an online delivery phase. One important aspect is the design of the placement phase in order to facilitate the delivery phase. There are several recent works that consider communication scenarios where user nodes have pre-cached information from a fixed library of possible files during the offline phase, in order to minimize the transmission from source during the delivery phase [9], [10]. There are only a limited number of works on the DoF with caching. In particular, [11], [12] study the DoF for the relay and interference channels with caching, respectively, under some assumptions and provide asymptotic results on the DoF as the solutions to some optimization problems. In [3] we showed that relay caching doubles the DoF of the two-way butterfly network with single-antenna nodes.

The contributions of this paper are summarized as follows: We generalize [3] to MIMO butterfly networks, where each node is equipped with multiple antennas. We provide several upper and lower bounds on the DoF for the general two-way MIMO butterfly network with arbitrary number of antennas at each

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node. In particular, a novel achievability scheme is proposed that performs interference neutralization and exploits side-information inherent to two-way communications. For the case where all source/destination nodes have M antennas and all relay nodes have N antennas, for some cases where N or M is the transmission bottleneck, we find the exact DoF. Specifically, if $N \leq M$, DoF $= 2N$, which is the DoF of the one-way MIMO butterfly network, i.e., the bidirectional transmission does not improve the DoF at all for this case; and if $N > 2M$, DoF $= 4M$, which is twice the one-way DoF. Moreover, we investigate the effect of relay caching on the DoF of MIMO two-way butterfly networks. In particular, for the case of M source/destination antennas and N relay antennas, we show that if $N \leq M$, relay caching doubles the DoF. But if $N > 2M$, then relay caching does not improve the DoF at all.

The remainder of this paper is organized as follows. In Section II, the two-way MIMO butterfly network model is given. In Sections III and IV, we present upper and lower bounds on the DoF of this network without caching, respectively. In Section V the special case where all source/destination nodes have M antennas and all relay nodes have N antennas is studied. In Section VI, we present the DoF results for the two-way MIMO butterfly network with relay caching. Finally, Section VII concludes this paper.

II. Channel Model

As shown in Fig. 1, the two-way butterfly MIMO interference network consists of four transceiver nodes and three relays R_1, R_2, and R_3. Transceiver node i is equipped with M_i antennas and consists of transmitter (source) S_i and receiver (destination) $D_{q(i)}$, where $q(i) = i + 2$ for $i = 1, 2$ and $q(i) = i - 2$ for $i = 3, 4$. Each transmitter S_i has one message that is intended for its designated receiver D_i, $i \in \{1, \ldots, 4\}$. The relay R_k comprises of N_k antennas, $k \in \{1, 2, 3\}$. Fig. 2 shows the two hops of this system separately. In the first hop (Fig. 2(a)), the signal received at relay R_k, $k \in \{1, 2, 3\}$, in time slot m is expressed as

$$\mathbf{y}_{R_1}[m] = \mathbf{H}_{1,R_1}\mathbf{x}_1[m] + \mathbf{H}_{4,R_1}\mathbf{x}_4[m] + \mathbf{z}_{R_1}[m], \quad (1)$$

$$\mathbf{y}_{R_2}[m] = \sum_{i=1}^{4} \mathbf{H}_{i,R_2}\mathbf{x}_i[m] + \mathbf{z}_{R_2}[m], \quad (2)$$

$$\mathbf{y}_{R_3}[m] = \mathbf{H}_{2,R_3}\mathbf{x}_2[m] + \mathbf{H}_{3,R_3}\mathbf{x}_3[m] + \mathbf{z}_{R_3}[m], \quad (3)$$

where $\mathbf{H}_{i,R_k}$ is the $N_k \times M_i$ complex channel matrix from transmitter S_i to relay R_k, $\mathbf{x}_i[m]$ is the $M_i \times 1$ signal vector transmitted from S_i, $\mathbf{y}_{R_k}[m]$ is the $N_k \times 1$ signal vector received at relay R_k and $\mathbf{z}_{R_k}[m]$ is the $N_k \times 1$ circularly symmetric complex Gaussian noise vector with i.i.d. zero mean and unit variance entries, $i \in \{1, 2, 3, 4\}$, $k \in \{1, 2, 3\}$. In the second hop (Fig. 2(b)), the signal received at receiver D_i in time slot m is given by

$$\mathbf{y}_i[m] = \mathbf{H}_{R_1,i}\mathbf{x}_{R_1}[m] + \mathbf{H}_{R_2,i}{}_{R_2}[m] + \mathbf{z}_i[m], \text{for } i \in \{2, 3\}, \quad (4)$$

$$\mathbf{y}_i[m] = \mathbf{H}_{R_2,i}\mathbf{x}_{R_2}[m] + \mathbf{H}_{R_3,i}\mathbf{X}_{R_3}[m] + \mathbf{z}_i[m], \text{for } i \in \{1, 4\}, \quad (5)$$

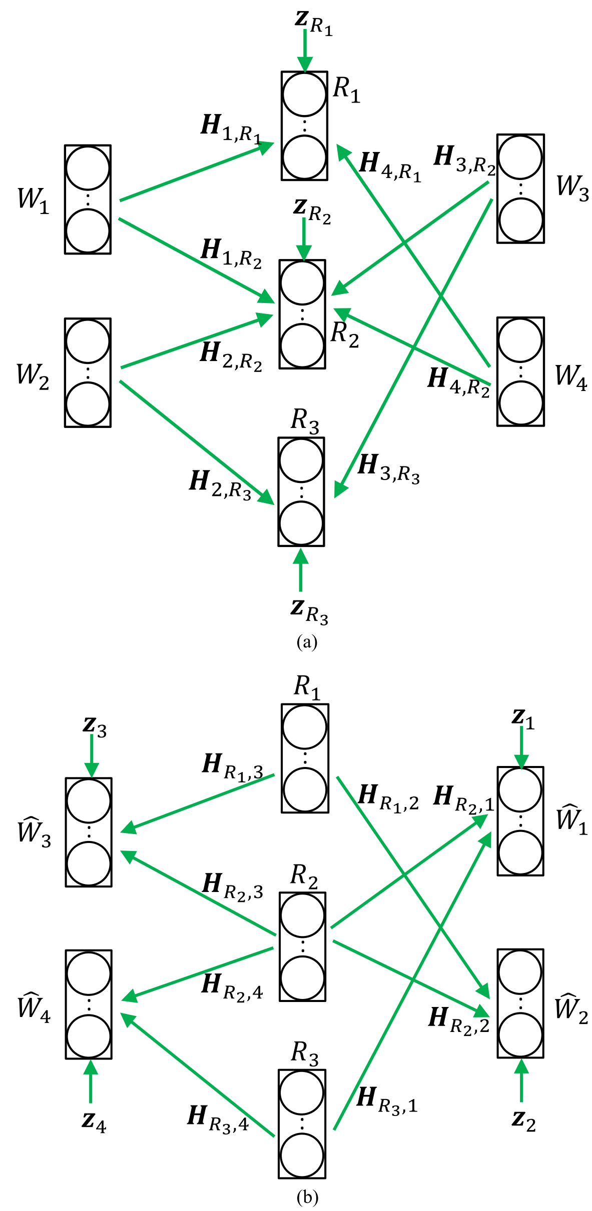

Fig. 2. The channels from and to relays in a two-way MIMO butterfly network. (a) The channels from transmitters to the relays. (b) The channels from relays to the receivers.

where $\mathbf{H}_{R_k,i}$ is the $M_{q(i)} \times N_k$ complex channel matrix from relay R_k to receiver D_i, $\mathbf{x}_{R_k}[m]$ is the $N_k \times 1$ signal vector transmitted from R_k, $\mathbf{y}_i[m]$ is the $M_{q(i)} \times 1$ signal received at receiver D_i and $\mathbf{z}_i[m]$ is the $M_{q(i)} \times 1$ circularly symmetric complex Gaussian noise vector with i.i.d. zero mean and unit variance entries, $i \in \{1, 2, 3, 4\}$, $k \in \{1, 2, 3\}$. We assume that the channel coefficient values are drawn i.i.d. from a continuous distribution and their magnitudes are bounded from below and above by $H_{\min}$ and $H_{\max}$ respectively as in [13]. The relays are assumed to be full-duplex and equipped with caches. Furthermore, the relays are assumed to be causal, which means that the signals transmitted from the relays depend only on the signals received in the past and not on the current received signals and can be described as

$$\mathbf{x}_{R_k}[m] = f(\mathbf{Y}_{R_k}^{m-1}, \mathbf{X}_{R_k}^{m-1}, C_{R_k}), \quad (6)$$

where $\mathbf{X}_{R_k}^{m-1} \triangleq (\mathbf{x}_{R_k}[1], \ldots, \mathbf{x}_{R_k}[m-1])$, $\mathbf{Y}_{R_k}^{m-1} \triangleq (\mathbf{y}_{R_k}[1], \ldots, \mathbf{y}_{R_k}[m-1])$ and C_{R_k} is the cached information in relay R_k. We assume that each source S_i knows only channels $\mathbf{H}_{i,R_k}$, $k \in \{1,2,3\}$; each relay knows all the channels; and each destination D_i knows only channels $\mathbf{H}_{R_k,i}$, $k \in \{1,2,3\}$.

The source S_i has a message W_i that is intended for destination D_i. $|W_i|$ denotes the size of the message W_i. The rates $\mathcal{R}_i = \frac{\log |W_i|}{n}$, $i \in \{1,2,3,4\}$ are achievable during n channel uses when n is large enough, if the probability of error can be arbitrarily small for all four messages simultaneously. The capacity region $\mathcal{C} = \{(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4)\}$ represents the set of all achievable quadruples. The sum-capacity is the maximum sum-rate that is achievable, i.e., $\mathcal{C}_\Sigma(P) = \sum_{i=1}^4 \mathcal{R}_i^c$ where $(\mathcal{R}_1^c, \mathcal{R}_2^c, \mathcal{R}_3^c, \mathcal{R}_4^c) = \arg\max_{(\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3,\mathcal{R}_4)\in\mathcal{C}} \sum_{i=1}^4 \mathcal{R}_i$ and P is the transmit power at each node (source or relay). The degrees of freedom (DoF) is defined as

$$DoF \triangleq \lim_{P\to\infty} \frac{\mathcal{C}_\Sigma(P)}{\log P} = \sum_{i=1}^{4} \lim_{P\to\infty} \frac{\mathcal{R}_i}{\log P} = \sum_{i=1}^{4} d_i, \quad (7)$$

where $d_i \triangleq \lim_{P\to\infty} \frac{\mathcal{R}_i}{\log P}$ is the DoF of source S_i, for $i \in \{1,2,3,4\}$. We denote DoF_C as the DoF for the case of with relay caching, and DoF_{NC} as the DoF for the case with no relay caching.

In this paper, for some special cases, we will obtain the exact DoF of both one-way and two-way MIMO butterfly networks and compare them. For one-way networks, all channels corresponding to directional links from right to left in Fig. 2 are zeros, and $d_3 = d_4 = 0$. We denote the one-way DoF by $\mathrm{DoF}_\rightarrow$.

III. DoF Upper Bounds

In this section, we present two upper bounds on the DoF of two-way MIMO butterfly network with no relay caching. The first theorem is a cut-set upper bound based on the genie-aided transmission through R_2.

Theorem 1: For the two-way MIMO butterfly network without caching, $\mathrm{DoF}_{NC} \le 2N_2$.

Proof: Consider S_1, R_1, and S_4 as one group of users and S_2, R_3, and S_3 as another group. As genie-aided side information, assume that the users in each group (super node) have access to all messages in the same group. Note that the first group has W_1 and W_4 needed by the second group and the second group has W_2 and W_3 needed by the first group. The genie-aided side information does not give the needed message to any destination, and the two groups can only communicate through R_2. The described channel can be seen in Fig. 3 where super nodes A_1 and A_2 both have three sets of antennas. Thus, the cut-set bound gives that $\mathrm{DoF}_{NC} \le 2N_2$, since R_2 is an N_2-antenna node and each of A_1 and A_2 can only decode N_2 DoF of information from it. ■

The next theorem is a cut-set bound on the number of receiver antennas.

Theorem 2: For the two-way MIMO butterfly network without caching, $\mathrm{DoF}_{NC} \le$
$2(\min\{M_1, M_3\} + \min\{M_2, M_4\})$.

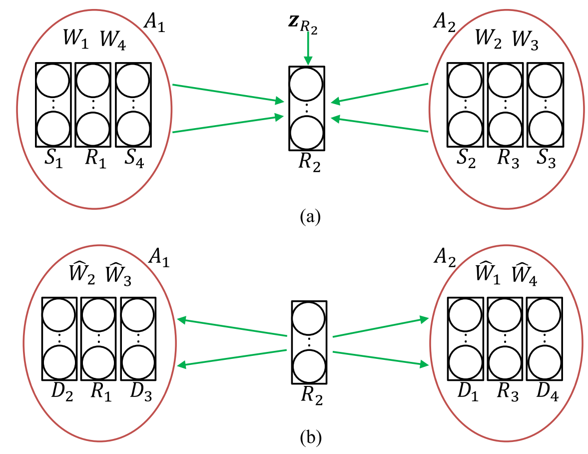

Fig. 3. The genie-aided MIMO butterfly network. (a) The channels from the transmitters to relay R_2. (b) The channels from relay R_2 to the receivers.

Proof: It follows from the cut-set bound on the total number of antennas (minimum number of transmit and receive antennas) for each user i which is $d_i = \min\{M_i, M_{q(i)}\}$. ■

With a similar argument, the following corollary holds for the one-way channel:

Corollary 1: For the one-way MIMO butterfly network without caching, $\mathrm{DoF}_{NC\rightarrow} \le \min\{M_1, M_3\} + \min\{M_2, M_4\}$.

IV. DoF Lower Bounds

The following theorem provides a lower bound on the DoF of the MIMO butterfly network without relay caching.

Theorem 3: If $d_1, \ldots, d_4$ are non-negative integers that satisfy the following conditions:

- $d_1, d_3 \le \min\{M_1, M_3, N_2\}$,
- $d_2, d_4 \le \min\{M_2, M_4, N_2\}$,
- $d_2 + d_3, d_1 + d_4 \le N_2$,
- $2(d_1d_4 + d_1d_2 + d_3d_4 + d_2d_3) \le (N_2^2 - 1) + \min\{N_1^2, d_3d_4 + d_1d_2\} + \min\{N_3^2, d_3d_4 + d_1d_2\}$,

then without relay caching the DoF of $\sum_{i=1}^4 d_i$ is achievable.

Proof: We show that if all conditions in the theorem statement hold, each source-destination pair $(S_i, D_i), i \in \{1, \ldots, 4\}$ can achieve the DoF of d_i. The first three conditions ensure that the DoF for each link is no more than the number of transmit antennas, the number of receive antennas, and also the number of antennas in the relay between them. In the following, we show that by adding the fourth condition, the DoF of (d_1, d_2, d_3, d_4) is achievable.

The received signals at relays are given by (1)–(3). Then, each relay R_k, $k \in \{1,2,3\}$, performs amplify-and-forward by transmitting $\mathbf{x}_{R_k}[m] = \mathbf{V}_k\mathbf{y}_{R_k}[m]$ using an $N_k \times N_k$ matrix $\mathbf{V}_k$, and the received signals at the destinations are given by:

$$\mathbf{y}_1[m] = \mathbf{H}_{R_3,1}\mathbf{V}_3\mathbf{y}_{R_3}[m] + \mathbf{H}_{R_2,1}\mathbf{V}_2\mathbf{y}_{R_2}[m] + \mathbf{z}_1[m], \quad (8)$$
$$\mathbf{y}_2[m] = \mathbf{H}_{R_1,2}\mathbf{V}_1\mathbf{y}_{R_1}[m] + \mathbf{H}_{R_2,2}\mathbf{V}_2\mathbf{y}_{R_2}[m] + \mathbf{z}_2[m], \quad (9)$$
$$\mathbf{y}_3[m] = \mathbf{H}_{R_1,3}\mathbf{V}_1\mathbf{y}_{R_1}[m] + \mathbf{H}_{R_2,3}\mathbf{V}_2\mathbf{y}_{R_2}[m] + \mathbf{z}_3[m], \quad (10)$$
$$\mathbf{y}_4[m] = \mathbf{H}_{R_3,4}\mathbf{V}_3\mathbf{y}_{R_3}[m] + \mathbf{H}_{R_2,4}\mathbf{V}_2\mathbf{y}_{R_2}[m] + \mathbf{z}_4[m]. \quad (11)$$

Substituting (1)–(3) into (8)–(11) results in:

$$\mathbf{y}_1[m] = \mathbf{H}_{R_3,1}\mathbf{V}_3\left(\mathbf{H}_{2,R_3}\mathbf{x}_2[m] + \mathbf{H}_{3,R_3}\mathbf{x}_3[m] + \mathbf{z}_{R_3}[m]\right) + \mathbf{H}_{R_2,1}\mathbf{V}_2\left(\sum_{i=1}^{4}\mathbf{H}_{i,R_2}\mathbf{x}_i[m] + \mathbf{z}_{R_2}[m]\right) + \mathbf{z}_1[m], \quad (12)$$

$$\mathbf{y}_2[m] = \mathbf{H}_{R_1,2}\mathbf{V}_1\left(\mathbf{H}_{1,R_1}\mathbf{x}_1[m] + \mathbf{H}_{4,R_1}\mathbf{x}_4[m] + \mathbf{z}_{R_1}[m]\right) + \mathbf{H}_{R_2,2}\mathbf{V}_2\left(\sum_{i=1}^{4}\mathbf{H}_{i,R_2}\mathbf{x}_i[m] + \mathbf{z}_{R_2}[m]\right) + \mathbf{z}_2[m], \quad (13)$$

$$\mathbf{y}_3[m] = \mathbf{H}_{R_1,3}\mathbf{V}_1\left(\mathbf{H}_{1,R_1}\mathbf{x}_1[m] + \mathbf{H}_{4,R_1}\mathbf{x}_4[m] + \mathbf{z}_{R_1}[m]\right) + \mathbf{H}_{R_2,3}\mathbf{V}_2\left(\sum_{i=1}^{4}\mathbf{H}_{i,R_2}\mathbf{x}_i[m] + \mathbf{z}_{R_2}[m]\right) + \mathbf{z}_3[m], \quad (14)$$

$$\mathbf{y}_4[m] = \mathbf{H}_{R_3,4}\mathbf{V}_3\left(\mathbf{H}_{2,R_3}\mathbf{x}_2[m] + \mathbf{H}_{3,R_3}\mathbf{x}_3[m] + \mathbf{z}_{R_3}[m]\right) + \mathbf{H}_{R_2,4}\mathbf{V}_2\left(\sum_{i=1}^{4}\mathbf{H}_{i,R_2}\mathbf{x}_i[m] + \mathbf{z}_{R_2}[m]\right) + \mathbf{z}_4[m]. \quad (15)$$

We assume that each transmitter S_i transmits signals from the top d_i antennas and nothing from the rest of the antennas. We will show the existence of $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$ such that each receiver D_i can decode the d_i information streams from its corresponding transmitter S_i, and then the proof of achievability will be complete.

Now, we analyze the interfering signals that should be nulled. For destination D_1, the signal $\mathbf{x}_1[m]$ is the intended signal and the receiver knows $\mathbf{x}_3[m]$ as it is the transmitter S_3 as well. Therefore, the interference from the signals $\mathbf{x}_2[m]$ and $\mathbf{x}_4[m]$ should be nulled at destination D_1.

1) The interfering signal from $\mathbf{x}_4[m]$ to D_1:

$$\mathbf{q}_{4\to1}[m] = \underbrace{\left(\mathbf{H}_{R_2,1}\mathbf{V}_2\mathbf{H}_{4,R_2}\right)}_{\triangleq\mathbf{G}_{4\to1}} \cdot \left[x_4^{(1)}[m], \ldots, x_4^{(d_4)}[m], 0, \ldots, 0\right]^T, \quad (16)$$

where $x_i^{(j)}[m]$ represents the jth entry of vector $\mathbf{x}_i[m]$.

2) The interfering signal from $\mathbf{x}_2[m]$ to D_1:

$$\mathbf{q}_{2\to1}[m] = \underbrace{\left(\mathbf{H}_{R_3,1}\mathbf{V}_3\mathbf{H}_{2,R_3} + \mathbf{H}_{R_2,1}\mathbf{V}_2\mathbf{H}_{2,R_2}\right)}_{\triangleq\mathbf{G}_{2\to1}} \cdot \left[x_2^{(1)}[m], \ldots, x_2^{(d_2)}[m], 0, \ldots, 0\right]^T. \quad (17)$$

We will choose $\mathbf{V}_1$ and $\mathbf{V}_2$ such that the top d_1 antennas at D_1 contain the d_1 intended data streams, by enforcing that first d_1 elements of both $\mathbf{q}_{4\to1}[m]$ and $\mathbf{q}_{2\to1}[m]$ does not contain elements of $\mathbf{x}_4[m]$ and $\mathbf{x}_2[m]$, respectively. That is, we force the corresponding submatrices in (16) and (17) to be zero, i.e.,

$$\mathbf{G}_{4\to1}[1:d_1, 1:d_4] = \mathbf{0}, \quad (18)$$

$$\mathbf{G}_{2\to1}[1:d_1, 1:d_2] = \mathbf{0}. \quad (19)$$

(18) consists of d_1d_4 linear equations of elements of $\mathbf{V}_2$ and (19) consists of d_1d_2 linear equations of elements of $\mathbf{V}_3$ and $\mathbf{V}_2$.

Also, the interference from the signals $\mathbf{x}_1[m]$ and $\mathbf{x}_3[m]$ should be nulled at destination D_2, which are defined as

$$\mathbf{q}_{3\to2}[m] = \underbrace{\left(\mathbf{H}_{R_2,2}\mathbf{V}_2\mathbf{H}_{3,R_2}\right)}_{\triangleq\mathbf{G}_{3\to2}} \cdot \left[x_3^{(1)}[m], \ldots, x_3^{(d_3)}[m], 0, \ldots, 0\right]^T, \quad (20)$$

$$\mathbf{q}_{1\to2}[m] = \underbrace{\left(\mathbf{H}_{R_1,2}\mathbf{V}_1\mathbf{H}_{1,R_1} + \mathbf{H}_{R_2,2}\mathbf{V}_2\mathbf{H}_{1,R_2}\right)}_{\triangleq\mathbf{G}_{1\to2}} \cdot \left[x_1^{(1)}[m], \ldots, x_1^{(d_1)}[m], 0, \ldots, 0\right]^T. \quad (21)$$

Therefore, the followings should hold:

$$\mathbf{G}_{3\to2}[1:d_2, 1:d_3] = \mathbf{0}, \quad (22)$$

$$\mathbf{G}_{1\to2}[1:d_2, 1:d_1] = \mathbf{0}. \quad (23)$$

(22) consists of d_2d_3 linear equations of elements of $\mathbf{V}_2$ and (23) consists of d_2d_1 linear equations of elements of $\mathbf{V}_1$ and $\mathbf{V}_2$.

Similarly, $\mathbf{x}_2[m]$ and $\mathbf{x}_4[m]$ should be nulled at destination D_3, which are defined as

$$\mathbf{q}_{2\to3}[m] = \underbrace{\left(\mathbf{H}_{R_2,3}\mathbf{V}_2\mathbf{H}_{2,R_2}\right)}_{\triangleq\mathbf{G}_{2\to3}} \cdot \left[x_2^{(1)}[m], \ldots, x_2^{(d_2)}[m], 0, \ldots, 0\right]^T, \quad (24)$$

$$\mathbf{q}_{4\to3}[m] = \underbrace{\left(\mathbf{H}_{R_1,3}\mathbf{V}_1\mathbf{H}_{4,R_1} + \mathbf{H}_{R_2,3}\mathbf{V}_2\mathbf{H}_{4,R_2}\right)}_{\triangleq\mathbf{G}_{4\to3}} \cdot \left[x_4^{(1)}[m], \ldots, x_4^{(d_4)}[m], 0, \ldots, 0\right]^T. \quad (25)$$

Therefore, the followings should hold:

$$\mathbf{G}_{2\to3}[1:d_3, 1:d_2] = \mathbf{0}, \quad (26)$$

$$\mathbf{G}_{4\to3}[1:d_3, 1:d_4] = \mathbf{0}. \quad (27)$$

(26) consists of d_3d_2 linear equations of elements of $\mathbf{V}_2$ and (27) consists of d_3d_4 linear equations of elements of $\mathbf{V}_1$ and $\mathbf{V}_2$.

Finally, $\mathbf{x}_1[m]$ and $\mathbf{x}_3[m]$ should be nulled at destination D_4, which are defined as

$$\mathbf{q}_{1\to4}[m] = \underbrace{(\mathbf{H}_{R_2,4}\mathbf{V}_2\mathbf{H}_{1,R_2})}_{\triangleq\mathbf{G}_{1\to4}} \cdot \left[x_1^{(1)}[m],\ldots,x_1^{(d_1)}[m],0,\ldots,0\right]^T, \quad (28)$$

$$\mathbf{q}_{3\to4}[m] = \underbrace{(\mathbf{H}_{R_3,4}\mathbf{V}_3\mathbf{H}_{3,R_3}+\mathbf{H}_{R_2,4}\mathbf{V}_2\mathbf{H}_{3,R_2})}_{\triangleq\mathbf{G}_{3\to4}} \cdot \left[x_3^{(1)}[m],\ldots,x_3^{(d_3)}[m],0,\ldots,0\right]^T. \quad (29)$$

Therefore, the followings should hold:

$$\mathbf{G}_{1\to4}[1:d_4,1:d_1] = \mathbf{0}, \quad (30)$$

$$\mathbf{G}_{3\to4}[1:d_4,1:d_3] = \mathbf{0}. \quad (31)$$

(30) consists of d_4d_1 linear equations of elements of $\mathbf{V}_2$ and (31) consists of d_4d_3 linear equations of elements of $\mathbf{V}_3$ and $\mathbf{V}_2$.

Combining the above equations, we have in total:

1) $2(d_1d_4+d_2d_3)$ linear equations of the form $\mathbf{G}_2\mathbf{v}_2=\mathbf{0}$ of the N_2^2 elements of $\mathbf{V}_2$, where $\mathbf{v}_2=\mathrm{vec}(\mathbf{V}_2)$ (see (18), (22), (26), (30)).
2) $(d_1d_2+d_3d_4)$ linear equations of the form $\mathbf{G}_1\mathbf{v}_1+\mathbf{G}_{1,2}\mathbf{v}_2$ of the N_1^2 elements of $\mathbf{V}_1$ and N_2^2 elements of $\mathbf{V}_2$, where $\mathbf{v}_1=\mathrm{vec}(\mathbf{V}_1)$ (see (23), (27)).
3) $(d_1d_2+d_3d_4)$ linear equations of the form $\mathbf{G}_3\mathbf{v}_3+\mathbf{G}_{3,2}\mathbf{v}_2$ of the N_3^2 elements of $\mathbf{V}_3$ and N_2^2 elements of $\mathbf{V}_2$, where $\mathbf{v}_3=\mathrm{vec}(\mathbf{V}_3)$ (see (19), (31)).

We set $(N_i^2-d_1d_2-d_3d_4)^+$ elements of $\mathbf{v}_i$ to zero, $i=1,3$[1] and solve for the remaining elements. Then when the fourth condition in the theorem holds, the number of unknowns is more than the number of equations. Given that the elements of the channel matrices are generically chosen, with high probability there exists a non-zero solution of $(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)$. Note that $\mathbf{v}_2\neq\mathbf{0}$ since otherwise by 2) and 3) we will have $\mathbf{v}_1=\mathbf{v}_2=\mathbf{0}$. Hence $\mathbf{v}_1$ and $\mathbf{v}_3$ are also non-zero by 2) and 3). ■

The next theorem provides another lower bound for the two-way MIMO butterfly network.

Theorem 4: For the two-way MIMO butterfly network without relay caching, $\mathrm{DoF}_{NC}\geq 2\min\{N_2,\max\{\min\{M_1,M_3\},\min\{M_2,M_4\}\}\}$.

Proof: First consider the case of $\min\{M_1,M_3\}\geq\min\{M_2,M_4\}$. If all nodes except for S_1, R_2, and S_3 in Fig. 1 are silent, then the channel can be seen as a two-way relay system formed by nodes S_1, R_2, and S_3 with numbers of antennas $\min\{M_1,M_3\}$, N_2 and $\min\{M_1,M_3\}$, respectively. This channel can achieve the DoF of $2\min\{N_2,\min\{M_1,M_3\}\}$ by simply forwarding the sum of the received signals at relay R_2, which is the sum of the two messages from S_1 and S_3.

If $\min\{M_1,M_3\}<\min\{M_2,M_4\}$, S_2 and S_4 communicate through S_2 in a similar way and the DoF of $2\min\{N_2,\min\{M_2,M_4\}\}$ is achievable. Therefore, the maximum of the above two DoFs is achievable.

The theorem can also be proven by using Theorem 3 with $d_1=d_3=\min\{N_2,\min\{M_1,M_3\}\}$, $d_2=d_4=0$ for the case of $\min\{M_1,M_3\}\geq\min\{M_2,M_4\}$, and with $d_2=d_4=\min\{N_2,\min\{M_2,M_4\}\}$, $d_1=d_3=0$ for the case of $\min\{M_1,M_3\}<\min\{M_2,M_4\}$. ■

[1]Given that we are proving an achievability, we can choose the beamforming parameters freely. Here, the purpose of setting to zero is to avoid unnecessary interfering signals beyond the desired amount.

V. Special Cases

In this section, we provide the DoF results on some special cases of the MIMO butterfly network using the theorems given in the previous section.

A. The Case of $M_i=M$ and $N_k=N$

In this subsection, we consider a special case where $M_i=M$, $i\in\{1,2,3,4\}$, and $N_k=N$, $k\in\{1,2,3\}$.

Theorem 5: If $N\leq M$, then $\mathrm{DoF}_{NC}=2N$. And if $N>2M$, then $\mathrm{DoF}_{NC}=4M$.

Proof: For $N\leq M$, the upper bound follows from Theorem 1 and the lower bound follows from Theorem 4. For $N>2M$, the upper bound follows from Theorem 2 and the lower bound follows from Theorem 3 by setting $d_i=M$, $i\in\{1,\ldots,4\}$. ■

The above theorem states that when $N\leq M$, then the bottleneck on the DoF is the number of relay antennas in R_2 and since it has N antennas the DoF in each direction is N (with a total of $2N$). On the other hand, when $N>2M$, then the bottleneck on the DoF is the number of source transmit antennas and since there are $4M$ transmit antennas in total, the DoF is $4M$.

The next theorem gives the DoF bounds for the case of $M<N\leq 2M$.

Theorem 6: If $M<N\leq 2M$, then $\min\{\lfloor\frac{N^2-1}{2M}\rfloor,\ 2(N-M)\}+2M\leq\mathrm{DoF}_{NC}\leq\min\{4M,2N\}$.

Proof: The upper bound follows from Theorems 1–2. For the lower bound, the conditions in Theorem 3 can be written as:

$$d_i\leq M,\quad i=1,\ldots,4,$$

$$d_2+d_3,d_1+d_4\leq N,$$

$$2(d_1d_4+d_2d_3)\leq N^2-1, \quad (32)$$

$$2(d_1d_4+d_1d_2+d_3d_4+d_2d_3)\leq 3N^2-1, \quad (33)$$

where (32) comes from $2(d_1d_4+d_1d_2+d_3d_4+d_2d_3)\leq N^2-1+2(d_3d_4+d_1d_2)$.

First, we note that $d_1=d_3=M$, $d_2=\min\{\lfloor\frac{N^2-1}{4M}\rfloor,\ N-M\}$ and $d_4=\min\{\lfloor\frac{N^2-1}{4M}+\frac{1}{2}\rfloor,N-M\}$ satisfies the conditions in (32) as

- $N\leq 2M \implies N^2-1\leq 4M^2-1<4M(M+\frac{1}{2}) \implies \frac{N^2-1}{4M}<M+\frac{1}{2}\implies \frac{N^2-1}{4M}+\frac{1}{2}<M+1 \implies \lfloor\frac{N^2-1}{4M}+\frac{1}{2}\rfloor\leq M$. Moreover, using Lemma 1 below, we have
- $2(d_1d_4+d_2d_3)=2M(\lfloor\frac{N^2-1}{4M}\rfloor+\lfloor\frac{N^2-1}{4M}+\frac{1}{2}\rfloor)=2M\lfloor\frac{N^2-1}{2M}\rfloor\leq N^2-1$,
- $2(d_1d_4+d_1d_2+d_3d_4+d_2d_3)\leq 4M(\lfloor\frac{N^2-1}{4M}\rfloor+\lfloor\frac{N^2-1}{4M}+\frac{1}{2}\rfloor)=4M\lfloor\frac{N^2-1}{2M}\rfloor=2(N^2-1)\leq 3N^2-1$.

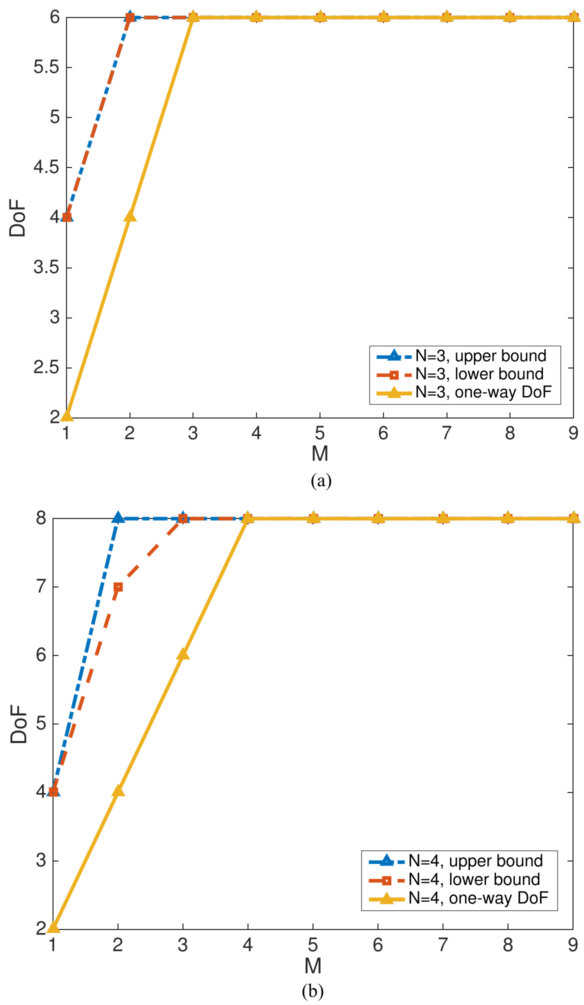

Fig. 4. The DoF bounds for the case of $N_k = N$ given by Theorems 5 and 6.

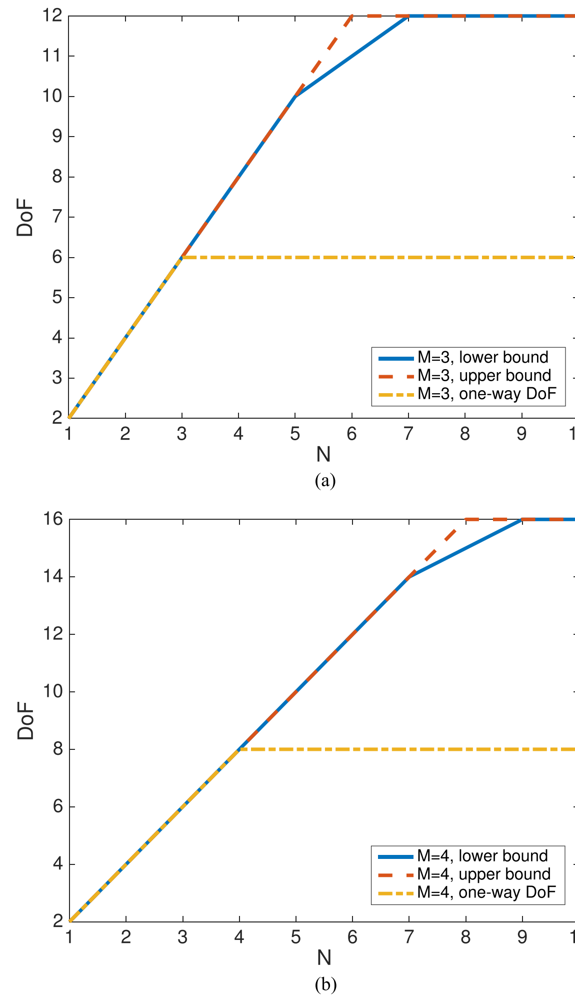

Fig. 5. The DoF bounds for the case of $M_i = M$ given by Theorems 5 and 6.

Hence the above $d_1, \ldots, d_4$ results in the DoF lower bound of $2M + \min\{\lfloor \frac{N^2-1}{4M} \rfloor, N - M\} + \min\{\lfloor \frac{N^2-1}{4M} + \frac{1}{2} \rfloor, N - M\} = 2M + \min\{\lfloor \frac{N^2-1}{2M} \rfloor, 2(N - M)\}$. ■

Lemma 1: For any real number x, we have $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$.

Moreover, the following lemma shows that the lower bound in Theorem 6 is the largest one obtained from Theorem 3.

Lemma 2: (d_1, d_2, d_3, d_4) obtained in the proof of Theorem 6 maximizes $\sum_{i=1}^{4} d_i$ with d_i being non-negative integers subject to (32).

Proof: The proof is given in Appendix A. ■

The DoF given in Theorem 5 and the bounds given in Theorem 6 are illustrated in Figs. 4–6 for different values of M and N.

Fig. 4 provides the DoF bounds for the case of $N_k = N$ given by Theorems 5 and 6. Fig. 5 provides the DoF bounds for the case of $M_i = M$ given by Theorems 5 and 6. In both Figs. 4 and 5 we observe that for the set of parameters where the condition of Theorem 5 holds, the upper and lower bounds are equal. Fig. 6 also provides the upper and lower bounds on DoF given in Theorems 5 and 6, in comparison with the one-way DoF.

The following corollary gives the exact DoF for the one-way MIMO butterfly network where each relay node has N antennas and each source/destination node has M antennas.

Corollary 2: For the case of $M_i = M$ and $N_k = N$, $\mathrm{DoF}_{NC\rightarrow} = \min\{2M, 2N\}$.

Proof: The upper bounds follow from Theorem 1 and Corollary 1. The lower bound follows from Theorem 3 by setting $d_1 = d_2 = \min\{M, N\}$, $d_3 = d_4 = 0$. ■

Remark 1: Comparing the results in Theorem 5 and Corollary 2, it is seen that when $N > 2M$, bidirectional transmission doubles the one-way DoF; but when $N < M$, then bidirectional links have no impact on the DoF.

Remark 1 can be simply verified in Figs. 4 and 5.

B. *The Case of $M_i = 1$, $N_k \geq 1$*

Next we consider the case of multi-antenna relay nodes and single-antenna source/destination nodes, i.e., $M_i = 1$, $N_k \geq k$.

Theorem 7: For the case of $M_i = 1$, $i = 1, 2, 3, 4$, $N_1 \geq 1$ and $N_3 \geq 1$, we have

- If $N_2 = 1$, then $\mathrm{DoF}_{NC} = 2$;
- If $N_2 = 2$, then $3 \leq \mathrm{DoF}_{NC} \leq 4$;
- If $N_2 \geq 3$, then $\mathrm{DoF}_{NC} = 4$.

Proof:

- $N_2 = 1$: The upper bound follows from Theorem 1 and the lower bound follows from Theorem 4.

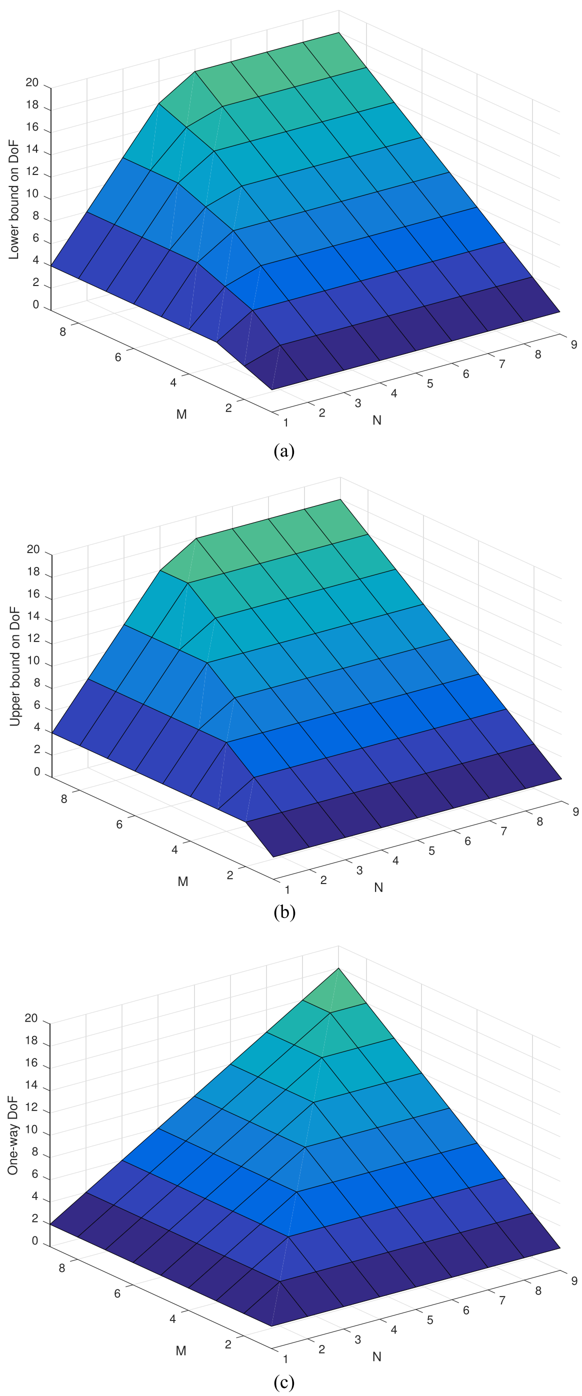

Fig. 6. The upper and lower bounds on DoF given in Theorems 5 and 6, in comparison with the one-way DoF. (a) The lower bound on DoF. (b) The upper bound on DoF. (c) The one-way DoF.

- $N_2 = 2$: The upper bound follows from Theorem 2 and the lower bound follows from Theorem 3 by setting $d_1 = d_2 = d_3 = 1$, and $d_4 = 0$.
- $N_2 \geq 3$: The upper bound follows from Theorem 2 and the lower bound follows from Theorem 3 by setting $d_i = 1$, $i \in \{1, \dots, 4\}$. ■

The following corollary gives the one-way DoF for the case of single-antenna source/destination nodes and multi-antenna relay nodes.

Corollary 3: For the case of $M_i = 1$, $N_j \geq 1$, $\mathrm{DoF}_{NC\rightarrow} = 2$.

Proof: The upper bounds follow from Corollary 1. The lower bound follows from Theorem 3 by setting $d_1 = d_2 = 1$, $d_3 = d_4 = 0$. ■

Remark 2: Comparing the results in Theorem 7 and Corollary 3, it is seen that for the butterfly network with single-antenna nodes, when relay node R_2 has a single antenna, bidirectional transmission does not improve the DoF. When R_2 has two antennas, bidirectional transmission improves the DoF by a factor of at least 1.5. And when R_2 has at least three antennas, bidirectional transmission doubles the DoF.

VI. Two-Way MIMO Butterfly Network With Caching

We now assume that each relay is equipped with a cache that can store the data from the sources. Our goal is to design strategies for caching and transmission so that the sum rate of all four source-destination pairs is maximized. The transmission comprises two phases. The first phase is the transmission from sources to the relays, as shown in Fig. 2(a), which is performed offline and is known as the placement phase. The second phase is the transmission from relays to the destinations, as shown in Fig. 2(b), which is performed online and is known as the delivery phase. We assume that the relays decode W_i, $i = 1, \dots, 4$ in the offline phase and store $W_1' \triangleq \{W_1, W_3\}$, $W_2' \triangleq \{W_2, W_4\}$ in their caches. The transmitted signals from the relays intend to make W_1' decodable at D_1 and D_3, and W_2' decodable at D_2 and D_4.

The next theorem provides an upper bound on the DoF of the two-way MIMO butterfly network with relay caching.

Theorem 8: For the two-way MIMO butterfly network with relay caching descried above, $\mathrm{DoF}_C \leq \min\{N_1 + N_2, M_1 + M_4\} + \min\{N_3 + N_2, M_3 + M_2\}$.

Proof: As in Fig. 7(a), for the transmission from relays R_1 and R_2 to receivers D_2 and D_3, assuming that relays have cached messages as side information, the DoF of $\min\{N_1 + N_2, M_1 + M_4\}$ is a cut-set bound on $d_2 + d_3$. Similarly, as in Fig. 7(b), in the transmission from relays R_2 and R_3 to receivers D_1 and D_4, assuming that relays have cached messages as side information, the DoF of $\min\{N_2 + N_3, M_3 + M_2\}$ is a cut-set bound on $d_1 + d_4$. Note that this theorem trivially holds for the case of no caching as well. ■

The following result characterizes the DoF of the two-way MIMO butterfly network with relay caching, when $M_i = M$ and $N_k = N$.

Theorem 9: For the case of $M_i = M$, $i = 1, 2, 3, 4$, and $N_k = N$, $k = 1, 2, 3$, we have $\mathrm{DoF}_C = 4\min\{N, M\}$.

Proof: First consider the case of $M \leq N$. The upper bound of $4M$ follows from Theorem 8. We now provide an achievability strategy. We do not use the last $N - M$ antennas of the relays and therefore effectively every transmitter, relay and receiver has only M antennas. The relays know the new messages W_1' and W_2', and the encoded signals in all relays for messages W_1' and W_2' at time $m = 1, 2, \dots, n$ are $M \times 1$ vectors, i.e., $\mathbf{a}[m] = f_m(W_1')$ and $\mathbf{b}[m] = f_m(W_2')$, $m = 1, 2, \dots, n$. At time m, the relays transmit the following signals

$$\mathbf{x}_{R_1}[m] = -\mathbf{H}_{R_1,2}{}^{-1}\mathbf{H}_{R_2,2}\mathbf{a}[m] - \mathbf{H}_{R_1,3}{}^{-1}\mathbf{H}_{R_2,3}\mathbf{b}[m],$$
$$\mathbf{x}_{R_2}[m] = \mathbf{a}[m] + \mathbf{b}[m],$$
$$\mathbf{x}_{R_3}[m] = -\mathbf{H}_{R_3,4}{}^{-1}\mathbf{H}_{R_2,4}\mathbf{a}[m] - \mathbf{H}_{R_3,1}{}^{-1}\mathbf{H}_{R_2,1}\mathbf{b}[m].$$

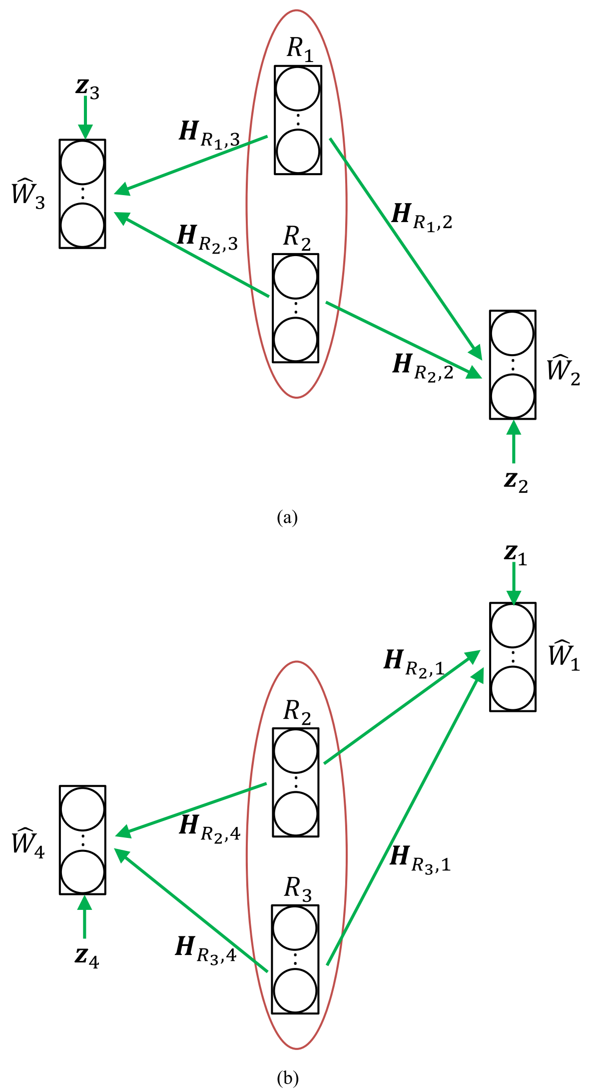

Fig. 7. The cut-set bounds from the relays to receivers in a two-way MIMO butterfly network. (a) The cut-set bound between R_1, R_2 and D_2, D_3. (b) The cut-set bound between R_2, R_3 and D_1, D_4.

Then the received signals at the destinations are as follows

$$\begin{aligned}
\mathbf{y}_1[m] &= \mathbf{H}_{R_2,1}\left(\mathbf{a}[m]+\mathbf{b}[m]\right)+\mathbf{H}_{R_3,1}\left(-\mathbf{H}_{R_3,4}{}^{-1}\right.\\
&\qquad \left.\mathbf{H}_{R_2,4}\mathbf{a}[m]-\mathbf{H}_{R_3,1}{}^{-1}\mathbf{H}_{R_2,1}\mathbf{b}[m]\right)+\mathbf{z}_1[m]\\
&= \left(\mathbf{H}_{R_2,1}-\mathbf{H}_{R_3,1}\mathbf{H}_{R_3,4}{}^{-1}\mathbf{H}_{R_2,4}\right)\mathbf{a}[m]+\mathbf{z}_1[m],\\
\mathbf{y}_2[m] &= \mathbf{H}_{R_2,2}\left(\mathbf{a}[m]+\mathbf{b}[m]\right)+\mathbf{H}_{R_1,2}\left(-\mathbf{H}_{R_1,2}{}^{-1}\right.\\
&\qquad \left.\mathbf{H}_{R_2,2}\mathbf{a}[m]-\mathbf{H}_{R_1,3}{}^{-1}\mathbf{H}_{R_2,3}\mathbf{b}[m]\right)+\mathbf{z}_2[m]\\
&= \left(\mathbf{H}_{R_2,2}-\mathbf{H}_{R_1,2}\mathbf{H}_{R_1,3}{}^{-1}\mathbf{H}_{R_2,3}\right)\mathbf{b}[m]+\mathbf{z}_2[m],\\
\mathbf{y}_3[m] &= \mathbf{H}_{R_2,3}\left(\mathbf{a}[m]+\mathbf{b}[m]\right)+\mathbf{H}_{R_1,3}\left(-\mathbf{H}_{R_1,2}{}^{-1}\right.\\
&\qquad \left.\mathbf{H}_{R_2,2}\mathbf{a}[m]-\mathbf{H}_{R_1,3}{}^{-1}\mathbf{H}_{R_2,3}\mathbf{b}[m]\right)+\mathbf{z}_3[m]\\
&= \left(\mathbf{H}_{R_2,3}-\mathbf{H}_{R_1,3}\mathbf{H}_{R_1,2}{}^{-1}\mathbf{H}_{R_2,2}\right)\mathbf{a}[m]+\mathbf{z}_3[m],\\
\mathbf{y}_4[m] &= \mathbf{H}_{R_2,4}\left(\mathbf{a}[m]+\mathbf{b}[m]\right)+\mathbf{H}_{R_3,4}\left(-\mathbf{H}_{R_3,4}{}^{-1}\right.\\
&\qquad \left.\mathbf{H}_{R_2,4}\mathbf{a}[m]-\mathbf{H}_{R_3,1}{}^{-1}\mathbf{H}_{R_2,1}\mathbf{b}[m]\right)+\mathbf{z}_4[m]\\
&= \left(\mathbf{H}_{R_2,4}-\mathbf{H}_{R_3,4}\mathbf{H}_{R_3,1}{}^{-1}\mathbf{H}_{R_2,1}\right)\mathbf{b}[m]+\mathbf{z}_4[m].
\end{aligned}$$

Note that the first and the third receivers receive noisy versions of $\mathbf{a}[m]$, from which they can decode W_1' and subtract the contribution of their own messages to obtain their desired messages. The argument is similar for the second and the fourth receivers using $\mathbf{b}[m]$ and W_2' and thus showing that a DoF of $4M$ can be achieved.

Now consider the case of $M > N$. The lower bound of $4N$ can be obtained with a similar approach. That is, we do not use the last $M - N$ antennas of the transceivers and effectively every transmitter, relay and receiver has only N antennas and transmit the $N \times 1$ vectors of $\mathbf{a}[m]$ and $\mathbf{b}[m]$. The upper bound again follows from Theorem 8. ∎

Remark 3: It is interesting to compare the results in this section with that in Section V-A. Specifically, when $N \leq M$, by Theorem 5 we have $\text{DoF}_{NC} = 2N$, and by Theorem 9, $\text{DoF}_C = 4N$. On the other hand, when $N > 2M$, we have $\text{DoF}_{NC} = \text{DoF}_C = 4M$. Hence depending on the number of antennas in each node, caching can either increase the DoF up to a factor of 2, or has no effect on the DoF at all. In particular, relay caching improves the DoF when the number of relay antennas is small compared with the number of source/destination antennas. As the relay antenna number increases, the effect of relay caching on the DoF becomes smaller and eventually vanishes, since the number of source/destination antennas becomes the DoF bottleneck.

Remark 4: For the butterfly network with multi-antenna relay nodes and single-antenna source/destination nodes with caching, i.e., $M_i = 1$, $i = 1, 2, 3, 4$, and $N_k \geq 1$, $k = 1, 2, 3$, we have $\text{DoF}_C = 4$. For the case of $N_1 = N_2 = N_3 = 1$, the proof is given in [3]. Trivially, the same achievability still holds if we increase the number of antennas in relays. The upper bound of 4 on DoF also holds as each source-destination pair can have the maximum DoF of 1.

Remark 5: Upper and lower bounds on the DoF of the two-way MIMO butterfly network with limited relay caching can be obtained by time-sharing between the corresponding bounds with and without caching.

Remark 6: In practice, the requested contents may or may not be cached in the relays. We can interpret our results in the following way: if the contents are not cached in the relays, then the link level DoF is given by DoF_{NC} discussed in Sections III–V; and if the contents are cached in the relays, the link level DoF is given by DoF_C discussed in Section VI.

VII. Conclusion

We have considered the two-way MIMO butterfly network, a class of two-way MIMO four-unicast networks. We have provided upper and lower bounds on the sum DoF of such network with any number of antennas in each node. For the special case that all source/destination nodes have M antennas and all relay nodes have N antennas, we have obtained the exact DoF for some special cases, i.e., $\text{DoF} = 2N$ if $N \leq M$, which is the same as the one-way DoF; and $\text{DoF} = 4M$ if $N > 2M$, which doubles the one-way DoF. Further, we have also studied the DoF of the two-way MIMO butterfly network wth caching at the relays. In particular, for the same special case of M source/destination antennas and N relay antennas, we have shown that when $N \leq M$, relay caching doubles the DoF; whereas when $N > 2M$, relay caching does not improve the DoF at all. Hence, for this network, in general, relay caching improves the DoF when the number of relay antennas is small compared with the number of source/destination antennas; whereas bidirectional transmission improves the DoF when the number of relay antennas is large.

This work is also a first step towards the study of general two-way relay-assisted networks. Another future work direction is studying the impact of practical considerations such as channel estimation error.

Appendix A
Proof of Lemma 2

First consider the case $\lfloor \frac{N^2-1}{4M} + \frac{1}{2} \rfloor \leq N - M$. The achievable DoF set (d_1, d_2, d_3, d_4) obtained in Theorem 6 is $d_1 = d_3 = M$, $d_2 = \lfloor \frac{N^2-1}{4M} \rfloor$ and $d_4 = \lfloor \frac{N^2-1}{4M} + \frac{1}{2} \rfloor$.

We show the optimality by contradiction. Assume that $(d_1^{'}, d_2^{'}, d_3^{'}, d_4^{'}) = (d_1 - p, d_2 + n, d_3 - q, d_4 + m)$ with integers m, n, p, q is achievable such that $m + n > p + q$ and thus $\sum d_i^{'} > \sum d_i$. Define $a \triangleq m + n$ and $b \triangleq p + q$ (and therefore $a > b$). Note that $p, q \geq 0$ (and therefore $b \geq 0$) since $d_1^{'}$ and $d_3^{'}$ cannot be more than M (see the first constraint in (32)). We show that this does not satisfy the third constraint in (32). We have

$$
\begin{aligned}
&2(d_1^{'} d_4^{'} + d_2^{'} d_3^{'}) \\
&= 2(M-p)\left(\left\lfloor \frac{N^2-1}{4M} + \frac{1}{2} \right\rfloor + m\right) \\
&\quad + 2(M-q)\left(\left\lfloor \frac{N^2-1}{4M} \right\rfloor + n\right) \\
&= 2M\left(\left\lfloor \frac{N^2-1}{4M} \right\rfloor + \left\lfloor \frac{N^2-1}{4M} + \frac{1}{2} \right\rfloor + m + n\right) \\
&\quad - 2p\left(\left\lfloor \frac{N^2-1}{4M} + \frac{1}{2} \right\rfloor + m\right) - 2q\left(\left\lfloor \frac{N^2-1}{4M} \right\rfloor + n\right) \\
&\overset{(a)}{=} 2M\left(\left\lfloor \frac{N^2-1}{2M} \right\rfloor + m + n\right) \\
&\quad - 2p\underbrace{\left(\left\lfloor \frac{N^2-1}{4M} + \frac{1}{2} \right\rfloor + m\right)}_{d_4^{'} \leq M} - 2q\underbrace{\left(\left\lfloor \frac{N^2-1}{4M} \right\rfloor + n\right)}_{d_2^{'} \leq M} \\
&\geq 2M\left(\left\lfloor \frac{N^2-1}{2M} \right\rfloor + \underbrace{m+n-p-q}_{a-b \geq 1}\right) \\
&> (N^2-1),
\end{aligned}
\tag{34}
$$

where (a) follows from Lemma 1.

Now consider the case $\lfloor \frac{N^2-1}{4M} + \frac{1}{2} \rfloor > N - M$. Consequently, $\lfloor \frac{N^2-1}{4M} \rfloor \geq N - M$. In this case, the achievable DoF set (d_1, d_2, d_3, d_4) obtained in Theorem 6 is $d_1 = d_3 = M$ and $d_2 = d_4 = N - M$ which is optimal given the second bound in (32).

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