A Characterization of Sampling Patterns for Union of Low-Rank Subspaces Retrieval Problem

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Abstract

In this paper, we characterize the deterministic conditions on the locations of the sampled entries, i.e., sampling pattern, that are equivalent to finite completability of a matrix that represents the union of several low-rank subspaces. To this end, in contrast with the existing analysis on Grassmannian manifold for conventional matrix completion problem, we propose a geometric analysis on the manifold structure for the union of several subspaces to incorporate all given rank constraints simultaneously. Then, using the developed tools for this analysis, we also derive a sufficient condition on the sampling pattern that ensures there exists only one completion of the sampled data.

Introduction

Low-rank matrix completion has received significant recent attention and finds applications in various areas including image or signal processing (Candès et al. 2013; Ji et al. 2010; Candès and Recht 2009), data mining (Eldén 2007), network coding (Harvey, Karger, and Murota 2005), bioinformatics (Ogundijo, Elmas, and Wang 2017), fingerprinting (Liu et al. 2016a), systems biology (Ogundijo, Elmas, and Wang), etc., and one of the main reasons of such versatility is that matrices consisting of the real-world data typically possess a low-rank structure. Recently, several approaches are proposed to tackle a more complicated version of the low-rank matrix completion problem named the union of low-rank subspaces completion problem, where each column belongs to a subspace among multiple low-rank subspaces, and therefore the whole matrix belongs to the union of those multiple low-rank subspaces (Eriksson, Balzano, and Nowak 2012; Gao et al. 2015). Also, in many applications, the subspace clustering problem is of importance (Elhamifar and Vidal 2009; Pimentel-Alarcón and Nowak 2016). However, in this paper, we consider the completion problem and not subspace clustering, where we assume that the subspace that each column is chosen from is specified.

In general, the existing methods in the literature on low-rank matrix and tensor completion can be categorized into several approaches, including those based on convex relaxation of matrix rank (Candès and Recht 2009; Candès and Tao 2010; Cai, Candès, and Shen 2010; Ashraphijuo, Madani, and Lavaei 2016; 2015; Candès et al. 2013) or different convex relaxations of tensor ranks (Gandy, Recht, and Yamada 2011; Tomioka, Hayashi, and Kashima 2010; Signoretto et al. 2014; Romera-Paredes and Pontil 2013; Kreimer, Stanton, and Sacchi 2013; Ashraphijuo and Wang 2017c), those based on alternating minimization (Wang, Aggarwal, and Aeron 2016; Liu et al. 2016b), and other heuristics (Liu et al. 2016a; Kressner, Steinlechner, and Vandereycken 2014; Krishnamurthy and Singh 2013; Goldfarb and Qin 2014). Note that the optimization-based approaches to low-rank data completion require strong assumptions on the correlations of the values of all entries (such as coherence). On the other hand, recently, deterministic conditions on the sampling patterns have been studied for subspace clustering in (Pimentel-Alarcón, Balzano, and Nowak 2016; Pimentel-Alarcón, Boston, and Nowak 2015; Pimentel-Alarcón et al. 2016; Pimentel-Alarcón, Nowak, and EDU 2016). In particular, fundamental conditions on the sampling pattern (independent from the values of entries) that guarantee the existence of finite or unique number of completions, have been investigated for single-view and multi-view matrix completion (Pimentel-Alarcón, Boston, and Nowak 2016; Ashraphijuo, Wang, and Aggarwal 2017c; 2017b), low canonical polyadic (CP) rank tensor completion (Ashraphijuo and Wang 2017b), low Tucker rank tensor completion (Ashraphijuo, Aggarwal, and Wang 2016; 2017), low tensor-train (TT) rank tensor completion (Ashraphijuo and Wang 2017a) and rank determination of low-rank data completion (Ashraphijuo, Wang, and Aggarwal 2017d; 2017a). In this paper, we study these fundamental conditions for matrices obtained from the union of several low-rank subspaces, i.e., we propose a geometric analysis on the manifold structure for union of low-rank subspaces to study the mentioned problem.

Problem Statement

Assume that $k \geq 2$ is a fixed integer and $n_1 < n_2 < \cdots < n_k$ are given integers. Let $\mathbf{U} \in \mathbb{R}^{m \times n_k}$ be a sampled matrix and denote the matrix consisting of the first n_i columns of $\mathbf{U}$ by $\mathbf{U}_i$, $i = 1, \dots, k$. Hence, note that $\mathbf{U} = \mathbf{U}_k$ and this is shown in Figure 1. Moreover, assume that $\text{rank}(\mathbf{U}_i) = r_i$, $i = 1, \dots, k$. For notational simplicity assume $n_0 = r_0 = 0$ and $\mathbf{U}_0 = \emptyset$. Let $\text{Gr}(r_i, \mathbb{R}^m)$ denote the Grassmannian of r_i-dimensional subspaces of $\mathbb{R}^m$ such that the space corresponding to r_i is a subspace of the space corresponding to r_{i+1}. Assume that $\mathbb{P}_{G_i}$ denotes the uniform measure on $\text{Gr}(r_i, \mathbb{R}^m)$ and $\mathbb{P}_{\theta_i}$ denotes the Lebesgue measure on $\mathbb{R}^{r_i \times s_i}$, where $s_i = n_i - n_{i-1}$ for $i = 1, \dots, k$. In this paper, we assume that the first n_1 columns of $\mathbf{U}$ are chosen generically from the manifold of $m \times n_1$ matrices of rank r_1, i.e., the entries of the first n_1 columns of $\mathbf{U}$ are drawn independently with respect to Lebesgue measure on the corresponding manifold. And in general the columns number $n_{i-1}+1$ to n_i of $\mathbf{U}$ are chosen generically from the manifold of $m \times (n_i - n_{i-1})$ matrices of rank r_i, i.e., the entries of the columns number $n_{i-1}+1$ to n_i of $\mathbf{U}$ are drawn independently with respect to Lebesgue measure $\mathbb{P}_{\theta_i}$ on the corresponding manifold, $i = 2, \ldots, k$. Also, in this paper the probability measure is $\Pi_{i=1}^{k}\mathbb{P}_{G_i}\mathbb{P}_{\theta_i}$.

Note that the problem of union of two low-rank subspaces ($k = 2$) is different from the multi-view matrix completion studied in (Ashraphijuo, Wang, and Aggarwal 2017c), as the multi-view matrix completion has one extra rank constraint that is independent from one of the rank constraints.

Let $\mathbf{\Omega}$ denote the binary sampling pattern matrix that is of the same size as $\mathbf{U}$. The entries of $\mathbf{\Omega}$ that correspond to the observed entries of $\mathbf{U}$ are equal to 1 and the rest of the entries are set as 0. Assume that the entries of $\mathbf{U}$ are sampled independently with some probability. This paper is mainly concerned with treating the following two problems.

Problem (i): Given the rank constraints $\text{rank}(\mathbf{U}_i) = r_i$, $i = 1, \ldots, k$, characterize the necessary and sufficient conditions on the sampling pattern $\mathbf{\Omega}$, under which there exist at most finitely many completions of $\mathbf{U}$ with probability one.

Problem (ii): Given the rank constraints $\text{rank}(\mathbf{U}_i) = r_i$, $i = 1, \ldots, k$, characterize sufficient conditions on the sampling pattern $\mathbf{\Omega}$, under which there exist only one completion of $\mathbf{U}$ with probability one.

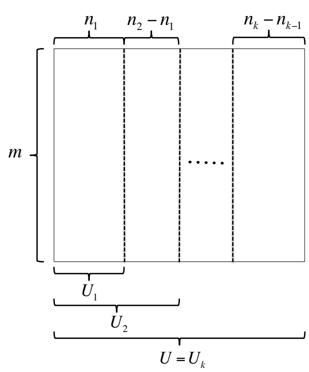

Figure 1: The structure of the sampled matrix $\mathbf{U}$.

Deterministic Conditions for Finite Completability

In this section, we first study the geometry of the manifold corresponding to the union of subspaces to define an equivalence class to classify the bases such that each basis of the sampled data belongs to exactly one of the defined classes. To this end we characterize the canonical structure of the bases and show the uniqueness of canonical basis for the sampled data with probability one. Then, we define a polynomial based on each observed entry and through studying the geometry of the manifold corresponding to the rank constraints, we transform the problem of finite completability of $\mathbf{U}$ to the problem of including a certain number of algebraically independent polynomials among the defined polynomials for the observed entries. A binary matrix is constructed based on the sampling pattern $\mathbf{\Omega}$, which allows us to study the algebraic independence of a subset of polynomials among all defined polynomials based on the samples. Finally, we characterize the necessary and sufficient condition on the sampling pattern for finite completability of the sampled data given the rank constraints.

Geometry

For each $\mathbf{U}$, there exist infinitely many rank decompositions, i.e., $\mathbf{V} \in \mathbb{R}^{m \times r_k}$ and $\mathbf{T} \in \mathbb{R}^{r_k \times n_k}$ such that $\mathbf{U} = \mathbf{V}\mathbf{T}$. However, we are interested in obtaining the canonical basis $\mathbf{V}$ such that there exists exactly one rank decomposition with canonical basis. In other words, we want a pattern on the basis that plays role as an equivalence class such that there exists exactly one basis $\mathbf{V}$ for $\mathbf{U}$ in each class. We start by the following lemma which will be used characterize such an equivalence class.

Lemma 1. *There exists a matrix $\mathbf{V} \in \mathbb{R}^{m \times r_k}$ such that $\mathbf{U}_i$ belongs to the column span of the first r_i columns of $\mathbf{V}$, $i = 1, \ldots, k$. Note that $\mathbf{V}$ is a basis for $\mathbf{U}$ and we call such basis an "appropriate basis".*

Proof. We construct such a matrix $\mathbf{V}$ by induction on i. In other words, in the i-th step, we construct $\mathbf{V}^i$ such that $\mathbf{U}_s$ belongs to the column span of the first r_s columns of $\mathbf{V}^i$, $s = 1, \ldots, i$. Note that for $i = 1$ it is straightforward to construct $\mathbf{V}^1$, which is simply a basis for $\mathbf{U}_1$. Induction hypothesis results in the matrix $\mathbf{V}^i$ with the mentioned properties and in order to complete the induction, we need to show the existence of a matrix $\mathbf{V}^{i+1}$ such that $\mathbf{U}_s$ belongs to the column span of the first r_s columns of $\mathbf{V}^i$, $s = 1, \ldots, i+1$.

We first claim that $\mathbf{V}^i$ belongs to the column span of $\mathbf{U}_{i+1}$. Note that according to the induction hypothesis, $\mathbf{V}^i$ is a basis for $\mathbf{U}_i$ and also $\mathbf{U}_i$ is a subset of columns of $\mathbf{U}_{i+1}$, which proves our claim. Let $\mathcal{S}_i$ denote the column span of $\mathbf{V}^i$, which is an r_i-dimensional space and $\mathcal{S}'_{i+1}$ denote the column span of $\mathbf{U}_{i+1}$, which is an r_{i+1}-dimensional space. As a result of our earlier claim, $\mathcal{S}_i$ is a subspace of $\mathcal{S}'_{i+1}$. Let $\mathcal{S}''_i$ denote the $(r_{i+1} - r_i)$-dimensional subspace of $\mathcal{S}'_{i+1}$ such that the union of $\mathcal{S}_i$ and $\mathcal{S}''_i$ is $\mathcal{S}'_{i+1}$.

Consider an arbitrary basis $\mathbf{V}^{i'} \in \mathbb{R}^{m \times (r_{i+1} - r_i)}$ for the space $\mathcal{S}''_i$. Observe that by putting together the columns of $\mathbf{V}^i$ and $\mathbf{V}^{i'}$, i.e., $\mathbf{V}^{i+1} = [\mathbf{V}^i | \mathbf{V}^{i'}]$, the new matrix $\mathbf{V}^{i+1} \in \mathbb{R}^{m \times r_{i+1}}$ is a basis for the space $\mathcal{S}'_{i+1}$. Therefore, $\mathbf{U}_{i+1}$ belongs to the column span of the first r_{i+1} columns of $\mathbf{V}^{i+1}$ since $\mathbf{V}^i$ has exactly r_{i+1} columns. Given the induction hypothesis, the proof is complete as $\mathbf{U}_s$ belongs to the column span of the first r_s columns of $\mathbf{V}^{i+1}$, $s = 1, \ldots, i+1$. ☐

Corollary 1. *There exists a rank decomposition $\mathbf{U} = \mathbf{V}\mathbf{T}$, where $\mathbf{V} \in \mathbb{R}^{m \times r_k}$, $\mathbf{T} \in \mathbb{R}^{r_k \times n_k}$, $\mathbf{T}(r_1 + 1 : r_k, 1 : n_1) = \mathbf{0}_{(r_k - r_1) \times n_1}$, $\mathbf{T}(r_2 + 1 : r_k, n_1 + 1 : n_2) = \mathbf{0}_{(r_k - r_2) \times (n_2 - n_1)}$, $\ldots$ and $\mathbf{T}(r_{k-1} + 1 : r_k, n_{k-1} + 1 : n_k) = \mathbf{0}_{(r_k - r_{k-1}) \times (n_k - n_{k-1})}$. We call such decomposition an "appropriate decomposition", which is shown in Figure 2.*

Proof. Note that $\mathbf{T} \in \mathbb{R}^{r_k \times n_k}$, $\mathbf{T}(r_1 + 1 : r_k, 1 : n_1) = \mathbf{0}_{(r_k - r_1) \times n_1}$ is equivalent to having that $\mathbf{U}_1$ belongs to the column span of the first r_1 columns of $\mathbf{V}$. Similarly, we can observe that the assumptions given in Corollary 1 are equivalent to the assumptions on the appropriate basis $\mathbf{V}$ in Lemma 1, and therefore according to Lemma 1, the proof is complete. ☐

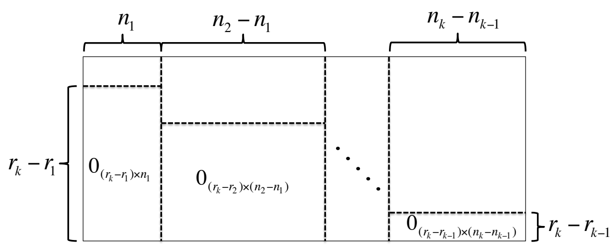

Figure 2: A matrix $\mathbf{T}$ that satisfies the properties of an appropriate decomposition.

From now on, we only consider appropriate decompositions. In fact, given Corollary 1, it is easy to verify that there exists infinitely many appropriate decompositions for $\mathbf{U}$. However, we are interested in having a canonical basis $\mathbf{V}$ so that for any $\mathbf{U}$ there exits exactly one appropriate decompositions satisfying the canonical structure.

Definition 1. *For notational simplicity, we divide the columns of a basis $\mathbf{V} \in \mathbb{R}^{m\times r_k}$ for $\mathbf{U}$ as $\mathbf{V} = [\mathbf{V}_1|\dots,|\mathbf{V}_k]$, where $\mathbf{V}_1 \in \mathbb{R}^{m\times r_1}$ denotes the first r_1 columns of $\mathbf{V}$, $\mathbf{V}_2 \in \mathbb{R}^{m\times(r_2-r_1)}$ denotes the next $(r_2 - r_1)$ columns of $\mathbf{V}$, $\dots$ and $\mathbf{V}_k \in \mathbb{R}^{m\times(r_k-r_{k-1})}$ denotes the next $(r_k - r_{k-1})$ columns of $\mathbf{V}$.*

Definition 2. *A basis $\mathbf{V} \in \mathbb{R}^{m\times r_k}$ for $\mathbf{U}$ has canonical structure if $\mathbf{V}(1:r_1, 1:r_1) = \mathbf{I}_{r_1}$, $\mathbf{V}(1:r_2, r_1+1, r_2) = [\mathbf{0}_{(r_2-r_1)\times r_1}|\mathbf{I}_{(r_2-r_1)}]^\top$, $\dots$ and $\mathbf{V}(1:r_k, r_{k-1}+1, r_k) = [\mathbf{0}_{(r_k-r_{k-1})\times r_{k-1}}|\mathbf{I}_{(r_k-r_{k-1})}]^\top$, as shown in Figure 3.*

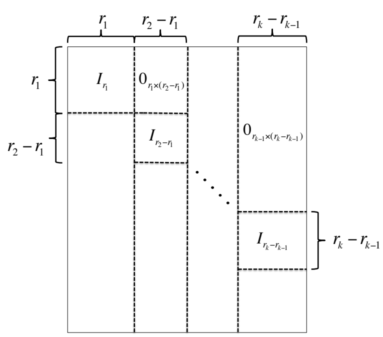

Figure 3: A canonical basis.

The following lemma characterizes the relationship between appropriate bases, which will be used in Lemmas 3 and 4.

Lemma 2. *Consider an appropriate basis $\mathbf{V} \in \mathbb{R}^{m\times r_k}$ for $\mathbf{U}$. Then, the full rank matrix $\mathbf{V}' \in \mathbb{R}^{m\times r_k}$ is an appropriate basis for $\mathbf{U}$ if and only if there exist matrices $\mathbf{A}_1 \in \mathbb{R}^{r_1\times r_1}$, $\mathbf{A}_2 \in \mathbb{R}^{r_2\times(r_2-r_1)}$, $\dots$ and $\mathbf{A}_k \in \mathbb{R}^{r_k\times(r_k-r_{k-1})}$ such that $\mathbf{V}'_1\mathbf{A}_1 = \mathbf{V}_1$, $[\mathbf{V}'_1|\mathbf{V}'_2]\mathbf{A}_2 = \mathbf{V}_2$, $\dots$ and $[\mathbf{V}'_1|\mathbf{V}'_2|\dots|\mathbf{V}'_k]\mathbf{A}_k = \mathbf{V}'\mathbf{A}_k = \mathbf{V}_k$.*

Proof. Assume that $\mathbf{V}'$ is an appropriate basis for $\mathbf{U}$. Then, the first r_1 columns of $\mathbf{V}'$, i.e., $\mathbf{V}'_1$, is a basis for the rank-r_1 matrix $\mathbf{U}_1$ and note that $\mathbf{V}$ is also an appropriate basis for $\mathbf{U}$. Therefore, $\mathbf{V}'_1$ and $\mathbf{V}_1$ span the same r_1-dimensional space, and therefore each column of $\mathbf{V}_1$ can be written as a linear combination of the columns of $\mathbf{V}'_1$, i.e., $\mathbf{V}'_1\mathbf{A}_1 = \mathbf{V}_1$ for some $\mathbf{A}_1 \in \mathbb{R}^{r_1\times r_1}$. Similarly, $[\mathbf{V}'_1|\mathbf{V}'_2]$ and $[\mathbf{V}_1|\mathbf{V}_2]$ span the same r_2-dimensional space since both of them are a basis for the rank-r_2 matrix $\mathbf{U}_2$. As a result, each column of $\mathbf{V}_2$ can be written as a linear combination of the columns of $[\mathbf{V}'_1|\mathbf{V}'_2]$, i.e., $[\mathbf{V}'_1|\mathbf{V}'_2]\mathbf{A}_2 = \mathbf{V}_2$ for some $\mathbf{A}_2 \in \mathbb{R}^{r_2\times(r_2-r_1)}$. Similarly, we can show $[\mathbf{V}'_1|\mathbf{V}'_2|\dots|\mathbf{V}'_k]\mathbf{A}_k = \mathbf{V}'\mathbf{A}_k = \mathbf{V}_k$ for some $\mathbf{A}_k \in \mathbb{R}^{r_k\times(r_k-r_{k-1})}$.

To prove the other direction of the statement, assume that there exist matrices $\mathbf{A}_1 \in \mathbb{R}^{r_1\times r_1}$, $\mathbf{A}_2 \in \mathbb{R}^{r_2\times(r_2-r_1)}$, $\dots$ and $\mathbf{A}_k \in \mathbb{R}^{r_k\times(r_k-r_{k-1})}$ such that $\mathbf{V}'_1\mathbf{A}_1 = \mathbf{V}_1$, $[\mathbf{V}'_1|\mathbf{V}'_2]\mathbf{A}_2 = \mathbf{V}_2$, $\dots$ and $[\mathbf{V}'_1|\mathbf{V}'_2|\dots|\mathbf{V}'_k]\mathbf{A}_k = \mathbf{V}'\mathbf{A}_k = \mathbf{V}_k$. Note that $\mathbf{V}$ is an appropriate basis for $\mathbf{U}$, and therefore the assumption $\mathbf{V}'_1\mathbf{A}_1 = \mathbf{V}_1$ results that $\mathbf{V}'_1$ and $\mathbf{V}_1$ span the same r_1-dimensional space. Hence, $\mathbf{V}'_1$ is basis for $\mathbf{U}_1$. The assumptions $\mathbf{V}'_1\mathbf{A}_1 = \mathbf{V}_1$ and $[\mathbf{V}'_1|\mathbf{V}'_2]\mathbf{A}_2 = \mathbf{V}_2$ together and the fact that $\mathbf{V}'$ is a full rank matrix results that $[\mathbf{V}'_1|\mathbf{V}'_2]$ and $[\mathbf{V}_1|\mathbf{V}_2]$ span the same r_2-dimensional space. Therefore, $[\mathbf{V}'_1|\mathbf{V}'_2]$ is a basis for $\mathbf{U}_2$. Similar reasoning results that $\mathbf{V}'$ is an appropriate basis for $\mathbf{U}$. ☐

Lemma 3. *There exists at most one appropriate decomposition $\mathbf{U} = \mathbf{V}\mathbf{T}$ such that $\mathbf{V}$ has the canonical structure.*

Proof. By contradiction assume that there exist two different canonical bases $\mathbf{V}$ and $\mathbf{V}'$. Then, according to Lemma 2, we have $\mathbf{V}'_1\mathbf{A}_1 = \mathbf{V}_1$, $[\mathbf{V}'_1|\mathbf{V}'_2]\mathbf{A}_2 = \mathbf{V}_2$, $\dots$ and $[\mathbf{V}'_1|\mathbf{V}'_2|\dots|\mathbf{V}'_k]\mathbf{A}_k = \mathbf{V}'\mathbf{A}_k = \mathbf{V}_k$ for some $\mathbf{A}_1 \in \mathbb{R}^{r_1\times r_1}$, $\mathbf{A}_2 \in \mathbb{R}^{r_2\times(r_2-r_1)}$, $\dots$ and $\mathbf{A}_k \in \mathbb{R}^{r_k\times(r_k-r_{k-1})}$. Since both $\mathbf{V}$ and $\mathbf{V}'$ are canonical bases, $\mathbf{V}(1:r_1,:) = \mathbf{V}'(1:r_1,:) = \mathbf{I}_{r_1}$, and therefore the equation $\mathbf{V}'(1:r_1,:)\mathbf{A}_1 = \mathbf{V}(1:r_1,:)$ results that $\mathbf{A}_1 = \mathbf{I}_{r_1}$. As a result, $\mathbf{V}_1 = \mathbf{V}'_1$. Moreover, we have $[\mathbf{V}'_1|\mathbf{V}'_2]\mathbf{A}_2 = \mathbf{V}_2$, which results $[\mathbf{V}'_1|\mathbf{V}'_2](1:r_1,:)\mathbf{A}_2 = \mathbf{V}_2(1:r_1,:) = \mathbf{0}_{r_1\times(r_2-r_1)}$. Since we have $\mathbf{V}'_2(1:r_1,:) = \mathbf{0}_{r_1\times(r_2-r_1)}$ and $\mathbf{V}'_1(1:r_1,:) = \mathbf{I}_{r_1}$, then $[\mathbf{V}'_1|\mathbf{V}'_2](1:r_1,:)\mathbf{A}_2 = \mathbf{0}_{r_1\times(r_2-r_1)}$ reduces to $\mathbf{I}_{r_1}\mathbf{A}_2 = \mathbf{0}_{r_1\times(r_2-r_1)}$, i.e., $\mathbf{A}_2(1:r_1,:) = \mathbf{0}_{r_1\times(r_2-r_1)}$. Therefore, $[\mathbf{V}'_1|\mathbf{V}'_2]\mathbf{A}_2 = \mathbf{V}_2$ reduces to $\mathbf{V}'_2\mathbf{A}_2(r_1+1:r_2,:) = \mathbf{V}_2$. Now, with the similar approach that we showed $\mathbf{V}_1 = \mathbf{V}'_1$, we can show $\mathbf{V}'_2 = \mathbf{V}_2$ since $\mathbf{V}'_2(r_1+1:r_2,:) = \mathbf{V}_2(r_1+1:r_2,:) = \mathbf{I}_{r_2}$. The similar approach results that $\mathbf{V}'_3 = \mathbf{V}_3$, $\dots$ and $\mathbf{V}'_k = \mathbf{V}_k$, and therefore $\mathbf{V}' = \mathbf{V}$, which contradicts the assumption. ☐

The following lemma shows the uniqueness of canonical structure in Definition 2.

Lemma 4. *With probability one, there exists a unique appropriate decomposition $\mathbf{U} = \mathbf{V}\mathbf{T}$ such that $\mathbf{V}$ has the canonical structure.*

Proof. As in Lemma 3 we showed that there exist at most one appropriate canonical basis, it suffices to show the existence of one appropriate canonical basis for $\mathbf{U}$ with probability one. According to Lemma 1, there exists an appropriate basis $\mathbf{V}'$ for $\mathbf{U}$ and we will construct an appropriate canonical basis based on $\mathbf{V}'$ to complete the proof. The genericity assumption results that the submatrix consisting of any r_1 rows of $\mathbf{V}'_1$ is full rank as each column of $\mathbf{U}_1$ is chosen generically from the Grassmannian manifold of $\text{Gr}(r_1, \mathbb{R}^m)$. As a result, $\mathbf{V}'_1(1:r_1,:)$ is full rank, i.e., $\mathbf{V}'_1(1:r_1,:)$ is nonsingular, with probability one with respect to the probability measure $\mathbb{P}_{G_1}\mathbb{P}_{\theta_1}$. Define $\mathbf{A}_1 = \mathbf{V}_1'(1:r_1,:)^{-1} \in \mathbb{R}^{r_1\times r_1}$ and $\mathbf{V}_1 = \mathbf{V}_1'\mathbf{A}_1 \in \mathbb{R}^{m\times r_1}$. Note that $\mathbf{V}_1(1:r_1,:) = \mathbf{I}_{r_1}$.

Similarly, $[\mathbf{V}_1'|\mathbf{V}_2'](1:r_2,:)$ is full rank with probability one with respect to the probability measure $\Pi_{i=1}^2\mathbb{P}_{G_i}\mathbb{P}_{\theta_i}$. Define $\mathbf{A}_2' = [\mathbf{V}_1'|\mathbf{V}_2'](1:r_2,:)^{-1} \in \mathbb{R}^{r_2\times r_2}$, $\mathbf{A}_2 = \mathbf{A}_2'(:,r_1+1:r_2) \in \mathbb{R}^{r_2\times(r_2-r_1)}$ and $\mathbf{V}_2 = [\mathbf{V}_1'|\mathbf{V}_2']\mathbf{A}_2 \in \mathbb{R}^{m\times(r_2-r_1)}$. Therefore, $\mathbf{V}_2(1:r_2,:) = [\mathbf{0}_{(r_2-r_1)\times r_1}|\mathbf{I}_{(r_2-r_1)}]^\top$. By repeating this procedure we construct $\mathbf{V} = [\mathbf{V}_1|\dots|\mathbf{V}_k]$ such that $\mathbf{V}$ has the canonical structure with probability one with respect to the probability measure $\Pi_{i=1}^k\mathbb{P}_{G_i}\mathbb{P}_{\theta_i}$. Moreover, according to Lemma 2, $\mathbf{V}$ is an appropriate basis for $\mathbf{U}$. □

As a result of Lemma 4, for each $\mathbf{U}$ there exists a unique appropriate decomposition with the canonical basis and observe that an arbitrary appropriate decomposition with the canonical basis results in a certain matrix $\mathbf{U}$ that satisfies the given rank constraints. Hence, the canonical structure plays the role of a bijective mapping from a generic member of the manifold corresponding to $\mathbf{U}$ to the appropriate decomposition with canonical basis and generic entries (excluding the entries of the canonical pattern). Consequently, those entries excluding the canonical pattern entries are chosen with respect to the Lebesgue measure on $\mathbb{R}$, i.e., are chosen generically.

Remark 1. *Similarly to the proof of Lemma 4, we can show the uniqueness of the bases having a structure of any permutation of the rows of the canonical structure given in Definition 2. Considering all these permutations of the canonical structure, we obtain some patterns that operate like an equivalence class such that with probability one, exactly one basis belongs to each class, i.e., exactly one basis satisfies a certain pattern, among all the bases for appropriate decompositions. This also leads to the fact that the dimension of all appropriate bases is equal to* $mr_k - \sum_{i=1}^k r_i(r_i - r_{i-1})$*, which is the number of unknown entries of the canonical structure.*

Polynomials and Finite Completability

We consider an appropriate decomposition $\mathbf{U} = \mathbf{VT}$, where $\mathbf{V} \in \mathbb{R}^{m\times r_k}$ and $\mathbf{T} \in \mathbb{R}^{r_k\times n_k}$. We are interested in obtaining all entries of $\mathbf{V}$ and $\mathbf{T}$ using the sampled entries of $\mathbf{U}$. Assuming that the unknown entries of $\mathbf{V}$ and $\mathbf{T}$ are variables, each sampled entry of $\mathbf{U}$ results in a polynomial in terms of these variables as the following,

$$\mathbf{U}(i,j) = \sum_{l=1}^{r_k} \mathbf{V}(i,l)\mathbf{T}(l,j). \quad (1)$$

Here, we briefly mention the following two notes to highlight the fundamentals of our proposed analysis.

- **Note** 1: As it can be observed from (1), any sampled entry $\mathbf{U}(i,j)$ results in a polynomial that involves the entries of the i-th row of $\mathbf{V}$ and the entries of the j-th column of $\mathbf{T}$. Moreover, for a sampled entry $\mathbf{U}(i,j)$, the values of i and j specify the location of the entries of $\mathbf{V}$ and $\mathbf{T}$ that are involved in the corresponding polynomial, respectively.
- **Note** 2: It can be concluded from Bernstein's theorem (Sturmfels 2002) that in a system of n polynomials in n variables with each consisting of a given set of monomials such that the coefficients are chosen with respect to the Lebesgue measure on the manifold corresponding to the basis of the given rank, the n polynomials are algebraically independent with probability one, and therefore there exist only finitely many solutions. However, in the structure of the polynomials in our model, the set of involved monomials are different for different set of polynomials, and therefore to ensure algebraically independency we need to have for any selected subset of the original n polynomials, the number of involved variables should be more than the number of selected polynomials.

The following assumption will be used frequently in this paper.

Assumption 1: Each column of $\mathbf{U}_i$ that does not belong to $\mathbf{U}_{i-1}$ includes at least r_i sampled entries, $i = 1,\dots,k$.

Lemma 5. *Given the basis* $\mathbf{V}$*, Assumption* 1 *holds if and only if* $\mathbf{T}$ *is uniquely solvable.*

Proof. We prove that Assumption 1 is necessary and sufficient condition for unique solvability of each column of $\mathbf{T}$. We show that the first column of $\mathbf{U}_1$ has less than r_1 sampled entries if and only if the first column of $\mathbf{T}$ is infinitely many solvable, and the same reasoning works for other columns as well. According to Note 1, only sampled entries of the first column of $\mathbf{U}_1$ result in a linear polynomial that involves the entries of the first column of $\mathbf{T}$ (since $\mathbf{V}$ is given the polynomials are linear). Note that as we consider appropriate decompositions, the first column of $\mathbf{T}$ includes r_1 unknown variables, and therefore exactly r_1 polynomials with generic coefficients results in a unique solution and less than r_1 polynomials results in infinitely many solutions. □

Definition 3. *For notational simplicity, define* $M = \sum_{i=1}^k r_i(n_i - n_{i-1})$ *(the number of non-zero entries of an appropriate* $\mathbf{T}$*, i.e., the number of sampled entries described in Assumption* 1*),* $M' = r_k n_k - M$ *(the number of zero entries of an appropriate* $\mathbf{T}$*),* $N = \sum_{i=1}^k r_i(r_i - r_{i-1})$ *(the number of fixed entries of a canonical basis) and* $N' = mr_k - N$ *(the number of entries of a canonical basis excluding the entries of the canonical pattern).*

As a result of Lemma 5, we specify the M sampled entries described in Assumption 1 to obtain $\mathbf{T}$ uniquely based on $\mathbf{V}$. Hence, we want to obtain the necessary and sufficient condition on the sampling pattern for finite solvability of $\mathbf{V}$ given $\mathbf{T}$.

Definition 4. *Let* $\mathcal{P}(\mathbf{\Omega})$ *denote the set of polynomials corresponding to the observed entries as in* (1) *excluding the* M *observed entries of Assumption* 1*. Note that since* $\mathbf{T}$ *is already solved in terms of* $\mathbf{V}$*, each polynomial in* $\mathcal{P}(\mathbf{\Omega})$ *is in terms of the entries of* $\mathbf{V}$*.*

The following lemma provides the necessary and sufficient condition on $\mathcal{P}(\mathbf{\Omega})$ for finite completability of the sampled matrix $\mathbf{U}$.

Lemma 6. *Suppose that Assumption* 1 *holds. With probability one, there exist only finitely many completions of* $\mathbf{U}$ *if and only if there exist* N' *algebraically independent polynomials in* $\mathcal{P}(\mathbf{\Omega})$*.*

Proof. The proof is omitted due to the similarity to the proof of Lemma 2 in (Ashraphijuo, Aggarwal, and Wang 2016). The only minor difference is that here the dimension is N' instead of $\left(\Pi_{i=1}^{j} n_i\right)\left(\Pi_{i=j+1}^{d} r_i\right) - \left(\sum_{i=j+1}^{d} r_i^2\right)$ which is the dimension of the core for Tucker decomposition. □

Having Lemma 6, we only need to obtain the maximum number of algebraically independent polynomials in $\mathcal{P}(\mathbf{\Omega})$ to determine if $\mathbf{U}$ is finitely many completable. Next, we construct a binary matrix based on the sampling pattern $\mathbf{\Omega}$ to obtain this number.

Constraint Matrix

In this section, we provide a procedure to construct a binary valued matrix based on the sampling pattern such that each column of it represents one polynomial, and therefore we can later obtain the maximum number of algebraically independent polynomials in $\mathcal{P}(\mathbf{\Omega})$ in terms of some combinatorial properties of the sampling pattern.

Let $l_i = N_{\mathbf{\Omega}}(\mathbf{U}_1(:,i))$ denote the number of observed entries in the i-th column of $\mathbf{U}_1$, where $i \in \{1,\dots,n_1\}$. Assumption 1 results that $l_i \geq r_1$. We construct a binary valued matrix $\breve{\mathbf{\Omega}}_1$ based on $\mathbf{\Omega}$ and r_1. Specifically, we construct $l_i - r_1$ columns with binary entries based on the locations of the observed entries in $\mathbf{U}_1(:,i)$ such that each column has exactly r_1+1 entries equal to one (if $l_i = r_1$ then $\breve{\mathbf{\Omega}}_1 = \emptyset$). Assume that $x_1,\dots,x_{l_i}$ are the row indices of all observed entries in this column. Let $\mathbf{\Omega}_1^i$ be the corresponding $m \times (l_i - r_1)$ matrix to this column which is defined as the following: for any $j \in \{1,\dots,l_i - r_1\}$, the j-th column has the value 1 in rows $\{x_1,\dots,x_{r_1},x_{r_1+j}\}$ and zeros elsewhere. Define the binary constraint matrix as $\breve{\mathbf{\Omega}}_1 = \left[\mathbf{\Omega}_1^1|\mathbf{\Omega}_1^2 \dots |\mathbf{\Omega}_1^{n_1}\right] \in \mathbb{R}^{m\times K_1}$ (Pimentel-Alarcón, Boston, and Nowak 2016), where $K_1 = N_{\mathbf{\Omega}}(\mathbf{U}_1) - n_1 r_1$. Similarly, we construct $\breve{\mathbf{\Omega}}_i$ for the matrix consisting of the columns of $\mathbf{U}_i$ that do not belong to $\mathbf{U}_{i-1}$ based on the corresponding sampling pattern and r_i, $i = 2,\dots,k$. Then, we put together all these k binary matrices $\breve{\mathbf{\Omega}} = \left[\breve{\mathbf{\Omega}}_1|\breve{\mathbf{\Omega}}_2|\dots|\breve{\mathbf{\Omega}}_k\right] \in \mathbb{R}^{m\times K}$ and call it the **constraint matrix**, where $K = N_{\mathbf{\Omega}}(\mathbf{U}) - M$.

In the next subsection, we characterize a relationship between the maximum number of algebraically independent polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}})$ and a combinatorial condition on the sampling pattern $\mathbf{\Omega}$.

Definition 5. *A submatrix of the constraint matrix is called a* **proper** *submatrix if its columns correspond to different columns of the sampling pattern.*

Algebraic Independence

In this subsection, we characterize the necessary and sufficient condition on the sampling pattern for finite completability of the sampled data given the rank constraints, i.e., the necessary and sufficient condition on the sampling pattern for having N' algebraically independent polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}) = \mathcal{P}(\mathbf{\Omega})$.

Definition 6. *Let $\breve{\mathbf{\Omega}}'$ be a subset of columns of the constraint matrix $\breve{\mathbf{\Omega}}$. Let $g(\breve{\mathbf{\Omega}}')$ denote the number of nonzero rows of $\breve{\mathbf{\Omega}}'$ and $\mathcal{P}(\breve{\mathbf{\Omega}}')$ denote the set of polynomials that correspond to the columns of $\breve{\mathbf{\Omega}}'$. Moreover, let $\breve{\mathbf{\Omega}}_i'$ denote the columns of $\breve{\mathbf{\Omega}}'$ that include exactly r_i+1 nonzero entries, i.e., correspond to the columns of $\mathbf{U}_i$ and not columns of $\mathbf{U}_{i-1}$.*

Lemma 7. *Let $\breve{\mathbf{\Omega}}' \in \mathbb{R}^{m\times t}$ be a proper subset of columns of the constraint matrix $\breve{\mathbf{\Omega}}$. Then, the maximum number of algebraically independent polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}')$ is at most*

$$\sum_{i=1}^{k}(r_i - r_{i-1})(g(\breve{\mathbf{\Omega}}_i') - r_i)^+. \tag{2}$$

Proof. Note that each observed entry of $\mathbf{U}_1$, i.e., each column of $\breve{\mathbf{\Omega}}_1'$, results in a polynomial that involves all r_1 entries of a row of $\mathbf{V}_1$. As a result, the number of entries of $\mathbf{V}_1$ that are involved in the polynomials is exactly $(r_1 - r_0)g(\breve{\mathbf{\Omega}}_1')$. However, the rows of the canonical pattern in $\mathbf{V}_1$ can be permuted, and therefore in the case of $\breve{\mathbf{\Omega}}_1' \neq \emptyset$ the number of known entries of the pattern in $\mathbf{V}_1$ is r_1^2 for a pattern. Hence, the minimum number of variables (unknown entries) of $\mathbf{V}_1$ is $(r_1-r_0)g(\breve{\mathbf{\Omega}}_1') - r_1^2 = (r_1-r_0)(g(\breve{\mathbf{\Omega}}_1') - r_1)^+$ since $\breve{\mathbf{\Omega}}_1' \neq \emptyset$ implies $g(\breve{\mathbf{\Omega}}_1') \geq r_1 + 1$. Moreover, clearly in the case of $\breve{\mathbf{\Omega}}_1' = \emptyset$ the number of variables (unknown entries) of $\mathbf{V}_1$ is $(r_1-r_0)(g(\breve{\mathbf{\Omega}}_1') - r_1)^+ = 0$. Similarly, we can show that the minimum number of variables (unknown entries) of $\mathbf{V}_1$ is $\sum_{i=1}^{k}(r_i - r_{i-1})(g(\breve{\mathbf{\Omega}}_i') - r_i)^+$. As a result, the maximum number of algebraically independent polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}')$ is at most equal to the number of involved variables in the polynomials, i.e., $\sum_{i=1}^{k}(r_i - r_{i-1})(g(\breve{\mathbf{\Omega}}_i') - r_i)^+$. □

A set of polynomials is called minimally algebraically dependent if the polynomials in that set are algebraically dependent but polynomials in every of its proper subset are algebraically independent. The next lemma which is Lemma 7 in (Ashraphijuo and Wang 2017b), states an important property of a set of minimally algebraically dependent among polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}})$. This lemma is needed to derive the maximum number of algebraically independent polynomials in any subset of $\mathcal{P}(\breve{\mathbf{\Omega}})$.

Lemma 8. *Let $\breve{\mathbf{\Omega}}' \in \mathbb{R}^{m\times t}$ be a proper subset of columns of the constraint matrix $\breve{\mathbf{\Omega}}$. Assume that polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}')$ are minimally algebraically dependent. Then, the number of variables (unknown entries) of $\mathbf{V}$ that are involved in $\mathcal{P}(\breve{\mathbf{\Omega}}')$ is equal to $t-1$.*

Lemma 9. *Given a proper subset of columns $\breve{\mathbf{\Omega}}' \in \mathbb{R}^{m\times t}$ of the constraint matrix, the polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}')$ are algebraically independent if and only if for any $t' \in \{1,\dots,t\}$ and any subset of columns $\breve{\mathbf{\Omega}}'' \in \mathbb{R}^{m\times t'}$ of $\breve{\mathbf{\Omega}}'$ we have*

$$\sum_{i=1}^{k}(r_i - r_{i-1})(g(\breve{\mathbf{\Omega}}_i'') - r_i)^+ \geq t'. \tag{3}$$

Proof. Assume that the polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}')$ are algebraically dependent. Then, there exists a subset of polynomials $\mathcal{P}(\breve{\mathbf{\Omega}}'')$ of the set $\mathcal{P}(\breve{\mathbf{\Omega}}')$ such that the polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}'')$ are minimally algebraically dependent. Let $\breve{\mathbf{\Omega}}'' \in \mathbb{R}^{m\times t'}$, where $t' \in \{1,\dots,t\}$. According to Lemma 8 the number of involved variables in $\mathcal{P}(\breve{\mathbf{\Omega}}'')$ is $t'-1$. However, in Lemma 7 we showed that the number of involved variables in $\mathcal{P}(\breve{\mathbf{\Omega}}'')$ is at least $\sum_{i=1}^{k}(r_i - r_{i-1})(g(\breve{\mathbf{\Omega}}''_i) - r_i)^+$, and therefore $\sum_{i=1}^{k}(r_i - r_{i-1})(g(\breve{\mathbf{\Omega}}''_i) - r_i)^+ \leq t'-1 < t'$.

In order to show the other direction, assume that the polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}')$ are algebraically independent, and therefore any subset of polynomials of $\mathcal{P}(\breve{\mathbf{\Omega}}')$ are also algebraically independent. By contradiction assume that there exists a subset of columns $\breve{\mathbf{\Omega}}'' \in \mathbb{R}^{m \times t'}$ of $\breve{\mathbf{\Omega}}'$ such that (3) does not hold. Hence, $\sum_{i=1}^{k}(r_i - r_{i-1})(g(\breve{\mathbf{\Omega}}''_i) - r_i)^+$ is less than the number of polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}'')$. On the other hand, according to Lemma (7), the maximum number of algebraically independent polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}'')$ is at most $\sum_{i=1}^{k}(r_i - r_{i-1})(g(\breve{\mathbf{\Omega}}''_i) - r_i)^+$, which is less than the number of polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}'')$, and this contradicts the assumption. □

The next theorem which is the main result of this subsection characterizes the necessary and sufficient condition on the sampling pattern for finite completability of $\mathbf{U}$.

Theorem 1. *Suppose that Assumption* 1 *holds. With probability one, the sampled data* $\mathbf{U}$ *is finitely many completable if and only if there exists a proper subset of columns* $\breve{\mathbf{\Omega}}' \in \mathbb{R}^{m \times N'}$ *of the constraint matrix* $\breve{\mathbf{\Omega}}$ *such that for any* $t' \in \{1, \dots, N'\}$ *and any subset of columns* $\breve{\mathbf{\Omega}}'' \in \mathbb{R}^{m \times t'}$ *of* $\breve{\mathbf{\Omega}}'$, (3) *holds.*

Proof. First we assume that there exists a subset of columns $\breve{\mathbf{\Omega}}' \in \mathbb{R}^{m \times N'}$ of the constraint matrix $\breve{\mathbf{\Omega}}$ such that for any $t' \in \{1, \dots, N'\}$ and any subset of columns $\breve{\mathbf{\Omega}}'' \in \mathbb{R}^{m \times t'}$, (3) holds and we need to show the finite completability of $\mathbf{U}$. According to Lemma 9, the N' polynomials corresponding to $\breve{\mathbf{\Omega}}'$ are algebraically independent, and therefore according to Lemma 6, $\mathbf{U}$ is finitely many completable.

In order to complete the proof, we assume that $\mathbf{U}$ is finitely many completable and show the existence of such $\breve{\mathbf{\Omega}}'$ described in the statement of theorem. According to Lemma 6, there exists N' algebraically independent polynomials in $\mathcal{P}(\breve{\Omega})$, and therefore according to Lemma 9, the submatrix corresponding to these N' polynomials satisfies the properties described in the statement of theorem. □

Remark 2. *One challenge of applying Theorem 1 is the exhaustive enumeration that it takes to check if* (3) *holds for all the corresponding subsets of columns. A sampling probability is proposed in complete version of this paper to ensure finite completability of the sampled data, which is not discussed in this paper due to the page limit. In fact, we obtain a lower bound on the sampling probability that ensures the combinatorial conditions in Theorem 1 hold with high probability. As a consequence of Theorem 1 and the mentioned analysis, we do not need to check combinatorial conditions but instead we can certify the above results with high probability and not deterministically anymore. A similar probabilistic analysis is proposed in (Ashraphijuo and Wang 2017b) for tensor completion problem.*

Deterministic Conditions for Unique Completability

In the previous section, we characterized the deterministic conditions on the sampling pattern for finite completability. In this section, we are interested in obtaining the deterministic conditions on the sampling pattern for unique completability. Note that for matrix completion problem (and therefore for our problem), finite completability does not necessarily imply unique completability (Ashraphijuo, Aggarwal, and Wang 2016). Unique completability simply means that, any completion of the sampled data obtained by any algorithm is exactly the original sampled data. We show that adding a set of mild assumptions to those stated in Theorem 1 leads to unique completability.

Recall that there exists at least one completion of $\mathbf{U}$ since the original matrix that is sampled satisfies the rank constraints. The following lemma is a re-statement of Lemma 25 in (Ashraphijuo and Wang 2017b).

Lemma 10. *Assume that Assumption* 1 *holds. Let* $\breve{\mathbf{\Omega}}'$ *be an arbitrary subset of columns of the constraint matrix* $\breve{\mathbf{\Omega}}$*. Assume that polynomials in* $\mathcal{P}(\breve{\mathbf{\Omega}}')$ *are minimally algebraically dependent. Then, all variables (unknown entries) of* $\mathbf{V}$ *that are involved in* $\mathcal{P}(\breve{\mathbf{\Omega}}')$ *can be determined uniquely.*

Theorem 2. *Suppose that Assumption* 1 *holds. With probability one, the sampled data* $\mathbf{U}$ *is uniquely completable if there exists disjoint subsets of columns* $\breve{\mathbf{\Omega}}' \in \mathbb{R}^{m \times N'}$ *and* $\breve{\mathbf{\Omega}}'_i \in \mathbb{R}^{m \times (m - r_i)}$ *(*$1 \leq i \leq k$*) of the constraint matrix* $\breve{\mathbf{\Omega}}$ *such that*

(i) for any $t' \in \{1, \dots, N'\}$ *and any subset of columns* $\breve{\mathbf{\Omega}}'' \in \mathbb{R}^{m \times t'}$ *of* $\breve{\mathbf{\Omega}}'$, (3) *holds.*

(ii) for any $t'_i \in \{1, \dots, m - r_i\}$ *and any subset of columns* $\breve{\mathbf{\Omega}}''_i \in \mathbb{R}^{m \times t'_i}$ *of* $\breve{\mathbf{\Omega}}'_i$ *we have*

$$(g(\breve{\mathbf{\Omega}}''_i) - r_i)^+ \geq t'_i, \tag{4}$$

$i = 1, 2, \dots, k$.

Proof. According to Theorem 1, condition (i) results that there are at most finitely many completions of $\mathbf{U}$. As we showed in the proof of Theorem 1, there exist N' algebraically independent polynomials $\{p_1, p_2, \dots, p_{N'}\}$ in $\mathcal{P}(\breve{\mathbf{\Omega}}')$. Note that any set of $N' + 1$ polynomials are algebraically dependent. Consider a single polynomial p_0 from the set of polynomials $\cup_{i=1}^{k} \mathcal{P}(\breve{\mathbf{\Omega}}'_i)$. Hence, $\{p_0, p_1, \dots, p_{N'}\}$ are algebraically dependent and since $\{p_1, p_2, \dots, p_{N'}\}$ are algebraically independent, there exist a set of polynomials $\mathcal{P}(p_0) \subseteq \{p_0, p_1, \dots, p_{N'}\}$ that is minimally dependent.

According to Lemma 10, all variables involved in $\mathcal{P}(p_0)$ and therefore all variables involved in p_0 can be determined uniquely, or in other words, we obtain r_i linear polynomials in terms of the entries of $\mathbf{V}_i$ given that $p_0 \in \mathcal{P}(\breve{\mathbf{\Omega}}'_i)$. It is easily verified that given (ii) and substituting p_0 by all of the polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}'_i)$ one by one, $\mathbf{V}_i$ can be determined uniquely, $i = 1, 2, \dots, k$. □

Remark 3. *As mentioned in Remark 2, a sampling probability can be obtained that ensures the combinatorial conditions in Theorem 2 hold with high probability.*

Conclusions

This paper is concerned with investigating the fundamental conditions on the sampling pattern for finite completability of a matrix that represents the union of several subspaces with given ranks. This investigation also leads to a lower bound on the sampling probability to ensure finite completability of the sampled data, which is not discussed in this paper due to the page limit. Furthermore, using the proposed geometric analysis for finite completability, we characterize sufficient conditions on the sampling pattern that ensure there exists only one completion for the sampled data.

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