A Characterization of Sampling Patterns for Low-Rank Multi-View Data Completion Problem

Morteza Ashraphijuo
Columbia University
Email: ashraphijuo@ee.columbia.edu

Xiaodong Wang
Columbia University
Email: wangx@ee.columbia.edu

Vaneet Aggarwal
Purdue University
Email: vaneet@purdue.edu

***Abstract*—In this paper, we consider the problem of completing a sampled matrix $\mathbf{U} = [\mathbf{U}_1|\mathbf{U}_2]$ given the ranks of $\mathbf{U}$, $\mathbf{U}_1$, and $\mathbf{U}_2$ which is known as the multi-view data completion problem. We characterize the deterministic conditions on the locations of the sampled entries that is equivalent (necessary and sufficient) to finite completability of the sampled matrix. To this end, in contrast with the existing analysis on Grassmannian manifold for a single-view matrix, i.e., conventional matrix completion, we propose a geometric analysis on the manifold structure for multi-view data to incorporate more than one rank constraint. Then, using the proposed geometric analysis, we propose sufficient conditions on the sampling pattern, under which there exists only one completion (unique completability) given the three rank constraints.**

I. Introduction

In this paper, we consider the multi-view low-rank data completion problem, where the ranks of the first and second views, as well as the rank of whole data consisting of both views together, are given. The single-view learning problem has plenty of applications in various areas. The multi-view learning problem also finds applications in signal processing [1], multi-label image classification [2, 3], image retrieval [4], data classification [5], etc.

Given a sampled matrix and a rank constraint, any matrix that agrees with the sampled entries and rank constraint is called a completion. A sampled matrix with a rank constraint is finitely completable if and only if there exist only finitely many completions of it. Most literature on matrix completion focus on developing optimization methods to obtain a completion. For single-view learning, methods including alternating minimization [6, 7], convex relaxation of rank [8–11], etc., have been proposed. Moreover, for multi-view learning, many optimization-based algorithms have been proposed recently [12–14].

Deterministic conditions on the locations of the sampled entries (sampling pattern) are obtained through algebraic geometry analyses on Grassmannian manifold that lead to finite/unique solutions to the matrix completion problem [15, 16]. Also, deterministic conditions on the sampling patterns have been studied for subspace clustering in [17–20]. In particular, in [15] by D. Pimentel-Alarcón *et. al.*, a deterministic sampling pattern is proposed that is necessary and sufficient for finite completability of the sampled matrix of the given rank. Such an algorithm-independent condition can lead to a much lower sampling rate than the one that is required by the optimization-based completion algorithms. In [21], [22] and [23], we proposed a geometric analysis on canonical polyadic (CP), Tucker and tensor-train (TT) manifolds for low-rank tensor completion problem and provided the necessary and sufficient conditions on sampling pattern for finite completability of tensor given its CP, Tucker and TT ranks, respectively. In this paper, we investigate the finite completability problem for multi-view data by proposing an analysis on the manifold structure for such data.

Consider a sampled data matrix $\mathbf{U}$ that is partitioned as $\mathbf{U} = [\mathbf{U}_1|\mathbf{U}_2]$, where $\mathbf{U}_1$ and $\mathbf{U}_2$ are the first and second views of $\mathbf{U}$. The multi-view matrix completion problem is to complete $\mathbf{U}$ given the ranks of $\mathbf{U}, \mathbf{U}_1$, and $\mathbf{U}_2$. Let $\mathbf{\Omega}$ be the sampling pattern matrix of $\mathbf{U}$, where $\mathbf{\Omega}(x, y) = 1$ if $\mathbf{U}(x, y)$ is sampled and $\mathbf{\Omega}(x, y) = 0$ otherwise. This paper is mainly concerned with the following two problems.

- **Problem (i)**: Characterize the necessary and sufficient conditions on $\mathbf{\Omega}$, under which there exist only finite completions of $\mathbf{U}$ that satisfy all three rank constraints.
- **Problem (ii)**: Characterize sufficient conditions on $\mathbf{\Omega}$, under which there exists only one completion of $\mathbf{U}$ that satisfy all three rank constraints.

II. Notations

Let $\mathbf{U}$ be the sampled matrix to be completed. Denote $\mathbf{\Omega}$ as the sampling pattern matrix that is of the same size as $\mathbf{U}$ and $\mathbf{\Omega}(x_1, x_2) = 1$ if $\mathbf{U}(x_1, x_2)$ is observed and $\mathbf{\Omega}(x_1, x_2) = 0$ otherwise. For each subset of columns $\mathbf{U}'$ of $\mathbf{U}$, define $N_{\mathbf{\Omega}}(\mathbf{U}')$ as the number of observed entries in $\mathbf{U}'$ according to the sampling pattern $\mathbf{\Omega}$. For any real number x, define $x^+ = \max\{0, x\}$. Also, $\mathbf{I}_n$ denotes an $n \times n$ identity matrix and $\mathbf{0}_{n\times m}$ denotes an $n \times m$ all-zero matrix.

The matrix $\mathbf{U} \in \mathbb{R}^{n\times(m_1+m_2)}$ is sampled. Denote a partition of $\mathbf{U}$ as $\mathbf{U} = [\mathbf{U}_1|\mathbf{U}_2]$ where $\mathbf{U}_1 \in \mathbb{R}^{n\times m_1}$ and $\mathbf{U}_2 \in \mathbb{R}^{n\times m_2}$ represent the first and second views of data, respectively. Given the rank constraints $\text{rank}(\mathbf{U}_1) = r_1$, $\text{rank}(\mathbf{U}_2) = r_2$ and $\text{rank}(\mathbf{U}) = r$, we are interested in characterizing the conditions on the sampling pattern matrix $\mathbf{\Omega}$ under which there are infinite, finite, or unique completions for the sampled matrix $\mathbf{U}$.

In [15] a necessary and sufficient condition on the sampling pattern is given for the finite completability of a matrix $\mathbf{U}$ given $\text{rank}(\mathbf{U}) = r$, based on an algebraic geometry analysis on the Grassmannian manifold. However, such analysis cannot

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be used to treat the above multi-view problem since it is not capable of incorporating the three rank constraints simultaneously. Moreover, the analysis on Tucker manifold in [24] is not capable of incorporating rank constraints for different views. In particular, if we obtain the conditions in [15] corresponding to $\mathbf{U}$, $\mathbf{U}_1$ and $\mathbf{U}_2$ respectively and then take the intersections of them, it will result in a *sufficient* condition (not necessary) on the sampling pattern matrix $\mathbf{\Omega}$ under which there are finite number of completions of $\mathbf{U}$.

III. Finite Completability

In Section III-A, we define an equivalence relation among all bases of the sampled matrix $\mathbf{U}$, where a basis is a set of r vectors ($r = \text{rank}(\mathbf{U})$) that spans the column space of $\mathbf{U}$. This equivalence relation leads to the manifold structure for multi-view data to incorporate all three rank constraints. We introduce a set of polynomials according to the sampled entries to analyze finite completability through analyzing algebraic independence of the defined polynomials.

In Section III-B, we analyze the required maximum number of algebraically independent polynomials that is necessary and sufficient for finite completability. Then, a relationship between the maximum number of algebraically independent polynomials (among the defined polynomials) and the sampling pattern (locations of the sampled entries) is characterized. Consequently, we obtain the necessary and sufficient condition on sampling pattern for finite completability.

A. Geometry of the Basis

Let define $r'_1 = r - r_2$, $r'_2 = r - r_1$ and $r' = r - r'_1 - r'_2 = r_1 + r_2 - r$. Observe that $r_1 \leq r$, $r_2 \leq r$ and $r \leq r_1 + r_2$. Suppose that the basis $\mathbf{V} \in \mathbb{R}^{n \times r}$ is such that its first r_1 columns is a basis for the first view $\mathbf{U}_1$, its last r_2 columns is a basis for the second view $\mathbf{U}_2$, and all columns of $\mathbf{V}$ is a basis for $\mathbf{U} = [\mathbf{U}_1|\mathbf{U}_2]$, as shown in Figure 1. Assume that $\mathbf{V} = [\mathbf{V}_1|\mathbf{V}_2|\mathbf{V}_3]$, where $\mathbf{V}_1 \in \mathbb{R}^{n \times r'_1}$, $\mathbf{V}_2 \in \mathbb{R}^{n \times r'}$ and $\mathbf{V}_3 \in \mathbb{R}^{n \times r'_2}$. Then, $[\mathbf{V}_1|\mathbf{V}_2]$ is a basis for $\mathbf{U}_1$ and $[\mathbf{V}_2|\mathbf{V}_3]$ is a basis for $\mathbf{U}_2$. As a result, there exist matrices $\mathbf{T}_1 \in \mathbb{R}^{r_1 \times m_1}$ and $\mathbf{T}_2 \in \mathbb{R}^{r_2 \times m_2}$ such that

$$\mathbf{U}_1 = [\mathbf{V}_1|\mathbf{V}_2] \cdot \mathbf{T}_1, \tag{1a}$$
$$\mathbf{U}_2 = [\mathbf{V}_2|\mathbf{V}_3] \cdot \mathbf{T}_2. \tag{1b}$$

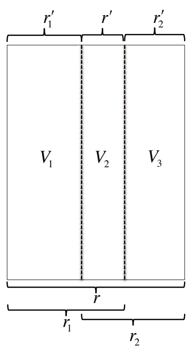

Fig. 1: A basis $\mathbf{V}$ for the sampled matrix $\mathbf{U}$.

For any $i_1 \in \{1, \ldots, m_1\}$ and $i_2 \in \{1, \ldots, m_2\}$, (1) can be written as

$$\mathbf{U}_1(:, i_1) = [\mathbf{V}_1|\mathbf{V}_2] \cdot \mathbf{T}_1(:, i_1), \tag{2a}$$
$$\mathbf{U}_2(:, i_2) = [\mathbf{V}_2|\mathbf{V}_3] \cdot \mathbf{T}_2(:, i_2). \tag{2b}$$

In the following, we list some useful facts that are instrumental to the subsequent analysis.

- **Fact** 1: Observe that each observed entry in $\mathbf{U}_1$ results in a scalar equation from (1a) or (2a) that involves only all r_1 entries of the corresponding row of $[\mathbf{V}_1|\mathbf{V}_2]$ and all r_1 entries of the corresponding column of $\mathbf{T}_1$ in (1a). Similarly, each observed entry in $\mathbf{U}_2$ results in a scalar equation from (1b) or (2b) that involves only all r_2 entries of the corresponding row of $[\mathbf{V}_2|\mathbf{V}_3]$ and all r_2 entries of the corresponding column of $\mathbf{T}_2$ in (1b). Treating the entries of $\mathbf{V}$, $\mathbf{T}_1$ and $\mathbf{T}_2$ as variables (right-hand sides of (1a) and (1b)), each observed entry results in a polynomial in terms of these variables.
- **Fact** 2: For any observed entry $\mathbf{U}_i(x_1, x_2)$, x_1 and x_2 specify the row index of $\mathbf{V}$ and the column index of $\mathbf{T}_i$, respectively, that are involved in the corresponding polynomial, $i = 1, 2$.
- **Fact** 3: It can be concluded from Bernstein's theorem [25] that in a system of n polynomials in n variables such that the coefficients are chosen generically, the n polynomials are algebraically independent with probability one, and therefore there exist only finitely many solutions. Moreover, in a system of n polynomials in $n-1$ variables (or less), the n polynomials are algebraically dependent with probability one. Also, given that a system of n polynomials (generically chosen coefficients) in $n-1$ variables (or less) has one solution, it can be concluded that it has a unique solution with probability one.

Given all observed entries $\{\mathbf{U}(x_1, x_2) : \mathbf{\Omega}(x_1, x_2) = 1\}$, we are interested in finding the number of possible solutions in terms of entries of $(\mathbf{V}, \mathbf{T}_1, \mathbf{T}_2)$ (infinite, finite or unique) via investigating the algebraic independence among the polynomials. Throughout this paper, we make the following assumption.

Assumption 1: *Any column of* $\mathbf{U}_1$ *includes at least* r_1 *observed entries and any column of* $\mathbf{U}_2$ *includes at least* r_2 *observed entries.*

Lemma 1. *Given a basis* $\mathbf{V} = [\mathbf{V}_1|\mathbf{V}_2|\mathbf{V}_3]$ *in* (1)*, if Assumption* 1 *holds, then there exists a unique solution* $(\mathbf{T}_1, \mathbf{T}_2)$*, with probability one. Moreover, if Assumption* 1 *does not hold, then there are infinite number of solutions* $(\mathbf{T}_1, \mathbf{T}_2)$*, with probability one.*

Proof. Note that only observed entries in the i-th column of $\mathbf{U}_1$ result in degree-1 polynomials in terms of the entries of $\mathbf{T}_1(:, i) \in \mathbb{R}^{r_1 \times 1}$. As a result, exactly r_1 generically chosen degree-1 polynomials in terms of r_1 variables are needed to ensure there is a unique solution $\mathbf{T}_1$ in (1), with probability one. Moreover, having less than r_1 polynomials in terms of r_1 variables results in infinite solutions of $\mathbf{T}_1$ in (1), with probability one. The same arguments apply to $\mathbf{T}_2$. $\square$

Definition 1. *Let $\mathcal{P}(\mathbf{\Omega})$ denote all polynomials in terms of the entries of $\mathbf{V}$ obtained through the observed entries excluding the $m_1r_1 + m_2r_2$ polynomials that were used to obtain $(\mathbf{T}_1, \mathbf{T}_2)$. Note that since $(\mathbf{T}_1, \mathbf{T}_2)$ is already solved in terms of $\mathbf{V}$, each polynomial in $\mathcal{P}(\mathbf{\Omega})$ is in terms of elements of $\mathbf{V}$.*

Consider two bases $\mathbf{V}$ and $\mathbf{V}'$ for the matrix $\mathbf{U}$ with the structure in (1). We say that $\mathbf{V}$ and $\mathbf{V}'$ span the same space if and only if: (i) the spans of the first r_1 columns of $\mathbf{V}$ and $\mathbf{V}'$ are the same, (ii) the spans of the last r_2 columns of $\mathbf{V}$ and $\mathbf{V}'$ are the same, (iii) the spans of all columns of $\mathbf{V}$ and $\mathbf{V}'$ are the same.

Therefore, $\mathbf{V}$ and $\mathbf{V}'$ span the same space if and only if: (i) each column of $\mathbf{V}_1$ is a linear combination of the columns of $[\mathbf{V}_1'|\mathbf{V}_2']$, (ii) each column of $\mathbf{V}_2$ is a linear combination of the columns of $\mathbf{V}_2'$, and (iii) each column of $\mathbf{V}_3$ is a linear combination of the columns of $[\mathbf{V}_2'|\mathbf{V}_3']$. The following equivalence class partitions all possible bases such that any two bases in a class span the same space, i.e., the above-mentioned properties (i), (ii) and (iii) hold.

Definition 2. *Define an equivalence class for all bases $\mathbf{V} \in \mathbb{R}^{n\times r}$ of the sampled matrix $\mathbf{U}$ such that two bases $\mathbf{V}$ and $\mathbf{V}'$ belong to the same class if there exist full rank matrices $\mathbf{A}_1 \in \mathbb{R}^{r_1\times r_1'}$, $\mathbf{A}_2 \in \mathbb{R}^{r'\times r'}$ and $\mathbf{A}_3 \in \mathbb{R}^{r_2\times r_2'}$ such that*

$$\mathbf{V}_1 = [\mathbf{V}_1'|\mathbf{V}_2'] \cdot \mathbf{A}_1, \tag{3a}$$

$$\mathbf{V}_2 = \mathbf{V}_2' \cdot \mathbf{A}_2, \tag{3b}$$

$$\mathbf{V}_3 = [\mathbf{V}_2'|\mathbf{V}_3'] \cdot \mathbf{A}_3, \tag{3c}$$

where $\mathbf{V} = [\mathbf{V}_1|\mathbf{V}_2|\mathbf{V}_3]$, $\mathbf{V}' = [\mathbf{V}_1'|\mathbf{V}_2'|\mathbf{V}_3']$, $\mathbf{V}_1, \mathbf{V}_1' \in \mathbb{R}^{n\times r_1'}$, $\mathbf{V}_2, \mathbf{V}_2' \in \mathbb{R}^{n\times r'}$ and $\mathbf{V}_3, \mathbf{V}_3' \in \mathbb{R}^{n\times r_2'}$.

Note that (3) leads to the fact that the dimension of all bases $\mathbf{V}$ is equal to $nr - r_1r_1' - r'r' - r_2r_2' = nr - r^2 - r_1^2 - r_2^2 + r(r_1 + r_2)$.

Definition 3. **(Canonical basis)** *As shown in Figure 2, denote*

$$\mathbf{B}_1 = \mathbf{V}\left(1 : r_1', 1 : r_1'\right) \in \mathbb{R}^{r_1'\times r_1'}, \tag{4a}$$

$$\mathbf{B}_2 = \mathbf{V}\left(1 : r_2', 1 + r_1 : r_2' + r_1\right) \in \mathbb{R}^{r_2'\times r_2'}, \tag{4b}$$

$$\mathbf{B}_3 = \mathbf{V}\left(1 + \max(r_1', r_2') : r' + \max(r_1', r_2'), 1 + r_1' : r' + r_1'\right) \in \mathbb{R}^{r'\times r'}, \tag{4c}$$

$$\mathbf{B}_4 = \mathbf{V}\left(1 + \max(r_1', r_2') : r' + \max(r_1', r_2'), 1 : r_1'\right) \in \mathbb{R}^{r'\times r_1'}, \tag{4d}$$

$$\mathbf{B}_5 = \mathbf{V}\left(1 + \max(r_1', r_2') : r' + \max(r_1', r_2'), 1 + r_1 : r_2' + r_1\right) \in \mathbb{R}^{r'\times r_2'}. \tag{4e}$$

Then, we call $\mathbf{V}$ a canonical basis if $\mathbf{B}_1 = \mathbf{I}_{r_1'}$, $\mathbf{B}_2 = \mathbf{I}_{r_2'}$, $\mathbf{B}_3 = \mathbf{I}_{r'}$, $\mathbf{B}_4 = \mathbf{0}_{r'\times r_1'}$ and $\mathbf{B}_5 = \mathbf{0}_{r'\times r_2'}$.

Remark 1. *In order to prove there are finitely many completions for the matrix $\mathbf{U}$, it suffices to prove there are finitely many canonical bases that fit in $\mathbf{U}$.*

It can be easily seen that according to the definition of the equivalence class in (3), any permutation of the rows of any of these patterns satisfies the property that in each class there exists exactly one basis with the permuted pattern.

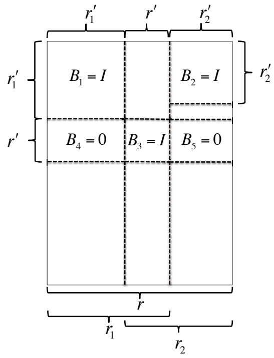

Fig. 2: A canonical basis.

B. Algebraic Independence

The following theorem provides a condition on the polynomials in $\mathcal{P}(\mathbf{\Omega})$ that is equivalent (necessary and sufficient) to finite completability of $\mathbf{U}$.

Theorem 1. *Assume that Assumption 1 holds. For almost every sampled matrix $\mathbf{U}$, there are at most finitely many bases that fit in $\mathbf{U}$ if and only if there exist $nr - r^2 - r_1^2 - r_2^2 + r(r_1 + r_2)$ algebraically independent polynomials in $\mathcal{P}(\mathbf{\Omega})$.*

Proof. According to Lemma 1, Assumption 1 results that $(\mathbf{T}_1, \mathbf{T}_2)$ can be determined uniquely (finitely). Let $\mathcal{P}(\mathbf{\Omega}) = \{p_1, \dots, p_t\}$ and define $\mathcal{S}_i$ as the set of all basis $\mathbf{V}$ that satisfy polynomials $\{p_1, \dots, p_i\}$, $i = 0, 1, \dots, t$, where $\mathcal{S}_0$ is the set of all bases $\mathbf{V}$ without any polynomial restriction. Each polynomial in terms of the entries of $\mathbf{V}$ reduces the degree of freedom or the dimension of the set of solutions by one. Therefore, $\dim(\mathcal{S}_i) = \dim(\mathcal{S}_{i-1})$ if the maximum number of algebraically independent polynomials in sets $\{p_1, \dots, p_i\}$ and $\{p_1, \dots, p_{i-1}\}$ are the same and $\dim(\mathcal{S}_i) = \dim(\mathcal{S}_{i-1}) - 1$ otherwise.

Observe that the number of variables is $\dim(\mathcal{S}_0) = nr - r^2 - r_1^2 - r_2^2 + r(r_1 + r_2)$ and the number of solutions of the system of polynomials $\mathcal{P}(\mathbf{\Omega})$ is $|\mathcal{S}_t|$. Therefore, using Fact 3, with probability one $|\mathcal{S}_t|$ is a finite number if and only if $\dim(\mathcal{S}_t) = 0$. As mentioned earlier, the dimension of the set of all bases without any polynomial restriction, i.e., $\dim(\mathcal{S}_0) = nr - r^2 - r_1^2 - r_2^2 + r(r_1 + r_2)$. Hence, we conclude that the existence of exactly $nr - r^2 - r_1^2 - r_2^2 + r(r_1 + r_2)$ algebraically independent polynomials in $\mathcal{P}(\mathbf{\Omega})$ is equivalent to having finitely many bases, i.e., finite completability of $\mathbf{U}$ with probability one. $\square$

We are interested in characterizing a relationship between the sampling pattern $\mathbf{\Omega}$ and the maximum number of algebraically independent polynomials in $\mathcal{P}(\mathbf{\Omega})$. To this end, we construct a **constraint matrix** $\breve{\mathbf{\Omega}}$ based on $\mathbf{\Omega}$ such that each column of $\breve{\mathbf{\Omega}}$ represents exactly one of the polynomials in $\mathcal{P}(\mathbf{\Omega})$.

Consider an arbitrary column of the first view $\mathbf{U}_1(:, i)$, where $i \in \{1, \dots, m_1\}$. Let $l_i = N_{\mathbf{\Omega}}(\mathbf{U}_1(:, i))$ denote the number of observed entries in the i-th column of the first view. Assumption 1 results that $l_i \geq r_1$.

We construct $l_i - r_1$ columns with binary entries based on the locations of the observed entries in $\mathbf{U}_1(:, i)$ such that each column has exactly $r_1 + 1$ entries equal to one. Assume that $x_1, \dots, x_{l_i}$ be the row indices of all observed entries in this column. Let $\mathbf{\Omega}_1^i$ be the corresponding $n \times (l_i - r_1)$ matrix to this column which is defined as the following: for any $j \in \{1, \dots, l_i - r_1\}$, the j-th column has the value 1 in rows $\{x_1, \dots, x_{r_1}, x_{r_1+j}\}$ and zeros elsewhere. Define the binary constraint matrix of the first view as $\breve{\mathbf{\Omega}}_1 = \left[\mathbf{\Omega}_1^1 | \mathbf{\Omega}_1^2 \dots | \mathbf{\Omega}_1^{m_1}\right] \in \mathbb{R}^{n \times K_1}$ [15], where $K_1 = N_{\mathbf{\Omega}}(\mathbf{U}_1) - m_1 r_1$.

Similarly, we construct the binary constraint matrix $\breve{\mathbf{\Omega}}_2 \in \mathbb{R}^{n \times K_2}$ for the second view, where $K_2 = N_{\mathbf{\Omega}}(\mathbf{U}_2) - m_2 r_2$. Define the constraint matrix of $\mathbf{U}$ as $\breve{\mathbf{\Omega}} = [\breve{\mathbf{\Omega}}_1 | \breve{\mathbf{\Omega}}_2] \in \mathbb{R}^{n \times (K_1 + K_2)}$. For any subset of columns $\breve{\mathbf{\Omega}}'$ of $\breve{\mathbf{\Omega}}$, $\mathcal{P}(\breve{\mathbf{\Omega}}')$ denotes the subset of $\mathcal{P}(\mathbf{\Omega})$ that corrseponds to $\breve{\mathbf{\Omega}}'$. In this paper, when we refer to a subset of columns of the constraint matrix, those columns are assumed to correspond to different columns of Ω.

Assume that $\breve{\mathbf{\Omega}}'$ is an arbitrary subset of columns of the constraint matrix $\breve{\mathbf{\Omega}}$. Then, $\breve{\mathbf{\Omega}}'_1$ and $\breve{\mathbf{\Omega}}'_2$ denote the columns that correspond to the first and second views, respectively. Similarly, assume that $\mathbf{\Omega}'$ is an arbitrary subset of columns of $\mathbf{\Omega}$. Then, $\mathbf{\Omega}'_1$ and $\mathbf{\Omega}'_2$ denote the columns that correspond to the first view and second view, respectively. Moreover, for any matrix $\mathbf{X}$, $c(\mathbf{X})$ denotes the number of columns of $\mathbf{X}$ and $g(\mathbf{X})$ denotes the number of nonzero rows of $\mathbf{X}$.

The following lemma gives an upper bound on the maximum number of algebraically independent polynomials in any subset of columns of the constraint matrix $\breve{\mathbf{\Omega}}$. Simply put, for a set of polynomials with coefficients chosen generically, the total number of involved variables in the polynomials is an upper bound for the maximum number of algebraically independent polynomials.

Lemma 2. *Assume that Assumption* 1 *holds. Let* $\breve{\mathbf{\Omega}}'$ *be an arbitrary subset of columns of the constraint matrix* $\breve{\mathbf{\Omega}}$*. Then, the maximum number of algebraically independent polynomials in* $\mathcal{P}(\breve{\mathbf{\Omega}}')$ *is upper bounded by*

$$r'_1(g(\breve{\mathbf{\Omega}}'_1) - r_1)^+ + r'_2(g(\breve{\mathbf{\Omega}}'_2) - r_2)^+ + r'(g(\breve{\mathbf{\Omega}}') - r')^+. \quad (5)$$

Proof. Please refer to Lemma 2 in [26]. □

The next lemma which is Lemma 3 in [24], states an important property of a set of minimally algebraically dependent among polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}})$. This lemma is needed to derive the the maximum number of algebraically independent polynomials in any subset of $\mathcal{P}(\breve{\mathbf{\Omega}})$. Note that $c(\breve{\mathbf{\Omega}}')$ is the number of polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}')$.

Lemma 3. *Assume that Assumption* 1 *holds. Let* $\breve{\mathbf{\Omega}}'$ *be an arbitrary subset of columns of the constraint matrix* $\breve{\mathbf{\Omega}}$*. Assume that polynomials in* $\mathcal{P}(\breve{\mathbf{\Omega}}')$ *are minimally algebraically dependent. Then, the number of variables (unknown entries) of* $\mathbf{V}$ *that are involved in* $\mathcal{P}(\breve{\mathbf{\Omega}}')$ *is equal to* $c(\breve{\mathbf{\Omega}}') - 1$.

The following lemma explicitly characterizes the relationship between the number of algebraically independent polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}})$ and the geometry of $\breve{\mathbf{\Omega}}$.

Lemma 4. *Assume that Assumption* 1 *holds. Let* $\breve{\mathbf{\Omega}}'$ *be an arbitrary subset of columns of the constraint matrix* $\breve{\mathbf{\Omega}}$*. The polynomials in* $\mathcal{P}(\breve{\mathbf{\Omega}}')$ *are algebraically dependent if and only if there exists* $\breve{\mathbf{\Omega}}'' \subseteq \breve{\mathbf{\Omega}}'$ *such that*

$$\begin{aligned} r'_1(g(\breve{\mathbf{\Omega}}''_1) - r_1)^+ + r'_2(g(\breve{\mathbf{\Omega}}''_2) - r_2)^+ \\ + r'(g(\breve{\mathbf{\Omega}}'') - r')^+ < c(\breve{\mathbf{\Omega}}''). \end{aligned} \quad (6)$$

Proof. Assume that there exists $\breve{\mathbf{\Omega}}'' \subseteq \breve{\mathbf{\Omega}}'$ such that (6) holds. Note that there are $c(\breve{\mathbf{\Omega}}'')$ polynomials in the set $\mathcal{P}(\breve{\mathbf{\Omega}}'')$. Hence, according to Lemma 2 and (6), the maximum number of algebraically independent polynomials is less than the number of polynomials, i.e., $c(\breve{\mathbf{\Omega}}'')$. Therefore, the polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}'')$ and therefore the polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}')$ are algebraically dependent.

For the converse, suppose that the polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}')$ are algebraically dependent. Hence, there exists a subset of these polynomials, $\mathcal{P}(\breve{\mathbf{\Omega}}'')$, such that the polynomials are minimally algebraically dependent. According to Lemma 3, the number of variables involved in the polynomials of $\mathcal{P}(\breve{\mathbf{\Omega}}'')$ is $c(\breve{\mathbf{\Omega}}'') - 1$.

On the other hand, as mentioned in the proof of Lemma 2, the minimum number of involved variables (unknown entries of $\mathbf{V}$) is equal to $r'_1(g(\breve{\mathbf{\Omega}}''_1) - r_1)^+ + r'_2(g(\breve{\mathbf{\Omega}}''_2) - r_2)^+ + r'(g(\breve{\mathbf{\Omega}}'') - r')^+$, which is therefore less than or equal to $c(\breve{\mathbf{\Omega}}'') - 1$ and the proof is complete. □

Finally, the next theorem which is the main result of this subsection gives the necessary and sufficient condition on $\breve{\mathbf{\Omega}}$ to ensure there exist $nr - r^2 - r_1^2 - r_2^2 + r(r_1 + r_2)$ algebraically independent polynomials in $\mathcal{P}(\mathbf{\Omega})$, and therefore it gives the necessary and sufficient condition on $\breve{\mathbf{\Omega}}$ for finite completability of $\mathbf{U}$.

Theorem 2. *Assume that Assumption* 1 *holds. For almost every* $\mathbf{U}$*, the sampled matrix* $\mathbf{U}$ *is finite completable if there exists a subset of columns* $\breve{\mathbf{\Omega}}' \in \mathbb{R}^{n \times m}$ *of the constraint matrix* $\breve{\mathbf{\Omega}}$ *such that* $m = nr - r^2 - r_1^2 - r_2^2 + r(r_1 + r_2)$ *and for any subset of columns* $\breve{\mathbf{\Omega}}''$ *of* $\breve{\mathbf{\Omega}}'$ *the following inequality holds*

$$\begin{aligned} r'_1(g(\breve{\mathbf{\Omega}}''_1) - r_1)^+ + r'_2(g(\breve{\mathbf{\Omega}}''_2) - r_2)^+ \\ + r'(g(\breve{\mathbf{\Omega}}'') - r')^+ \geq c(\breve{\mathbf{\Omega}}''). \end{aligned} \quad (7)$$

Proof. According to Theorem 1, with probability one, the sampled matrix $\mathbf{U}$ is finitely completable if and only if there exist $nr - r^2 - r_1^2 - r_2^2 + r(r_1 + r_2)$ algebraically independent polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}})$. On the other hand, according to Lemma 4, there exist $nr - r^2 - r_1^2 - r_2^2 + r(r_1 + r_2)$ algebraically independent polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}})$ if and only if there exists a subset of columns $\breve{\mathbf{\Omega}}'$ with $nr - r^2 - r_1^2 - r_2^2 + r(r_1 + r_2)$ columns of the constraint matrix $\breve{\mathbf{\Omega}}$ that satisfies (7) for any of its subset of columns. □

IV. Unique Completability

Theorem 2 gives the necessary and sufficient condition on sampling pattern for finite completability. As we showed in an example in [24], finite completability can be different from unique completability. We show that adding a mild condition to the conditions obtained in the analysis for Problem (i) leads to unique completability.

Theorem 3. *Suppose that Assumption* 1 *holds. Moreover assume that there exist disjoint subsets of columns* $\breve{\mathbf{\Omega}}' \in \mathbb{R}^{n \times m}$, $\breve{\mathbf{\Omega}}'_1 \in \mathbb{R}^{n \times m'}$ *and* $\breve{\mathbf{\Omega}}'_2 \in \mathbb{R}^{n \times m''}$ *of the constraint matrix* $\breve{\mathbf{\Omega}}$ *such that the following properties hold*

(i) $m = nr - r^2 - r_1^2 - r_2^2 + r(r_1 + r_2)$ *and for any subset of columns* $\breve{\mathbf{\Omega}}''$ *of the matrix* $\breve{\mathbf{\Omega}}'$, (7) *holds.*

(ii) $\breve{\mathbf{\Omega}}'_1$ *is a subset of columns of* $\breve{\mathbf{\Omega}}_1$ *(constraint matrix of the first view),* $m' = n - r_1$ *and for any subset of columns* $\breve{\mathbf{\Omega}}''_1$ *of the matrix* $\breve{\mathbf{\Omega}}'_1$

$$g(\breve{\mathbf{\Omega}}''_1) - r_1 \geq c(\breve{\mathbf{\Omega}}''_1). \tag{8}$$

(iii) $\breve{\mathbf{\Omega}}'_2$ *is a subset of columns of* $\breve{\mathbf{\Omega}}_2$ *(constraint matrix of the first view),* $m'' = n - r_2$ *and for any subset of columns* $\breve{\mathbf{\Omega}}''_2$ *of the matrix* $\breve{\mathbf{\Omega}}'_2$

$$g(\breve{\mathbf{\Omega}}''_2) - r_2 \geq c(\breve{\mathbf{\Omega}}''_2). \tag{9}$$

Then, with probability one, there exists exactly one completion of $\mathbf{U}$ *that satisfies the rank constraints.*

Proof. Please refer to Theorem 4 in [26]. □

V. Conclusions

This paper characterizes fundamental algorithm-independent conditions on the sampling pattern for finite completability of a low-rank multi-view matrix through an algebraic geometry analysis on the manifold structure of multi-view data. A set of polynomials is defined based on the sample locations and we characterize the number of maximum algebraically independent polynomials. Then, we transform the problem of characterizing the finite or unique completability of the sampled data to the problem of finding the maximum number of algebraically independent polynomials among the defined polynomials. Using these developed tools, we have obtained the following results: (i) The necessary and sufficient conditions on the sampling pattern, under which there are only finite completions given the three rank constraints, (ii) Sufficient conditions on the sampling pattern, under which there exists only one completion given the three rank constraints.

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