2016 IEEE 55th Conference on Decision and Control (CDC)
ARIA Resort & Casino
December 12-14, 2016, Las Vegas, USA

A Strong Semidefinite Programming Relaxation of the Unit Commitment Problem

Morteza Ashraphijuo, Salar Fattahi, Javad Lavaei, and Alper Atamtürk

Abstract— **The unit commitment (UC) problem aims to find an optimal schedule of generating units subject to the demand and operating constraints for an electricity grid. The majority of existing algorithms for the UC problem rely on solving a series of convex relaxations by means of branch-and-bound or cutting-planning methods. In this paper, we develop a strengthened semidefinite program (SDP) for the UC problem by first deriving certain valid quadratic constraints and then relaxing them to linear matrix inequalities. These valid inequalities are obtained by the multiplication of the linear constraints of the UC problem such as the flow constraints of two different lines. The performance of the proposed convex relaxation is evaluated on several instances of the UC problem. For most of the instances, globally optimal integer solutions are obtained by solving a single convex problem. Since the proposed technique leads to a large number of valid quadratic inequalities, an iterative procedure is devised to impose a small number of such valid inequalities. For the cases where the strengthened SDP does give a global integer solution, we incorporate other valid inequalities, including a set of Boolean quadric polytope constraints. The proposed relaxations are extensively tested on various IEEE power systems in simulations.**

I. Introduction

The unit commitment (UC) problem is concerned with finding an optimal schedule of generating units in a power system, by minimizing the operational cost of power generators subject to forecasted energy demand and operating constraints. The operating constraints include physical limits and security constraints. In a mixed-integer programming (MIP) formulation of the UC problem, discrete variables model the on/off status of each generator and the continuous variables account for the amount of production for each generator. The UC problem is NP-complete [1], and large instances of UC are computationally challenging to solve.

The UC problem has a vital role in the operation of electricity grids and been studied extensively. The existing optimization techniques for UC include Lagrangian relaxation (LR) methods, branch-and-bound (BB) methods, dynamic programming (DP) methods, simulated-annealing (SA) methods, and cutting-plane methods [2]. The LR method provides an approximation for the optimal value of an intractable optimization problem by solving a simpler problem. Ongsakul *et al.* [3] propose an enhanced adaptive LR method by defining new decision variables. Moreover, there are several papers that propose a unit decommitment procedure for solving the UC problem [4], [5]. Turgeon designs an algorithm based on the BB method by recursively splitting the search space into smaller branches [6]. Furthermore, Rajan *et al.* [7] propose a set of valid inequalities (turn on/off) instead of the simple minimum up- and down-time constraints to be able to solve hard cases of the UC problem by adopting a branch-and-cut technique.

The Mixed-Integer Linear Programming (MILP) reformulation of the UC problem was first proposed by Garver [8]. In addition, Morales-Espana *et al.* [9] provide new mixed-integer linear reformulations for start-up and shut-down constraints in the UC problem, which lead to tighter relaxations. Furthermore, Ostrowski *et al.* [10] propose a class of facet inequalities, including upper bounds for the generating powers and ramp down and up constraints, to provide smaller feasible operating schedules for the generators. The work by Muckstadt *et al.* [11] designs a BB algorithm based on the LR method, which breaks down the UC problem into several simpler UC problems with one generator.

Pang *et al.* [12] propose DP-based methods by decomposing the problem into a set of smaller subproblems, which are then solved iteratively one at a time. Since the pure SA method would give an infeasible solution with a high probability, advanced SA-based methods aim to address this issue. For instance, Purushothama *et al.* [13] improve the rate of the feasible output by providing a heuristic local search in the neighborhood of the best solution for the UC problem. The work by Madrigal *et al.* [14] proposes an interior-point/cutting-plane method to solve the UC problem, which attempts to emend a proposed set repeatedly to ultimately find the optimal solution by solving the problem over a tighter feasible set.

In this paper, we adopt a semidefinite programming (SDP) relaxation scheme combined with valid inequalities based on the Sherali-Adams method [15]. The SDP technique aims to find a strong convex model that returns a global minimum of the UC problem. This mathematical programming method has received significant attention due to numerous applications in many fields, including combinatorial and non-convex optimization, control theory, and power systems [16]–[19].

In this paper, we provide a set of valid inequalities to attain a tighter description of the feasible operating schedules for the generators in the UC problem. In order to obtain the above-mentioned inequalities, we use the Sherali-Adams method to generate valid non-convex quadratic inequalities and then relax them to valid convex inequalities in a lifted space. For instance, we multiply the flow constraints over two

The authors are with the Department of Industrial Engineering and Operations Research, University of California, Berkeley (e-mail: ashraphijuo@berkeley.edu, fattahi@berkeley.edu, lavaei@berkeley.edu, atamturk@berkeley.edu). This work was supported by DARPA YFA, ONR YIP Award, NSF CAREER Award 1351279 and NSF EECS Award 1406865. A. Atamtürk was supported, in part, by grant FA9550-10-1-0168 from the Office of the Assistant Secretary of Defense for Research and Engineering.

different lines to obtain a valid non-convex constraint and then resort to SDP for convexification. The proposed convex program is called a strengthened SDP, which contrasts with the traditional SDP relaxation without valid inequalities. The above procedure is used for producing valid inequalities and its impact on the feasible set of mixed-integer optimization problems is broadly studied in the literature [15], [20]–[23]. In this work, we will demonstrate that the strengthened SDP problem is able to find discrete solutions for almost all test cases. Since the strengthened SDP problem is computationally prohibitive for large power systems, its complexity is reduced through the following steps:

1) Relaxing the high-order SDP constraint to lower-order conic constraints;
2) Adopting a multi-stage approach for imposing a subset of the developed valid inequalities on the SDP problem.

As shown in simulations, the above steps significantly reduce the complexity of the strengthened SDP problem without affecting its solution in most of the test systems. In the case where the SDP relaxation is not exact, we employ a number of valid inequalities including the triangle inequalities and a special case of variable upper bound (VUB) ramp constraints [21], [24], [25]. The total number of the valid inequalities deployed in this paper is polynomial in the size of the problem. Similar to the methods surveyed above, this work studies the UC problem for a linear model of the power flow equations, known as a DC model. However, the results can be applied to a nonlinear AC model of power systems by combining the proposed technique for handling discrete variables with the convexification method [19] for tackling the nonlinearity of continuous variables.

Notations: The symbol $\operatorname{rank}\{\cdot\}$ denotes the rank of a matrix and the notation $(\cdot)^\top$ represents the transpose operator. Vectors and matrices are shown by bold lower case and bold upper case letters, respectively. The notation W_{ij} denotes the $(i,j)^{\text{th}}$ entry of a matrix $\mathbf{W}$, and w_i denotes the i^{th} entry of a vector $\mathbf{w}$. The symbols $\mathbb{R}$ and $\mathbb{S}^n$ represent the sets of real numbers and $n \times n$ real symmetric matrices, respectively. The relation $\mathbf{u} \geq \mathbf{v}$ indicates that the vector $\mathbf{v}$ is less than or equal to the vector $\mathbf{u}$ entry-wise (the same relation is used for matrices). Given two sets of natural numbers $\mathcal{V}_1$ and $\mathcal{V}_2$ as well as a matrix $\mathbf{W}$, the notation $\mathbf{W}\{\mathcal{V}_1, \mathcal{V}_2\}$ denotes the submatrix of $\mathbf{W}$ that is obtained by keeping only those rows of $\mathbf{W}$ that correspond to the elements of the set $\mathcal{V}_1$ and those columns of $\mathbf{W}$ that are associated with the elements of the set $\mathcal{V}_2$. Given a vector $\mathbf{w}$, the notation $\mathbf{w}\{\mathcal{V}_1\}$ denotes the subvector of $\mathbf{w}$ that is obtained by keeping only those elements of $\mathbf{w}$ corresonding to the elements of $\mathcal{V}_1$. The notation $\mathbf{W} \succeq 0$ indicates that $\mathbf{W}$ is a symmetric and positive semidefinite matrix.

II. Problem Formulation

Consider a power grid with n_b buses, n_g generators, and n_l lines. Assume that $\mathcal{B} = \{1, \ldots, n_b\}$, $\mathcal{G} = \{1, \ldots, n_g\}$ and $\mathcal{L} = \{1, \ldots, n_l\}$ denote the bus set, generator set and line set, respectively. Moreover, suppose that $\mathcal{T} = \{0, 1, \ldots, t_0, t_0 + 1\}$ is the set of time slots over which the UC problem needs to be solved. Let $p_{i;t}$ and $x_{i;t}$ denote the amount of generation and the status of the generator i at time t, respectively, for all $i \in \mathcal{G}$ and $t \in \mathcal{T}$. Assume that the initial $(t = 0)$ and terminal $(t = t_0 + 1)$ statuses of all generators are off, implying that $p_{i;0} = x_{i;0} = p_{i;t_0+1} = x_{i;t_0+1} = 0$ for all $i \in \mathcal{G}$. The set of the decision variables consists of the continuous variables $p_{i;t}$ and the binary variables $x_{i;t}$ for all $i \in \mathcal{G}$ and $t \in \mathcal{T}$. Let $f_{q;t}$ denote the flow of line $q \in \mathcal{L}$ (in an arbitrary direction) at time $t \in \mathcal{T}$. For the sake of notational simplicity, define $\mathbf{x_t}$ as the vector of all commitment statuses and $\mathbf{p_t}$ as the vector of all generator outputs at time $t \in \mathcal{T}$:

$$\mathbf{x}_t \triangleq [x_{1;t}, \ldots, x_{n_g;t}]^\top, \quad \mathbf{p}_t \triangleq [p_{1;t}, \ldots, p_{n_g;t}]^\top.$$

The objective function of the UC problem is the sum of the operational costs of all generating units, which consist of the power generation, startup and shutdown costs. The power generation cost is modeled as a quadratic function of the amount of generation:

$$g_{i;t}(p_{i;t}, x_{i;t}) \triangleq a_i \times p_{i;t}^2 + b_i \times p_{i;t} + c_{i;\,\text{fixed}} \times x_{i;t}, \quad (1)$$

where $a_i, b_i,$ and $c_{i;\,\text{fixed}}$ are constant coefficients for generator i, and $a_i \geq 0$. Note that the term $c_{i;\text{fixed}} \times x_{i;t}$ accounts for a fixed cost if the generator is on and becomes zero otherwise. The startup and shutdown costs are both assumed to be identical and modeled as

$$h_{i;t}(x_{i;t+1}, x_{i;t}) \triangleq c_{i;\,\text{start}} \cdot (x_{i;t+1} - x_{i;t})^2, \quad (2)$$

where $c_{i;\,\text{start}}$ is the amount of startup or shutdown cost. Note that since all generators are assumed to be off at the beginning and the end of the horizon (i.e., $t = 0$ and $t = t_0 + 1$), if the startup and shutdown costs have different values, we can precisely model the problem using the expression (2) after setting $c_{i;\,\text{start}}$ equal to the average of those two different costs.

The cost associated with turning on or off a generator induces a coupling between the decision variables at different times. There are some operating restrictions for the UC problem, such as the physical limits and the security constraints. Physical limits include unit capacity and line capacity constraints, ramping constraints, and minimum up- and down-time constraints. A unit capacity constraint ensures that the unit operates within certain limits. A line capacity constraint enforces the flow on each transmission line not to exceed its thermal or stability limit. Due to the physical design of a generator, it may be impossible to significantly change the production level within a short time interval. These restrictions are referred to as the ramping constraints. In addition, each generator may have minimum up-time and down-time constraints, which prohibit the status of a generator from changing fast. In order to formulate the UC problem, we need to define several parameters below.

Define the vector of demands at time t as $\mathbf{d}_t$, where its j^{th} entry is equal to the demand at bus $j \in \mathcal{B}$ at time $t \in \mathcal{T}$ (shown as d_{t_j}). Let $\mathbf{f}_{\max}$ denote the the maximum flow vector for all transmission lines, where its q^{th} entry is equal to the flow limit for the line $q \in \mathcal{L}$ (shown as $f_{\max_q}$). Assume that $p_{i;\max}$ and $p_{i;\min}$ represent the upper and lower bounds on the generation of unit $i \in \mathcal{G}$, respectively. Furthermore, define s_i as the maximum amount of generation for the startup and shutdown of generator $i \in \mathcal{G}$. Moreover, r_i denotes the maximum difference between the generations at two adjacent operating time slots for generator i. Furthermore, suppose that U_i and D_i denote the minimum up-time and down-time for generator i, respectively. Let $\mathbf{H}$ be the power transfer distribution factors (PTDF) or shift factor matrix and $\mathbf{C}_g \in \mathbb{R}^{n_b \times n_g}$ be the bus-to-generator incidence matrix. Note that $C_{g_{ji}} = 1$ if and only if generator i is connected to bus j and $C_{g_{ji}} = 0$, otherwise. Since we adopt the DC modeling of the UC problem, the flow of each line q at time t (shown as $f_{q;t}$) can be expressed as a linear combination of all generations at time t. Therefore, the UC problem can be formulated as follows:

$$\underset{\substack{\{x_{i;t}\}_{i\in\mathcal{G};t\in\overline{\mathcal{T}}}\\ \{p_{i;t}\}_{i\in\mathcal{G};t\in\overline{\mathcal{T}}}}}{\text{minimize}} \quad \sum_{\substack{i\in\mathcal{G}\\ t\in\overline{\mathcal{T}}}} g_{i;t}(p_{i;t}, x_{i;t}) + \sum_{\substack{i\in\mathcal{G}\\ t\in\overline{\mathcal{T}}_0}} h_{i;t}(x_{i;t+1}, x_{i;t}), \tag{3a}$$

$$\text{subject to} \quad x_{i;t} \in \{0,1\}, \tag{3b}$$

$$p_{i;\min}.x_{i;t} \leq p_{i;t} \leq p_{i;\max}.x_{i;t}, \tag{3c}$$

$$\sum_{i=1}^{n_g} p_{i;t} = \sum_{j=1}^{n_b} d_{t_j}, \tag{3d}$$

$$|\mathbf{H}(\mathbf{d}_t - \mathbf{C}_g \mathbf{p}_t)| \leq \mathbf{f}_{\max}, \tag{3e}$$

$$|p_{i;t+1} - p_{i;t}| \leq (2s_i - r_i) + (r_i - s_i)(x_{i;t+1} + x_{i;t}), \tag{3f}$$

$$x_{i;t+1} - x_{i;t} \leq x_{i;\tau}, \quad \forall \tau \in \{t+1, \ldots, \min(t+U_i, t_0)\}, \tag{3g}$$

$$x_{i;t-1} - x_{i;t} \leq 1 - x_{i;\tau}, \quad \forall \tau \in \{t \ldots, \min(t-1+D_i, t_0)\}, \tag{3h}$$

where:

- $\overline{\mathcal{T}} \triangleq \{1, 2, \ldots, t_0\}$ and $\overline{\mathcal{T}}_0 \triangleq \{0, 1, 2, \ldots, t_0\}$.
- (3b) imposes that status of each generator to be binary and holds for all $i \in \mathcal{G}$ and $t \in \overline{\mathcal{T}}$.
- (3c) is the unit capacity constraint and holds for all $i \in \mathcal{G}$ and $t \in \overline{\mathcal{T}}$.
- (3d) represents the power balance equation and holds for all $i \in \mathcal{G}$ and $t \in \overline{\mathcal{T}}$.
- (3e) indicates the line capacity constraint and holds for all $t \in \overline{\mathcal{T}}$.
- (3f) formulates the ramping constraint and holds for all $i \in \mathcal{G}$ and $t \in \overline{\mathcal{T}}_0$.
- (3g) is the minimum up-time constraint and holds for all $i \in \mathcal{G}$ and $t \in \overline{\mathcal{T}}_0$.
- (3h) is the minimum down-time constraint and holds for all $i \in \mathcal{G}$ and $t \in \overline{\mathcal{T}}_0$.

Note that the security constraints have not been modeled explicitly in order to streamline the presentation. However, the results to be presented in this work are valid in presence of linear security constraints obtained using line outage distribution factors.

Remark 1. *The inequality* (3f) *encapsulates two types of ramping constraints. More precisely, it imposes the inequality* $|p_{i;t+1} - p_{i;t}| \leq r_i$ *in the case* $x_{i;t+1} = x_{i;t} = 1$ *and the inequality* $|p_{i;t+1} - p_{i;t}| \leq s_i$ *in the case* $x_{i;t+1} \neq x_{i;t}$

Remark 2. *Note that the constraints* (3c)-(3h) *can all be formulated linearly in terms of the decision variables.*

III. Convex Relaxation of UC Problem

In what follows, the main results of this work will be developed.

A. SDP Relaxations

By relaxing the integrality (3b) to the linear constraints

$$0 \leq x_{i;t} \leq 1, \tag{4}$$

the resulting optimization problem becomes convex, which is referred to as the **basic quadratic programming (QP) relaxation of the UC problem**. As shown in Section IV, the solution of this convex problem is almost always fractional for the test systems. Motivated by this observation, the objective is to design stronger relaxations. Consider the vector

$$\mathbf{w} \triangleq [\mathbf{x}_1^\top, \ldots, \mathbf{x}_{t_0}^\top, \mathbf{p}_1^\top, \ldots, \mathbf{p}_{t_0}^\top]^\top \tag{5}$$

The constraint (4) together with the constraints of the UC problem except for (3b) can all be merged into a single linear vector constraint $\mathbf{M}\mathbf{w} \geq \mathbf{m}$, for some constant matrix $\mathbf{M}$ and vector $\mathbf{m}$. Furthermore, the condition (3b) can be expressed as the quadratic equation

$$x_{i;t}(x_{i;t} - 1) = 0. \tag{6}$$

Therefore, the UC problem can be stated as follows:

$$\underset{\mathbf{w}\in\mathbb{R}^{2t_0}}{\text{minimize}} \quad c(\mathbf{w}) \tag{7a}$$

$$\text{subject to} \quad \mathbf{M}\mathbf{w} \geq \mathbf{m}, \tag{7b}$$

$$w_k(w_k - 1) = 0, \qquad k = 1, 2, \ldots, n_g t_0, \tag{7c}$$

where $c(\mathbf{w})$ is equivalent to the total cost of the UC problem. It is straightforward to verify that $c(\mathbf{w})$ is a convex function with respect to $\mathbf{w}$.

Remark 3. *Let* $\mathbf{0}_{a\times b}$ *and* $\mathbf{1}_{a\times b}$ *denote* $a \times b$ *matrices with their entries all equal to* 0 *and* 1, *respectively. Moreover, let* $\mathbf{I}_n$ *be the* $n\times n$ *identity matrix. Given a vector* $\mathbf{p}$, *the notation* $\text{diag}\{\mathbf{p}\}$ *represents a diagonal matrix such that the* $(i,i)^{th}$ *entry equals* p_i. *Assume that the* i^{th} *entries of the vectors* $\mathbf{p}_{\max}$ *and* $\mathbf{p}_{\min}$ *represent the upper and lower bounds on the generation of unit* $i \in \mathcal{G}$, *respectively. In order to elaborate on the reformulation* (7) *and the structure of its parameters,* *note that*

$$\mathbf{M} = \begin{bmatrix} \mathbf{I}_{n_g} & \mathbf{0}_{n_g \times n_g} \\ -\mathbf{I}_{n_g} & \mathbf{0}_{n_g \times n_g} \\ -\text{diag}\{\mathbf{p}_{\min}\} & \mathbf{I}_{n_g} \\ \text{diag}\{\mathbf{p}_{\max}\} & -\mathbf{I}_{n_g} \\ \mathbf{0}_{1\times n_g} & \mathbf{1}_{1\times n_g} \\ \mathbf{0}_{1\times n_g} & -\mathbf{1}_{1\times n_g} \\ \mathbf{0}_{n_l\times n_g} & \mathbf{H}.\mathbf{C}_g \\ \mathbf{0}_{n_l\times n_g} & -\mathbf{H}.\mathbf{C}_g \end{bmatrix}, \quad \mathbf{m} = \begin{bmatrix} \mathbf{0}_{n_g\times 1} \\ -\mathbf{1}_{n_g\times 1} \\ \mathbf{0}_{n_g\times 1} \\ \mathbf{0}_{n_g\times 1} \\ \sum_{j=1}^{n_b} d_j \\ -\sum_{j=1}^{n_b} d_j \\ \mathbf{H}.\mathbf{d} - \mathbf{f}_{\max} \\ -\mathbf{H}.\mathbf{d} - \mathbf{f}_{\max} \end{bmatrix}.$$

in the case $t_0 = 1$.

Consider a matrix variable $\mathbf{W}$ and set it to $\mathbf{w}\mathbf{w}^\top$. The constraints of the UC problem can all be written as inequalities in terms of $\mathbf{W}$ and $\mathbf{w}$. This leads to a reformulation of the UC problem, where $\mathbf{W} = \mathbf{w}\mathbf{w}^\top$ is the only non-convex constraint. An SDP relaxation of the UC problem can be obtained by relaxing $\mathbf{W} = \mathbf{w}\mathbf{w}^\top$ to the conic constraint $\mathbf{W} \succeq \mathbf{w}\mathbf{w}^\top$. This yields the convex program

$$\begin{aligned} \underset{\substack{\mathbf{w}\in\mathbb{R}^{2n_g t_0} \\ \mathbf{W}\in\mathbb{S}^{2n_g t_0}}}{\text{minimize}} \quad & c_r(\mathbf{w}, \mathbf{W}) && \text{(8a)} \\ \text{subject to} \quad & \mathbf{M}\mathbf{w} \geq \mathbf{m}, && \text{(8b)} \\ & W_{kk} - w_k = 0, \qquad k = 1, 2, \ldots, n_g t_0, && \text{(8c)} \\ & \mathbf{W} \succeq \mathbf{w}\mathbf{w}^\top, && \text{(8d)} \end{aligned}$$

where

$$\begin{aligned} c_r(\mathbf{w}, \mathbf{W}) = & \sum_{\substack{i\in\mathcal{G} \\ t\in\mathcal{T}}} \big(a_i W_{n_g t_0 + n_g(t-1)+i,\, n_g t_0 + n_g(t-1)+i} \\ & + b_i w_{n_g t_0 + n_g(t-1)+i} + c_{i;\text{fixed}} w_{n_g(t-1)+i}\big) \\ & + \sum_{\substack{i\in\mathcal{G} \\ t\in\mathcal{T}_0}} c_{i;\text{start}}.\big(W_{n_g t+i,\, n_g t+i} + W_{n_g(t-1)+i,\, n_g(t-1)+i} \\ & - W_{n_g t+i,\, n_g(t-1)+i} - W_{n_g(t-1)+i,\, n_g t+i}\big) \end{aligned} \tag{9}$$

Note that (8d) can be written as a linear matrix inequality with respect to $\mathbf{w}$ and $\mathbf{W}$. The above problem is called the **SDP relaxation of the UC problem**.

Theorem 1. *The optimal objective values of the SDP relaxation* (8) *and the basic QP relaxation of the UC problem are the same when* $t_0 = 1$.

Proof. Assume that $(\mathbf{w}^*, \mathbf{W}^*)$ denotes an optimal solution of the SDP relaxation (8). First, we aim to show that $\mathbf{w}^*$ is a feasible point of the basic QP relaxation. Consider an index k corresponding to an element of $\mathbf{w}$ associated with a generator status. The constraint (8b) is the same as (7b). Moreover, (8d) implies that $W^*_{kk} \geq {w^*_k}^2$, which together with the constraint (8c) leads to the relation $0 \leq w^*_k \leq 1$. As a result, $\mathbf{w}^*$ is a point feasible for the basic QP problem. Due to the definition of $c_r(\mathbf{w}, \mathbf{W})$ and $c(\mathbf{w})$ and the fact that $W^*_{kk} \geq {w^*_k}^2$, one can verify that $c_r(\mathbf{w}^*, \mathbf{W}^*) \geq c(\mathbf{w}^*)$. Therefore, the optimal cost of the SDP relaxation is greater than or equal to the cost of the QP relaxation.

In order to complete the proof, it suffices to show that the optimal cost of the QP relaxation is greater than or equal to the optimal cost of the SDP relaxation. Suppose that $\hat{\mathbf{w}}$ denotes the optimal solution of QP relaxation of the UC problem. One can build a matrix $\hat{\mathbf{W}}$ such that $(\hat{\mathbf{w}}, \hat{\mathbf{W}})$ is a feasible point of the SDP relaxation with a cost equal to the optimal cost of the QP relaxation. The constraint (8b) is a reformulation of the linear constraints and therefore it holds true. Furthermore, the constraint $0 \leq \hat{w}_k \leq 1$ implies that $\hat{w}_k^2 \leq \hat{w}_k$. Therefore, we can construct a non-negative diagonal matrix $\mathbf{W}_0$ such that $(\hat{W}_{0_{kk}} + \hat{w}_k^2) - \hat{w}_k = 0$. As a result, $(\hat{\mathbf{w}}, \hat{\mathbf{W}})$ is feasible for SDP relaxation, where $\hat{\mathbf{W}} = \hat{\mathbf{w}}\hat{\mathbf{w}}^T + \hat{\mathbf{W}}_\mathbf{0}$. Note that the only possibly required positive elements of $\mathbf{W}_0$ are the diagonal elements corresponding to the status of generators. Furthermore, notice that these diagonal elements do not appear in the objective when $t_0 = 1$. Therefore, one can verify that $c_r(\hat{\mathbf{w}}, \hat{\mathbf{W}}) = c(\hat{\mathbf{w}})$. This completes the proof. □

Remark 4. *Note that* (8) *is indeed a relaxation of the UC problem. This is due to the fact that if* $\mathbf{w}$ *defined in* (5) *is an optimal solution of the UC problem, then* $(\mathbf{w}, \mathbf{w}\mathbf{w}^T)$ *is feasible for* (8) *and has the same objective value as the optimal cost of UC. Furthermore, the proposed SDP relaxation solves the UC problem if and only if it has an optimal solution* $(\mathbf{w}^*, \mathbf{W}^*)$ *for which the matrix*

$$\left[\begin{array}{c|c} 1 & {\mathbf{w}^*}^\top \\ \hline \mathbf{w}^* & \mathbf{W}^* \end{array}\right]$$

has rank 1. From a different perspective, in the case where $x^*_{i;t}$*'s are all binary numbers at an optimal solution of* (8)*, the relaxation is exact.*

As shown in Section IV, the solution of the convex problem (8) is almost always fractional for the test systems.

B. Valid Inequalities

Let S denote the set of feasible points of the UC problem (3). An inequality is said to be valid if it is satisfied by all points in S. The SDP relaxation (8) can be strengthened by adding valid inequalities to it. Consider two scalar inequalities of the UC problem, namely

$$\mathbf{u}^\top\mathbf{w} - m_1 \geq 0, \quad \mathbf{v}^\top\mathbf{w} - m_2 \geq 0,$$

for fixed coefficients $\mathbf{u}$, $\mathbf{v}$, m_1 and m_2. Since both of these inequalities hold for all points $\mathbf{w}$ in S, the quadratic inequality

$$\mathbf{u}^\top\mathbf{w}\mathbf{w}^\top\mathbf{v} - (\mathbf{v}^\top m_1 + \mathbf{u}^\top m_2)\mathbf{w} + m_1 m_2 \geq 0,$$

is also satisfied for every $\mathbf{w} \in S$. The above quadratic inequality can be relaxed to the linear inequality

$$\mathbf{u}^\top\mathbf{W}^\top\mathbf{v} - (\mathbf{v}^\top m_1 + \mathbf{u}^\top m_2)\mathbf{w} + m_1 m_2 \geq 0.$$

C. Strengthened SDP Relaxation

In this part, we construct a set of valid inequalities via the multiplication of all linear inequalities of the UC problem, using the strategy delineated in Section III-B. The resulting quadratic inequalities obtained from (8b) can be expressed as the matrix constraint $(\mathbf{M}\mathbf{w}-\mathbf{m})(\mathbf{M}\mathbf{w}-\mathbf{m})^\top \geq 0$, or equivalently,

$$\mathbf{M}\mathbf{w}\mathbf{w}^\top\mathbf{M}^\top - \mathbf{m}\mathbf{w}^\top\mathbf{M}^\top - \mathbf{M}\mathbf{w}\mathbf{m}^\top + \mathbf{m}\mathbf{m}^\top \geq 0.$$

The relaxation of this non-convex inequality yields the linear matrix inequality

$$\mathbf{M}\mathbf{W}\mathbf{M}^\top - \mathbf{m}\mathbf{w}^\top\mathbf{M}^\top - \mathbf{M}\mathbf{w}\mathbf{m}^\top + \mathbf{m}\mathbf{m}^\top \geq 0 \tag{10}$$

Replacing the non-convex constraint (7c) in the UC formulation (7) with the linear constraint (10) leads to a **Reformulation-Linearization Technique (RLT) relaxation of the UC problem**. Although it has been proven in [15] that this relaxation outperforms the basic QP relaxation, it will be shown in Section IV that this method often fails to generate feasible solutions for the UC problem.

The addition of the constraint (10) to the SDP relaxation leads to the convex program:

$$\begin{aligned}
\underset{\substack{\mathbf{w}\in\mathbb{R}^{22n_g t_0}\\ \mathbf{W}\in\mathbb{S}^{2n_g t_0}}}{\text{minimize}} \quad & c_r(\mathbf{w},\mathbf{W}) && \text{(11a)}\\
\text{subject to} \quad & \mathbf{M}\mathbf{w} \geq \mathbf{m}, && \text{(11b)}\\
& \mathbf{M}\mathbf{W}\mathbf{M}^\top - \mathbf{m}\mathbf{w}^\top\mathbf{M}^\top - \mathbf{M}\mathbf{w}\mathbf{m}^\top + \mathbf{m}\mathbf{m}^\top \geq 0, && \text{(11c)}\\
& W_{kk} - w_k = 0, \qquad k = 1,2,\ldots,n_g t_0, && \text{(11d)}\\
& \mathbf{W} \succeq \mathbf{w}\mathbf{w}^\top. && \text{(11e)}
\end{aligned}$$

This problem is referred to as the **strengthened SDP relaxation of the UC problem**. In Section IV, it will be shown that the strengthened SDP (11) is exact and significantly improves the standard SDP and RLT relaxations in most test cases.

Real-world UC problems are large-scale due to the size of power grids and the number of time slots. Hence, the strengthened SDP relaxation (11) would be computationally expensive for practical systems. In the next section, the constraint (11e) will be replaced by a number of lower-order conic constraints. To further reduce the computational complexity of the proposed relaxation due to the large number of valid inequalities, an iterative method will be developed to incorporate only a subset of the inequalities into the relaxation.

D. Weakly-Strengthened SDP Relaxation

In this subsection, we design a weakly-strengthened SDP relaxation whose complexity is lower than that of the strengthened SDP relaxation.

1) Relaxing the conic constraint: Define the sets

$$\mathcal{V}_{x_t} \triangleq \{n_g(t-1)+1, n_g(t-1)+2,\ldots,n_g(t+1)\},$$
$$\mathcal{V}_{p_t} \triangleq$$
$$\{n_g(t_0+t-1)+1, n_g(t_0+t-1)+2,\ldots,n_g(t_0+t+1)\},$$
$$\mathcal{V}_t \triangleq \mathcal{V}_{x_t} \cup \mathcal{V}_{p_t}$$

for every $t \in \{1,\ldots,t_0-1\}$. Observe that $\mathcal{V}_{x_t}$ and $\mathcal{V}_{p_t}$ are the index sets of those elements of $\mathbf{w}$ that correspond to $\{x_{1;t},\ldots,x_{n_g;t},x_{1;t+1},\ldots,x_{n_g;t+1}\}$ and $\{p_{1;t},\ldots,p_{n_g;t},p_{1;t+1},\ldots,p_{n_g;t+1}\}$, respectively. There are constant matrices $\mathbf{Y}_1,\ldots,\mathbf{Y}_{t_0-1}$ and vectors $\mathbf{y}_1,\ldots,\mathbf{y}_{t_0-1}$ such that, for every $t\in\{1,\ldots,t_0-1\}$, the inequality

$$\mathbf{Y}_t\mathbf{w}\{\mathcal{V}_t\} \geq \mathbf{y}_t \tag{12}$$

is equivalent to the collection of those inequalities in (8b) that only include the decision variables $x_{i;t}, p_{i;t}, x_{i;t+1}$, and $p_{i;t+1}$ for all $i\in\mathcal{G}$. Note that the inequalities given in (12) for $t\in\{1,\ldots,t_0-1\}$ cover all inequalities in (8b) except for the minimum up-time and down-time constraints.

To handle the minimum up- and down-time constraints, define the set $\mathcal{V}_{t_0} \triangleq \{1,\ldots,n_g t_0\}$. Note that $\mathcal{V}_{t_0}$ is the index set of those elements of $\mathbf{w}$ that correspond to the statuses of the generators over different time slots. There are a matrix $\mathbf{Y}_{t_0}$ and a vector $\mathbf{y}_{t_0}$ such that the inequality

$$\mathbf{Y}_{t_0}\mathbf{w}\{\mathcal{V}_{t_0}\} \geq \mathbf{y}_{t_0} \tag{13}$$

is equivalent to the minimum up- and down-time constraints (3g) and (3h). Note that these constraints are inherently linear functions of the variables $x_{i;t}$'s.

So far, it has been shown that the condition (8b) can be replaced by (12) and (13) for $t=1,\ldots,t_0$. Based on this fact, we introduce a relaxation of the strengthened SDP problem as follows:

$$\begin{aligned}
\underset{\substack{\mathbf{w}\in\mathbb{R}^{2n_g t_0}\\ \mathbf{W}\in\mathbb{S}^{2n_g t_0}}}{\text{minimize}} \quad & c_r(\mathbf{w},\mathbf{W}) && \text{(14a)}\\
\text{subject to} \quad & \mathbf{Y}_t\mathbf{w}\{\mathcal{V}_t\} \geq \mathbf{y}_t, \qquad t=1,2,\ldots,t_0, && \text{(14b)}\\
& \mathbf{Y}_t\mathbf{W}\{\mathcal{V}_t,\mathcal{V}_t\}\mathbf{Y}_t^\top - \mathbf{y}_t\mathbf{w}\{\mathcal{V}_t\}^\top\mathbf{Y}_t^\top \\
& - \mathbf{Y}_t\mathbf{w}\{\mathcal{V}_t\}\mathbf{y}_t^\top + \mathbf{y}_t\mathbf{y}_t^\top \geq 0, \\
& \qquad t=1,2,\ldots,t_0, && \text{(14c)}\\
& W_{kk}-w_k=0, \qquad k=1,2,\ldots,n_g t_0, && \text{(14d)}\\
& \mathbf{W} \succeq \mathbf{w}\mathbf{w}^T. && \text{(14e)}
\end{aligned}$$

After this relaxation, the exactness of the proposed relaxation can be certified if and only if the variables $x_{i;t}$'s take binary values at optimality. The next theorem shows that the large conic constraint (14e) can be broken down into smaller conic constraints.

Theorem 2. *The conic constraint* $\mathbf{W} \succeq \mathbf{w}\mathbf{w}^T$ *in the relaxation of strengthened SDP problem* (14) *is equivalent to the following set of smaller conic constraints:*

$$\mathbf{W}\{\mathcal{V}_t,\mathcal{V}_t\} \succeq \mathbf{w}\{\mathcal{V}_t\}\mathbf{w}\{\mathcal{V}_t\}^T, \qquad t=1,2,\ldots,t_0 \tag{15}$$

in the absence of minimum up- and down-time constraints.

Proof. Assume that the minimum up- and down-time constraints do not exist. It can be observed that in (14b), (14c), and (14d), the decision variables at each time instance are coupled only with the decision variables of the next and previous time slots. Using the chordal extension technique [26], it is easy to verify that relaxing the constraint (14e) to (15) does not affect the optimal cost. This is due to the fact that the tree decomposition of the above problem is a path. The details are omitted for brevity. □

2) Imposing a subset of valid inequalities: The strengthened SDP problem is obtained from the SDP problem by adding a large number of valid inequalities. However, many of those inequalities may not be binding at optimality. As an effort to avoid such unnecessary constraints, we propose the following procedure: (i) solve the problem (8) and denote its solution with $\mathbf{W}_0^*$, (ii) detect violated valid inequalities by substituting $\mathbf{W}_0^*$ for the matrix variable $\mathbf{W}$ in (14c), (iii) add only the violated valid inequalities to the problem rather than all valid inequalities. This problem is referred to as the *first-order weakly-strengthened SDP relaxation*. One can solve the new problem, check the constraint (14c) at a solution of the first-order relaxation, and add all of the violated inequalities to the problem. This leads to a tighter relaxation, named *second-order weakly-strengthened SDP*. By continuing this procedure, one will be able to k^{th}-order weakly-strengthened SDP for $k = 1, 2, ...,$ until the relaxation becomes equivalent to the strengthened SDP problem in terms of the satisfaction of all constraints.

E. Triangle and VUB Constraints

It will be shown in simulations that the proposed SDP relaxations are able to find a global solution of the UC problem for many test systems under various conditions. However, there are cases for which the relaxations are not exact. To further improve the relaxations for such systems, the so-called triangle inequalities are incorporated in the UC problem. The efficacy of these valid inequalities has been studied by Burer *et al.* [24] and Anstreicher *et al.* [21]. These triangle inequalities are

$$x_{i;t}x_{j;t} + x_{k;t} \geq x_{i;t}x_{k;t} + x_{j;t}x_{k;t}$$
$$x_{i;t}x_{j;t} + x_{i;t}x_{k;t} + x_{j;t}x_{k;t} + 1 \geq x_{i;t} + x_{j;t} + x_{k;t}$$

for every $i, j, k \in \mathcal{G}$ and $t \in \mathcal{T}$. Moreover, the proposed method can be reinforced by adding a special case of the VUB ramp constraints developed by Damcı-Kurt *et al.* [25]. These constraints are

$$p_{i;t} \leq p_{i;\max}.x_{i;t} - (p_{i;\max} - s_i).(x_{i;t} - x_{i;t-1})$$
$$p_{i;t} \leq p_{i;\max}.x_{i;t} - (p_{i;\max} - s_i).(x_{i;t} - x_{i;t+1})$$

which can be added to (8b). Note that the above valid inequalities are the VUB ramp constraints for only two adjacent time slots. Although the number of all VUB ramp constraints is exponential in the size of the UC problem, the number of the inequalities considered above (for two adjacent time slots) has a polynomial size.

IV. Numerical Results

In this section, the numerical results for evaluating the proposed relaxations on IEEE case systems are provided. To generate multiple UC problems for each test case, we multiply all loads of each IEEE system by a load factor α chosen from a discrete set $\{\alpha_1, \alpha_2, ..., \alpha_k\}$. For each IEEE system, we plot four curves for k load profiles: (i) the optimal cost of the (weakly) strengthened SDP, (ii) the optimality gaps for three different relaxations. As the load factor changes from α_1 to α_k, the optimal statuses of the generators may change multiple times. Whenever the statuses of the generators for a load scenario varies from those of the previous load scenario, the corresponding scenario is marked on the curve by a red cross. Hence, if there is no mark on the SDP cost curve for a particular load scenario, it means that the statuses of the generators are the same as those for the previous load scenario. Each red cross is accompanied by an integer number, which can be interpreted as follows: if this number is converted from base 10 to 2, it is the concatenation of the globally optimal status of all generators. For example, for a case with 3 generators, the number 5 on the SDP cost curve indicates that the first and third generators are active while the second generator is off at a globally optimal solution of UC (note that $5 = (101)_2$). Moreover, for every scenario that at least one of generator statuses found by the strengthened SDP is neither 0 nor 1, we write "Not Rank-1" on the curve instead of the an integer number encoding the optimal generator statuses.

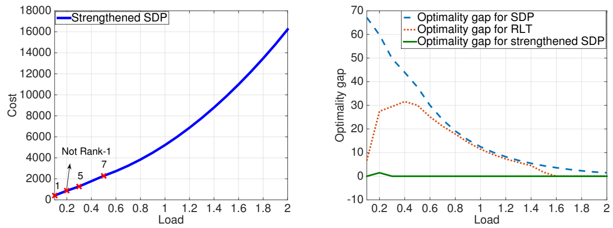
Figure 1(a) shows the solutions found by the strengthened SDP for 20 load scenarios for the IEEE 9-bus system with 3 generators over one time slot ($t_0 = 1$). The load factors are $\alpha_i = 0.1 \times i$ for $i = 1, 2, ..., 20$. It can be observed that the proposed convex relaxation has found a global solution of the UC problem for 19 out of 20 scenarios. We define the optimality gap for any relaxation of the UC problem as

$$\text{Optimality gap} \triangleq \frac{\text{upper bound} - \text{lower bound}}{\text{upper bound}} \times 100,$$

where "upper bound" and "lower bound" denote the globally optimal cost of the UC problem (found using an extensive search) and the optimal cost of the relaxation, respectively. The optimality gaps for the SDP, RLT and strengthened SDP relaxations are compared in Figure 1(b). Notice that the SDP and RLT relaxations perform poorly and the proposed valid inequalities are essential for obtaining integer solutions.

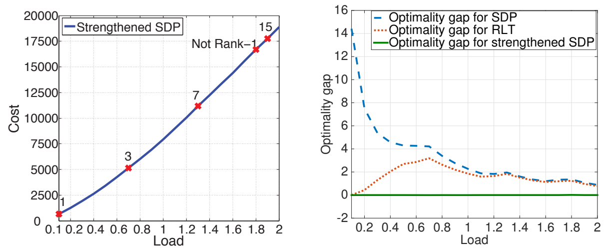
Figure 2 shows the solutions found by the strengthened SDP for 20 load scenarios for the IEEE 14-bus system with 5 generators over one time slot. The load factors are $\alpha_i = 0.1 \times i$ for $i = 1, 2, ..., 20$. The relaxation is exact in 19 load scenarios. For both of the IEEE 9- and 14-bus systems, there is only one load scenario for which the strengthened SDP does not find the globally optimal solution of the UC problem. After adding the triangle and VUB ramp constraints to the formulation, the relaxation becomes exact and it retrieves the optimal solution of the UC problem. Figure 3 illustrates the results of the strengthened SDP for 13 load scenarios for the IEEE 30-bus system with 6 generators over one time slot. The load factors are $\alpha_i = 0.1 \times i$ for $i = 1, 2, ..., 13$. It can be observed that the proposed convex relaxation is exact and finds the globally optimal solution of the problem for all scenarios. If the load factor is greater than or equal to 1.4, the UC problem becomes infeasible for this case since the total load exceeds the total capacity of generators.

Figure 4 shows the output of the strengthened SDP for 15 load scenarios for the IEEE 57-bus system with 7 generators



(a) Strengthened SDP (b) Optimality Gap

Fig. 1: 20 load scenarios for the IEEE 9-bus system with 3 generators over one time slot.



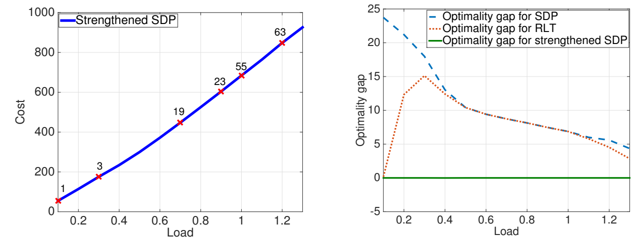
(a) Strengthened SDP (b) Optimality Gap

Fig. 2: 20 load scenarios for the IEEE 14-bus system with 5 generators over one time slot.

over one time slot. The load factors are $\alpha_i = 0.1 \times i$ for $i = 1, 2, ..., 15$. As before, the proposed convex relaxation obtains the globally optimal solution of the problem for all scenarios. Furthermore, the UC problem becomes infeasible if load factor is greater than or equal to 1.6 since the total load exceeds the total generation capacity.

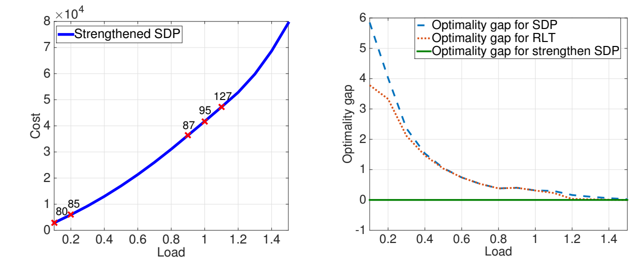
Consider 10 load scenarios for the IEEE 118-bus system with 54 generators over one time slot. The load factors are $\alpha_i = 0.1 \times i$ for $i = 1, 2, ..., 10$. Since the strengthened SDP problem is time-consuming to solve for this system, we resort to weakly-strengthened SDP relaxations with lower-order conic constraints. The results are plotted for the first-order and fourth-order weakly-strengthened SDP problems in Figure 5. It can be observed that the first-order relaxation solves half of the cases successfully, whereas the fourth-order relaxation solves all trials correctly. Figure 6 illustrates the results of the strengthened SDP (14) with low-order conic constraints for 10 load scenarios for the IEEE 30-bus system with 6 generators over $t_0 = 5$ time slots. The load factors are $\alpha_i = 0.8 + 0.02 \times i$ for $i = 1, 2, ..., 10$. Observe that SDP relaxation fails in only two cases. Note that each red cross in Figure 6(a) is accompanied by a vertical array of 5 numbers, each showing the commitment parameters (in base 10) for different time instances.

Figure 7 shows the solutions of the strengthened SDP (14) for 10 load scenarios for the IEEE 57-bus system with 7 generators over 6 time slots. The load factors are $\alpha_i = 0.1 \times i$ for $i = 1, 2, ..., 10$. The proposed relaxation is exact for all load scenarios. Consider the IEEE 300-bus system with 69 generators over one time slot and for the single load factor of 1. The first-order weakly-strengthened SDP achieves the global minimum of the UC problem. The number natural 18338481760792186850 encodes the optimal



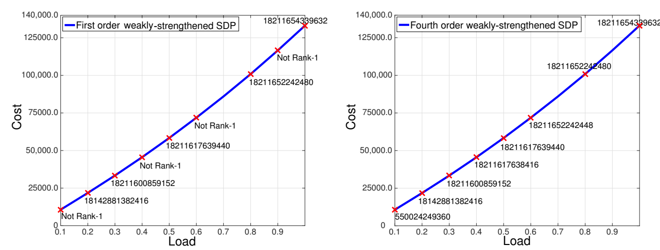
(a) Strengthened SDP (b) Optimality Gap

Fig. 3: 13 load scenarios for the IEEE 30-bus system with 6 generators over one time slot.



(a) Strengthened SDP (b) Optimality Gap

Fig. 4: 15 load scenarios for the IEEE 57-bus system with 7 generators over one time slot.



(a) First-order Weakly-strengthened SDP (b) Fourth-order Weakly-strengthened SDP

Fig. 5: 10 load scenarios for the IEEE 118-bus system with 54 generators over one time slot.

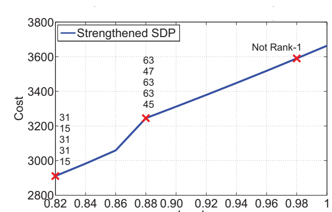
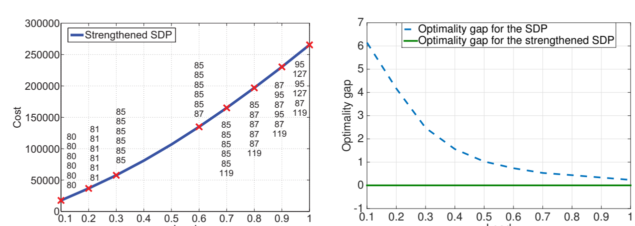


Fig. 6: 10 load scenarios for the IEEE 30-bus system with 6 generators over $t_0 = 5$ time slot.



(a) Strengthened SDP (b) Optimality Gap

Fig. 7: 10 load scenarios for the IEEE 57-bus system with 7 generators over $t_0 = 6$ time slot.

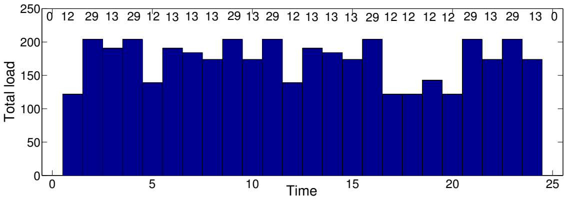


Fig. 8: IEEE 14-bus system with 5 generators over 24 time slots.

statues of all generators in base 10. After converting this number to a binary vector, it can be seen that 53 generators are on and 16 generators are off at optimality. Finally, consider the IEEE 14-bus system with 5 generators over 24 time slots. As before, the proposed convex model (14) achieves the globally optimal solution of the UC problem for this scenario. Figure 8 displays the total load distribution over this horizon. Furthermore, the integer number on top of each column represents the optimal configuration of the generators at each time slot. The optimal costs associated with the strengthened SDP (14) and its first-order weakly-strengthened SDP are 210159 and 205838, respectively. However, the optimal cost for the SDP relaxation without the proposed valid inequalities is 162600.

V. Conclusions

The objective of this paper is to design a convex model for the unit commitment (UC) problem, under the DC modeling assumption. Finding a global solution to the UC problem is a daunting challenge. In this paper, we develop a strengthened SDP relaxation for the UC problem. This is achieved by generating valid constraints and then relaxing them to linear matrix inequalities. These valid inequalities are obtained by the multiplication of the linear constraints of the UC problem such as the flow constraints of two different lines. Since the proposed technique incorporates a large number of valid quadratic inequalities, an iterative algorithm is developed to select only a subset of such inequalities. The proposed relaxations are extensively tested on benchmark systems. Unlike the branch-and-bound and cutting-planning methods used in the power industry, the technique developed in this work can be readily generalized to handle an AC nonlinear model of power flow equations.

References

[1] X. Guan, Q. Zhai, and A. Papalexopoulos, "Optimization based methods for unit commitment: Lagrangian relaxation versus general mixed integer programming," in *IEEE Power Engineering Society General Meeting*, vol. 2, 2003.

[2] B. F. Hobbs, *The next generation of electric power unit commitment models*. Springer Science & Business Media, 2001, vol. 36.

[3] W. Ongsakul and N. Petcharaks, "Unit commitment by enhanced adaptive lagrangian relaxation," *IEEE Transactions on Power Systems*, vol. 19, no. 1, pp. 620–628, 2004.

[4] C.-L. Tseng, S. S. Oren, A. J. Svoboda, and R. B. Johnson, "A unit decommitment method in power system scheduling," *International Journal of Electrical Power & Energy Systems*, vol. 19, no. 6, pp. 357–365, 1997.

[5] C. Tseng, C. Li, and S. Oren, "Solving the unit commitment problem by a unit decommitment method," *Journal of Optimization Theory and Applications*, vol. 105, no. 3, pp. 707–730, 2000.

[6] A. Turgeon, "Optimal unit commitment," *IEEE Transactions on Automatic Control*, vol. 22, no. 2, pp. 223–227, 1977.

[7] D. Rajan and S. Takriti, "Minimum up/down polytopes of the unit commitment problem with start-up costs," *IBM Res. Rep*, no. RC23628 (W0506-050), 2005.

[8] L. L. Garver, "Power generation scheduling by integer programming-development of theory," *Power Apparatus and Systems, Part III. Transactions of the American Institute of Electrical Engineers*, vol. 81, no. 3, pp. 730–734, 1962.

[9] G. Morales-Espana, J. M. Latorre, and A. Ramos, "Tight and compact MILP formulation of start-up and shut-down ramping in unit commitment," *IEEE Transactions on Power Systems*, vol. 28, no. 2, pp. 1288–1296, 2013.

[10] J. Ostrowski, M. F. Anjos, and A. Vannelli, "Tight mixed integer linear programming formulations for the unit commitment problem," *IEEE Transactions on Power Systems*, vol. 27, no. 1, p. 39, 2012.

[11] J. A. Muckstadt and S. A. Koenig, "An application of lagrangian relaxation to scheduling in power-generation systems," *Operations Research*, vol. 25, no. 3, pp. 387–403, 1977.

[12] C. Pang, G. B. Sheblé, and F. Albuyeh, "Evaluation of dynamic programming based methods and multiple area representation for thermal unit commitments," *IEEE Transactions on Power Apparatus and Systems*, no. 3, pp. 1212–1218, 1981.

[13] G. Purushothama and L. Jenkins, "Simulated annealing with local search-a hybrid algorithm for unit commitment," *IEEE Transactions on Power Systems*, vol. 18, no. 1, pp. 273–278, 2003.

[14] M. Madrigal and V. H. Quintana, "An interior-point/cutting-plane method to solve unit commitment problems," in *Proceedings of the 21st IEEE International Conference Power Industry Computer Applications (PICA)*, 1999, pp. 203–209.

[15] H. D. Sherali and W. P. Adams, *A reformulation-linearization technique for solving discrete and continuous nonconvex problems*. Springer Science & Business Media, 2013, vol. 31.

[16] M. X. Goemans, "Semidefinite programming in combinatorial optimization," *Mathematical Programming*, vol. 79, no. 1-3, pp. 143–161, 1997.

[17] Y. Nesterov, "Semidefinite relaxation and nonconvex quadratic optimization," *Optimization Methods and Software*, vol. 9, no. 1-3, pp. 141–160, 1998.

[18] A. Kalbat, R. Madani, G. Fazelnia, and J. Lavaei, "Efficient convex relaxation for stochastic optimal distributed control problem," in *52nd Annual Allerton Conference on Communication, Control, and Computing (Allerton)*. IEEE, 2014, pp. 589–596.

[19] J. Lavaei and S. H. Low, "Zero duality gap in optimal power flow problem," *IEEE Transactions on Power Systems*, vol. 27, no. 1, pp. 92–107, 2012.

[20] M. Kojima and L. Tunçel, "Cones of matrices and successive convex relaxations of nonconvex sets," *SIAM Journal on Optimization*, vol. 10, no. 3, pp. 750–778, 2000.

[21] K. M. Anstreicher, "On convex relaxations for quadratically constrained quadratic programming," *Mathematical Programming*, vol. 136, no. 2, pp. 233–251, 2012.

[22] S. Burer and A. Saxena, "Old wine in a new bottle: The MILP road to MIQCP," *Optimization Online*, 2009.

[23] H. D. Sherali and A. Alameddine, "A new reformulation-linearization technique for bilinear programming problems," *Journal of Global optimization*, vol. 2, no. 4, pp. 379–410, 1992.

[24] S. Burer and A. N. Letchford, "On nonconvex quadratic programming with box constraints," *SIAM Journal on Optimization*, vol. 20, no. 2, pp. 1073–1089, 2009.

[25] P. Damcı-Kurt, S. Küçükyavuz, D. Rajan, and A. Atamtürk, "A polyhedral study of ramping in unit commitment," *Univ. California-Berkeley, BCOL Res. Rep. 13.02*, 2013.

[26] R. Grone, C. R. Johnson, E. M. Sá, and H. Wolkowicz, "Positive definite completions of partial hermitian matrices," *Linear algebra and its applications*, vol. 58, pp. 109–124, 1984.