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Characterization of Rank-Constrained Feasibility Problems via a Finite Number of Convex Programs

Morteza Ashraphijuo, Ramtin Madani and Javad Lavaei

Abstract— **In this paper, the rank-constrained matrix feasibility problem is considered, where an unknown positive semidefinite (PSD) matrix is to be found based on a set of linear specifications. First, we consider a scenario for which the number of given linear specifications is at least equal to the dimension of the corresponding space of rank-constrained matrices. Given a nominal symmetric and PSD matrix, we design a convex program with the property that every arbitrary matrix could be recovered by this convex program based on its specifications if: i) the unknown matrix has the same size and rank as the nominal matrix, and ii) the distance between the nominal and unknown matrices is less than a positive constant number. It is also shown that if the number of specifications is nearly doubled, then it is possible to recover all rank-constrained PSD matrices through a finite number of convex programs. The results of this paper are demonstrated on many randomly generated matrices.**

I. Introduction

In this paper, we consider the problem of finding an $n\times n$ symmetric and positive semidefinite (PSD) matrix $\mathbf{X}$ of rank k that satisfies a modest number of linear specifications of the form:

$$\langle \mathbf{M}_r, \mathbf{X}\rangle = y_r, \quad r = 1,\ldots,m, \tag{1}$$

where $\mathbf{M}_1,\ldots,\mathbf{M}_m$ are some known $n\times n$ symmetric matrices, the notation $\langle\cdot,\cdot\rangle$ denotes the Frobenius inner product, and the scalars $y_1,\ldots,y_m$ are some given specifications corresponding to the unknown matrix $\mathbf{X}$.

Let $\mathcal{S}^+_{n,k}$ denote the set of $n\times n$ symmetric and PSD matrices of rank k. The problem of finding a matrix $\mathbf{X}\in\mathcal{S}^+_{n,k}$ based on a set of specifications $\{y_1,\ldots,y_m\}$ is NP-hard in general, and captures a wide range of other problems including quadratic feasibility and matrix completion [1]. Special cases of the above rank-constrained feasibility problem have been successfully addressed in the literature, such as the case where the number of specifications is relatively high [2] or whenever the matrices $\mathbf{M}_1,\ldots,\mathbf{M}_m$ admit a graph-theoretic structure [3], [4].

In this work, we aim to design a class of convex optimization problems to find the matrix $\mathbf{X}\in\mathcal{S}^+_{n,k}$, given the input vector $(y_1,\ldots,y_m)$. This paper is built upon our recent work [5], which calculates the inverse of polynomial functions using semidefinite programming (SDP). The inverse function theorem states that the inverse of a polynomial function exists at a neighborhood of any nominal point at which the Jacobian of the function is invertible. In [5], we have proven that this inverse function can be found locally using convex optimization. More precisely, infinitely many SDPs are proposed in [5], each of which finds the inverse function at a neighborhood of the nominal point. We have also designed a convex optimization in [5] to check the existence of an SDP problem that finds the inverse of the polynomial function at multiple nominal points and a neighborhood around each point.

Every optimization problem proposed in this work is in the form of a semidefinite program. SDP is a subdiscipline of convex optimization, which has received a significant amount of attention in the past two decades [6]–[9]. The SDP relaxation technique is a powerful method for tackling nonlinearity, which has been proven to be effective in the convexification of several hard optimization problems in various areas, including graph theory, approximation theory, quantum mechanics, neural networks, communication networks, and power systems [10]–[15]. SDP relaxation methods have been successfully used for real-world applications such as radar code design, multiple-input and multiple-output beamforming, error-correcting codes, magnetic resonance imaging (MRI), data training, and portfolio selection, among many others [16]–[19]. Several papers have evaluated the performance of SDP relaxations for various problems, by investigating the approximation ratio and the maximum rank of SDP solutions [3], [20]–[23]. Moreover, different global optimization techniques for polynomial optimization have been built upon SDP relaxations [14], [24]–[27].

A. Notations

The symbols $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{S}_n$ denote the sets of real numbers, nonnegative reals and $n\times n$ real symmetric matrices, respectively. $\mathrm{rank}\{\cdot\}$, $\mathrm{trace}\{\cdot\}$, and $\det\{\cdot\}$ denote the rank, trace, and determinant of a given scalar/matrix. $\|\cdot\|_2$ denotes the Euclidean norm and $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm of a matrix. Matrices are shown by capital and bold letters. The symbol $(\cdot)^T$ denotes the transpose operator. The notation $\langle \mathbf{A},\mathbf{B}\rangle$ represents $\mathrm{trace}\{\mathbf{A}^{\mathrm{T}}\mathbf{B}\}$, which is the inner product of $\mathbf{A}$ and $\mathbf{B}$. The notation $\mathbf{W}\succeq 0$ means that $\mathbf{W}$ is a symmetric and PSD matrix. Moreover, $\mathbf{W}\succ 0$ means that it is symmetric and positive definite. The (i,j) entry of $\mathbf{W}$ is denoted as W_{ij}. The interior of a set $\mathcal{D}\in\mathbb{R}^n$ is denoted by $\mathrm{int}\{\mathcal{D}\}$. The notation $\mathrm{null}(\cdot)$ denotes the null space of a matrix. The notation $\mathrm{diag}\{v_1,\ldots,v_n\}$ denotes the $n\times n$ square matrix whose diagonal values are given by $v_1,\ldots,v_n$.

Morteza Ashraphijuo and Javad Lavaei are with the Department of Industrial Engineering & Operations Research, University of California-Berkeley, Berkeley (e-mail: ashraphijuo@berkeley.edu, lavaei@berkeley.edu). Ramtin Madani is with the Electrical Engineering Department at the University of Texas at Arlington (e-mail: ramtin.madani@uta.edu). This work was supported by DARPA YFA, ONR YIP Award, NSF CAREER Award 1351279 and NSF EECS Award 1406865.

$\mathbf{0}_n$ and $\mathbf{1}_n$ denote the $n \times 1$ vectors of zeros and ones, respectively. $\mathcal{S}^{+}_{n;k}$ denotes the space of $n \times n$ symmetric PSD matrices of rank k. Let $\mathcal{R}_{n,k}$ denote the set of all $n \times n$ symmetric PSD matrices of rank less than or equal to k. In addition, $\mathrm{dist}_{\mathrm{P}}(\cdot,\cdot)$ denotes the projection distance. The cardinality of a set $\mathcal{V}$ is denoted as $|\mathcal{V}|$. Given two sets of natural numbers $\mathcal{V}_1$ and $\mathcal{V}_2$ as well as a matrix $\mathbf{W}$, the notation $\mathbf{W}[\mathcal{V}_1,\mathcal{V}_2]$ denotes the submatrix of $\mathbf{W}$ that is obtained by keeping only those rows of $\mathbf{W}$ that correspond to the elements of the set $\mathcal{V}_1$ and those columns of $\mathbf{W}$ that are associated with the elements of the set $\mathcal{V}_2$.

II. Problem Statement and Preliminaries

This paper is concerned with the feasibility problem

$$\begin{aligned}
&\text{find} && \mathbf{X} \in \mathbb{S}_n \\
&\text{subject to} && \langle \mathbf{M}_r, \mathbf{X}\rangle = y_r, \quad r = 1,\ldots,m, && \text{(2a)}\\
& && \mathbf{X} \succeq 0, && \text{(2b)}\\
& && \mathrm{rank}\{\mathbf{X}\} = k. && \text{(2c)}
\end{aligned}$$

Definition 1: Associated to the feasibility problem (2), define the mapping $\mathcal{F} : \mathcal{S}^{+}_{n;k} \to \mathbb{R}^m$ as

$$\mathcal{F}(\mathbf{X}) \triangleq [\langle \mathbf{M}_r, \mathbf{X}\rangle]_{r=1,\ldots,m}.$$

Denote the set of non-critical points of $\mathcal{F}$ as $\mathcal{I}_{\mathcal{F}} \subseteq \mathcal{S}^{+}_{n;k}$.

Observe that the dimension of $\mathcal{S}^{+}_{n;k}$ is $nk - k(k-1)/2$. In this paper, we first investigate the case where there are exactly $nk - k(k-1)/2$ linear specifications and then generalize to the over-specified case $m > nk - k(k-1)/2$. In order to reduce the complexity of the nonconvex feasibility problem (2), we aim to drop the rank constraint (2c) and penalize its effect into the objective function via a linear term. This idea will be explained below.

Definition 2: Consider the optimization problem

$$\begin{aligned}
&\underset{\mathbf{X}\in\mathbb{S}_n}{\text{minimize}} && \langle \mathbf{N}, \mathbf{X}\rangle && \text{(3a)}\\
&\text{subject to} && \langle \mathbf{M}_r, \mathbf{X}\rangle = y_r, \quad r = 1,\ldots,m, && \text{(3b)}\\
& && \mathbf{X} \succeq 0, && \text{(3c)}
\end{aligned}$$

for some constant matrix $\mathbf{N}$. This is referred to as an **SDP relaxation problem with the parameters** $(\mathbf{N}, [y_r]_{r=1}^{m})$.

A solution of the rank-constrained problem (2) is said to be recoverable through the SDP problem (3) if it is the unique optimal solution of (3). Let $\mathcal{R}_{\mathcal{F}}(\mathbf{N})$ denotes the recoverable region for the optimization problem (3), i.e., the set of all matrices $\mathbf{X}$ in $\mathcal{S}^{+}_{n;k}$ that are the unique solution of the above problem for some input vector $(y_1,\ldots,y_k)$. Indeed, the minimum number of linear specifications necessary to recover the matrix $\mathbf{X}$ would be the dimension of the manifold $\mathcal{S}^{+}_{n,k}$, which is equal to

$$d(n,k) \triangleq nk - k(k-1)/2. \tag{4}$$

In this work, we generate the matrix $\mathbf{N}$ in (3a) based on an initial guess $\mathbf{X}_0 \in \mathcal{S}^{+}_{n;k}$ for the solution. One interesting property of the proposed convex relaxation scheme is that if the two subspaces spanned by the columns of the unknown solution $\mathbf{X}$ and the initial guess $\mathbf{X}_0$ are relatively close, then the relaxation is guaranteed to be exact, even if $\mathbf{X}_0$ and $\mathbf{X}$ are distanced from each other. In order to quantify the distance between the subspaces spanned by the columns of matrices, it is necessary to define the notion of projection metric.

Definition 3: For every $\mathbf{X} \in \mathcal{S}^{+}_{n,k}$, define $\mathrm{P}_{\mathbf{X}} \in \mathcal{S}^{+}_{n,k}$ as the unique matrix representing the orthogonal projection onto the space spanned by the columns of $\mathbf{X}$. Accordingly, define the projection metric $\mathrm{dist}_{\mathrm{P}}(\cdot,\cdot) : \mathcal{S}^{+}_{n,k} \times \mathcal{S}^{+}_{n,k} \to \mathbb{R}^{+}$ as

$$\mathrm{dist}_{\mathrm{P}}(\mathbf{X},\mathbf{Y}) \triangleq \|\mathrm{P}_{\mathbf{X}} - \mathrm{P}_{\mathbf{Y}}\|_2. \tag{5}$$

Moreover, for every $\mathbf{X}_0 \in \mathcal{S}^{+}_{n;k}$, define

$$\mathcal{B}_{\mathbf{X}_0;\varepsilon} = \{\mathbf{X} \in \mathcal{S}^{+}_{n;k} \,|\, \mathrm{dist}_{\mathrm{P}}(\mathbf{X},\mathbf{X}_0) < \varepsilon\}.$$

Suppose that $\mathbf{X} \in \mathcal{S}^{+}_{n,k}$ admits the eigenvalue decomposition $\mathbf{X} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{\mathrm{T}}$, where $\mathbf{\Lambda} = \mathrm{diag}\{\lambda_1,\ldots,\lambda_k\} \in \mathbb{R}^{k\times k}$ and $\mathbf{Q} = [\mathbf{v}_1,\ldots,\mathbf{v}_k] \in \mathrm{R}^{n\times k}$ collect nonzero eigenvalues and the corresponding eigenvectors of $\mathbf{X}$, respectively. In order to observe that the notation $\mathrm{P}_{\mathbf{X}}$ is well-defined, one can easily verify that

$$\mathrm{P}_{\mathbf{X}} = \mathbf{Q}\mathbf{Q}^{\mathrm{T}} \tag{6}$$

is the unique projection matrix described by Definition 3 (see [28] for further details on projection metric).

The following is a summery of the contributions of this work:

1) Given an arbitrary nominal point $\mathbf{X_0} \in \mathcal{S}^{+}_{n,k}$ that is not singular for the mapping $\mathcal{F}$, we design a convex program that recovers every member of $\mathcal{S}^{+}_{n,k}$ in a neighborhood around $\mathbf{X_0}$. The set of recoverable matrices can be explicitly characterized through a nonlinear matrix inequality and it contains the cone
$$\{\mathbf{X} \in \mathcal{S}^{+}_{n,k} \,|\, \mathrm{dist}_{\mathrm{P}}(\mathbf{X},\mathbf{X_0}) < \epsilon\}, \tag{7}$$
for some $\epsilon > 0$, where $\mathrm{dist}_{\mathrm{P}}(\cdot,\cdot)$ denotes the projection distance.
2) Consider the case where the matrices $\mathbf{M}_1,\ldots,\mathbf{M}_m$ are chosen generically and
$$m > k(2n-k). \tag{8}$$
We show that the parameter ϵ in (7) is independent of the nominal point $\mathbf{X}_0$. Then, we prove that a finite number of nominal points can be chosen such that every matrix in $\mathcal{S}^{+}_{n,k}$ is recoverable via the convex program associated with at least one of the nominal points.

The "generic" assumption in the above statement implies that our result is true for almost every choice of $\mathbf{M}_1,\ldots,\mathbf{M}_m \in \mathbb{S}_n$. A formal definition for the phrase "generically chosen" will be provided next.

Definition 4: A property (Q) is said to hold for every "generically chosen" member of a topological space $\mathcal{O}$ if there exists an open dense subset of $\mathcal{O}$ whose members all satisfy (Q).

The dual of the convex program (3) is regarded as the **dual SDP problem with the parameters** $(\mathbf{N}, [y_r]_{r=1}^m)$, which can be written as

$$\underset{\mathbf{u}\in\mathbb{R}^m}{\text{minimize}} \qquad \mathbf{y}^{\mathrm{T}}\mathbf{u} \tag{9a}$$

$$\text{subject to} \qquad \mathbf{B}_{\mathcal{F}}(\mathbf{N},\mathbf{u}) \succeq 0, \tag{9b}$$

where $\mathbf{u} \in \mathbb{R}^m$ is the vector of dual variables and $\mathbf{B}_{\mathcal{F}} : \mathbb{S}_n \times \mathbb{R}^m \to \mathbb{S}_n$ is defined as

$$\mathbf{B}_{\mathcal{F}}(\mathbf{N},\mathbf{u}) \triangleq \mathbf{N} + \sum_{i=1}^{m} u_i \mathbf{M}_i. \tag{10}$$

The main results of this paper will be developed in the next section.

III. Main Results

The next theorem states that a matrix $\mathbf{N}$ can be systematically designed such that the region of recoverable matrices $\mathcal{R}_{\mathcal{F}}(\mathbf{N})$ contains a neighborhood around a given nominal point $\mathbf{X}_0$.

Theorem 1: Let $\mathbf{X_0} \in \mathcal{S}^+_{n,k}$ be an arbitrary nominal point that is not singular for the mapping $\mathcal{F}$. For every arbitrary matrix $\mathbf{N} \in \mathcal{S}^+_{n;n-k}$ satisfying the relation $\mathbf{N}\mathbf{X}_0 = \mathbf{0}$, there exists a constant $\varepsilon > 0$ such that every matrix $\mathbf{X} \in \mathcal{B}_{\mathbf{X}_0;\varepsilon}$ is the unique solution to the SDP relaxation problem (3) with the parameters $(\mathbf{N}, [\langle \mathbf{M}_r, \mathbf{X}\rangle]_{r=1}^m)$.

Proof: The proof is given in Section IV. ∎

According to Theorem 1, we can design a matrix $\mathbf{N}$ such that the region $\mathcal{R}_{\mathcal{F}}(\mathbf{N})$ includes a neighborhood around the nominal point $\mathbf{X}_0$. Roughly speaking, each of these convex programs covers a cone containing the nominal point. The next theorem states that a lower bound on the radius of these balls could be found for choices of $\mathbf{N}$.

Theorem 2: Let

$$m > k(2n-k), \tag{11}$$

and suppose that the matrices $\mathbf{M}_1, \ldots, \mathbf{M}_m$ are chosen generically from the set $\mathbb{S}_n$. If $\mathbf{N} \in \mathcal{S}^+_{n;n-k}$ satisfies the equation $\mathbf{N}\mathbf{X}_0 = 0$ for a matrix $\mathbf{X}_0 \in \mathcal{S}^+_{n;k}$ and all nonzero eigenvalues of $\mathbf{N}$ are equal to 1, then there exists a constant $\varepsilon > 0$ in terms of $\mathbf{M}_1, \ldots, \mathbf{M}_m$ such that every matrix $\mathbf{X} \in \mathcal{S}^+_{n;k}$ with the property

$$\mathrm{dist}_{\mathrm{P}}(X, X_0) \leq \varepsilon, \tag{12}$$

is recoverable through the SDP relaxation problem (3) with the parameters $(\mathbf{N}, [\langle \mathbf{M}_r, \mathbf{X}\rangle]_{r=1}^m)$.

Proof: The proof follows immediately from Lemma 3 (to be presented later) and Theorem 1. ∎

Theorem 2 proves the existence of a local recovery region with a non-demenishing volume. A question arises as to whether there exists a finite number of SDPs whose recovery regions altogether cover the whole space of rank-k positive semidefinite matrices. This problem will be addressed below.

Theorem 3: Assume that

$$m > k(2n-k), \tag{13}$$

and that the matrices $\mathbf{M}_1, \ldots, \mathbf{M}_m$ are generic members of $\mathbb{S}_n$. Then, there exists a **finite** number of matrices $\mathbf{N}_1, \ldots, \mathbf{N}_q \in \mathcal{S}^+_{n,n-k}$ such that every matrix $\mathbf{X} \in \mathcal{S}^+_{n;k}$ is recoverable through the SDP relaxation problem (3) with the parameters $(\mathbf{N}_s, [\langle \mathbf{M}_r, \mathbf{X}\rangle]_{r=1}^m)$ for at least one index $s \in \{1, \ldots, q\}$.

Proof: The proof is given in Section IV. ∎

Theorem 3 states that if the number of measurements is not too small, then there is a finite number of SDPs such that the feasibility problem (2) with an arbitrary input $(y_1, ..., y_r)$ can be solved via at least one of those SDPs.

IV. Proofs

The proofs of the results presented in the preceding section will be provided below. Define $\mathcal{N}_n \triangleq \{1, \ldots, n\}$ and let $\mathcal{A}_{n;k}$ denote the set of subsets of $\mathcal{N}_n$ of size k:

$$\mathcal{A}_{n;k} \triangleq \{A \subseteq \mathcal{N}_n \,|\, |A| = k\}.$$

Let $A = \{a_1, \ldots, a_k\} \in \mathcal{A}_{n;k}$ and $B = \{b_1, \ldots, b_{n-k}\} = \mathcal{N}_n \setminus A$ such that $\{a_i\}_{i=1,\ldots,k}$ and $\{b_i\}_{i=1,\ldots,n-k}$ are in ascending order. Define the $n \times n$ permutation matrix $\Pi_{n;A}$ as

$$\Pi_{n;A} = \begin{bmatrix} [\delta_{a_i,j}]_{i=1,\ldots,k;j=1,\ldots,n} \\ [\delta_{b_i,j}]_{i=1,\ldots,n-k;j=1,\ldots,n} \end{bmatrix},$$

where δ_{ij} denotes the Kronecker delta function. This matrix consists of two block submatrices of dimensions $k \times n$ and $(n-k) \times n$. The set of $n \times k$ lower triangular matrices is denoted by

$$\mathcal{L}_{n;k} \triangleq \{\mathbf{V} \in \mathbb{R}^{n\times k} \,|\, V_{i,j} = 0 \quad \text{if} \quad i < j\}.$$

Moreover, for every $A \in \mathcal{A}_{n;k}$, define

$$\mathcal{S}^+_{n;A} \triangleq \{\mathbf{X} \in \mathcal{S}^+_{n;|A|} \,|\, \mathbf{X}[A,A] \succ 0\},$$

as the set of matrices in $\mathcal{S}^+_{n;|A|}$ whose principal submatrix corresponding to the rows and columns in A has full rank. Observe that

$$\mathcal{S}^+_{n;k} = \bigcup_{A\in\mathcal{A}_{n;k}} \mathcal{S}^+_{n;A}.$$

In order to continuously map the manifold $\mathcal{S}^+_{n;k}$ into $\mathbb{R}^{d(n,k)}$, we build a family of mappings each defined from a subset $\mathcal{S}^+_{n;A} \subset \mathcal{S}^+_{n;k}$ to the linear space $\mathcal{L}_{n;k}$ for some $A \in \mathcal{A}_{n;k}$.

Definition 5: (Cholesky embeddings) For every $A \in \mathcal{A}_{n;k}$, define the function $\mathcal{C}_{n;A} : \mathcal{S}^+_{n;A} \to \mathcal{L}_{n;k}$ as

$$\mathcal{C}_{n;A}(\mathbf{X}) \triangleq \begin{bmatrix} \mathbf{L}^{\mathrm{T}}, & \mathbf{L}^{-1}\mathbf{X}[A, \mathcal{N}_n \setminus A] \end{bmatrix}^{\mathrm{T}},$$

where $\mathbf{L}$ is the lower triangular matrix obtained from the Cholesky decomposition of $\mathbf{X}[A,A]$, i.e., $\mathbf{L}\mathbf{L}^{\mathrm{T}} = \mathbf{X}[A,A]$.

Note that

$$\mathbf{X} = \Pi_{n;A}^{\mathrm{T}} \mathcal{C}_{n;A}(\mathbf{X}) \mathcal{C}_{n;A}(\mathbf{X})^{\mathrm{T}} \Pi_{n;A},$$

for every $\mathbf{X} \in \mathcal{S}_{n;k}^{+}$. Throughout the paper, we employ the collection $\{(\mathcal{S}_{n;A}^{+}, \mathcal{C}_{n;A})\}_{A \in \mathcal{A}_{n;k}}$ as an atlas for the manifold $\mathcal{S}_{n;k}^{+}$.

In order to differentiate the mapping $\mathcal{F} : \mathcal{S}_{n;k}^{+} \to \mathbb{R}^{m}$, we need to characterize the tangent spaces of $\mathcal{S}_{n;k}^{+}$. The tangent space of $\mathcal{S}_{n;k}^{+}$ at point $\mathbf{X} \in \mathcal{S}_{n;A}^{+}$ is an equivalence class on all curves $\Gamma : (-1, +1) \to \mathcal{S}_{n;A}^{+}$ that pass through $\mathbf{X}$ at point 0. Two curves Γ_1 and Γ_2 are equivalent if they admit the same derivative at 0, i.e.,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{C}_{n;A} \circ \Gamma_1(t)\right|_{t=0} = \left.\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{C}_{n;A} \circ \Gamma_2(t)\right|_{t=0}.$$

Definition 6: Let $\mathcal{T}_{\mathbf{X}} \mathcal{S}_{n;k}^{+}$ denote the tangent space for $\mathcal{S}_{n;k}^{+}$ at point $\mathbf{X}$. Given $A \in \mathcal{A}_{n;k}$ such that $\mathbf{X} \in \mathcal{S}_{n;A}^{+}$, the following curve can serve as a representative for the equivalence class that it belongs to in $\mathcal{T}_{\mathbf{X}} \mathcal{S}_{n;k}^{+}$:

$$\begin{aligned} \Gamma_{\mathbf{X};A}\{\mathbf{V}\}(t) \triangleq \Pi_{n;A}^{\mathrm{T}} & \left(\mathcal{C}_{n;A}(\mathbf{X}) + t\mathbf{V}\right) \\ & \left(\mathcal{C}_{n;A}(\mathbf{X}) + t\mathbf{V}\right)^{\mathrm{T}} \Pi_{n;A}, \end{aligned} \tag{14}$$

where $\mathbf{V} \in \mathcal{L}_{n;k}$.

In order to perform a singularity analysis, the pushforward (Jacobian) of the mapping $\mathcal{F}$ can be formed, by differentiating the curves defined in (14):

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}t} \Gamma_{\mathbf{X};A}\{\mathbf{V}\}(t)\right|_{t=0} &= \Pi_{n;A}^{\mathrm{T}} \mathcal{C}_{n;A}(\mathbf{X}) \mathbf{V}^{\mathrm{T}} \Pi_{n;A} \\ &\quad + \Pi_{n;A}^{\mathrm{T}} \mathbf{V} \mathcal{C}_{n;A}(\mathbf{X})^{\mathrm{T}} \Pi_{n;A}, \end{aligned}$$

Notation 1: Define the lower triangular vectorization operator $\mathcal{V}_{n;k} : \mathbb{R}^{n \times k} \to \mathbb{R}^{d(n,k)}$ as

$$\begin{aligned} \mathcal{V}_{n;k}(\mathbf{V}) \triangleq [& V_{1,1}, V_{2,1}, \ldots, V_{n,1}, V_{2,2}, V_{3,2}, \ldots, V_{n,2}, \ldots \\ & V_{k,k}, V_{k+1,k}, \ldots, V_{n,k}]^{\mathrm{T}} \in \mathbb{R}^{d(n,k)}. \end{aligned}$$

Definition 7: Define $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_{d(n,k)}$ as a basis for $\mathcal{L}_{n;k}$ with the property that

$$\mathcal{V}_{n;k}(\mathbf{E}_r) = e_r, \quad \text{for} \quad r = 1, \ldots, d(n,k),$$

where $\{e_r\}_{r=1}^{d(n,k)}$ is the standard basis for $\mathbb{R}^{d(n,k)}$.

Notice that

$$\begin{aligned} & \left.\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}_r\left(\Gamma_{\mathbf{X};A}\{\mathbf{V}\}(t)\right)\right|_{t=0} = \langle \mathbf{M}_r, \left.\frac{\mathrm{d}}{\mathrm{d}t}\left(\Gamma_{\mathbf{X};A}\{\mathbf{V}\}(t)\right)\right|_{t=0} \rangle \\ & = \langle \mathbf{M}_r, \Pi_{n;A}^{\mathrm{T}} \mathcal{C}_{n;A}(\mathbf{X}) \mathbf{V}^{\mathrm{T}} \Pi_{n;A} + \Pi_{n;A}^{\mathrm{T}} \mathbf{V} \mathcal{C}_{n;A}(\mathbf{X})^{\mathrm{T}} \Pi_{n;A} \rangle \\ & = 2 \langle \Pi_{n;A} \mathbf{M}_r \Pi_{n;A}^{\mathrm{T}} \mathcal{C}_{n;A}(\mathbf{X}), \mathbf{V} \rangle. \end{aligned}$$

If $\{\mathbf{E}_r\}_{r=1}^{d(n,k)}$ is adopted as a basis for $\mathcal{L}_{n;k}$, then the pushforward function $\mathcal{J}_{\mathcal{F},A}(\mathbf{X}) : \mathcal{T}_{\mathbf{X}} \mathcal{S}_{n;k}^{+} \to \mathbb{R}^{d(n,k)}$ at a point $\mathbf{X} \in \mathcal{S}_{n;A}^{+}$ can be parameterized as

$$\mathcal{J}_{\mathcal{F},A}(\mathbf{X}) = 2 \begin{bmatrix} \mathcal{V}_{n;k}(\Pi_{n;A} \mathbf{M}_1 \Pi_{n;A}^{\mathrm{T}} \mathcal{C}_{n;A}(\mathbf{X}))^{\mathrm{T}} \\ \mathcal{V}_{n;k}(\Pi_{n;A} \mathbf{M}_2 \Pi_{n;A}^{\mathrm{T}} \mathcal{C}_{n;A}(\mathbf{X}))^{\mathrm{T}} \\ \vdots \\ \mathcal{V}_{n;k}(\Pi_{n;A} \mathbf{M}_m \Pi_{n;A}^{\mathrm{T}} \mathcal{C}_{n;A}(\mathbf{X}))^{\mathrm{T}} \end{bmatrix}.$$

Lemma 1: There exists a point $\mathbf{X} \in \mathcal{S}_{n;k}^{+}$ at which $\mathcal{F}$ is singular if and only if there exists a nonzero matrix $\mathbf{Y} \in \mathbb{R}^{n \times n}$ (not necessarily symmetric) of rank less than or equal to k satisfying the equations

$$\langle \mathbf{M}_r, \mathbf{Y} \rangle = 0, \quad r = 1, \ldots, m. \tag{15}$$

Proof: Let $\mathbf{X} \in \mathcal{S}_{n;A}^{+}$ be a singularity point of $\mathcal{F}$. There exists a nonzero $\mathbf{C}_0 \in \mathcal{L}_{n,A}$ such that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}\left([\mathcal{C}_{n;A}(\mathbf{X}) + t\mathbf{C}_0][\mathcal{C}_{n;A}(\mathbf{X}) + t\mathbf{C}_0]^{\mathrm{T}}\right)\right|_{t=0} = 0, \tag{16}$$

which means that

$$\langle \mathbf{M}_r, \mathcal{C}_{n;A}(\mathbf{X}) \mathbf{C}_0 \rangle = 0, \quad r = 1, \ldots, m. \tag{17}$$

Now, suppose that there exists a $\mathbf{Y} \in \mathbb{R}^{n \times n}$ of rank less than or equal to k that satisfies (15). It can be easily seen that there exist $\mathbf{C}_1, \mathbf{C}_2 \in \mathcal{L}_{n,A}$ such that $\mathbf{C}_1$ has full column rank and $Y = \mathbf{C}_1 \mathbf{C}_2^{\mathrm{T}}$. Therefore, $\mathbf{C}_1 \mathbf{C}_1^{\mathrm{T}}$ is a singularity point in the domain of $\mathcal{F}$. ∎

Definition 8: Denote the Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$ as $\mathcal{G}_{n;k}$. Define $\overset{G}{\sim}$ as the equivalency relation on $\mathcal{S}_{n;k}^{+}$, associated to $\mathcal{G}_{n;k}$, such that $\mathbf{X}_1 \overset{G}{\sim} \mathbf{X}_2$ if and only if the columns of $\mathbf{X}_1$ and $\mathbf{X}_2$ span the same k-dimensional subspaces. Moreover, denote the equivalence class of a matrix $\mathbf{X} \in \mathcal{S}_{n;k}^{+}$ with $[\mathbf{X}]_G \in \mathcal{G}_{n;k}$.

Lemma 2: If

$$m > k(2n - k), \tag{18}$$

then every generically chosen $\mathcal{F}$ does not have any singular point.

Proof: A general mapping $\mathcal{F}$ is characterized by the matrices $\mathbf{M}_1, \ldots, \mathbf{M}_m$. Define $\mathcal{Z}_{\mathcal{F}} \subseteq \mathbb{S}_n$ as the null space of the mapping

$$\mathbf{Z} \in \mathbb{S}_n \quad \to \quad [\langle \mathbf{M}_r, \mathbf{Z} \rangle]_{r=1}^{m} \in \mathbb{R}^m. \tag{19}$$

The linear space $\mathcal{Z}_{\mathcal{F}}$ has the dimension $n(n+1)/2 - m$, while $\mathbb{S}_n$ is of dimension $n(n+1)/2$. Therefore, a generic choice of $\mathcal{F}$ results in a generic choice of $\mathcal{Z}_{\mathcal{F}}$, which makes it an arbitrary member of $\mathcal{G}_{n(n+1)/2;\,[n(n+1)/2-m]}$ that is of dimension

$$d_1 = m[n(n+1)/2 - m]. \tag{20}$$

We aim to show that if (18) holds, then the set of all linear spaces $\mathcal{Z}_{\mathcal{F}}$ such that $\mathcal{F}$ has a nonzero singularity is of a dimension smaller than d_1. To this end, let $\mathcal{Y}_{n,k} \subset \mathbb{R}^{n \times n}$ be the set of matrices $\mathbf{Y} \in \mathbb{R}^{n \times n}$ of rank less than or equal to k, which is of dimension

$$d_2 = k(2n - k). \tag{21}$$

According to Lemma 1, the mapping $\mathcal{F}$ has a nonzero singularity if and only if there exists a nonzero $\mathbf{Y} \in \mathcal{Y}_{n,k}$ such that $\mathbf{Y} + \mathbf{Y}^{\mathrm{T}} \in \mathcal{Z}_{\mathcal{F}}$. For every $\mathbf{Y} \in \mathcal{Y}_{n,k}$, define

$$G_{\mathcal{F}}(\mathbf{Y}) \triangleq \{\mathcal{Z}_{\mathcal{F}} \,|\, \mathbf{Y} + \mathbf{Y}^{\mathrm{T}} \in \mathcal{Z}_{\mathcal{F}}\}. \tag{22}$$

Observe that for every nonzero $\mathbf{Y} \in \mathcal{Y}_{n,k}$, the manifold $G_{\mathcal{F}}(\mathbf{Y})$ is isomorphic to $\mathcal{G}_{[n(n+1)/2-1];\,[n(n+1)/2-m-1]}$ that is of dimension

$$d_3 = m[n(n+1)/2 - m - 1]. \tag{23}$$

Consider the manifold of every $\mathcal{Z}_{\mathcal{F}}$, where $\mathcal{F}$ has a nonzero singularity. According to (1), this can be represented as:

$$Z \triangleq \bigcup_{\mathbf{Y} \in \mathcal{Y}_{n,k}} G_{\mathcal{F}}(\mathbf{Y}). \tag{24}$$

Now, it remains to investigate the dimension of Z, which is clearly not greater than $d_2 + d_3$. Moreover, according to (18) we have $d_2 + d_3 < d_1$, which completes the proof. ■

Let $(\mathbf{X}, \mathbf{u}) \in \mathbb{S}_n^+ \times \mathbb{R}^m$ be a pair of primal and dual optimal points for the SDP relaxation problem (3) and the dual SDP problem (9). The Karush-Kuhn-Tucker (KKT) conditions impose the relationship

$$\left(\mathbf{N} + \sum_{r=1}^{m} u_r \mathbf{M}_r\right) \mathbf{X} = 0. \tag{25}$$

Let $\mathbf{X}_1, \mathbf{X}_2 \in \mathcal{S}_{n;k}^+$. According to the definition $\mathbf{X}_1 \overset{G}{\sim} \mathbf{X}_2$, we have

$$\mathbf{X}_1 \in \mathcal{I}_{\mathcal{F}} \iff \mathbf{X}_2 \in \mathcal{I}_{\mathcal{F}}.$$

Therefore, define

$$\mathcal{I}_{\mathcal{F};G} \triangleq \{[\mathbf{X}]_G \,|\, \mathbf{X} \in \mathcal{I}_{\mathcal{F}}\}.$$

Moreover, given $[\mathbf{X}]_G \in \mathcal{I}_{\mathcal{F};G}$, the equation (25) can be solved for $\mathbf{u}$ as follows:

$$\mathbf{u} = -2(\mathcal{J}_{\mathcal{F},\mathcal{A}}^{-1})^{\mathrm{T}} \mathcal{V}_{n;k}(\Pi_{n;A}\, \mathbf{N}\, \Pi_{n;A}^{\mathrm{T}} \mathcal{C}_{n;A}(\mathbf{X}_0)),$$

where $\mathbf{X}_0 \in \mathcal{S}_{n,A}^+$ is an arbitrary member of $[\mathbf{X}]_G$. Accordingly, define $\mathcal{U}_{\mathcal{F}} : \mathcal{G}_{n;k} \times \mathbb{S}_n^+ \to \mathbb{R}^m$ as follows

$$\mathcal{U}_{\mathcal{F}}(G_0, \mathbf{N}) \triangleq -2(\mathcal{J}_{\mathcal{F},\mathcal{A}}^{-1})^{\mathrm{T}} \mathcal{V}_{n;k}(\Pi_{n;A}\, \mathbf{N}\, \Pi_{n;A}^{\mathrm{T}} \mathcal{C}_{n;A}(\mathbf{X}_0))$$

where $\mathbf{X}_0 \in \mathcal{S}_{n,A}^+$ is an arbitrary member of G_0. Observe that the function $\mathcal{U}_{\mathcal{F}}(\cdot,\cdot)$ does not depend on $A \in \mathcal{A}_{n;k}$, due to the invariance of pushforward with respect to the choice of chart.

Lemma 3: Define $\mathcal{T}_{n,n-k}^+$ as the set of matrices in $\mathcal{S}_{n,n-k}^+$ with all nonzero eigenvalues equal to 1. If

$$m > k(2n-k), \tag{26}$$

and $\mathbf{M}_1, \ldots, \mathbf{M}_m$ are generic, then there exists a constant number $\sigma \geq 0$ in terms of the matrices $\mathbf{M}_1, \ldots, \mathbf{M}_m$ such that the inequality

$$\|\mathcal{U}_{\mathcal{F}}(G_1, \mathbf{N}) - \mathcal{U}_{\mathcal{F}}(G_2, \mathbf{N})\|_2 \leq \sigma \times \mathrm{dist}_{\mathrm{P}}(G_1, G_2) \tag{27}$$

holds for every pair $G_1, G_2 \in \mathcal{G}_{n;k}$ and matrix $\mathbf{N} \in \mathcal{T}_{n,n-k}^+$.

Proof: For every arbitrary matrix $\mathbf{N} \in \mathcal{T}_{n,n-k}^+$, the mapping $\mathcal{U}_{\mathcal{F}}(\cdot, \mathbf{N}) : \mathcal{G}_{n;k} \to \mathbb{R}^m$ is well defined and continuous according to Lemma 2. Moreover, since the domain of $\mathcal{U}_{\mathcal{F}}(\cdot, \mathbf{N})$ is compact, it is Lipschitz continuous. Thus, for every $\mathbf{N}$, there exists a constant number $\sigma_{\mathbf{N}}$ such that the following inequality holds for every pair $G_1, G_2 \in \mathcal{G}_{n;k}$:

$$\|\mathcal{U}_{\mathcal{F}}(G_1, \mathbf{N}) - \mathcal{U}_{\mathcal{F}}(G_2, \mathbf{N})\|_2 \leq \sigma_{\mathbf{N}} \times \mathrm{dist}_{\mathrm{P}}(G_1, G_2). \tag{28}$$

Now, observe that the set $\mathcal{T}_{n,n-k}^+$ is isomorphic to $\mathcal{G}_{n,k}$ and therefore is a compact set. The compactness of $\mathcal{T}_{n,n-k}^+$ ensures that $\sigma_{\mathbf{N}}$ attains its maximum over $\mathcal{T}_{n,n-k}^+$ and the following constant exists:

$$\sigma \triangleq \max_{\mathbf{N} \in \mathcal{T}_{n,n-k}^+} \sigma_{\mathbf{N}}. \tag{29}$$

This satisfies the equation (27). ■

Lemma 4: Let $\mathbf{X} \in \mathcal{I}_{\mathcal{F}}$ and suppose that $\mathbf{N} \in \mathcal{S}_{n;n-k}^+$ satisfies the relation $\mathbf{N}\mathbf{X}_0 = \mathbf{0}$. Then, strong duality holds between the SDP relaxation problem (3) and the dual SDP problem (9) with the parameters $(\mathbf{N}, [\langle \mathbf{M}_r, \mathbf{X} \rangle]_{r=1}^m)$.

Proof: In order to prove strong duality, it suffices to construct a strictly feasible point for the dual problem. Let $\tilde{\mathbf{u}} \triangleq \mathcal{U}_{\mathcal{F}}([\mathbf{X}_0]_G, -\mathbf{I}_n)$. According to the definition of $\mathcal{U}_{\mathcal{F}}$, we have:

$$\left(-\mathbf{I}_n + \sum_{r=1}^{m} \hat{u}_r \mathbf{M}_r\right) \mathbf{X}_0 = 0.$$

Therefore, a sufficiently small $\delta > 0$ satisfies

$$\mathbf{N} + \delta \left(\mathbf{I}_n - \mathbf{I}_n + \sum_{r=1}^{m} \hat{u}_r \mathbf{M}_r\right) \succ 0,$$

which concludes that $\delta \times \tilde{\mathbf{u}}$ is strictly feasible for the dual problem and Slater's condition is satisfied. ■

Proof of Theorem 1: Since $\mathbf{N}\mathbf{X}_0 = \mathbf{0}$, complementary slackness yields that $u_r = 0$ for $r = 1, \ldots, m$ and consequently the dual feasibility holds. Therefore, $\mathbf{X}_0$ is the unique solution to the SDP relaxation problem (3). Moreover, $\mathbf{X}_0 \in \mathcal{I}_{\mathcal{F}}$ and $\mathbf{N}\mathbf{X}_0 = \mathbf{0}$ imply that for a $\delta > 0$ there exists an $\varepsilon > 0$ such that for every $\mathbf{X} \in \mathcal{B}_{\mathbf{X}_0;\varepsilon}$, complementary slackness gives unique dual variables $\tilde{\mathbf{u}}$ and $|\tilde{u}_r| < \delta$ for $r = 1, \ldots, m$. In order to complete the proof, it suffices to show that by choosing a sufficiently small $\delta > 0$, the dual feasibility holds, i.e.,

$$\tilde{\mathbf{A}} = \mathbf{N} + \sum_{r=1}^{m} \tilde{u}_r \mathbf{M}_r \succeq 0.$$

Due to the complementary slackness, $\tilde{\mathbf{A}}$ has k zero eigenvalues. From the eigenvalue perturbation analysis and by choosing a sufficiently small $\delta > 0$, the remaining $n-k$ eigenvalues of $\tilde{\mathbf{A}}$ would be close enough to the $n-k$ positive eigenvalues of $\mathbf{N}$. ■

Proof of Theorem 3: According to Theorem 1, for every $\mathbf{X}_0 \in \mathcal{S}_{n;k}^+$, there exist $\mathbf{N} \in \mathcal{S}_{n;n-k}^+$ and $\varepsilon > 0$ such that every member of $\mathcal{B}_{\mathbf{X}_0;\varepsilon}$ is recoverable using $\mathbf{N}$ by the SDP relaxation problem (3). Since recoverability depends on $\mathbf{X} \in \mathcal{S}^+_{n;k}$ only through the equivalence class $[\mathbf{X}]_G$, there exists $\varepsilon_0 > 0$ such that every member of the following set is recoverable as well:

$$\{\mathbf{X} \in \mathcal{S}^+_{n;k} \,|\, \mathrm{dist}_{\mathrm{P}}(\mathbf{X}, \mathbf{X}_0) < \varepsilon_0\}.$$

Now, according to Lemma 2, condition (13) implies that the mapping $\mathcal{F}$ has no singularity. Therefore, we can design infinitely many convex programs so that the union of their recoverable regions covers all of the defined equivalence classes in $\mathcal{G}_{n,k}$. Due to the compactness of $\mathcal{G}_{n,k}$, there exists a finite number of matrices $\mathbf{N}_1, \ldots, \mathbf{N}_q \in \mathbb{S}_n$ such that the union of their recoverable regions covers all members of $\mathcal{G}_{n,k}$ and, thus, the entire set $\mathcal{S}^+_{n;k}$. ■

V. Simulations

In this section, we examine the performance of the proposed convex programs for finding a rank-k positive semidefinite matrix satisfying a set of linear constraints. In the simulations, we generate a random symmetric and PSD nominal matrix $\mathbf{X}_0$ and then find a corresponding matrix $\mathbf{N}$ based on Theorem 1. More precisely, we first produce an $n \times k$ matrix with i.i.d. standard normal distribution entries and denote it by $\mathbf{X}_{\mathrm{D}}$. Then, we build the random matrix $\mathbf{X}_0$ as $\mathbf{X}_{\mathrm{D}}\mathbf{X}_{\mathrm{D}}^{\mathrm{T}}$ that is an element of $\mathcal{S}^+_{n,k}$. In order to assess the recoverable region for the SDP problem (3), random matrices in $\mathcal{S}^+_{n,k}$ are selected and then it is checked whether those matrices belong to the recovery region. To construct such a random matrix, we first generate an $n \times k$ matrix with i.i.d. zero mean normal distribution entries with the variance of σ^2 and denote it by $\mathbf{X}_{\sigma,\mathrm{D}}$. Then, we form the matrix

$$\mathbf{X}_\sigma = (\mathbf{X}_{\mathrm{D}} + \mathbf{X}_{\sigma,\mathrm{D}})(\mathbf{X}_{\mathrm{D}} + \mathbf{X}_{\sigma,\mathrm{D}})^{\mathrm{T}},$$

where the value $\|\mathbf{X}_0 - \mathbf{X}_\sigma\|_{\mathrm{F}}$ can be controlled by changing the variance σ^2. To generate the given specifications, we uniformly sample a random subset of m entries of $\mathbf{X}_\sigma$ from all $(n^2 + n)/2$ entries excluding those below the diagonal (since the matrix is symmetric).

In all figures offered in this section, the x-axis represents the number of sampled entries and the y-axis (that is logarithmic) represents the value of variance. The number of sampled entries or specifications (shown as m) varies from $m_{\min} = nk - k(k-1)/2$ to $m_{\max} = k(2n-k)+1$. Moreover, for each pair (σ, m), we repeat the aforesaid procedure 50 times, where the tested values for σ are different for each of the provided simulations. The color of each point in the figures reflects the recovery rate of the randomly generated matrix $\mathbf{X}_\sigma$ in the 50 runs, which is scaled between 0 and 1 such that white and black colors indicate success and failure, respectively.

Figure 1 shows the result for the recovery of 20×20 matrices with rank of 1 (i.e., $n = 20$ and $k = 1$). In this case, the value of σ changes from 10^{-2} to 1, and the number of given specifications changes from $nk - k(k-1)/2 = 20$ to $k(2n-k)+1 = 40$. Notice that the recovered neighborhood becomes larger by increasing the number of sampled entries. Consider now the case where n is still 20, but the rank k is changed to 2. The results are provided in Figure 2. In this case, the value of σ varies from 10^{-3} to 1, and the number of sampled entries changes from $nk - k(k-1)/2 = 39$ to $k(2n-k)+1 = 77$.

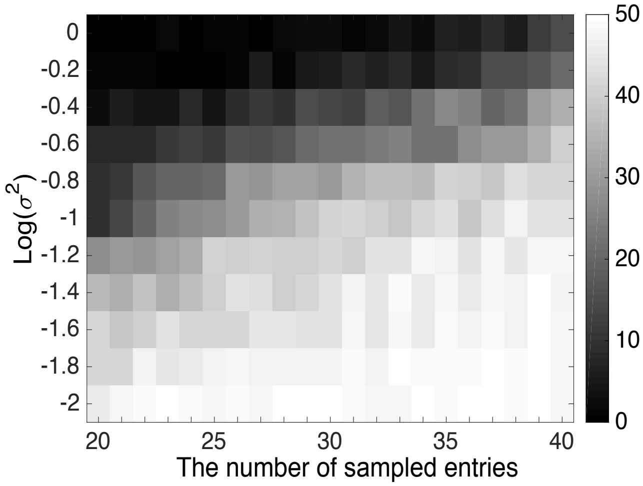

Fig. 1: Recovery of 20×20 matrices with rank of 1.

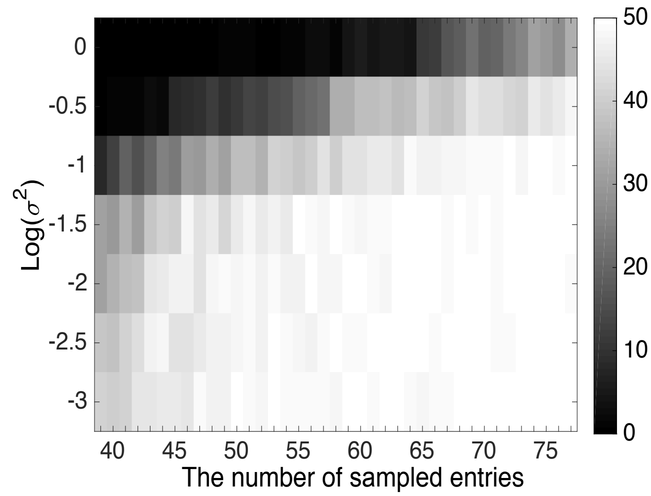

Fig. 2: Recovery of 20×20 matrices with rank of 2.

Figure 3 corresponds to the case where $n = 20$ and $k = 5$. In this scenario, the value of σ changes from 10^{-3} to 1 and the number of sampled entries varies from $nk - k(k-1)/2 = 90$ to $k(2n-k)+1 = 176$. By comparing all of the above plots, it can be observed that as the rank increases, the recoverable region becomes wider.

VI. Conclusions

This paper is concerned with the problem of finding an unknown matrix, given a modest number of linear specifications in the set of $n \times n$ symmetric and positive semidefinite matrices of rank k, denoted by $\mathcal{S}^+_{n;k}$. Based on a given nominal point in $\mathcal{S}^+_{n;k}$ as an initial guess, we design a convex

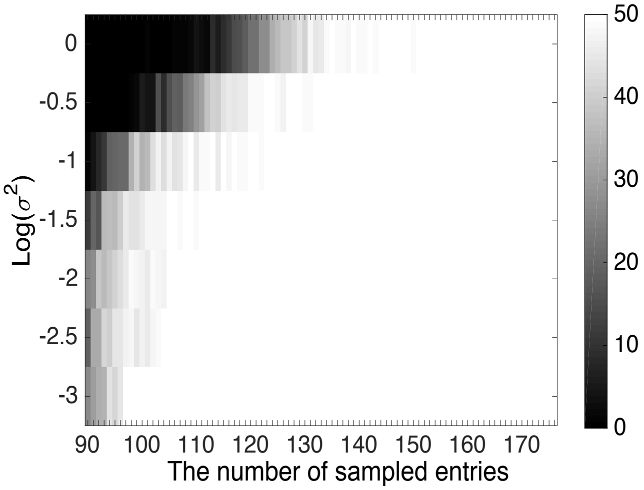

Fig. 3: Recovery of 20×20 matrices with rank of 5.

program in order to recover the unknown solution. Every arbitrary unknown member of $\mathcal{S}^+_{n;k}$ can be found from its specifications if the distance between the nominal point and the unknown matrix is less than a positive constant. We show that if the number of linear specifications is greater than a specific number (in terms of the size and rank of the unknown matrix), then there is a finite number of convex programs such that every unknown rank-constrained matrix can be found via one of these convex problems. The results are demonstrated on many randomly generated systems of equations.

References

[1] M. Fazel, "Matrix rank minimization with applications," Ph.D. dissertation, Stanford University, 2002.

[2] E. J. Candès and B. Recht, "Exact matrix completion via convex optimization," *Foundations of Computational mathematics*, vol. 9, no. 6, pp. 717–772, 2009.

[3] R. Madani, G. Fazelnia, S. Sojoudi, and J. Lavaei, "Low-rank solutions of matrix inequalities with applications to polynomial optimization and matrix completion problems," in *IEEE 53rd Annual Conference on Decision and Control (CDC)*, 2014, pp. 4328–4335.

[4] M. Laurent and A. Varvitsiotis, "A new graph parameter related to bounded rank positive semidefinite matrix completions," *Mathematical Programming*, vol. 145, no. 1-2, pp. 291–325, 2014.

[5] M. Ashraphijuo, R. Madani, and J. Lavaei, "Inverse function theorem for polynomial equations using semidefinite programming," in *IEEE 54th Annual Conference on Decision and Control (CDC)*, 2015, pp. 6589–6596.

[6] L. Vandenberghe and S. Boyd, "Semidefinite programming," *SIAM review*, vol. 38, no. 1, pp. 49–95, 1996.

[7] F. Alizadeh, "Interior point methods in semidefinite programming with applications to combinatorial optimization," *SIAM Journal on Optimization*, vol. 5, no. 1, pp. 13–51, 1995.

[8] F. Alizadeh, J.-P. A. Haeberly, and M. L. Overton, "Complementarity and nondegeneracy in semidefinite programming," *Mathematical Programming*, vol. 77, no. 1, pp. 111–128, 1997.

[9] Z. Li, S. He, and S. Zhang, *Approximation methods for polynomial optimization: Models, Algorithms, and Applications*. Springer Science & Business Media, 2012.

[10] M. X. Goemans and D. P. Williamson, "Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming," *Journal of the ACM (JACM)*, vol. 42, no. 6, pp. 1115–1145, 1995.

[11] S. Sojoudi and J. Lavaei, "Exactness of semidefinite relaxations for nonlinear optimization problems with underlying graph structure," *SIAM Journal on Optimization*, vol. 24, no. 4, pp. 1746–1778, 2014.

[12] Y. Nesterov, "Semidefinite relaxation and nonconvex quadratic optimization," *Optimization methods and software*, vol. 9, no. 1-3, pp. 141–160, 1998.

[13] S. Zhang and Y. Huang, "Complex quadratic optimization and semidefinite programming," *SIAM Journal on Optimization*, vol. 87, pp. 871–890, 2006.

[14] Z.-Q. Luo and S. Zhang, "A semidefinite relaxation scheme for multivariate quartic polynomial optimization with quadratic constraints," *SIAM Journal on Optimization*, vol. 20, no. 4, pp. 1716–1736, 2010.

[15] A. Beck, Y. Drori, and M. Teboulle, "A new semidefinite programming relaxation scheme for a class of quadratic matrix problems," *Operations Research Letters*, vol. 40, no. 4, pp. 298–302, 2012.

[16] B. Jiang, Z. Li, and S. Zhang, "Approximation methods for complex polynomial optimization," *Computational Optimization and Applications*, vol. 59, no. 1-2, pp. 219–248, 2014.

[17] S. He, Z. Li, and S. Zhang, "Approximation algorithms for discrete polynomial optimization," *Journal of the Operations Research Society of China*, vol. 1, no. 1, pp. 3–36, 2013.

[18] S. He, Z. Li, and S. Zhang, "Approximation algorithms for homogeneous polynomial optimization with quadratic constraints," *Mathematical Programming*, vol. 125, no. 2, pp. 353–383, 2010.

[19] Y. Huang, A. De Maio, and S. Zhang, "Semidefinite programming, matrix decomposition, and radar code design," *Convex Optimization in Signal Processing and Communications*, pp. 192–228, 2010.

[20] Y. Huang and S. Zhang, "Approximation algorithms for indefinite complex quadratic maximization problems," *Science China Mathematics*, vol. 53, no. 10, pp. 2697–2708, 2010.

[21] S. He, Z.-Q. Luo, J. Nie, and S. Zhang, "Semidefinite relaxation bounds for indefinite homogeneous quadratic optimization," *SIAM Journal on Optimization*, vol. 19, no. 2, pp. 503–523, 2008.

[22] Z.-Q. Luo, N. D. Sidiropoulos, P. Tseng, and S. Zhang, "Approximation bounds for quadratic optimization with homogeneous quadratic constraints," *SIAM Journal on optimization*, vol. 18, no. 1, pp. 1–28, 2007.

[23] R. Madani, M. Ashraphijuo, and J. Lavaei, "Promises of conic relaxation for contingency-constrained optimal power flow problem," *IEEE Transactions on Power Systems*, vol. 31, no. 2, pp. 1297–1307, March 2016.

[24] P. A. Parrilo, "Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization," Ph.D. dissertation, 2000.

[25] A. Beck and M. Teboulle, "Global optimality conditions for quadratic optimization problems with binary constraints," *SIAM Journal on Optimization*, vol. 11, no. 1, pp. 179–188, 2000.

[26] S. Prajna, A. Papachristodoulou, P. Seiler, and P. A. Parrilo, "New developments in sum of squares optimization and SOSTOOLS," in *Proceedings of the American Control Conference*, 2004, pp. 5606–5611.

[27] J. B. Lasserre, "Global optimization with polynomials and the problem of moments," *SIAM Journal on Optimization*, vol. 11, no. 3, pp. 796–817, 2001.

[28] N. I. Akhiezer and I. M. Glazman, *Theory of linear operators in Hilbert space*. Courier Corporation, 2013.