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Characterization Of sampling patterns for low-tt-rank tensor retrieval

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Abstract

In this paper, we analyze the fundamental conditions for low-rank tensor completion given the separation or tensor-train (TT) rank, i.e., ranks of TT unfoldings. We exploit the algebraic structure of the TT decomposition to obtain the deterministic necessary and sufficient conditions on the locations of the samples to ensure finite completability. Specifically, we propose an algebraic geometric analysis on the TT manifold that can incorporate the whole rank vector simultaneously in contrast to the existing approach based on the Grassmannian manifold that can only incorporate one rank component. Our proposed technique characterizes the algebraic independence of a set of polynomials defined based on the sampling pattern and the TT decomposition, which is instrumental to obtaining the deterministic condition on the sampling pattern for finite completability. In addition, based on the proposed analysis, assuming that the entries of the tensor are sampled independently with probability p, we derive a lower bound on the sampling probability p, or equivalently, the number of sampled entries that ensures finite completability with high probability. Moreover, we also provide the deterministic and probabilistic conditions for unique completability.

Keywords Data completion · Tensor retrieval · Low-rank tensor completion · Tensor-train decomposition · Finite completability · Unique completability

Mathematics Subject Classification (2010) 68W01

1 Introduction

Most of the literature on low-rank data completion (either matrix or tensor) propose optimization-based algorithms to construct a completion that matches the given samples and rank. For example, for the two-way tensor, i.e., matrix, many algorithms have been proposed that are based on convex relaxation of rank [3, 9–12] or alternating minimization

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[17, 21]. Similarly, for higher dimensional data a number of tensor completion algorithms exist that are based on different convex relaxations of the tensor ranks [16, 39, 42, 46] or other heuristics [5, 18, 25, 26, 29, 30, 48]. The low-rank tensor completion problem has various applications, including compressed sensing [16, 27, 41], visual data reconstruction [28, 30], seismic data processing [15, 24, 48], RF fingerprinting [29, 31], etc.

Existing works on optimization-based matrix or tensor completion usually make a set of strong assumptions on the correlations of the values of either the sampled or non-sampled entries (such as coherence) in order to provide a tensor that approximately fits in the sampled tensor. In contrast, here we are interested in investigating fundamental conditions on the sampling pattern that guarantee the existence of finite or unique number of completions. Such conditions are "fundamental" in the sense that they are independent of either the optimization formulation or the optimization algorithm used to compute the completion. The matrix version of this problem has been treated in [37]. Also, the noisy scenario and the tensor version of this problem have been treated in [1] and [4], respectively. In this paper, we investigate this problem for tensors under the tensor-train (TT) rank.

There are a number of tensor decompositions available, including Tucker decomposition or higher-order singular value decomposition [19, 23, 25], polyadic decomposition [43, 45], tubal rank decomposition [22] and several other representations [13, 14, 36]. TT decomposition (also known as tree-tensor decomposition) was proposed in the field of quantum physics about 20 years ago [8, 40]. Later it was used in the area of tensor analysis [33–35] and there are several works on the problem of tensor completion in the TT format [38, 49]. A comprehensive survey on TT decomposition and the manifold of tensors of fixed TT rank can be found in [47] and [20] also includes a comparison between the TT and Tucker decompositions for a better understanding of the advantages of TT decomposition. The well-known TT-cross method proposed in [32] provides the condition for the existence of a tensor with given TT-rank that is within a certain distance from the original tensor, as well as an algorithm to find such a low-TT-rank approximation.

Let $\mathcal{U}$ denote the sampled tensor and Ω denote the binary sampling pattern tensor that is of the same dimension and size as $\mathcal{U}$. The entries of Ω that correspond to the observed entries of $\mathcal{U}$ are equal to 1 and the rest of the entries are set as 0. This paper is mainly concerned with the following three problems.

Problem (i) Given the TT rank, characterize the necessary and sufficient conditions on the sampling pattern Ω, under which $\mathcal{U}$ admits only a finite number of completions.

We define a polynomial for each sampled entry such that the variables of the polynomial are the entries of the core tensors of the TT decomposition. Then, we propose a geometric method on the TT manifold to obtain the maximum number of algebraically independent polynomials (among all the defined polynomials for any of the sampled entries) in terms of the geometric structure of the sampling pattern Ω. Finally, we show that if the maximum number of algebraically independent polynomials meets a threshold, which depends on the structure of the sampling pattern Ω, the sampled tensor $\mathcal{U}$ is finitely completable. We emphasize the fact that the proposed algebraic geometry analysis on the TT manifold is not a simple generalization of the existing analysis on the Grassmannian [37] or Tucker manifold [2] as almost every step needs to be developed anew.

Problem (ii) Given the TT rank, characterize a sufficient conditions on the sampling pattern Ω, under which there exists only one completion of $\mathcal{U}$.

We use the developed tools for solving Problem (i) and in addition to the condition for finite completability, we add more polynomials (samples) in a way such that the corresponding minimally algebraically dependent set of polynomials leads to that all involved variables can be determined uniquely.

Problem (iii) Provide a lower bound on the total number of sampled entries such that the proposed conditions on the sampling pattern Ω for finite and unique completability are satisfied with high probability.

Assuming that the entries of $\mathcal{U}$ are sampled independently with probability p, we develop lower bounds on p such that the deterministic conditions for Problems (i) and (ii) are met with high probability.

The remainder of this paper is organized as follows. In Section 2, the preliminaries and problem statement are presented. Problems (i), (ii) and (iii) are treated in Sections 3, 4 and 5, respectively. Some numerical results are provided in Section 6. Finally, Section 7 concludes the paper.

2 Background

2.1 Preliminaries and notations

In this paper, it is assumed that a d-way tensor $\mathcal{U} \in \mathbb{R}^{n_1 \times \cdots \times n_d}$ is sampled. For the sake of simplicity in notation, define $N_i \triangleq \left(\Pi_{j=1}^{i} n_j\right)$ and $\bar{N}_i \triangleq \left(\Pi_{j=i+1}^{d} n_j\right)$. Also, for any real number x, define $x^+ \triangleq \max\{0, x\}$.

Define the matrix $\widetilde{\mathbf{U}}_{(i)} \in \mathbb{R}^{N_i \times \bar{N}_i}$ as the i-th TT unfolding of the tensor $\mathcal{U}$, such that $\mathcal{U}(\mathbf{x}) = \widetilde{\mathbf{U}}_{(i)}(\widetilde{M}_i(x_1, \dots, x_i), \widetilde{M}_{-i}(x_{i+1}, \dots, x_d))$, where $\widetilde{M}_i : (x_1, \dots, x_i) \rightarrow \{1, 2, \dots, N_i\}$ and $\widetilde{M}_{-i} : (x_{i+1}, \dots, x_d) \rightarrow \{1, 2, \dots, \bar{N}_i\}$ are two bijective mappings and $\mathcal{U}(\mathbf{x})$ represents an entry of tensor $\mathcal{U}$ with coordinate $\mathbf{x} = (x_1, \dots, x_d)$.

The separation or tensor-train (TT) rank of a tensor is defined as $\text{rank}(\mathcal{U}) = (r_1, \dots, r_{d-1})$ where $r_i = \text{rank}(\widetilde{\mathbf{U}}_{(i)})$, $i = 1, \dots, d-1$. Note that $r_i \leq \max\{N_i, \bar{N}_i\}$ in general and also r_1 is simply the conventional matrix rank when $d = 2$. The TT decomposition of a tensor $\mathcal{U}$ with TT-rank of $\text{rank}(\mathcal{U}) = (r_1, \dots, r_{d-1})$ consists of d "core tensors" $\mathcal{U}^{(i)} \in \mathbb{R}^{r_{i-1} \times n_i \times r_i}$ for $i = 2, \dots, d-1$, $\mathcal{U}^{(1)} \in \mathbb{R}^{n_1 \times r_1}$, and $\mathcal{U}^{(d)} \in \mathbb{R}^{r_{d-1} \times n_d}$ and for each entry of the tensor we have

$$\mathcal{U}(\mathbf{x}) = \sum_{k_1=1}^{r_1} \cdots \sum_{k_{d-1}=1}^{r_{d-1}} \mathcal{U}^{(1)}(x_1, k_1) \left(\prod_{i=2}^{d-1} \mathcal{U}^{(i)}(k_{i-1}, x_i, k_i) \right) \mathcal{U}^{(d)}(k_{d-1}, x_d). \tag{1}$$

For notational simplicity, we denote $\mathbb{U} = (\mathcal{U}^{(1)}, \dots, \mathcal{U}^{(d)})$. Given the order d and dimension sizes $n_1, \dots, n_d$, the space of all tensors of fixed TT rank vector $r = (r_1, \dots, r_{d-1})$ is a embedded manifold of dimension [20]

$$\sum_{i=1}^{d} r_{i-1} n_i r_i - \sum_{i=1}^{d-1} r_i^2, \tag{2}$$

where $r_0 = r_d \triangleq 1$. As $\mathcal{U}^{(1)}$ and $\mathcal{U}^{(d)}$ are two-way tensors, we can also denote them by $\mathbf{U}^{(1)}$ and $\mathbf{U}^{(d)}$ in this paper.

Denote Ω as the binary sampling pattern tensor that is of the same size as $\mathcal{U}$ and $\Omega(\mathbf{x}) = 1$ if $\mathcal{U}(\mathbf{x})$ is observed and $\Omega(\mathbf{x}) = 0$ otherwise. $\mathbf{X}(1:m,:)$ denotes the first m rows of the matrix $\mathbf{X}$ and $\mathbf{X}^\top$ denotes the transpose of $\mathbf{X}$.

Let $\mathbf{U}_{(i)}$ be the i-th Tucker unfolding of the tensor $\mathcal{U}$, i.e., the matrix $\mathbf{U}_{(i)}$ has n_i rows and $\frac{N_d}{n_i}$ columns such that $\mathcal{U}(\mathbf{x}) = \mathbf{U}_{(i)}(x_i, M_i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_d))$, where $M_i : (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_d) \rightarrow \{1,2,\ldots,\frac{N_d}{n_i}\}$ is a bijective mapping. Observe that for any arbitrary tensor $\mathcal{A}$, the first Tucker unfolding and the first TT unfolding are the same, i.e., $\mathbf{A}_{(1)} = \widetilde{\mathbf{A}}_{(1)}$.

2.2 Summary of results and main steps of analysis

An executive summary of the steps in our finite completability analysis is as follows:

(i) Deterministic analysis (Section 3) - Necessary and sufficient conditions on the sampling pattern Ω:

Tensor Completion $\Longleftrightarrow$ Solving polynomial equations in terms of the entries of TT core tensors $\mathcal{U}^{(1)},\ldots,\mathcal{U}^{(d)}$.
(1) Finite completability of $\mathcal{U}^{(d)}$ given $\mathcal{U}^{(1)},\ldots,\mathcal{U}^{(d-1)}$ (with $n_d r_{d-1}$ entries) $\Longleftrightarrow$ Assumption 1. (2) Finite completability of $\mathcal{U}^{(1)},\ldots,\mathcal{U}^{(d-1)}$ $\Longleftrightarrow$ Existence of $\sum_{i=1}^{d-1} r_{i-1}n_i r_i - \sum_{i=1}^{d-1} r_i^2$ algebraically independent polynomials in $\mathcal{P}(\Omega)$ (Lemma 1) $\Longleftrightarrow$ Relationship between Ω (via the constraint tensor $\breve{\Omega}$) and the maximum number of algebraically independent polynomials (Lemma 4).
(1) and (2) $\Longleftrightarrow$ Geometric patterns for finite completability of $\mathcal{U}$ (Theorem 1).

(ii) Probabilistic analysis (Section 4) - Minimum uniform sampling rate p to ensure finite completability:

(1) Connecting conditions on the number of samples and geometry of Ω (Lemma 7)
(2) Connecting geometries of Ω and $\breve{\Omega}$ (Lemma 8).
(1) and (2) and Theorem 1 (geometry of $\breve{\Omega}$) $\Longrightarrow$ Probabilistic condition (Theorem 3).

Moreover, in Section 5, for unique completability, we obtain a deterministic sufficient condition in Theorem 4, which has one additional constraint in comparison with Theorem 1 (i.e., (33)). And we provide the corresponding probabilistic condition in Theorem 5 which requires slightly more samples in comparison with Theorem 3.

3 Deterministic conditions for finite completability

This section characterizes the connection between the sampling pattern and the number of solutions of a low-rank tensor completion problem. In Section 3.1, we define a polynomial based on each observed entry. Then, given the rank vector, we transform the problem of finite completability of $\mathcal{U}$ into the problem of including enough number of algebraically independent polynomials among the polynomials defined for the observed entries. In Section 3.2, we construct a constraint tensor based on the sampling pattern Ω.

This tensor is useful for analyzing the algebraic independency of a subset of polynomials among all defined polynomials. In Section 3.3, we show the relationship between the number of algebraically independent polynomials in the mentioned set of polynomials and finite completability of the sampled tensor.

Note that as we showed in [2], for matrix and tensor completion problems, finite completability does not necessarily imply unique completability. In particular, we found an example such that there are exactly two completions of the sampled data of the given rank.

3.1 Geometry of TT manifold

Define $\mathbb{P}_1$ as the Lebesgue measure on $\mathbb{R}^{n_1 \times r_1}$, $\mathbb{P}_2$ as the Lebesgue measure on $\mathbb{R}^{r_1 \times n_2 \times r_2}$, ... and $\mathbb{P}_d$ as the Lebesgue measure on $\mathbb{R}^{r_{d-1} \times n_d}$. We assume that $\mathcal{U}$ is chosen generically from the manifold corresponding to $(r_1, \ldots, r_{d-1})$, or in other words, the entries of $\mathcal{U}$ are drawn independently with respect to Lebesgue measure on the corresponding manifold. Hence, any statement that holds for this "generic" $\mathcal{U}$, it holds with probability one with respect to the product measure $\mathbb{P}_1 \times \mathbb{P}_2 \times \cdots \times \mathbb{P}_d$.

Here, we briefly mention two facts. Recall that $r_0 = r_d = 1$.

- **Fact 1**: As it can be seen from (1), any observed entry $\mathcal{U}(\mathbf{x})$ results in an equation that involves $r_{i-1}r_i$ entries of $\mathcal{U}^{(i)}$, $i = 1, \ldots, d$. Considering the entries of $\mathbb{U}$ as variables (right-hand side of (1)), each observed entry results in a polynomial in terms of these variables. We can further visualize this fact by looking at the following equation

$$\mathcal{U}(\mathbf{x}) = \sum_{k_1=1}^{r_1} \cdots \sum_{k_{d-1}=1}^{r_{d-1}} \left(\mathcal{U}^{(1)}(x_1, k_1) \ldots \mathcal{U}^{(i)}(k_{i-2}, x_{i-1}, k_{i-1})\right)$$
$$\mathcal{U}^{(i)}(k_{i-1}, x_i, k_i) \left(\mathcal{U}^{(i)}(k_i, x_{i+1}, k_{i+1}) \ldots \mathcal{U}^{(d)}(k_{d-1}, x_d)\right), \tag{3}$$

  where all entries $\mathcal{U}^{(i)}(k_{i-1}, x_i, k_i)$ are involved for $1 \leq k_{i-1} \leq r_{i-1}$ and $1 \leq k_i \leq r_i$.
- **Fact 2**: As it can be seen from (1), for any observed entry $\mathcal{U}(\mathbf{x})$, the value of x_i specifies the "slice" of $\mathcal{U}^{(i)}$ or in other words the location of the $r_{i-1}r_i$ entries of $\mathcal{U}^{(i)}$ that are involved in the corresponding polynomial, $i = 1, \ldots, d$. In other words, the value of x_i specifies the row number of the second Tucker unfolding of $\mathcal{U}^{(i)}$ whose $r_{i-1}r_i$ entries are involved in the corresponding polynomial, $i = 2, \ldots, d$. Note that the value of x_1 specifies the row number of the first Tucker unfolding of $\mathcal{U}^{(1)}$ whose r_1 entries are involved in the corresponding polynomial.

Note that it can be concluded from Bernstein's theorem [44] (also used in [37] and [4]) that in a system of n polynomials in n variables with each consisting of a given set of monomials such that the coefficients are chosen generically (i.e., according to the Lebesgue measure on TT-manifold), the n polynomials are algebraically independent with probability one with respect to the corresponding product measure, and therefore there exist only finitely many solutions (all given probabilities in this paper are with respect to this product measure). However, in the structure of the polynomials in our model, the set of involved monomials are different for different set of polynomials, and therefore to ensure algebraically independency we need to have for any selected subset of the original n polynomials, the number of involved variables should be more than the number of selected polynomials.

Given all observed entries $\{\mathcal{U}(\mathbf{x}) : \Omega(\mathbf{x}) = 1\}$, we are interested in finding the number of possible solutions in terms of entries of $\mathbb{U}$ (infinite or finite) via investigating the algebraic independence among these polynomials. Note that in this paper the dimension of the TT manifold is a lower bound on the number of samples, otherwise there are infinitely many completions with probability one.

Assumption 1 *Each column of $\widetilde{\mathbf{U}}_{(d-1)}$ includes at least r_{d-1} observed entries.*

Remark 1 Note that given $\mathcal{U}^{(1)}, \ldots, \mathcal{U}^{(d-1)}$, polynomials in (1) are degree-1 in terms of the entries of $\mathcal{U}^{(d)}$. Hence, Assumption 1 results in $n_d r_{d-1}$ degree-1 polynomials in terms of the entries of $\mathcal{U}^{(d)}$. As a result, the entries of $\mathcal{U}^{(d)}$ can be determined uniquely in terms of the entries of $(\mathcal{U}^{(1)}, \ldots, \mathcal{U}^{(d-1)})$.

Remark 2 It is easily verified that as a result of Remark 1, Assumption 1 is necessary for finite completability and therefore, we need Assumption 1 in the rest of this paper.

Definition 1 Let $x_1^{l,v}, x_2^{l,v}, \ldots, x_d^{l,v}$ for $1 \leq l \leq n_d$ and $1 \leq v \leq r_{d-1}$ denote the coordinates of the r_{d-1} sampled entries belonging to the l-th slice, i.e., the l-th column of $\widetilde{\mathbf{U}}_{(d-1)}$, that we consider for Assumption 1.

Definition 2 Let $\mathcal{P}(\Omega)$ denote the set of polynomials corresponding to the observed entries as in (1) excluding the $n_d r_{d-1}$ observed entries of Assumption 1. Note that since $\mathcal{U}^{(d)}$ is already solved in terms of $(\mathcal{U}^{(1)}, \ldots, \mathcal{U}^{(d-1)})$, each polynomial in $\mathcal{P}(\Omega)$ is in terms of elements of $(\mathcal{U}^{(1)}, \ldots, \mathcal{U}^{(d-1)})$.

The following lemma provides the necessary and sufficient condition on $\mathcal{P}(\Omega)$ for finite completability of the sampled tensor $\mathcal{U}$.

Lemma 1 *Suppose that Assumption 1 holds. With probability one, there exist only finitely many completions of $\mathcal{U}$ if and only if there exist $m = \sum_{i=1}^{d-1} r_{i-1} n_i r_i - \sum_{i=1}^{d-1} r_i^2$ algebraically independent polynomials in $\mathcal{P}(\Omega)$.*

Proof Given $\mathcal{U}^{(1)}, \ldots, \mathcal{U}^{(d-1)}$ of the TT decomposition of the sampled tensor $\mathcal{U}$, assumption 1 results that there exists only one possible $\mathcal{U}^{(d)}$. Let $\mathcal{P}(\Omega) = \{p_1, \ldots, p_m\}$ and define $\mathcal{S}_i$ as the set of all tuples $(\mathcal{U}^{(1)}, \ldots, \mathcal{U}^{(d-1)})$ that satisfy polynomial restrictions $\{p_1, \ldots, p_i\}$, $i = 0, \ldots, m$ ($\mathcal{S}_0$ is the set of all tuples $(\mathcal{U}^{(1)}, \ldots, \mathcal{U}^{(d-1)})$ without any polynomial restriction). Note that $\dim(\mathcal{S}_0) = \sum_{i=1}^{d-1} r_{i-1} n_i r_i - \sum_{i=1}^{d-1} r_i^2$ as we assume $\mathcal{U}^{(d)}$ is given [20].

Observe that each algebraically independent polynomial reduces the dimension (degree of freedom) of the set of solutions by one. In other words, $\dim(\mathcal{S}_i) = \dim(\mathcal{S}_{i-1})$ if the maximum number of algebraically independent polynomials in sets $\{p_1, \ldots, p_i\}$ and $\{p_1, \ldots, p_{i-1}\}$ are the same and $\dim(\mathcal{S}_i) = \dim(\mathcal{S}_{i-1}) - 1$ otherwise. Moreover, with probability one, $|\mathcal{S}_m|$ is finite if and only if there are $\dim(\mathcal{S}_0)$ algebraically independent polynomial restrictions on the entries of $\mathcal{U}^{(1)}, \ldots, \mathcal{U}^{(d-1)}$, i.e., $|\mathcal{S}_m|$ is finite if and only if $\dim(\mathcal{S}_m) = 0$. Hence, there are finitely many completions of the sampled tensor $\mathcal{U}$ if and only if there exist $\sum_{i=1}^{d-1} r_{i-1} n_i r_i - \sum_{i=1}^{d-1} r_i^2$ algebraically independent polynomials in $\mathcal{P}(\Omega)$. $\square$

3.2 Constraint tensor

In order to ensure that Lemma 1 holds, we need $\sum_{i=1}^{d-1} r_{i-1} n_i r_i - \sum_{i=1}^{d-1} r_i^2$ algebraically independent polynomials in $\mathcal{P}(\Omega)$, i.e., among all polynomials excluding the polynomials corresponding to the sampled entries of Assumption 1. Then, in order to count the number of algebraically independent polynomials, we construct the "constraint tensor" $\breve{\Omega}$ based on the sampling pattern tensor Ω such that each $(d-1)$-dimensional slice of $\breve{\Omega}$ includes $r_{d-1}+1$ entries equal to one (i.e., $r_{d-1}+1$ non-zero entries). Moreover, these $r_{d-1}+1$ non-zero entries are located such that one of them represents the location of one of the sampled entries corresponding to a polynomial in $\mathcal{P}(\Omega)$, and the other r_{d-1} non-zero entries represent the location of the r_{d-1} sampled entries in the same column of $\widetilde{\mathbf{U}}_{(d-1)}$ (from Assumption 1) that have been used for obtaining $\mathcal{U}^{(d)}$. This structure helps us to characterize the number of involved variables in a subset of polynomials based on the geometry of non-zero entries of $\breve{\Omega}$, and we count the number of algebraically independent polynomials based on the number of involved variables in the corresponding polynomials. Then, we characterize the geometrical pattern on $\breve{\Omega}$ that is equivalent with existence of $\sum_{i=1}^{d-1} r_{i-1}n_ir_i - \sum_{i=1}^{d-1} r_i^2$ algebraically independent polynomials (excluding the polynomials corresponding to the sampled entries of Assumption 1).

In the following, we provide a procedure to construct a binary tensor $\breve{\Omega}$ based on Ω such that $\mathcal{P}(\breve{\Omega}) = \mathcal{P}(\Omega)$ and each polynomial can be represented by one d-way slice of $\breve{\Omega}$ that belongs to $\mathbb{R}^{n_1\times\cdots\times n_{d-1}\times 1}$. Using $\breve{\Omega}$, we are able to recognize the observed entries that have been used to obtain the $\mathcal{U}^{(d)}$ in terms of the entries of $\mathcal{U}^{(1)},\ldots,\mathcal{U}^{(d-1)}$, and we can easily verify if two polynomials in $\mathcal{P}(\Omega)$ are in terms of the same set of variables. Then, in Section 3.3, we characterize the relationship between the maximum number of algebraically independent polynomials in $\mathcal{P}(\breve{\Omega})$ and $\breve{\Omega}$.

For each slice $\mathcal{Y}$ of the sampled tensor $\mathcal{U}$, let $N_\Omega(\mathcal{Y})$ denote the number of sampled entries in $\mathcal{Y}$. Specifically, consider any slice $\mathcal{Y}\in\mathbb{R}^{n_1\times n_2\times\cdots\times n_{d-1}\times 1}$ of the tensor $\mathcal{U}$. Then, $\mathcal{Y}$ contributes $N_\Omega(\mathcal{Y}) - r_{d-1}$ polynomial equations in terms of the entries of $\mathcal{U}^{(1)},\ldots,\mathcal{U}^{(d-1)}$ among all $N_\Omega(\mathcal{U}) - r_{d-1}n_d$ polynomials in $\mathcal{P}(\Omega)$.

Note that the sampled tensor $\mathcal{U}$ includes n_d slices, each belonging to $\mathbb{R}^{n_1\times n_2\times\cdots\times n_{d-1}\times 1}$. Let $\mathcal{Y}_i$ denote these n_d slices for $1\le i\le n_d$. Define a binary valued tensor $\breve{\mathcal{Y}}_i\in\mathbb{R}^{n_1\times n_2\times\cdots\times n_{d-1}\times k_i}$, where $k_i = N_\Omega(\mathcal{Y}_i) - r_{d-1}$ and its entries are described as follows. We can look at $\breve{\mathcal{Y}}_i$ as k_i slices each belongs to $\mathbb{R}^{n_1\times n_2\times\cdots\times n_{d-1}\times 1}$. For each of the mentioned k_i slices in $\breve{\mathcal{Y}}_i$, we set the entries corresponding to the r_{d-1} observed entries that are used to obtain $\mathcal{U}^{(d)}$ equal to 1. In other words, for each of the mentioned k_i slices in $\breve{\mathcal{Y}}_i$, we set the r_{d-1} entries with coordinates $(x_1^{i,l}, x_2^{i,l},\ldots,x_{d-1}^{i,l},1)$ for $1\le l\le r_{d-1}$ equal to 1. For each of the k_i observed entries (those remained after excluding the r_{d-1} sampled entries of $\breve{\mathcal{Y}}_i$), we pick one of the k_i tensors of $\breve{\mathcal{Y}}_i$ and set its corresponding entry (the same location as that specific observed entry) equal to 1 and set the rest of the entries equal to 0. In the case that $k_i=0$ we simply ignore $\breve{\mathcal{Y}}_i$, i.e., $\breve{\mathcal{Y}}_i=\emptyset$.

By putting together all n_d tensors in dimension d, we construct a binary valued tensor $\breve{\Omega}\in\mathbb{R}^{n_1\times n_2\times\cdots\times n_{d-1}\times K}$, where $K=\sum_{i=1}^{n_d} k_i = N_\Omega(\mathcal{U}) - r_{d-1}n_d$ and call it the **constraint tensor**. Observe that each slice of the constraint tensor that belongs to $\mathbb{R}^{n_1\times n_2\times\cdots\times n_{d-1}\times 1}$, i.e., each column of the $(d-1)$-th TT unfolding of $\breve{\Omega}$, includes exactly $r_{d-1}+1$ nonzero entries, where r_{d-1} of them correspond to the r_{d-1} observed entries that are used to obtain $\mathcal{U}^{(d)}$ and the other one corresponds to one of the other k_i observed entries.

Note that each slice of $\breve{\Omega}$ that belongs to $\mathbb{R}^{n_1\times\cdots\times n_{d-1}\times 1}$ represents one of the polynomials in $\mathcal{P}(\Omega)$ besides showing the polynomials that have been used to obtain $\mathcal{U}^{(d)}$. More specifically, consider a slice of $\breve{\Omega}$ that belongs to $\mathbb{R}^{n_1\times\cdots\times n_{d-1}\times 1}$ with $r_{d-1}+1$ nonzero entries. As we can observe from Facts 1 and 2 exactly r_{d-1} of them correspond to the observed entries that have been used to obtain the corresponding column of $\mathcal{U}^{(d)}$. As a result, the r_{d-1} entries of each column of $\mathcal{U}^{(d)}$ can be obtained in terms of the entries of

$\mathcal{U}^{(1)}, \ldots, \mathcal{U}^{(d-1)}$ using the corresponding r_{d-1} sampled entries. Hence, this slice represents a polynomial after replacing entries of $\mathcal{U}^{(d)}$ by the expressions in terms of entries of $\mathcal{U}^{(1)}, \ldots, \mathcal{U}^{(d-1)}$, i.e., a polynomial in $\mathcal{P}(\Omega)$.

The structure of the constraint tensor in this paper is similar the one in [4]. Note that the purpose of introducing the constraint tensor in this paper is being able to characterize the geometric pattern provided in Theorem 1. The constraint tensor is helpful to consider all the dependencies among polynomials and provide a relatively simple pattern as provided in Theorem 1. Due to the notational and fundamental differences between this paper and [4] we reintroduced the constraint tensor instead of referring to [4].

3.3 Algebraic independence

In Lemma 1, we obtained the required number of algebraically independent polynomials in $\mathcal{P}(\Omega)$ for finite completability, and therefore we can certify finite completability based on the maximum number of algebraically independent polynomials in $\mathcal{P}(\Omega) = \mathcal{P}(\breve{\Omega})$. In this subsection, a sampling pattern on the constraint tensor is proposed to obtain the maximum number of algebraically independent polynomials in $\mathcal{P}(\breve{\Omega})$ based on the structure of the nonzero entries of $\breve{\Omega}$.

Definition 3 Let $\breve{\Omega}' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t}$ be a subtensor of the constraint tensor $\breve{\Omega}$. Let $m_i(\breve{\Omega}')$ denote the number of nonzero rows of $\breve{\Omega}'_{(i)}$. Also, let $\mathcal{P}(\breve{\Omega}')$ denote the set of polynomials that correspond to nonzero entries of $\breve{\Omega}'$.

Recall Facts 1 and 2 regarding the number of involved entries of core tensors of the TT decomposition in a set of polynomials. According to Lemma 17, some of the entries of $\mathcal{U}^{(i)}$'s are known, i.e., $(\mathbf{P}_1, \ldots, \mathbf{P}_{d-1})$ that satisfy properties (i) and (ii) in Definition 4. Therefore, in order to find the number of variables (unknown entries of $\mathcal{U}^{(i)}$'s) in a set of polynomials, we should subtract the number of known entries in the corresponding pattern from the total number of involved entries.

For any subtensor $\breve{\Omega}' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t}$ of the constraint tensor, the next lemma gives an upper bound on the number of algebraically independent polynomials in the set $\mathcal{P}(\breve{\Omega}')$. Recall that $\mathcal{P}(\breve{\Omega}')$ includes exactly t polynomials.

Lemma 2 *Suppose that Assumption 1 holds. For any subtensor $\breve{\Omega}' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t}$ of the constraint tensor, the maximum number of algebraically independent polynomials in $\mathcal{P}(\breve{\Omega}')$ is upper bounded by*

$$\sum_{i=1}^{d-1} \left(r_{i-1} r_i m_i(\breve{\Omega}') - r_i^2 \right)^+ . \tag{4}$$

Proof Recall Fact 1 which states that any of the t polynomials in $\mathcal{P}(\breve{\Omega}')$ involves exactly $r_{i-1}r_i$ entries of $\mathcal{U}^{(i)}$, $i = 1, 2, \ldots, d$. Moreover, we use Fact 2 in order to find the number of entries of tuple $(\mathcal{U}^{(1)}, \ldots, \mathcal{U}^{(d-1)})$ that are involved in at least one of the polynomials in $\mathcal{P}(\breve{\Omega}')$. Note that according to Fact 2, if an entry $(x_1, \ldots, x_d)$ is observed such that $x_i = l$, then all $r_{i-1}r_i$ entries of the l-th row of the second (first) Tucker unfolding of $\mathcal{U}^{(i)}$ are involved in the polynomial corresponding to this observed entry, $i = 2, \ldots, d-1$

$(i = 1)$. Hence, it is easily verified that the total number of involved entries of the tuple $(\mathcal{U}^{(1)}, \ldots, \mathcal{U}^{(d-1)})$ in the t polynomials in $\mathcal{P}(\breve{\Omega}')$ is $\sum_{i=1}^{d-1} r_{i-1} r_i m_i(\breve{\Omega}')$.

On the other hand, among the $\sum_{i=1}^{d-1} r_{i-1} r_i m_i(\breve{\Omega}')$ known entries corresponding to $(\mathbf{P}_1, \ldots, \mathbf{P}_{d-1})$ in TT decomposition, some of them are involved in polynomials of $\mathcal{P}(\breve{\Omega}')$. Among all TT decompositions, consider the one that has the maximum number of known entries (resulted from the degree of freedom of TT decomposition) that are involved in the polynomials in $\mathcal{P}(\breve{\Omega}')$. For $\mathcal{U}^{(i)}$, this number (the maximum number of known entries) is $\min\{r_i^2, r_{i-1} r_i m_i(\breve{\Omega}')\}$. Hence, the number of variables that are involved in the set of polynomials $\mathcal{P}(\breve{\Omega}')$ is $\sum_{i=1}^{d-1} \left(r_{i-1} r_i m_i(\breve{\Omega}') - r_i^2\right)^+$. The proof is complete since the number of algebraically independent polynomials in a subset of polynomials of $\mathcal{P}(\breve{\Omega}')$ is at most equal to the total number of variables that are involved in the corresponding polynomials. □

We are interested in obtaining the maximum number of algebraically independent polynomials in $\mathcal{P}(\breve{\Omega}')$ as Lemma 2 only provides an upper bound. A subset of polynomials $\mathcal{P}(\breve{\Omega}')$ is minimally algebraically dependent if the polynomials in $\mathcal{P}(\breve{\Omega}')$ are algebraically dependent but polynomials in every of its proper subset are algebraically independent. The next lemma is Lemma 7 in [4] and will be used to determine if the polynomials in the set $\mathcal{P}(\breve{\Omega}')$ are algebraically dependent.

Lemma 3 *Suppose that Assumption 1 holds. Suppose that $\breve{\Omega}' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t}$ is a subtensor of the constraint tensor such that $\mathcal{P}(\breve{\Omega}')$ is minimally algebraically dependent. Then, with probability one, the number of variables that are involved in the set of polynomials $\mathcal{P}(\breve{\Omega}')$ is $t - 1$.*

The next lemma characterizes a relationship between the number of algebraically independent polynomials in $\mathcal{P}(\breve{\Omega})$ and the structure of the nonzero entries of $\breve{\Omega}$.

Lemma 4 *Suppose that Assumption 1 holds and consider a subtensor $\breve{\Omega}' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t}$ of the constraint tensor $\breve{\Omega}$. The polynomials in the set $\mathcal{P}(\breve{\Omega}')$ are algebraically dependent if and only if $\sum_{i=1}^{d-1} \left(r_{i-1} r_i m_i(\breve{\Omega}'') - r_i^2\right)^+ < t'$ for some subtensor $\breve{\Omega}'' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t'}$ of $\breve{\Omega}'$.*

Proof First assume that $\sum_{i=1}^{d-1} \left(r_{i-1} r_i m_i(\breve{\Omega}'') - r_i^2\right)^+ < t'$ for some subtensor $\breve{\Omega}'' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t'}$ of the tensor $\breve{\Omega}'$. Recall that t' is the number of polynomials in $\mathcal{P}(\breve{\Omega}')$. On the other hand, according to Lemma 2, $\sum_{i=1}^{d-1} \left(r_{i-1} r_i m_i(\breve{\Omega}'') - r_i^2\right)^+$ is the maximum number of algebraically independent polynomials, and therefore the polynomials in $\mathcal{P}(\breve{\Omega}'')$ are not algebraically independent.

Now, assume that the polynomials in set $\mathcal{P}(\breve{\Omega}')$ are algebraically dependent. Then, there exists a subset of the polynomials that are minimally algebraically dependent. According to Lemma 3, if $\breve{\Omega}'' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t'}$ is the corresponding subtensor to this minimally algebraically dependent set of polynomials, the number of variables that are involved in $\mathcal{P}(\breve{\Omega}'') = \{p_1, , p_2 \ldots, p_{t'}\}$ is equal to $t' - 1$. On the other hand, $\sum_{i=1}^{d-1} \left(r_{i-1} r_i m_i(\breve{\Omega}'') - r_i^2\right)^+$ is the minimum possible number of involved variables in

$\mathcal{P}(\breve{\Omega}'')$ since $\sum_{i=1}^{d-1} \min\{r_i^2, r_{i-1}r_i m_i(\breve{\Omega}'')\}$ is the maximum number of known entries that are involved in $\mathcal{P}(\breve{\Omega}'')$. Hence, we have $\sum_{i=1}^{d-1} \left(r_{i-1}r_i m_i(\breve{\Omega}'') - r_i^2\right)^+ \leq t-1$. □

Finally, the following theorem characterizes the necessary and sufficient condition on the sampling patterns for finite completability of the sampled tensor $\mathcal{U}$ given its TT rank.

Theorem 1 *Suppose that Assumption 1 holds. Given that $\mathcal{U}$ is chosen generically from the corresponding TT manifold then with probability one the following statement holds true. There are only finitely many tensors that fit in the sampled tensor $\mathcal{U}$, and have TT rank $(r_1, r_2, \ldots, r_{d-1})$ if the following two conditions hold:*

(i) there exists a subtensor $\breve{\Omega}' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times M}$ of the constraint tensor such that $M = \sum_{i=1}^{d-1} r_{i-1}n_i r_i - \sum_{i=1}^{d-1} r_i^2$, and

(ii) for any $t \in \{1, \ldots, M\}$ and any subtensor $\breve{\Omega}'' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t}$ of the tensor $\breve{\Omega}'$, the following inequality holds

$$\sum_{i=1}^{d-1} \left(r_{i-1}r_i m_i(\breve{\Omega}'') - r_i^2\right)^+ \geq t. \tag{5}$$

Proof As a result of Lemma 4, the polynomials in $\mathcal{P}(\breve{\Omega}')$ are algebraically independent if and only if condition (ii) in the statement of the theorem holds. On the other hand, Lemma 1 concludes that with probability one, there are finitely many completions of $\mathcal{U}$ if and only if there exist $\sum_{i=1}^{d-1} r_{i-1}n_i r_i - \sum_{i=1}^{d-1} r_i^2$ algebraically independent polynomials in $\mathcal{P}(\breve{\Omega})$. Therefore, with probability one, there are finitely many completions of $\mathcal{U}$ if and only if conditions (i) and (ii) hold. □

Note that the condition given in Theorem 1 is combinatorial in nature and hard to verify in practice. In the next section, we provide a lower bound on the number of samples so that the combinatorial condition given in Theorem 1 hold true with high probability (not deterministically, i.e., with probability one anymore).

4 Probabilistic conditions for finite completability

In this section, consider a d-way sampled tensor $\mathcal{U} \in \mathbb{R}^{\overbrace{n \times \cdots \times n}^{d}}$ with TT rank vector $r = (r_1, \ldots, r_{d-1})$. Assume that the entries of $\mathcal{U}$ are independently sampled with probability p. Under a set of mild assumptions, we bound the sampling probability, or equivalently, the number of needed samples such that the corresponding constraint tensor satisfies conditions (i) and (ii) in the statement of Theorem 1 with high probability. In other words, satisfying the bound on the number of samples guarantees that the sampled tensor $\mathcal{U}$ is finitely completable with high probability. Assume that the entries of the sampling pattern are assumed to be independent from the entries of the tensor as well as from each other and the entries of the tensor $\mathcal{U}$ are sampled independently with probability p.

We note that this problem was considered for the matrix case in [37]. Hence, one may apply the result of this problem for matrix on each TT unfolding (since TT unfolding ranks are given), which is discussed in Section 4.1. Then, in Section 4.2, we will develop a combinatorial method in terms of the number of samples to verify if Theorem 1 holds.

4.1 TT Unfolding approach

In this section, we apply the analysis in [37] for each of the TT unfoldings to obtain a bound on the number of samples that ensures finite/unique completability. First, we restate Theorem 3 in [37], which is the basis of the TT unfolding approach.

Theorem 2 *Consider an $n \times N$ matrix with the given rank r and let $0 < \epsilon < 1$ be given. Suppose $r \leq \frac{n}{6}$ and that each* ***column*** *of the sampled matrix is observed in at least l entries, distributed uniformly at random and independently across entries, where*

$$l > \max \left\{ 12 \ \log \left(\frac{n}{\epsilon} \right) + 12, 2r \right\}. \tag{6}$$

Also, assume that $r(n-r) \leq N$. Then, with probability at least $1 - \epsilon$, the sampled matrix will be finitely completable.

Observe that in the case that $1 < r < n - 1$, the assumption $r(n - r) \leq N$ results that $n < N$ which is very important to check when we apply this theorem. We can simply apply Theorem 2 to each TT unfolding of the sampled tensor to obtain the following.

Corollary 1 *Assume that $i \leq \frac{d-1}{2}$, $1 < r_i \leq \frac{n^i}{6}$ and let $0 < \epsilon < 1$ be given. Note that $n^i \leq r_i n^{d-i}$ and $n^{d-i} > r_i(n^i - r_i)$ hold. Suppose that each* ***column*** *of the i-th TT unfolding of the sampled tensor is observed in at least l entries, distributed uniformly at random and independently across entries, where*

$$l > \max \left\{ 12 \ \log \left(\frac{n^i}{\epsilon} \right) + 12, 2r_i \right\}. \tag{7}$$

Then, since $n^i \leq r_i n^{d-i}$ and according to Theorem 2, with probability at least $1 - \epsilon$, the sampled tensor (TT unfolding matrix) is finitely completable. This results in $n^{d-i} \max \left\{ 12 \ \log \left(\frac{n^i}{\epsilon} \right) + 12, 2r_i \right\}$ samples in total.

Now, assume that $i \geq \frac{d+1}{2}$ and $1 < r_i \leq \frac{n^{d-i}}{6}$. Note that $n^{d-i} \leq r_i n^i$ and $n^i > r_i(n^{d-i} - r_i)$ hold. Suppose that each ***row*** *of the i-th TT unfolding of the sampled tensor is observed in at least l entries, distributed uniformly at random and independently across entries, where*

$$l > \max \left\{ 12 \ \log \left(\frac{n^{d-i}}{\epsilon} \right) + 12, 2r_i \right\}. \tag{8}$$

Then, since $n^{d-i} \leq r_i n^i$ and according to Theorem 2, with probability at least $1 - \epsilon$, the sampled tensor (TT unfolding matrix) is finitely completable. This results in $n^i \max \left\{ 12 \ \log \left(\frac{n^{d-i}}{\epsilon} \right) + 12, 2r_i \right\}$ samples in total.

Assume that $i = \frac{d}{2}$ and $1 < r_i$. Then, as the i-th TT unfolding of the sampled tensor is an $n^i \times n^i$ matrix, we can simply verify that Theorem 2 is not applicable due the assumption $r(n - r) \leq N$.

Remark 3 Consider a tensor $\mathcal{U}$ that satisfies $1 < r_i \leq \frac{n^i}{6}$ for $i \leq \frac{d-1}{2}$ and $1 < r_i \leq \frac{n^{d-i}}{6}$ for $i \geq \frac{d+1}{2}$. According to Corollary 1, the sampled tensor $\mathcal{U}$ requires more than

$$n^{\lceil \frac{d+1}{2} \rceil} \max \left\{ 12 \ \log \left(\frac{n^{\lfloor \frac{d-1}{2} \rfloor}}{\epsilon} \right) + 12, 2r_{\lfloor \frac{d-1}{2} \rfloor} \right\} \tag{9}$$

samples to be finitely completable with probability at least $1 - \epsilon$.

4.2 TT approach

In this section, we are interested in obtaining a better bound on the number of samples than the one given in Section 4.1 based on our deterministic analysis of the TT manifold. In this approach, instead of simply using Theorem 2 which is taken from [37], we are interested in finding the number of sampled entries which ensures conditions (i) and (ii) in the statement of Theorem 1 to hold with high probability. In particular, in this section, under the uniform sampling assumption, we find a lower bound on the number of samples to ensure that all the combinatorial patterns in Theorem 1 (i.e., conditions (i) and (ii)) hold with high probability, and consequently the tensor admits only a finite number of completions with high probability.

According to the above-mentioned target for this section, we are interested in finding a lower bound on the number of sampled entries to ensure that inequalities similar to condition (ii) in the statement of Theorem 1 hold true. The following lemma serves the role of connecting the number of sampled entries to such inequalities and will be used later to obtain Lemma 7.

Lemma 5 *Assume that $r' \leq \frac{n}{6}$ and also each column of $\Omega_{(1)}$ (first Tucker unfolding of Ω) includes at least l nonzero entries, where*

$$l > \max \left\{ 9 \ \log \left(\frac{n}{\epsilon} \right) + 3 \ \log \left(\frac{k}{\epsilon} \right) + 6, 2r' \right\}. \tag{10}$$

Let $\Omega'_{(1)}$ be an arbitrary set of $n - r'$ columns of $\Omega_{(1)}$ and $0 < \epsilon < 1$ be given. Then, with probability at least $1 - \frac{\epsilon}{k}$, every subset $\Omega''_{(1)}$ of columns of $\Omega'_{(1)}$ satisfies

$$m_1(\Omega'') - r' \geq t, \tag{11}$$

where t is the number of columns of $\Omega''_{(1)}$ and $m_1(\Omega'')$ is the number of nonzero rows of $\Omega''_{(1)}$.

Proof Please refer to the proof of [37, Lemma 9]. Note that the only difference is that the last inequalities of (16) and (18) in [37] should now be upper bounded by $\frac{\epsilon}{rd}$ instead of $\frac{\epsilon}{d^2}$. □

Note that (11) is still very different from the inequalities required for condition (ii) in Theorem 1. Hence, in this section, starting from (11) we will obtain inequalities that are more similar to inequalities in condition (ii) in Theorem 1.

The following lemma provides a bound on the number of sampled entries in each column of the j-th TT unfolding of the sampled tensor such that the i-th Tucker unfolding of the subtensor corresponding to a columns of the j-th TT unfolding includes more than the RHS of (10) observed entries of Ω with different values of the i-th coordinate.

Lemma 6 *Assume that $r' \leq \frac{n}{6}$ and let $j \in \{1, 2, \ldots, d-1\}$ be a fixed number and also $0 < \epsilon < 1$ be given. Consider an arbitrary set $\widetilde{\Omega}'_{(j)}$ of $n - r'$ columns of $\widetilde{\Omega}_{(j)}$ (j-th TT unfolding of Ω). Assume that $n > \max\{200, \sum_{k=1}^{d-1} r_{k-1}r_k\}$, and also each column of $\widetilde{\Omega}_{(j)}$ includes at least l nonzero entries, where*

$$l > \max\left\{27\ \log\left(\frac{n}{\epsilon}\right) + 9\ \log\left(\frac{2r}{\epsilon}\right) + 18, 6r'\right\}, \tag{12}$$

where $r \leq \sum_{k=1}^{d-1} r_{k-1}r_k$ (recall that $r_0 = r_d = 1$). Then, with probability at least $1 - \frac{\epsilon}{2r}$, each column of $\widetilde{\Omega}'_{(j)}$ includes more than $l_0 \triangleq \max\left\{9\ \log\left(\frac{n}{\epsilon}\right) + 3\ \log\left(\frac{2r}{\epsilon}\right) + 6, 2r'\right\}$ observed entries of Ω with different values of the i-th coordinate, i.e., the i-th Tucker unfolding of the tensor Ω' that corresponds to $\widetilde{\Omega}'_{(j)}$ includes more than l_0 nonzero rows, $1 \leq i \leq j$.

Proof Each column of $\widetilde{\Omega}'_{(j)}$ includes n^j entries and they can be represented by $(x_1, \ldots, x_j)$ for $1 \leq x_k \leq n$ and $1 \leq k \leq j$, where x_k denotes the k-th coordinate of the corresponding entry. Let $P(\zeta)$ be the probability that at least one of the columns of $\widetilde{\Omega}'_{(j)}$ includes at most l_0 observed entries of Ω with different values of the i-th coordinate. Also, let $P(\zeta_s)$ denote the probability that the s-th column of $\widetilde{\Omega}'_{(j)}$ includes at most l_0 observed entries of Ω with different values of the i-th coordinate, $1 \leq s \leq n - r'$. Then, we have $P(\zeta) \leq (n - r')P(\zeta_1)$.

By assumption, each column of $\widetilde{\Omega}'_{(j)}$ includes more than $3l_0$ observed entries. In the case that the first column of $\widetilde{\Omega}'_{(j)}$ includes at most l_0 observed entries of Ω with different values of the i-th coordinate, we conclude the set of i-th coordinates of all observed entries of this column (which are more than $3l_0$ entries) belong to a set with at most l_0 numbers. As it is assumed to have the uniform random sampling, we have

$$P(\zeta_1) \leq \binom{n}{l_0}\left(\frac{l_0}{n}\right)^{3l_0}. \tag{13}$$

Furthermore, we have

$$\binom{n}{l_0} = \frac{n(n-1)\ldots(n-l_0+1)}{l_0!} \leq \frac{n^{l_0}}{l_0!} \leq \left(\frac{ne}{l_0}\right)^{l_0}, \tag{14}$$

where the last inequality holds since $e^{l_0} = \sum_{k=0}^{\infty} \frac{l_0{}^k}{k!} \geq \frac{l_0{}^{l_0}}{l_0!}$. Having (13) and (14), we can conclude

$$P(\zeta_1) \leq e^{l_0}\left(\frac{l_0}{n}\right)^{2l_0} = \left(\frac{e^{\frac{1}{2}} l_0}{n}\right)^{2l_0}, \tag{15}$$

and therefore

$$\begin{aligned}\log\left(P(\zeta)\right) \ &\leq\ \log\left((n-r')P(\zeta_1)\right) \overset{(a)}{<} 2l_0\left(\frac{1}{2} + \log(l_0) - \log(n)\right) + \log(n)\\ &\overset{(b)}{\leq}\ 2l_0\left(\frac{1}{2} + \log(l_0) - \log(n)\right) + \frac{1}{9}l_0 = 2l_0\left(\frac{13}{18} + \log(l_0) - \log(n)\right) - \frac{l_0}{3},\end{aligned} \tag{16}$$

where (a) follows from the fact that $\log(n - r') < \log(n)$ and (b) follows from $l_0 \geq 9\log(n) - 9\log(\epsilon) \geq 9\log(n)$ which is easy to verify having the definition of l_0. On the other hand, we have

$$-\frac{l_0}{3} \leq -3\ \log\left(\frac{n}{\epsilon}\right) - \ \log\left(\frac{2r}{\epsilon}\right) - 2 \ = \ 4\log(\epsilon) - 3\log(n) - \log(2r) - 2$$
$$\overset{(c)}{<} \log(\epsilon) - \log(2r) = \log\left(\frac{\epsilon}{2r}\right), \quad (17)$$

where (c) follows from $3\log(\epsilon) - 3\log(n) - 2 < 0$ since $\log(\epsilon) < 0 < \log(n)$. Moreover, for the term $\frac{13}{18} + \log(l_0) - \log(n)$, there are following two possibilities:

(i) $l_0 = 2r'$: We conclude $\frac{13}{18} + \log(l_0) - \log(n) < \log(2.06) + \log(2r') - \log(n) = \log\left(\frac{4.12\ r'}{n}\right) < 0$, where the last inequality is a simple result of the assumption $r' \leq \frac{n}{6}$.

(ii) $l_0 = 9\ \log\left(\frac{n}{\epsilon}\right) + 3\ \log\left(\frac{2r}{\epsilon}\right) + 6$: Recall that $r \leq \sum_{k=1}^{d-1} r_{k-1} r_k < n$, and therefore $l_0 \leq 12\log(n) + 6 + 3\log(2)$. Then, having the assumption $200 < n$, we simply conclude $\frac{13}{18} + \log(l_0) - \log(n) \leq \frac{13}{18} + \log(12\log(n) + 6 + 3\log(2)) - \log(n) < 0$.

Therefore, the assumptions $\max\{200, \sum_{k=1}^{d-1} r_{k-1} r_k\} < n$ and $r' \leq \frac{n}{6}$ result in

$$\frac{13}{18} + \log(l_0) - \log(n) \leq 0. \quad (18)$$

Having (16), (17), and (18) result that $\log(P(\zeta)) < \log\left(\frac{\epsilon}{2r}\right)$, and the proof is complete.

□

The following lemma exploits Lemma 5 and Lemma 6 to provide a bound on the number of sampled entries in each column of the j-th TT unfolding of the sampled tensor such that the i-th Tucker unfolding of the subtensor corresponding to a subset of columns of the j-th TT unfolding satisfies the property in the statement of Lemma 5 with high probability.

Lemma 7 *Let $j \in \{1, 2, \ldots, d-1\}$ be a fixed number and $0 < \epsilon < 1$ be given. Assume that $r_i' \leq \frac{n}{6}$, where $i \in \{1, \ldots, j\}$. Consider an arbitrary set $\widetilde{\Omega}'_{(j)}$ of $n - r_i'$ columns of $\widetilde{\Omega}_{(j)}$. Assume that $n > \max\{200, \sum_{k=1}^{d-1} r_{k-1} r_k\}$, and also each column of $\widetilde{\Omega}_{(j)}$ includes at least l nonzero entries, where*

$$l > \max\left\{27\ \log\left(\frac{n}{\epsilon}\right) + 9\ \log\left(\frac{2r}{\epsilon}\right) + 18, 6r_i'\right\}, \quad (19)$$

where $r \leq \sum_{k=1}^{d-1} r_{k-1} r_k$ (recall that $r_0 = r_d = 1$). Then, with probability at least $1 - \frac{\epsilon}{r}$, every subset $\widetilde{\Omega}''_{(j)}$ of columns of $\widetilde{\Omega}'_{(j)}$ satisfies

$$m_i(\Omega'') - r_i' \geq t, \quad (20)$$

where t is the number of columns of $\widetilde{\Omega}''_{(j)}$ and Ω'' is the corresponding tensor such that $\widetilde{\Omega}''_{(j)}$ is the j-th TT unfolding of Ω''.

Proof Each column of $\widetilde{\Omega}_{(j)}$ includes n^j entries and they can be represented by $(x_1, \ldots, x_j)$ for $1 \leq x_k \leq n$ and $1 \leq k \leq j$, where x_k denotes the k-th coordinate of the corresponding entry. According to Lemma 6, with probability at least $1 - \frac{\epsilon}{2r}$, each column of $\widetilde{\Omega}_{(j)}$ includes more than $\max\left\{9\ \log\left(\frac{n}{\epsilon}\right) + 3\ \log\left(\frac{2r}{\epsilon}\right) + 6, 2r'\right\}$ observed entries with different values of the i-th coordinate. Therefore, according to Lemma 5, with probability at least $\left(1 - \frac{\epsilon}{2r}\right)^2$, every subset $\widetilde{\Omega}''_{(j)}$ of columns of $\widetilde{\Omega}'_{(j)}$ satisfies (20). The proof is complete as $\left(1-\frac{\epsilon}{2r}\right)^2 \geq 1-\frac{\epsilon}{r}$. $\square$

Note that we are interested in obtaining a condition in terms of the number of samples to ensure finite completability, i.e., to certify the given conditions on the constraint tensor $\breve{\Omega}$ (not the sampling pattern Ω) in Theorem 1 hold, with high probability. However, it is clear to observe given that the number of samples satisfies some inequalities, it is less complicated to verify the mentioned combinatorial conditions on the sampling pattern. Then, the following lemma connects the conditions in terms of the number of samples and the combinatorial conditions on the constraint tensor. In particular, the following lemma states that, if the property in Lemma 5 holds for the sampling pattern Ω, then it will be satisfied for $\breve{\Omega}$ as well. The following lemma is taken from [4, Lemma 18].

Lemma 8 *Let r' be a given nonnegative integer and $1 \leq i \leq j \leq d-1$. Assume that there exists an $n^j \times (n-r')$ matrix $\widetilde{\Omega}'_{(j)}$ composed of $n-r'$ columns of $\widetilde{\Omega}_{(j)}$ such that each column of $\widetilde{\Omega}'_{(j)}$ includes at least $r'+1$ nonzero entries and satisfies the following property:*

– *Denote an $n^j \times t$ matrix (for any $1 \leq t \leq n-r'$) composed of any t columns of $\widetilde{\Omega}'_{(j)}$ by $\widetilde{\Omega}''_{(j)}$. Then*

$$m_i(\Omega'') - r' \geq t. \tag{21}$$

Then, there exists an $n^j \times (n-r')$ matrix $\breve{\widetilde{\Omega}}'_{(j)}$ such that: each column has exactly $r'+1$ entries equal to one, and, if $\breve{\widetilde{\Omega}}'_{(j)}(x,y)=1$, then we have $\widetilde{\Omega}'_{(j)}(x,y)=1$. Moreover, $\breve{\widetilde{\Omega}}'_{(j)}$ satisfies the above-mentioned property.

The following lemma makes use of Lemma 8 to extend the result of Lemma 7 to some combinatorial properties on the constraint tensor.

Lemma 9 *Assume that $1 \leq i \leq j \leq d-1$ and consider r' matrices $\widetilde{\Omega}'_{(j)_k}$ each composed from $n-r'_i$ columns of $\widetilde{\Omega}_{(j)}$ indexed by disjoint sets for $1 \leq k \leq r'$, where $r'_i \leq \frac{n}{6}$ and $r' \leq r \leq \sum_{k=1}^{d-1} r_{k-1}r_k$. Let $0 < \epsilon < 1$ be given and $\widetilde{\Omega}'_{(j)}$ denote the union of all r' sets of columns $\widetilde{\Omega}'_{(j)_k}$'s, and therefore it includes $r'(n-r'_i)$ columns. Assume that $n > \max\{200, \sum_{k=1}^{d-1} r_{k-1}r_k\}$, and also each column of $\widetilde{\Omega}_{(j)}$ includes at least l nonzero entries, where*

$$l > \max\left\{27\ \log\left(\frac{n}{\epsilon}\right) + 9\ \log\left(\frac{2r}{\epsilon}\right) + 18, 6r'_i\right\}. \tag{22}$$

Then, there exists an $n^j \times r'(n-r'_i)$ matrix $\breve{\widetilde{\Omega}}'_{(j)}$ such that each column has exactly r'_i+1 entries equal to one, and if $\breve{\widetilde{\Omega}}'_{(j)}(x,y)=1$ then we have $\widetilde{\Omega}'_{(j)}(x,y)=1$ and also it satisfies the following property: with probability at least $1-\frac{\epsilon r'}{r}$, every subset $\breve{\widetilde{\Omega}}''_{(j)}$ of columns of $\breve{\widetilde{\Omega}}'_{(j)}$ satisfies the following inequality

$$r'\left(m_i(\breve{\Omega}'') - r'_i\right) \geq t, \tag{23}$$

where t is the number of columns of $\breve{\widetilde{\Omega}}''_{(j)}$ and $\breve{\Omega}''$ is the corresponding tensor such that $\breve{\widetilde{\Omega}}''_{(j)}$ is the j-th TT unfolding of $\breve{\Omega}''$.

Proof Consider any subset $\widetilde{\Omega}''_{(j)_k}$ of columns of $\widetilde{\Omega}'_{(j)_k}$ and consider its corresponding tensor Ω''_k such that the j-th TT unfolding of Ω''_j is $\widetilde{\Omega}''_{(j)_k}$. First of all, according to Lemma 7, Ω''_k satisfies the following inequality with probability at least $1 - \frac{\epsilon}{r}$

$$m_i(\Omega''_k) - r'_i \geq t_k, \tag{24}$$

where t_k is the number of columns of $\widetilde{\Omega}''_{(j)_k}$.

According to Lemma 8, there exists an $n^j \times (n - r'_i)$ matrix $\widetilde{\breve{\Omega}}'_{(j)_k}$ such that each column has exactly $r'_i + 1$ entries equal to one, and if $\widetilde{\breve{\Omega}}'_{(j)_k}(x, y) = 1$ then we have $\widetilde{\Omega}'_{(j)_k}(x, y) = 1$ and also it satisfies the following property: with probability at least $1 - \frac{\epsilon}{r}$, every subset $\widetilde{\breve{\Omega}}''_{(j)_k}$ of columns of $\widetilde{\breve{\Omega}}'_{(j)_k}$ satisfies (24). Define the union of the columns of $\widetilde{\breve{\Omega}}'_{(j)_k}$'s as $\widetilde{\breve{\Omega}}'_{(j)} = [\widetilde{\breve{\Omega}}'_{(j)_1}|\widetilde{\breve{\Omega}}'_{(j)_2}|\ldots|\widetilde{\breve{\Omega}}'_{(j)_{r'}}]$. In order to complete the proof it suffices to show that with probability at least $1 - \epsilon$, the tensor $\breve{\Omega}''$ corresponding to any subset $\widetilde{\breve{\Omega}}''_{(j)}$ of columns of $\widetilde{\breve{\Omega}}'_{(j)}$ satisfies (23).

Let $\widetilde{\breve{\Omega}}''_{(j)_k}$ denote those columns of $\widetilde{\breve{\Omega}}''_{(j)}$ that belong to $\widetilde{\breve{\Omega}}'_{(j)_k}$ and define s_k as the number of columns of $\widetilde{\breve{\Omega}}''_{(j)_k}$, $1 \leq k \leq r'$, and define s as the number of columns of $\widetilde{\breve{\Omega}}''_{(j)}$. Without loss of generality, assume that $s_1 \geq s_2 \geq \cdots \geq s_{r'}$. Also, assume that all Ω''_k's satisfy (24). Hence, we have

$$s = \sum_{k=1}^{r'} s_k \leq r's_1 \leq r'\left(m_i(\breve{\Omega}''_1) - r'_i\right) \leq r'\left(m_i(\breve{\Omega}'') - r'_i\right). \tag{25}$$

Observe that each Ω''_k satisfies (24) with probability at least $1 - \frac{\epsilon}{r}$. Therefore, all Ω''_k's $(1 \leq k \leq r')$ satisfy (24) with probability at least $1 - \frac{\epsilon r'}{r}$. □

Finally, the following theorem exploits Lemma 9 and Theorem 1 to obtain a bound on the number of sampled entries to ensure finite completability of the sampled tensor, with high probability.

Theorem 3 *Define $m = \sum_{k=1}^{d-2} r_{k-1}r_k$, $M = n\sum_{k=1}^{d-2} r_{k-1}r_k - \sum_{k=1}^{d-2} r_k^2$ and $r' = \max\left\{\frac{r_1}{r_0}, \ldots, \frac{r_{d-2}}{r_{d-3}}\right\}$. Assume that $n > \max\{m, 200\}$ and $r' \leq \min\{\frac{n}{6}, r_{d-2}\}$ hold and also let $0 < \epsilon < 1$ be given. Moreover, assume that each column of $\widetilde{\Omega}_{(d-2)}$ includes at least l nonzero entries, where*

$$l > \max\left\{27 \log\left(\frac{n}{\epsilon}\right) + 9 \log\left(\frac{2M}{\epsilon}\right) + 18, 6r_{d-2}\right\}. \tag{26}$$

Then, with probability at least $1 - \epsilon$, the sampled tensor admits only a finite number of completions with separation rank $(r_1, r_2, \ldots, r_{d-1})$.

Proof Define the $(d-1)$-way tensor $\mathcal{U}' \in \mathbb{R}^{\overbrace{n \times \cdots \times n}^{d-2} \times n^2}$ which is obtained through merging the $(d-1)$-th and d-th dimensions of the tensor $\mathcal{U}$. Observe that the finiteness of the number of completions of the tensor $\mathcal{U}'$ with rank vector $(r_1, r_2, \ldots, r_{d-2})$ ensures the finiteness of the number of completions of the tensor $\mathcal{U}$ with rank vector $(r_1, r_2, \ldots, r_{d-1})$.

According to Theorem 1, it suffices to show that with probability at least $1-\epsilon$, conditions (i) and (ii) in the statement of Theorem 1 hold for the tensor $\mathcal{U}'$ with rank vector $(r_1, r_2, \ldots, r_{d-2})$.

Note that the assumption $m < n$ results that $M < n^2$, and therefore $\widetilde{\Omega}_{(d-2)}$ has least M columns. Hence, for any $1 \leq i \leq d-2$ we can choose $r_{i-1}r_i n - r_i^2$ arbitrary columns of $\widetilde{\Omega}_{(d-2)}$ and denote it by $\widetilde{\Omega}'_{(d-2)_i}$ such that $\widetilde{\Omega}'_{(d-2)_i}$'s are composed from columns of $\widetilde{\Omega}_{(d-2)}$ indexed by disjoint sets. Define $r'_i = \lceil \frac{r_i}{r_{i-1}} \rceil$ and note that the assumption $r' \leq r_{d-2}$ results in that $r'_i \leq r_{d-2}$. As a result, the assumption $r'_i \leq r_{d-2}$ and (26) results in $l > \max\left\{27\ \log\left(\frac{n}{\epsilon}\right) + 9\ \log\left(\frac{2M}{\epsilon}\right) + 18, 6r'_i\right\}$. Therefore, according to Lemma 9, there exists a matrix $\widetilde{\breve{\Omega}}'_{(d-2)_i}$ with n^{d-2} rows and $r_{i-1}r_i\left(n - \frac{r_i}{r_{i-1}}\right) = r_{i-1}r_i n - r_i^2$ columns such that: each column has exactly $r'_i + 1$ entries equal to one, and if $\widetilde{\breve{\Omega}}'_{(d-2)_i}(x, y) = 1$ then we have $\widetilde{\Omega}'_{(d-2)_i}(x, y) = 1$ and also it satisfies the following property: with probability at least $1 - \frac{\epsilon r_{i-1} r_i}{M}$, every subset $\widetilde{\breve{\Omega}}''_{(d-2)_i}$ of columns of $\widetilde{\breve{\Omega}}'_{(d-2)_i}$ satisfies the following

$$r_{i-1}r_i m_i(\breve{\Omega}''_i) - r_i^2 \geq t, \tag{27}$$

where t is the number of columns of $\widetilde{\breve{\Omega}}''_{(d-2)_i}$ and $\breve{\Omega}''_i$ is the corresponding tensor such that $\widetilde{\breve{\Omega}}''_{(d-2)_i}$ is the $(d-2)$-th TT unfolding of $\breve{\Omega}''_i$. Moreover, as we have $r'_i \leq r_{d-2}$, by changing $r_{d-2} - r'_i$ entries from zero to one in each column, we can assume that $\widetilde{\breve{\Omega}}'_{(d-2)_i}$ has exactly $r_{i-1}r_i n - r_i^2$ columns of the $(d-2)$-th TT unfolding of the constraint tensor $\breve{\Omega}$ and satisfies the above properties. Let $\breve{\Omega}'_i$ denote the subtensor of the constraint tensor corresponding to $\widetilde{\breve{\Omega}}'_{(d-2)_i}$.

Let $\widetilde{\breve{\Omega}}'_{(d-2)} = [\widetilde{\breve{\Omega}}'_{(d-2)_1} | \ldots | \widetilde{\breve{\Omega}}'_{(d-2)_{d-2}}]$ denote the union of $\widetilde{\breve{\Omega}}'_{(d-2)_i}$'s and $\breve{\Omega}'$ denote its corresponding subtensor of the constraint tensor. Hence, $\breve{\Omega}'$ satisfies condition (i) in the statement of Theorem 1 for tensor $\mathcal{U}'$ with rank vector $(r_1, r_2, \ldots, r_{d-2})$ since $\widetilde{\breve{\Omega}}'_{(d-2)}$ has $\sum_{k=1}^{d-2} r_{k-1}r_k n - \sum_{k=1}^{d-2} r_k^2$ columns. Furthermore, with probability at least $1 - \frac{\epsilon}{M}\sum_{k=1}^{d-2} r_{k-1}r_k = 1-\epsilon$, any subtensor $\breve{\Omega}'' \in \mathbb{R}^{\overbrace{n \times \cdots \times n}^{d-2} \times t}$ of tensor $\breve{\Omega}'$ satisfies

$$\sum_{i=1}^{d-2}\left(r_{i-1}r_i m_i(\breve{\Omega}'') - r_i^2\right)^+ \geq \sum_{i=1}^{d-2}\left(r_{i-1}r_i m_i(\breve{\Omega}''_i) - r_i^2\right)^+ \geq \sum_{i=1}^{d-2} t_i = t, \tag{28}$$

where $\breve{\Omega}''_i \in \mathbb{R}^{\overbrace{n \times \cdots \times n}^{d-2} \times t_i}$ are such that $\breve{\Omega}'' = [\breve{\Omega}''_1 | \ldots | \breve{\Omega}''_{d-2}]$ and $\breve{\Omega}''_i$ is a subtensor of $\breve{\Omega}'_i$, $1 \leq i \leq d-2$. The proof is complete as condition (ii) in the statement of Theorem 1 holds. $\square$

Remark 4 A tensor $\mathcal{U}$ that satisfies the properties in the statement of Theorem 3 requires

$$n^2 \max\left\{27\ \log\left(\frac{n}{\epsilon}\right) + 9\ \log\left(\frac{2M}{\epsilon}\right) + 18, 6r_{d-2}\right\} \tag{29}$$

samples to be finitely completable with probability at least $1-\epsilon$ since $\widetilde{\Omega}_{(d-2)}$ has n^2 columns, with $M = n\sum_{k=1}^{d-2} r_{k-1}r_k - \sum_{k=1}^{d-2} r_k^2$, in contrast to the number of samples required by the TT unfolding approach given in Remark 3.

The following lemma is taken from [2] and is used in Lemma 2 to derive a lower bound on the sampling probability that results (26) with high probability.

Lemma 10 *Consider a vector with n entries where each entry is observed with probability p independently from the other entries. If $p > p' = \frac{k}{n} + \frac{1}{\sqrt[4]{n}}$, then with probability at least $\left(1 - \exp(-\frac{\sqrt{n}}{2})\right)$, more than k entries are observed.*

Corollary 2 *Define $m = \sum_{k=1}^{d-2} r_{k-1}r_k$, $M = n\sum_{k=1}^{d-2} r_{k-1}r_k - \sum_{k=1}^{d-2} r_k^2$ and $r' = \max\left\{\frac{r_1}{r_0}, \ldots, \frac{r_{d-2}}{r_{d-3}}\right\}$. Assume that $n > \max\{m, 200\}$ and $r' \leq \min\{\frac{n}{6}, r_{d-2}\}$ hold and also let $0 < \epsilon < 1$ be given. Moreover, assume that the sampling probability satisfies*

$$p > \frac{1}{n^{d-2}} \max\left\{27 \log\left(\frac{n}{\epsilon}\right) + 9 \log\left(\frac{2M}{\epsilon}\right) + 18, 6r_{d-2}\right\} + \frac{1}{\sqrt[4]{n^{d-2}}} \tag{30}$$

Then, with probability at least $(1-\epsilon)\left(1 - \exp\left(-\frac{\sqrt{n^{d-2}}}{2}\right)\right)^{n^2}$, $\mathcal{U}$ is finitely completable.

Proof According to Lemma 10, assumption (30) results that each column of $\widetilde{\Omega}_{(d-2)}$ includes at least l nonzero entries, where l satisfies (26) with probability at least $\left(1 - \exp\left(-\frac{\sqrt{n^{d-2}}}{2}\right)\right)$. Therefore, with probability at least $\left(1 - \exp\left(-\frac{\sqrt{n^{d-2}}}{2}\right)\right)^{n^2}$, all n^2 columns of $\widetilde{\Omega}_{(d-2)}$ satisfy (26). Hence, according to Theorem 3, with probability at least $(1-\epsilon)\left(1 - \exp\left(-\frac{\sqrt{n^{d-2}}}{2}\right)\right)^{n^2}$, $\mathcal{U}$ is finitely completable. □

5 Deterministic and probabilistic conditions for unique completability

As we showed in [2], for matrix and tensor completion problems, finite completability does not necessarily imply unique completability. Theorem 1 and Theorem 3 characterize the deterministic and probabilistic conditions on the sampling pattern Ω for finite completability, respectively. In this section, we add some additional mild restrictions on Ω and the number of samples to ensure unique completability. To this end, we obtain multiple sets of minimally algebraically dependent polynomials and show that the variables involved in these polynomials can be determined uniquely, and therefore entries of $\mathcal{U}$ can be determined uniquely. The following lemma is a re-statement of Lemma 25 in [4].

Lemma 11 *Suppose that Assumption 1 holds. Let $\breve{\Omega}' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t}$ be an arbitrary subtensor of the constraint tensor $\breve{\Omega}$. Assume that polynomials in $\mathcal{P}(\breve{\Omega}')$ are minimally algebraically dependent. Then, all variables (unknown entries) of $\mathcal{U}^{(1)}$, $\mathcal{U}^{(2)}, \ldots,$ and $\mathcal{U}^{(d-1)}$ that are involved in $\mathcal{P}(\breve{\Omega}')$ can be determined uniquely.*

In the following theorem, we use the fundamental analysis and techniques proposed in previous sections (for finite completability) to extend the result of Theorem 1 to unique completability. In particular, we characterize a sufficient condition on the constraint tensor to ensure reduce the number of completions from a finite number to one, with probability one. We explain the key point behind the proof of the following theorem. Condition (i) results in $\sum_{i=1}^{d-1} r_{i-1}n_ir_i - \sum_{i=1}^{d-1} r_i^2$ algebraically independent polynomials in terms of the entries of $\mathcal{U}^{(1)}, \mathcal{U}^{(2)}, \ldots,$ and $\mathcal{U}^{(d-1)}$, i.e., results in finite completability. As a result, adding any single polynomial to these $\sum_{i=1}^{d-1} r_{i-1}n_ir_i - \sum_{i=1}^{d-1} r_i^2$ algebraically independent polynomials results in a set of algebraically dependent polynomials and according to Lemma 11 some of the entries of $\mathcal{U}^{(1)}, \mathcal{U}^{(2)}, \ldots,$ and $\mathcal{U}^{(d-1)}$ can be determined uniquely. Then, condition (ii) results in more polynomials such that all entries of $\mathcal{U}^{(1)}, \mathcal{U}^{(2)}, \ldots,$ and $\mathcal{U}^{(d-1)}$ can be determined uniquely.

Theorem 4 *Suppose that Assumption 1 holds. Also, assume that there exist disjoint subtensors $\breve{\Omega}' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times M}$ and $\breve{\Omega}'^{i} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times M_i}$ (for $1 \leq i \leq d-1$) of the constraint tensor such that the following conditions hold:*

(i) $M = \sum_{k=1}^{d-1} r_{k-1}n_kr_k - \sum_{k=1}^{d-1} r_k^2$, *and for any $t \in \{1, \ldots, M\}$ and any subtensor $\breve{\Omega}'' \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t}$ of the tensor $\breve{\Omega}'$, the following inequality holds*

$$\sum_{k=1}^{d-1} \left(r_{k-1}r_km_k(\breve{\Omega}'') - r_k^2 \right)^+ \geq t. \tag{31}$$

(ii) *for each $i \in \{1, \ldots, d-1\}$ we have $M_i = n_i - \lfloor \frac{r_i}{r_{i-1}} \rfloor$, and for any $t_i \in \{1, \ldots, M_i\}$ and any subtensor $\breve{\Omega}''^{i} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{d-1} \times t_i}$ of the tensor $\breve{\Omega}'^{i}$, the following inequality holds*

$$m_i(\breve{\Omega}'') - \frac{r_i}{r_{i-1}} \geq t_i - \frac{r_i}{r_{i-1}}(t_i - M_i + 1)^+. \tag{32}$$

Then, with probability one, there exists only a unique tensor that fits in the sampled tensor $\mathcal{U}$, and has TT rank $(r_1, r_2, \ldots, r_{d-1})$.

Proof According to the proof of Theorem 1, $\mathcal{P}(\breve{\Omega}')$ includes $M = \sum_{i=1}^{d-1} r_{i-1}n_ir_i - \sum_{i=1}^{d-1} r_i^2$ algebraically independent polynomials which results the finite completability of the sampled tensor $\mathcal{U}$ and let $\{p_1, \ldots, p_M\}$ denote these M algebraically independent polynomials. Also, M is the number of total variables among the polynomials, and therefore adding any polynomial p_0 to $\{p_1, \ldots, p_M\}$ results in a set of algebraically dependent polynomials. As a result, there exists a set of polynomials $\mathcal{P}(\breve{\Omega}'')$ such that $\mathcal{P}(\breve{\Omega}'') \subset \{p_1, \ldots, p_M\}$ and also polynomials in $\mathcal{P}(\breve{\Omega}'') \cup p_0$ are minimally algebraically dependent polynomials. Hence, according to Lemma 11, all the variables involved in the polynomials $\mathcal{P}(\breve{\Omega}'') \cup p_0$ can be determined uniquely. As a result, all variables involved in p_0 can be determined uniquely.

We can repeat the above procedure for any polynomial $p_0 \in \mathcal{P}(\breve{\Omega}'^{i})$ to determine the involved variables uniquely with the help of $\{p_1, \ldots, p_M\}$, $i = 1, \ldots, d-1$. Hence, we obtain $\sum_{k=1}^{d-1} r_{k-1}r_k$ polynomials but some of the entries of TT decomposition are elements of the $\mathbf{Q}_i$ matrices (in the statement of Lemma 17). In order to complete the proof, we need to show that condition (ii) with the above procedure using $\{p_1, \ldots, p_M\}$ results in obtaining all variables uniquely. In particular, we show that polynomials in $\mathcal{P}(\breve{\Omega}'^{i})$ result in obtaining all variables of the i-th element of TT decomposition uniquely.

Note that since $t_i \leq M_i$, we have $(t_i - M_i + 1)^+ = 1$ if $t_i - M_i = 0$ and $(t_i - M_i + 1)^+ = 0$ otherwise. Hence, if $t_i < M_i$ condition (ii) can be written as

$$r_{i-1} r_i m_i(\breve{\Omega}'') - {r_i}^2 \geq r_{i-1} r_i t_i, \tag{33}$$

which certifies the algebraically independence of the corresponding polynomials obtained by the mentioned procedure. Observe that we need $r_{i-1} r_i n_i - r_i^2$ algebraically independent polynomials and in the case that $t_i = M_i$, condition (ii) results in $r_{i-1} r_i n_i - r_i^2$ algebraically independent polynomials. □

Theorem 4 provides the deterministic condition on the sampling pattern Ω for unique completability. Using Theorem 4 we provide a bound on the number of samples to ensure unique completability with high probability. We first need to extended some of the lemmas in Section 4 to obtain a condition on the number of samples to ensure condition (ii) in the statement of Theorem 4 holds with high probability. Note that Condition (i) is the same condition for finite completability and we already have the corresponding bound.

In the rest of this section, for the sake of simplicity, as in Section 4 we consider the sampled tensor $\mathcal{U} \in \mathbb{R}^{\overbrace{n \times \cdots \times n}^{d}}$.

Lemma 12 *Assume that $r' \leq \frac{n}{6}$ and also each column of $\Omega_{(1)}$ (first Tucker unfolding of Ω) includes at least l nonzero entries, where*

$$l > \max \left\{ 21 \ \log\left(\frac{n}{\epsilon}\right) + 3 \ \log\left(\frac{k}{\epsilon}\right) + 6, 2r' \right\}. \tag{34}$$

*Let $\Omega'_{(1)}$ be an arbitrary set of $n - r' + 1$ columns of $\Omega_{(1)}$ and $0 < \epsilon < 1$ be given. Then, with probability at least $1 - \frac{\epsilon}{k}$, every **proper** subset $\Omega''_{(1)}$ of columns of $\Omega'_{(1)}$ satisfies*

$$m_1(\Omega'') - r' \geq t, \tag{35}$$

where t is the number of columns of $\Omega''_{(1)}$ and $m_1(\Omega'')$ is the number of nonzero rows of $\Omega''_{(1)}$.

Proof Note that (34) results the following

$$l > \max \left\{ 9 \ \log\left(\frac{n}{\frac{\epsilon}{n}}\right) + 3 \ \log\left(\frac{k}{\frac{\epsilon}{n}}\right) + 6, 2r' \right\}. \tag{36}$$

Consider $n - r'$ columns of $\Omega'_{(1)}$. According to Lemma 5, with probability at least $1 - \frac{\epsilon}{nk}$, any subset of columns $\Omega''_{(1)}$ of these $n - r'$ particular columns of $\Omega'_{(1)}$ satisfies (35). Since there are n possible subsets of columns of $\Omega'_{(1)}$ with $n - r'$ columns, with probability at least $1 - \frac{\epsilon}{k}$, every **proper** subset $\Omega''_{(1)}$ of columns of $\Omega'_{(1)}$ satisfies (35). □

Lemma 13 *Assume that $r' \leq \frac{n}{6}$ and let $j \in \{1, 2, \ldots, d-1\}$ be a fixed number and also $0 < \epsilon < 1$ be given. Consider an arbitrary set $\widetilde{\Omega}'_{(j)}$ of $n - r'$ columns of $\widetilde{\Omega}_{(j)}$ (j-th TT unfolding of Ω). Assume that $n > \max\{400, \sum_{k=1}^{d-1} r_{k-1} r_k\}$, and also each column of $\widetilde{\Omega}_{(j)}$ includes at least l nonzero entries, where*

$$l > \max \left\{ 63 \ \log\left(\frac{n}{\epsilon}\right) + 9 \ \log\left(\frac{2r}{\epsilon}\right) + 18, 6r' \right\}, \tag{37}$$

where $r \leq \sum_{k=1}^{d-1} r_{k-1}r_k$ *(recall that* $r_0 = r_d = 1$*). Then, with probability at least* $1 - \frac{\epsilon}{2r}$*, each column of* $\widetilde{\Omega}'_{(j)}$ *includes more than* $\max\left\{21 \log\left(\frac{n}{\epsilon}\right) + 3 \log\left(\frac{2r}{\epsilon}\right) + 6, 2r'\right\}$ *observed entries of* Ω *with different values of the* i*-th coordinate,* $1 \leq i \leq j$*.*

Proof The proof is similar to the proof of Lemma 6 and the only difference is in the calculations of $P(\zeta)$, where for this lemma $n > 400$ is needed instead of $n > 200$. $\square$

Lemma 14 *Let* $j \in \{1, 2, \ldots, d-1\}$ *be a fixed number and also* $0 < \epsilon < 1$ *be given. Assume that* $r'_i \leq \frac{n}{6}$*, where* r'_i *is rational and non-integer and also* $i \in \{1, \ldots, j\}$*. Consider a matrix* $\widetilde{\Omega}'_{(j)}$ *composed from* $n - \lfloor r'_i \rfloor$ *arbitrary columns of* $\widetilde{\Omega}_{(j)}$*. Assume that* $n > \max\{400, \sum_{k=1}^{d-1} r_{k-1}r_k\}$*, and also each column of* $\widetilde{\Omega}_{(j)}$ *includes at least* l *nonzero entries, where*

$$l > \max\left\{63 \log\left(\frac{n}{\epsilon}\right) + 9 \log\left(\frac{2r}{\epsilon}\right) + 18, 6\lceil r'_i \rceil\right\}, \tag{38}$$

where $r \leq \sum_{k=1}^{d-1} r_{k-1}r_k$ *(recall that* $r_0 = r_d = 1$*). Then, with probability at least* $1 - \frac{\epsilon}{r}$*, every* ***proper*** *subset* $\widetilde{\Omega}''_{(j)}$ *of columns of* $\widetilde{\Omega}'_{(j)}$ *satisfies*

$$m_i(\Omega'') - \lceil r'_i \rceil \geq t, \tag{39}$$

where t *is the number of columns of* $\widetilde{\Omega}''_{(j)}$ *and* Ω'' *is the corresponding tensor such that* $\widetilde{\Omega}''_{(j)}$ *is the* j*-th TT unfolding of* Ω''*.*

Proof Each column of $\widetilde{\Omega}_{(j)}$ includes n^j entries and they can be represented by $(x_1, \ldots, x_j)$ for $1 \leq x_k \leq n$ and $1 \leq k \leq j$, where x_k denotes the k-th coordinate of the corresponding entry. According to Lemma 13, with probability at least $1 - \frac{\epsilon}{2r}$, each column of $\widetilde{\Omega}_{(j)}$ includes more than $\max\left\{21 \log\left(\frac{n}{\epsilon}\right) + 3 \log\left(\frac{2r}{\epsilon}\right) + 6, 2\lceil r'_i \rceil\right\}$ observed entries with different values of the i-th coordinate. Therefore, as $\lceil r'_i \rceil = \lfloor r'_i \rfloor + 1$ and according to Lemma 12, with probability at least $\left(1 - \frac{\epsilon}{2r}\right)^2$ which is more than $1 - \frac{\epsilon}{r}$, every **proper** subset $\widetilde{\Omega}''_{(j)}$ of columns of $\widetilde{\Omega}'_{(j)}$ satisfies (39). $\square$

Theorem 5 *Define* $m = \sum_{k=1}^{d-2} r_{k-1}r_k$*,* $M = n\sum_{k=1}^{d-2} r_{k-1}r_k - \sum_{k=1}^{d-2} r_k^2$ *and* $r' = \max\left\{\frac{r_1}{r_0}, \ldots, \frac{r_{d-2}}{r_{d-3}}\right\}$*. Assume that* $n > \max\{m + d, 400\}$ *and* $r' \leq \min\{\frac{n}{6}, r_{d-2}\}$ *hold and also let* $0 < \epsilon < 1$ *be given. Moreover, assume that each column of* $\widetilde{\Omega}_{(d-2)}$ *includes at least* l *nonzero entries, where*

$$l > \max\left\{63 \log\left(\frac{4n}{\epsilon}\right) + 9 \log\left(\frac{8M}{\epsilon}\right) + 18, 6r_{d-2}\right\}. \tag{40}$$

Then, with probability at least $1 - \epsilon$*, there exists only one completion of the sampled tensor* $\mathcal{U}$ *with rank vector* $(r_1, r_2, \ldots, r_{d-1})$*.*

Proof According to Theorem 3, with probability at least $1 - \frac{\epsilon}{4}$, condition (i) in the statement of Theorem 4 holds true. Moreover, as $M > d$ and according to Lemma 14, with probability at least $1 - \frac{\epsilon}{2d}$, condition (ii) is satisfied for each i. Therefore, with probability at least $1 - \epsilon$, conditions (i) and (ii) in the statement of Theorem 4 hold. $\square$

Remark 5 A tensor $\mathcal{U}$ that satisfies the properties in the statement of Theorem 5 requires

$$n^2 \max\left\{63\ \log\left(\frac{4n}{\epsilon}\right) + 9\ \log\left(\frac{8M}{\epsilon}\right) + 18, 6r_{d-2}\right\} \tag{41}$$

samples to be uniquely completable with probability at least $1-\epsilon$ since $\widetilde{\Omega}_{(d-2)}$ has n^2 columns, where $M = n\sum_{k=1}^{d-2} r_{k-1}r_k - \sum_{k=1}^{d-2} r_k^2$. Note that the number of samples given in Theorem 3 of [37] results in both finite and unique completability, and therefore the number of samples required by the TT unfolding approach given in Remark 3 is for both finite and unique completability.

Corollary 3 *Define* $m = \sum_{k=1}^{d-2} r_{k-1}r_k$, $M = n\sum_{k=1}^{d-2} r_{k-1}r_k - \sum_{k=1}^{d-2} r_k^2$ *and* $r' = \max\left\{\frac{r_1}{r_0}, \ldots, \frac{r_{d-2}}{r_{d-3}}\right\}$. *Assume that* $n > \max\{m+d, 400\}$ *and* $r' \leq \min\{\frac{n}{6}, r_{d-2}\}$ *hold and also let* $0 < \epsilon < 1$ *be given. Moreover, assume that the sampling probability satisfies*

$$p > \frac{1}{n^{d-2}} \max\left\{63\ \log\left(\frac{4n}{\epsilon}\right) + 9\ \log\left(\frac{8M}{\epsilon}\right) + 18, 6r_{d-2}\right\} + \frac{1}{\sqrt[4]{n^{d-2}}} \tag{42}$$

Then, with probability at least $(1-\epsilon)\left(1 - \exp(-\frac{\sqrt{n^{d-2}}}{2})\right)^{n^2}$, $\mathcal{U}$ *is uniquely completable.*

Proof Using Theorem 5, the proof is similar to the proof of Lemma 2. □

6 Numerical comparisons

In this section, we compute the total number of samples that is required for finiteness/uniqueness using an example to compare the TT unfolding approach and the TT

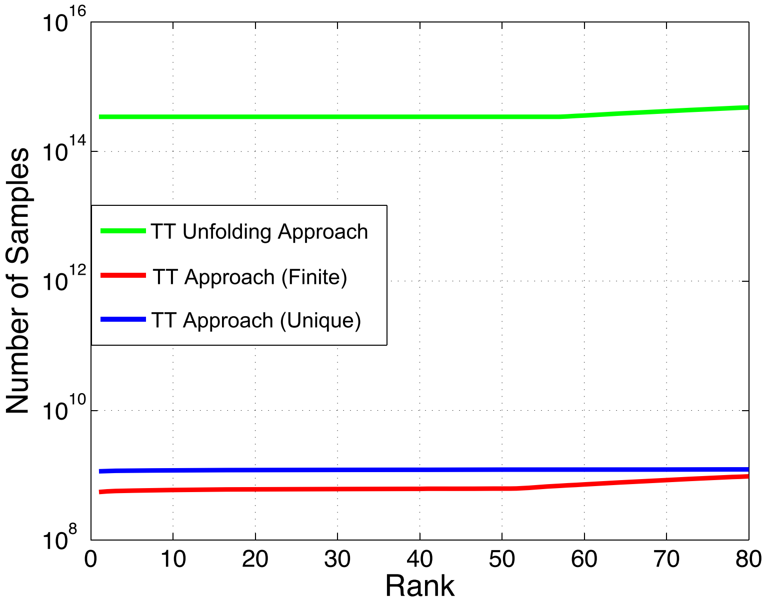

Fig. 1 Lower bounds on the number of samples for a 7-way tensor with rank vector $(r, 2r, 3r, 3r, 2r, r)$

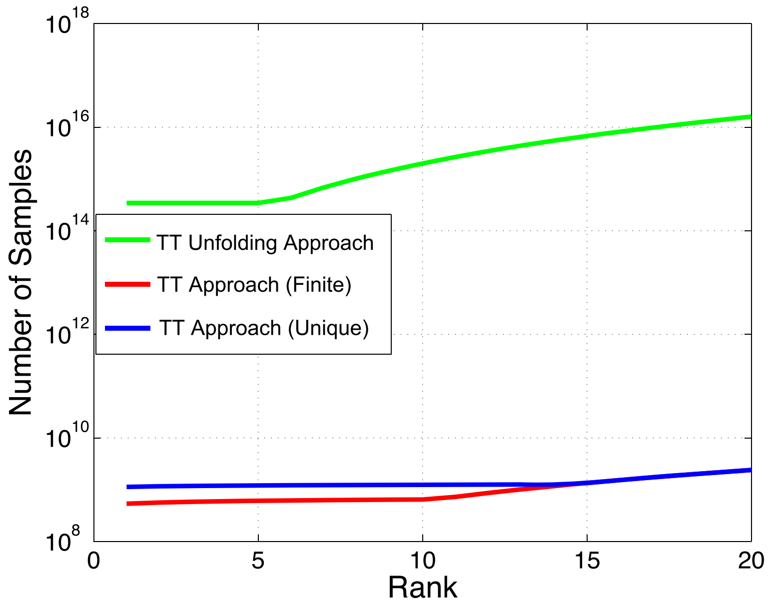

Fig. 2 Lower bounds on the number of samples for a 7-way tensor with rank vector $(r, r^2, r^3, r^3, r^2, r)$

approach. In this numerical example, we consider a 7-way tensor $\mathcal{U}$ $(d = 7)$ such that each dimension size is $n = 10^3$. We also consider the TT rank $(r_1, r_2, \ldots, r_6) = (r, 2r, 3r, 3r, 2r, r)$ and $(r_1, r_2, \ldots, r_6) = (r, r^2, r^3, r^3, r^2, r)$ in Figs. 1 and 2, respectively. Figures 1 and 2 plot the bounds given in Remark 3 (TT unfolding approach for either finite or unique completability), Remark 4 (TT approach for finite completability), and Remark 5 (TT approach for unique completability) for the corresponding rank vector, where $\epsilon = 0.001$. We change the value of r from 1 to 80 which is denoted by "rank" in Fig. 1 and from 1 to 20 in Fig. 2. It is seen that the number of samples required by the proposed TT approach is substantially lower than that is required by the TT unfolding approach.

Note that our probabilistic conditions in Theorems 3 and 5 provides lower bounds on the number of samples to guarantee finite and unique completability with high probability under uniform sampling. Although $\mathcal{O}(D)$, where D is the dimension of the TT-manifold, is an intuitive lower bound on the number of samples needed for completion, there is no theoretical guarantee.

Therefore, Figs. 1 and 2 indicate that our probabilistic conditions are tight (close to the absolute lower bound $\mathcal{O}(D)$) in some cases (e.g., all cases in Fig. 2 and when $r \geq 70$ in Fig. 1), and there might be room for potential improvement in other cases (e.g., when $r \leq 50$ in Fig. 1).

7 Conclusions

This paper characterizes fundamental conditions on the sampling pattern for finite completability of a low TT rank and partially sampled tensor through a new algebraic geometry analysis on the TT manifold. We defined a polynomial based on each sampled entry and exploited the structure of the TT decomposition to study the algebraic independence of these polynomials based on the locations of the samples. We also developed a canonical structure on the TT decomposition, which can be treated as an equivalence class that partitions all TT decompositions of one particular tensor to different classes. This equivalence class is helpful to study the algebraic independence of the defined polynomials. Using the developed tools on the TT manifold, we characterized the maximum number of algebraically independent polynomials among all the defined polynomials in terms of a simple geometric structure of the sampling pattern. Our analysis results in the following fundamental conditions for low-TT-rank tensor completion: (i) The necessary and sufficient deterministic conditions on the sampling pattern, under which there are only finite completions given the TT rank, (ii) Deterministic sufficient conditions on the sampling pattern, under which there exists exactly one completion given the TT rank, (iii) Lower bounds on the number of samples that leads to finite/unique completability with high probability. Although this work provides only lower bounds on the sampling rate to ensure unique (finite) completability, in [7], we developed completion algorithms based on Newton's method to solve the system of polynomials resulted from the rank decomposition and the sampled entries. It was observed that the proposed approach performs much better than the conventional completion methods such as alternating minimization and nuclear norm minimization when the sampling rate is very low and close to the lower bound given by the probabilistic analysis. Moreover, the analysis in this work has been used in [6] to approximate the TT-rank of a sampled tensor.

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Appendix: A Canonical Decomposition and The Degree of Freedom

We are interested in providing a structure on the decomposition $\mathbb{U}$ such that there is one decomposition among all possible decompositions of the sampled tensor $\mathcal{U}$ that captures the structure. Before describing such a structure on TT decomposition, we start with a similar structure for matrix decomposition.

Lemma 15 *Let $\mathbf{X}$ denote a generically chosen matrix from the manifold of $n_1 \times n_2$ matrices of rank r. Then, there exists a unique decomposition $\mathbf{X} = \mathbf{YZ}$ such that $\mathbf{Y} \in \mathbb{R}^{n_1 \times r}$, $\mathbf{Z} \in \mathbb{R}^{r \times n_2}$ and $\mathbf{Y}(1:r, 1:r) = \mathbf{I}_r$, where $\mathbf{Y}(1:r, 1:r)$ represents the submatrix of $\mathbf{Y}$ consists of the first r columns and the first r rows and $\mathbf{I}_r$ denotes the $r \times r$ identity matrix.*

Proof Weshow that there exists exactly one decomposition $\mathbf{X} = \mathbf{YZ}$ such that $\mathbf{Y}(1:r, 1:r) = \mathbf{I}_r$ with probability one. Considering the first r rows of $\mathbf{X} = \mathbf{YZ}$, we conclude $\mathbf{X}(1:r, :) = \mathbf{I}_r\mathbf{Z} = \mathbf{Z}$. Therefore, we need to show that there exists exactly one $\mathbf{Y}(r+1:n_1, :)$ such that $\mathbf{X}(r+1:n_1, :) = \mathbf{Y}(r+1:n_1, :)\mathbf{Z}$ or equivalently $\mathbf{X}(r+1:n_1, :)^\top = \mathbf{X}(1:r, :)^\top \mathbf{Y}(r+1:n_1, :)^\top$. It suffices to show that each column of $\mathbf{Y}(r+1:n_1, :)$ can be determined uniquely having $\mathbf{x} = \mathbf{X}(1:r, :)^\top \mathbf{y}$ where $\mathbf{x} \in \mathbb{R}^{n_2 \times 1}$ and $\mathbf{y} \in \mathbb{R}^{r \times 1}$. As $\mathbf{X}$ is a generically chosen $n_1 \times n_2$ matrix of rank r, we have rank $(\mathbf{X}(1:r, :)) = r$ with probability one. Hence, $\mathbf{x}(1:r) = \mathbf{X}(1:r, 1:r)^\top \mathbf{y}$ results in r independent degree-1 equations in terms of the r variables (entries of $\mathbf{y}$), and therefore $\mathbf{y}$ has exactly one solution with probability one. $\square$

Remark 6 Note that the genericity assumption is necessary as we can find counter examples for Lemma 15 in the absence of genericity assumption, e.g., it is easily verified that the following decomposition is not possible:

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 1 & 3 \\ \hline 1 & 2 & 1 & 3 \\ \hline 1 & 2 & 2 & 3 \\ \hline 1 & 2 & 2 & 3 \\ \hline \end{array} = \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline y_1 & y_2 \\ \hline y_3 & y_4 \\ \hline \end{array} \times \begin{array}{|c|c|c|c|}\hline x_1 & x_2 & x_3 & x_4 \\ \hline x_5 & x_6 & x_7 & x_8 \\ \hline \end{array}$$

Remark 7 Assume that $\mathbf{Q} \in \mathbb{R}^{r\times r}$ is an arbitrary given full rank matrix. Then, for any submatrix[1] $\mathbf{P} \in \mathbb{R}^{r\times r}$ of $\mathbf{Y}$, Lemma 15 also holds if we replace $\mathbf{Y}(1 : r, 1 : r) = \mathbf{I}_r$ by $\mathbf{P} = \mathbf{Q}$ in the statement. The proof is similar to the proof of Lemma 15 and thus it is omitted.

As mentioned earlier, similar to the matrix case, we are interested in obtaining a structure on TT decomposition of a tensor such that there exists one decomposition among all possible TT decompositions of a tensor that captures the structure. Hence, we define the following structure on the decomposition in order to characterize a condition on the sampling pattern to study the algebraic independency of the above-mentioned polynomials.

Definition 4 Consider any $d-1$ submatrices $\mathbf{P}_1, \ldots, \mathbf{P}_{d-1}$ of $\mathbf{U}^{(1)}, \mathbf{U}^{(2)}_{(2)}, \mathbf{U}^{(3)}_{(2)}, \ldots, \mathbf{U}^{(d-1)}_{(2)}$, respectively such that (i) $\mathbf{P}_i \in \mathbb{R}^{r_i\times r_i}$, $i = 1, \ldots, d-1$, (ii) the r_i columns of $\mathbf{U}^{(i)}_{(2)}$ corresponding to columns of $\mathbf{P}_i$ belong to r_i distinct rows of $\mathbf{U}^{(i)}_{(3)}$, $i = 2, \ldots, d-1$. Then, $\mathbb{U}$ is said to have a proper structure if $\mathbf{P}_i$ is full rank, $i = 1, \ldots, d$.[2]

Define the matrices $\mathbf{P}^{\text{can}}_1, \ldots, \mathbf{P}^{\text{can}}_{d-1}$ such that for any $1 \leq x_i \leq r_i$ and any $1 \leq x'_i \leq r_i$ we have:

$$\mathbf{P}^{\text{can}}_i(x_i, k_i) = \mathcal{U}^{(i)}(1, x_i, k_i) \in \mathbb{R}^{r_i\times r_i}, \qquad i = 2, \ldots, d-1, \tag{43}$$

and

$$\mathbf{P}^{\text{can}}_1(x_1, k_1) = \mathcal{U}^{(1)}(x_1, k_1) \in \mathbb{R}^{r_1\times r_1}. \tag{44}$$

It is easy to verify that $\mathbf{P}^{\text{can}}_1, \ldots, \mathbf{P}^{\text{can}}_{d-1}$ satisfy properties (i) and (ii) in Definition 4.

Definition 5 (Canonical basis) We call $\mathbb{U}$ a canonical decomposition if for $i = 1, \ldots, d$ we have $\mathbf{P}^{\text{can}}_i = \mathbf{I}_{r_i}$, where $\mathbf{I}_{r_i}$ is the $r_i \times r_i$ identity matrix.

Lemma 16 *Consider the TT decomposition in* (1)*. Then,* $\mathbf{U}^{(1)} \in \mathbb{R}^{n_1\times r_1}$, $\mathbf{U}^{(d)} \in \mathbb{R}^{r_{d-1}\times n_d}$, $\mathbf{U}^{(i)}_{(1)} \in \mathbb{R}^{r_{i-1}\times n_i r_i}$ *and* $\mathbf{U}^{(i)}_{(3)} \in \mathbb{R}^{r_i\times r_{i-1}n_i}$, $i = 2, \ldots, d-1$, *are full rank matrices.*

Proof In general, besides the separation rank $(r_1, \ldots, r_{d-1})$, we may be able to obtain a TT decomposition for other vectors $(r'_1, \ldots, r'_{d-1})$ as well. However, according to [20] among all possible TT decomposition for different values of r'_i's, $r'_i = \text{rank}(\widetilde{\mathbf{U}}_{(i)}) = r_i$, $i = 1, \ldots, d-1$, is minimal, in the sense that there does not exist any decomposition with

[1]Specified by a subset of rows and a subset of columns (not necessarily consecutive).

[2]Since $\mathcal{U}^{(1)}$ and $\mathcal{U}^{(d)}$ are two-way tensors, i.e., matrices we also denote them by $\mathbf{U}^{(1)}$ and $\mathbf{U}^{(d)}$. Moreover, since $\mathcal{U}^{(i)}$ is a three-way tensor, $\widetilde{\mathbf{U}}^{(i)}_{(2)} = \mathbf{U}^{(i)\top}_{(3)}$, $i = 2, \ldots, d-1$.

r_i''s such that $r_i' \leq r_i$ for $i = 1, \ldots, d-1$ and $r_i' < r_i$ for at least one $i \in \{1, \ldots, d-1\}$. By contradiction, assume that $\mathbf{U}_{(1)}^{(i+1)}$ is not full rank. Then, rank $\left(\widetilde{\mathbf{U}}_{(2)}^{(i)} \mathbf{U}_{(1)}^{(i+1)}\right) < r_i$.

Let $\mathbf{X}$ denote the matrix $\widetilde{\mathbf{U}}_{(2)}^{(i)} \mathbf{U}_{(1)}^{(i+1)}$. Since rank $(\mathbf{X}) = r_i' < r_i$, there exists a decomposition $\mathbf{X} = \widetilde{\mathbf{U}}_{(2)}^{(i)'} \mathbf{U}_{(1)}^{(i+1)'}$ such that $\widetilde{\mathbf{U}}_{(2)}^{(i)'} \in \mathbb{R}^{r_{i-1}n_i \times r_i'}$ and also $\mathbf{U}_{(1)}^{(i+1)'} \in \mathbb{R}^{r_i' \times n_{i+1} r_{i+1}}$. Hence, the existence of the TT decomposition with $\mathcal{U}^{(i)}$ and $\mathcal{U}^{(i+1)}$ replaced by $\mathcal{U}^{(i)'}$ and $\mathcal{U}^{(i+1)'}$ contradicts the above-mentioned minimum property of the separation rank. Note that for a three-way tensor, the second TT unfolding is the transpose of the third Tucker unfolding, and therefore rank $\left(\widetilde{\mathbf{U}}_{(2)}^{(i)}\right) = $ rank $\left(\mathbf{U}_{(3)}^{(i)}\right)$ and the rest of the cases can be verified similarly. □

Lemma 17 *Assume that* $\mathbf{Q}_i \in \mathbf{R}^{r_i \times r_i}$ *is an arbitrary given full rank matrix,* $1 \leq i \leq d-1$. *Consider a set of matrices* $\mathbf{P}_1, \ldots, \mathbf{P}_{d-1}$ *that satisfy properties (i) and (ii) in Definition 4. Then, there exists exactly one decomposition* $\mathbb{U}$ *of the sampled tensor* $\mathcal{U}$ *such that* $\mathbf{P}_i = \mathbf{Q}_i$, $i = 1, \ldots, d-1$.

Proof Consider an arbitrary decomposition $\mathbb{U}$ of the sampled tensor $\mathcal{U}$. Let $\mathcal{A}^{(i)} = \mathcal{U}^{(i)} \mathcal{U}^{(i+1)} \in \mathbb{R}^{r_{i-1} \times n_i \times n_{i+1} \times r_{i+1}}$, $i = 1, \ldots, d-1$, where the above multiplication is the same tensor multiplication in TT decomposition (1). Note that for a three-way tensor, the second TT unfolding is the transpose of the third Tucker unfolding, and therefore their ranks are the same. According to Lemma 16, rank $\left(\mathbf{U}^{(1)}\right) = $ rank $\left(\mathbf{U}_{(1)}^{(2)}\right) = r_1$, rank $\left(\widetilde{\mathbf{U}}_{(2)}^{(2)}\right) = $ rank $\left(\mathbf{U}_{(1)}^{(3)}\right) = r_2, \ldots,$ and rank $\left(\widetilde{\mathbf{U}}_{(2)}^{(d-1)}\right) = $ rank $\left(\mathbf{U}^{(d)}\right) = r_{d-1}$.

As a result, we have rank $\left(\mathbf{U}^{(1)} \mathbf{U}_{(1)}^{(2)}\right) = r_1$, rank $\left(\widetilde{\mathbf{U}}_{(2)}^{(2)} \mathbf{U}_{(1)}^{(3)}\right) = r_2, \ldots,$ rank $\left(\widetilde{\mathbf{U}}_{(2)}^{(d-1)} \mathbf{U}^{(d)}\right) = r_{d-1}$. Observe that $\widetilde{\mathbf{U}}_{(2)}^{(i)} \mathbf{U}_{(1)}^{(i+1)} = \widetilde{\mathbf{A}}_{(2)}^{(i)}$, and therefore rank $\left(\widetilde{\mathbf{A}}_{(2)}^{(i)}\right) = r_i$ for $i = 2, \ldots, d-2$ and similarly rank $\left(\widetilde{\mathbf{A}}_{(1)}^{(1)}\right) = r_1$ and rank $\left(\widetilde{\mathbf{A}}_{(2)}^{(d)}\right) = r_d$. According to Lemma 15 and Remark 7, for an $n_1 \times n_2$ matrix $\mathbf{X}$ of rank r there exists a unique decomposition $\mathbf{X} = \mathbf{X}_1 \mathbf{X}_2$ such that $\mathbf{X}_1 \in \mathbb{R}^{n_1 \times r}$ and $\mathbf{X}_2 \in \mathbb{R}^{r \times n_2}$ and an arbitrary $r \times r$ submatrix of $\mathbf{X}_1$ is equal to the given $r \times r$ full rank matrix.

We claim that there exist $(\mathcal{V}^{(i)}, \mathcal{V}^{(i+1)})$ such that $\mathcal{V}^{(i)} \mathcal{V}^{(i+1)} = \mathcal{A}^{(i)}$ and the corresponding submatrix $\mathbf{P}_i$ is equal to the given full rank matrix $\mathbf{Q}_i$, $i = 1, \ldots, d-1$. We repeat this procedure for each $i = 1, \ldots, d-1$ and update two core tensors of TT decomposition $(\mathcal{V}^{(i)}, \mathcal{V}^{(i+1)})$ at iteration i and at the end, we obtain a TT decomposition that has the mentioned structure in the statement of Lemma 17. In the following we show the existence of such $(\mathcal{V}^{(i)}, \mathcal{V}^{(i+1)})$ at each iteration. At step one, we find $(\mathcal{V}^{(1)}, \mathcal{V}^{(2)})$ such that $\mathcal{V}^{(1)} \mathcal{V}^{(2)} = \mathcal{A}^{(1)}$ and the corresponding submatrix $\mathbf{P}_1$ of $\mathcal{V}^{(1)}$ is equal to $\mathbf{Q}_1$. We update the decomposition with $\mathcal{U}^{(1)}$ and $\mathcal{U}^{(2)}$ replaced by $\mathcal{V}^{(1)}$ and $\mathcal{V}^{(2)}$, and therefore we obtain a new decomposition $\mathbb{U}^1$ of the sampled tensor $\mathcal{U}$ such that the submatrix of $\mathcal{V}^{(1)}$ corresponding to $\mathbf{P}_1$ is equal to $\mathbf{Q}_1$. Then, in step 2 we consider $\mathcal{A}^{(2)}$ and similarly we update the second and third factor of the decomposition obtained in the last step. Eventually after $d-1$ steps, we obtain a decomposition of the sampled tensor $\mathcal{U}$ that $\mathbf{P}_i = \mathbf{Q}_i$, $i = 1, \ldots, d-1$. To show the uniqueness of such decomposition, we show that each core tensor of the TT decomposition can be determined uniquely. Remark 7 for rank component r_1 results that $\mathcal{U}^{(1)}$ and the multiplication of the rest of the core tensors of the TT decomposition can be determined uniquely. By repeating this procedure for other rank components the uniqueness of such decomposition can be verified by showing the uniqueness of the core tensors one by one. □

Lemma 17 leads to the fact that given $\mathcal{U}^{(d)}$, the dimension of all tuples $(\mathcal{U}^{(1)}, \ldots, \mathcal{U}^{(d-1)})$ that satisfy TT decomposition is $\sum_{i=1}^{d-1} r_{i-1} n_i r_i - \sum_{i=1}^{d-1} r_i^2$, as $\sum_{i=1}^{d-1} r_{i-1} n_i r_i$ is the total number of entries of $(\mathcal{U}^{(1)}, \ldots, \mathcal{U}^{(d-1)})$ and $\sum_{i=1}^{d-1} r_i^2$ is the total number of the entries of the pattern or structure that is equivalent to the uniqueness of TT decomposition. We make the following assumption which will be referred to, when it is needed.

Note that for Lemma 17, we need the strong low-rankness assumption $r_i \leq n_i$. However, these results are also consequences of the analysis in [20]. The purpose is to present a simple and intuitive proof.

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