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Fundamental conditions on the sampling pattern for union of low-rank subspaces retrieval

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Abstract

This paper is concerned with investigating the fundamental conditions on the locations of the sampled entries, i.e., sampling pattern, for finite completability of a matrix that represents the union of several subspaces with given ranks. In contrast with the existing analysis on Grassmannian manifold for the conventional matrix completion, we propose a geometric analysis on the manifold structure for the union of several subspaces to incorporate all given rank constraints simultaneously. In order to obtain the deterministic conditions on the sampling pattern, we characterizes the algebraic independence of a set of polynomials defined based on the sampling pattern, which is closely related to finite completion. We also give a probabilistic condition in terms of the number of samples per column, i.e., the sampling probability, which leads to finite completability with high probability. Furthermore, using the proposed geometric analysis for finite completability, we characterize sufficient conditions on the sampling pattern that ensure there exists only one completion for the sampled data.

Keywords Low-rank data completion · Matrix completion · Manifold · Union of subspaces · Finite completability · Unique completability

Mathematics Subject Classification (2010) 68W01

1 Introduction

Low-rank matrix completion has received significant recent attention and finds applications in various areas including image or signal processing [12, 13, 24], data mining [15], network coding [23], power systems [20, 21], etc., and one of the main reasons of such versatility is that matrices consisting of the real-world data typically possess a low-rank structure.

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Recently, several approaches are proposed to tackle a more complicated version of the low-rank matrix completion problem named the union of low-rank subspaces completion problem, where each column belongs to a subspace among multiple low-rank subspaces, and therefore the whole matrix belongs to the union of those multiple low-rank subspaces [17, 19]. Also, in many applications, the subspace clustering problem is of importance [10, 16, 30–32, 35]. However, in this paper, we consider the completion problem and not subspace clustering, where we assume that the subspace that each column is chosen from is specified.

In general, the existing methods in the literature on low-rank matrix and tensor completion can be categorized into several approaches, including those based on convex relaxation of matrix rank [3, 11–14] or different convex relaxations of tensor ranks [18, 25, 36, 37, 39], those based on alternating minimization [29, 40], and other heuristics [2, 9, 22, 26–28]. Note that the optimization-based approaches to low-rank data completion require strong assumptions on the correlations of the values of all entries (such as coherence). On the other hand, recently, fundamental conditions on the sampling pattern (independent from the values of entries) that guarantee the existence of finite or unique number of completions, have been investigated for single-view and multi-view matrix completion [5, 7, 34], low canonical polyadic (CP) rank tensor completion [4], low Tucker rank tensor completion [1], data clustering [6, 33], and rank determination for low-rank data completion [8]. In this paper, we study these fundamental conditions for matrices obtained from the union of several low-rank subspaces, i.e., we propose a geometric analysis on the manifold structure for union of low-rank subspaces to study the mentioned problem. This work is inspired by [34], where the analysis on Grassmannian manifold is proposed to solve similar problems for a matrix. Specifically, in [34] a novel approach is proposed to consider the rank factorization of a matrix and to treat each observed entry as a polynomial in terms of the entries of the components of the rank factorization. Then, the algebraic independence among the mentioned polynomials is studied. In this paper, we consider the union of subspaces with special structure. One may apply the method in [34] on each of the subspaces, but we propose an efficient method to obtain stronger conditions on the sampling pattern by incorporating all the rank constraints at the same time instead of applying the matrix analysis several times separately.

The remainder of this paper is organized as follows. In Section 2, the preliminaries and problem statements are presented. In Section 3, the deterministic sampling patterns that ensure finite completability are found. In Section 4, we provide the sampling probability that ensures the obtained deterministic sampling patterns in Section 3 hold with high probability. The deterministic sampling patterns and the sampling probability that ensure unique completability are characterized in Section 5. Some numerical results are provided in Section 6. Finally, Section 7 concludes the paper.

2 Preliminaries

2.1 Problem statement

Assume that $k \geq 2$ is a fixed integer and $n_1 < n_2 < \cdots < n_k$ are given integers. Let $\mathbf{U} \in \mathbb{R}^{m \times n_k}$ be a sampled matrix and denote the matrix consisting of the first n_i columns of $\mathbf{U}$ by $\mathbf{U}_i$, $i = 1, \ldots, k$. Hence, note that $\mathbf{U} = \mathbf{U}_k$ and this is shown in Fig. 1. Moreover, assume that $\text{rank}(\mathbf{U}_i) = r_i$, $i = 1, \ldots, k$. For notational simplicity assume $n_0 = r_0 = 0$ and $\mathbf{U}_0 = \emptyset$. Let $\text{Gr}(r_i, \mathbb{R}^m)$ denote the Grassmannian of r_i-dimensional subspaces of $\mathbb{R}^m$ such

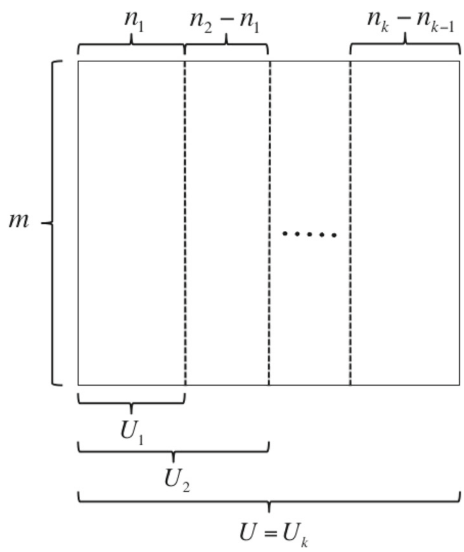


Fig. 1 The structure of the sampled matrix $\mathbf{U}$

that the space corresponding to r_i is a subspace of the space corresponding to r_{i+1}. Assume that $\mathbb{P}_{G_i}$ denotes the uniform measure on $\mathrm{Gr}(r_i, \mathbb{R}^m)$ and $\mathbb{P}_{\theta_i}$ denotes the Lebesgue measure on $\mathbb{R}^{r_i \times s_i}$, where $s_i = n_i - n_{i-1}$ for $i = 1, \dots, k$. In this paper, we assume that the first n_1 columns of $\mathbf{U}$ are chosen generically from the manifold of $m \times n_1$ matrices of rank r_1, i.e., the entries of the first n_1 columns of $\mathbf{U}$ are drawn independently with respect to Lebesgue measure on the corresponding manifold. And in general the columns number $n_{i-1} + 1$ to n_i of $\mathbf{U}$ are chosen generically from the manifold of $m \times (n_i - n_{i-1})$ matrices of rank r_i, i.e., the entries of the columns number $n_{i-1} + 1$ to n_i of $\mathbf{U}$ are drawn independently with respect to Lebesgue measure $\mathbb{P}_{\theta_i}$ on the corresponding manifold, $i = 2, \dots, k$. Also, in this paper the probability measure is $\Pi_{i=1}^{k} \mathbb{P}_{G_i} \mathbb{P}_{\theta_i}$.

Note that the problem of union of two low-rank subspaces ($k = 2$) is different from the multi-view matrix completion studied in [7], as the multi-view matrix completion has one extra rank constraint that is independent from one of the rank constraints.

Let $\mathbf{\Omega}$ denote the binary sampling pattern matrix that is of the same size as $\mathbf{U}$. The entries of $\mathbf{\Omega}$ that correspond to the observed entries of $\mathbf{U}$ are equal to 1 and the rest of the entries are set as 0. Assume that the entries of $\mathbf{U}$ are sampled independently with probability p. This paper is mainly concerned with treating the following three problems.

Problem (i): Given the rank constraints $\mathrm{rank}(\mathbf{U}_i) = r_i$, $i = 1, \dots, k$, characterize the conditions on the sampling pattern $\mathbf{\Omega}$, under which there exist at most finitely many completions of $\mathbf{U}$ with probability one.

Problem (ii): Given the rank constraints $\mathrm{rank}(\mathbf{U}_i) = r_i$, $i = 1, \dots, k$, characterize sufficient conditions on the sampling pattern $\mathbf{\Omega}$, under which there exist only one completion of $\mathbf{U}$ with probability one.

Problem (iii): Provide a lower bound on the sampling probability p such that the deterministic conditions on the sampling pattern $\mathbf{\Omega}$ for finite/unique completability for Problems (i) and (ii) are satisfied with high probability (not with probability one anymore).

2.2 A motivating example

Note that applying the existing analysis on the Grassmannian manifold for each of the rank constraints individually results in a "*weak*" sufficient condition for finite/unique completability. Next, we provide an example to motivate our proposed analysis in this paper and to emphasize the exigency of our proposed analysis. Assume that $k = 2$, $\mathbf{U}_1 \in \mathbb{R}^{4\times 2}$, $\mathbf{U}_2 \in \mathbb{R}^{4\times 4}$, $r_1 = 1$ and $r_2 = 2$. Moreover, assume that

$$\Omega = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Then, we compare the following two approaches on this example: (i) applying the existing analysis on the Grassmannian manifold for each of the rank constraints individually, and (ii) applying our proposed analysis to take advantage of all rank constraints simultaneously.

Approach (i): We show that $\mathbf{U}_1$ and $\mathbf{U}_2$ are infinitely many completable under the rank constraints $r_1 = 1$ and $r_2 = 2$, respectively.

First, consider $\mathbf{U}_1$ under the constraint $\text{rank}(\mathbf{U}_1) = 1$. Observe that for any arbitrary value of $\mathbf{U}_1(4, 1)$, the second column of $\mathbf{U}_1$ can be obtained uniquely. Hence, there are infinitely many completions of $\mathbf{U}_1$. Second, consider $\mathbf{U}_2$ under the constraint $\text{rank}(\mathbf{U}_2) = 2$. Observe that for any arbitrary value of $\mathbf{U}_2(2, 2)$, the second column of $\mathbf{U}_2$ can be obtained uniquely. Hence, there are infinitely many completions of $\mathbf{U}_2$.

Approach (ii): We show that $\mathbf{U}_1$ and $\mathbf{U}_2$ are finitely many completable under the two rank constraints $r_1 = 1$ and $r_2 = 2$ simultaneously (this claim can be later verified using Theorem 1 as well). Genericity assumption results that the third and fourth columns of $\mathbf{U}_2$ are linearly independent, i.e., $\text{rank}(\mathbf{U}_2(:, 3 : 4)) = 2$, and therefore $\mathbf{U}_2(:, 3 : 4)$ is a basis for $\mathbf{U}$. Note that $\mathbf{U}_2(:, 3 : 4)$ is given, and therefore having $\mathbf{U}_2(1, 1)$ and $\mathbf{U}_2(2, 1)$ we can obtain the first column of $\mathbf{U}_2$ uniquely. Now that the first column of $\mathbf{U}_2$ (which is also the first column of $\mathbf{U}_1$) is obtained uniquely, using the fact $\text{rank}(\mathbf{U}_1) = 1$ we can obtain the second column of $\mathbf{U}_1$ uniquely. Hence, the sampled matrix $\mathbf{U}$ is uniquely completable.

Hence, this example illustrates that collapsing the problem of completability of $\mathbf{U}$ into several matrices analyses (for each rank constraint individually) results in loss of information and thus motivates the investigation of the manifold corresponding to the rank constraints $\text{rank}(\mathbf{U}_i) = r_i$ for $i = 1, \ldots, k$.

3 Deterministic conditions for finite completability

In Section 3.1, we study the geometry of the manifold corresponding to the union of subspaces to define an equivalence class to classify the bases such that each basis of the sampled data belongs to exactly one of the defined classes. To this end we characterize the canonical structure of the bases and show the uniqueness of canonical basis for the sampled data with probability one. In Section 3.2, we define a polynomial based on each observed entry and through studying the geometry of the manifold corresponding to the rank constraints, we transform the problem of finite completability of $\mathbf{U}$ to the problem of including a certain number of algebraically independent polynomials among the defined polynomials for the observed entries. In Section 3.3, a binary matrix is constructed based on the sampling pattern $\mathbf{\Omega}$, which allows us to study the algebraic independence of a subset of polynomials among all defined polynomials based on the samples. Finally, in Section 3.4, we characterize the condition on the sampling pattern for finite completability of the sampled data given the rank constraints.

3.1 Geometry

For each $\mathbf{U}$, there exist infinitely many rank decompositions, i.e., $\mathbf{V} \in \mathbb{R}^{m \times r_k}$ and $\mathbf{T} \in \mathbb{R}^{r_k \times n_k}$ such that $\mathbf{U} = \mathbf{VT}$. However, we are interested in obtaining the canonical basis $\mathbf{V}$ such that there exists exactly one rank decomposition with canonical basis. In other words, we want a pattern on the basis that plays role as an equivalence class such that there exists exactly one basis $\mathbf{V}$ for $\mathbf{U}$ in each class. We start by the following lemma which will be used characterize such an equivalence class.

Lemma 1 *There exists a matrix $\mathbf{V} \in \mathbb{R}^{m \times r_k}$ such that $\mathbf{U}_i$ belongs to the column span of the first r_i columns of $\mathbf{V}$, $i = 1, \ldots, k$. Note that $\mathbf{V}$ is a basis for $\mathbf{U}$ and we call such basis an "appropriate basis".*

Proof We construct such a matrix $\mathbf{V}$ by induction on i. In other words, in the i-th step, we construct $\mathbf{V}^i$ such that $\mathbf{U}_s$ belongs to the column span of the first r_s columns of $\mathbf{V}^i$, $s = 1, \ldots, i$. Note that for $i = 1$ it is straightforward to construct $\mathbf{V}^1$, which is simply a basis for $\mathbf{U}_1$. Induction hypothesis results in the matrix $\mathbf{V}^i$ with the mentioned properties and in order to complete the induction, we need to show the existence of a matrix $\mathbf{V}^{i+1}$ such that $\mathbf{U}_s$ belongs to the column span of the first r_s columns of $\mathbf{V}^i$, $s = 1, \ldots, i+1$.

We first claim that $\mathbf{V}^i$ belongs to the column span of $\mathbf{U}_{i+1}$. Note that according to the induction hypothesis, $\mathbf{V}^i$ is a basis for $\mathbf{U}_i$ and also $\mathbf{U}_i$ is a subset of columns of $\mathbf{U}_{i+1}$, which proves our claim. Let $\mathcal{S}_i$ denote the column span of $\mathbf{V}^i$, which is an r_i-dimensional space and $\mathcal{S}'_{i+1}$ denote the column span of $\mathbf{U}_{i+1}$, which is an r_{i+1}-dimensional space. As a result of our earlier claim, $\mathcal{S}_i$ is a subspace of $\mathcal{S}'_{i+1}$. Let $\mathcal{S}''_i$ denote the $(r_{i+1} - r_i)$-dimensional subspace of $\mathcal{S}'_{i+1}$ such that the union of $\mathcal{S}_i$ and $\mathcal{S}''_i$ is $\mathcal{S}'_{i+1}$.

Consider an arbitrary basis $\mathbf{V}^{i'} \in \mathbb{R}^{m \times (r_{i+1} - r_i)}$ for the space $\mathcal{S}''_i$. Observe that by putting together the columns of $\mathbf{V}^i$ and $\mathbf{V}^{i'}$, i.e., $\mathbf{V}^{i+1} = [\mathbf{V}^i | \mathbf{V}^{i'}]$, the new matrix $\mathbf{V}^{i+1} \in \mathbb{R}^{m \times r_{i+1}}$ is a basis for the space $\mathcal{S}'_{i+1}$. Therefore, $\mathbf{U}_{i+1}$ belongs to the column span of the first r_{i+1} columns of $\mathbf{V}^{i+1}$ since $\mathbf{V}^i$ has exactly r_{i+1} columns. Given the induction hypothesis, the proof is complete as $\mathbf{U}_s$ belongs to the column span of the first r_s columns of $\mathbf{V}^{i+1}$, $s = 1, \ldots, i+1$. □

Corollary 1 *There exists a rank decomposition $\mathbf{U} = \mathbf{VT}$, where $\mathbf{V} \in \mathbb{R}^{m \times r_k}$, $\mathbf{T} \in \mathbb{R}^{r_k \times n_k}$, $\mathbf{T}(r_1 + 1 : r_k, 1 : n_1) = \mathbf{0}_{(r_k - r_1) \times n_1}$, $\mathbf{T}(r_2 + 1 : r_k, n_1 + 1 : n_2) = \mathbf{0}_{(r_k - r_2) \times (n_2 - n_1)}$, ... and $\mathbf{T}(r_{k-1} + 1 : r_k, n_{k-1} + 1 : n_k) = \mathbf{0}_{(r_k - r_{k-1}) \times (n_k - n_{k-1})}$. We call such decomposition an "appropriate decomposition", which is shown in Fig. 2.*

Proof Note that $\mathbf{T} \in \mathbb{R}^{r_k \times n_k}$, $\mathbf{T}(r_1 + 1 : r_k, 1 : n_1) = \mathbf{0}_{(r_k - r_1) \times n_1}$ is equivalent to having that $\mathbf{U}_1$ belongs to the column span of the first r_1 columns of $\mathbf{V}$. Similarly, we can observe that the assumptions given in Corollary 1 are equivalent to the assumptions on the appropriate basis $\mathbf{V}$ in Lemma 1, and therefore according to Lemma 1, the proof is complete. □

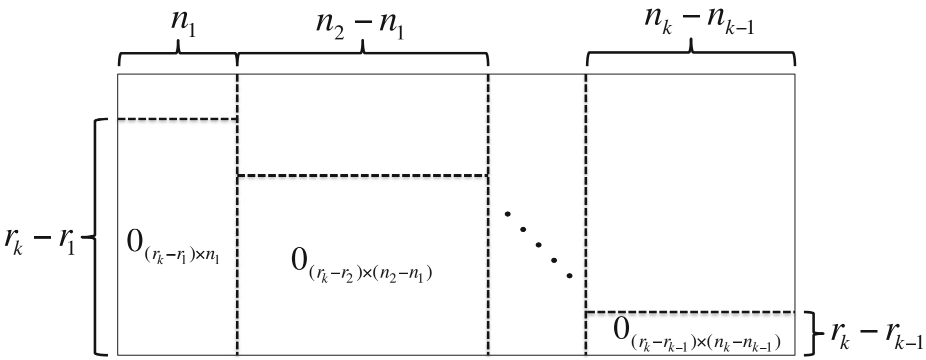
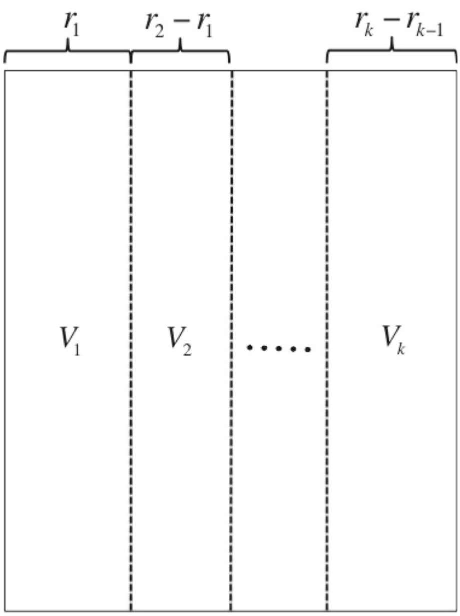


Fig. 2 A matrix $\mathbf{T}$ that satisfies the properties of an appropriate decomposition

From now on, we only consider appropriate decompositions. In fact, given Corollary 1, it is easy to verify that there exists infinitely many appropriate decompositions for $\mathbf{U}$. However, we are interested in having a canonical basis $\mathbf{V}$ so that for any $\mathbf{U}$ there exits exactly one appropriate decompositions satisfying the canonical structure.

Definition 1 For notational simplicity, we divide the columns of a basis $\mathbf{V} \in \mathbb{R}^{m\times r_k}$ for $\mathbf{U}$ as $\mathbf{V} = [\mathbf{V}_1|\ldots,|\mathbf{V}_k]$, where $\mathbf{V}_1 \in \mathbb{R}^{m\times r_1}$ denotes the first r_1 columns of $\mathbf{V}$, $\mathbf{V}_2 \in \mathbb{R}^{m\times(r_2-r_1)}$ denotes the next $(r_2 - r_1)$ columns of $\mathbf{V}$, ... and $\mathbf{V}_k \in \mathbb{R}^{m\times(r_k-r_{k-1})}$ denotes the next $(r_k - r_{k-1})$ columns of $\mathbf{V}$. This structure is shown in Fig. 3.

Fig. 3 An appropriate basis $\mathbf{V} = [\mathbf{V}_1|\ldots,|\mathbf{V}_k]$



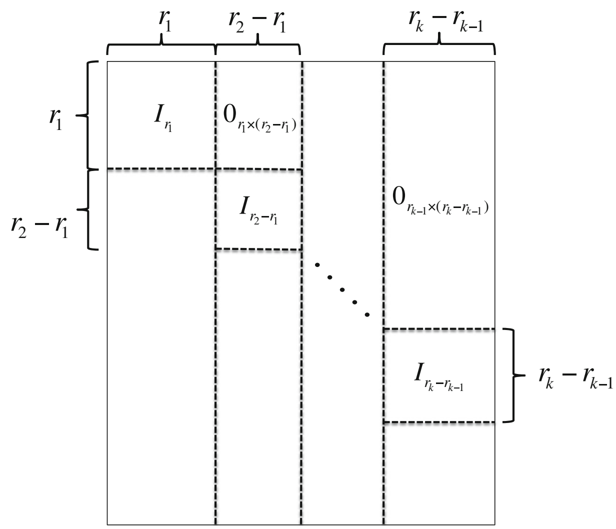


Fig. 4 A canonical basis

Definition 2 A basis $\mathbf{V} \in \mathbb{R}^{m \times r_k}$ for $\mathbf{U}$ has canonical structure if $\mathbf{V}(1 : r_1, 1 : r_1) = \mathbf{I}_{r_1}$, $\mathbf{V}(1 : r_2, r_1 + 1, r_2) = [\mathbf{0}_{(r_2-r_1)\times r_1} | \mathbf{I}_{(r_2-r_1)}]^\top$, … and $\mathbf{V}(1 : r_k, r_{k-1} + 1, r_k) = [\mathbf{0}_{(r_k-r_{k-1})\times r_{k-1}} | \mathbf{I}_{(r_k-r_{k-1})}]^\top$, as shown in Fig. 5.

The following lemma characterizes the relationship between appropriate bases, which will be used in Lemmas 3 and 4 (Fig. 4).

Lemma 2 *Consider an appropriate basis* $\mathbf{V} \in \mathbb{R}^{m \times r_k}$ *for* $\mathbf{U}$. *Then, the full rank matrix* $\mathbf{V}' \in \mathbb{R}^{m \times r_k}$ *is an appropriate basis for* $\mathbf{U}$ *if and only if there exist matrices* $\mathbf{A}_1 \in \mathbb{R}^{r_1 \times r_1}$, $\mathbf{A}_2 \in \mathbb{R}^{r_2 \times (r_2-r_1)}$, … *and* $\mathbf{A}_k \in \mathbb{R}^{r_k \times (r_k-r_{k-1})}$ *such that* $\mathbf{V}'_1\mathbf{A}_1 = \mathbf{V}_1$, $[\mathbf{V}'_1|\mathbf{V}'_2]\mathbf{A}_2 = \mathbf{V}_2$, … *and* $[\mathbf{V}'_1|\mathbf{V}'_2|\dots|\mathbf{V}'_k]\mathbf{A}_k = \mathbf{V}'\mathbf{A}_k = \mathbf{V}_k$.

Proof Assume that $\mathbf{V}'$ is an appropriate basis for $\mathbf{U}$. Then, the first r_1 columns of $\mathbf{V}'$, i.e., $\mathbf{V}'_1$, is a basis for the rank-r_1 matrix $\mathbf{U}_1$ and note that $\mathbf{V}$ is also an appropriate basis for $\mathbf{U}$. Therefore, $\mathbf{V}'_1$ and $\mathbf{V}_1$ span the same r_1-dimensional space, and therefore each column of $\mathbf{V}_1$ can be written as a linear combination of the columns of $\mathbf{V}'_1$, i.e., $\mathbf{V}'_1\mathbf{A}_1 = \mathbf{V}_1$ for some $\mathbf{A}_1 \in \mathbb{R}^{r_1 \times r_1}$. Similarly, $[\mathbf{V}'_1|\mathbf{V}'_2]$ and $[\mathbf{V}_1|\mathbf{V}_2]$ span the same r_2-dimensional space since both of them are a basis for the rank-r_2 matrix $\mathbf{U}_2$. As a result, each column of $\mathbf{V}_2$ can be written as a linear combination of the columns of $[\mathbf{V}'_1|\mathbf{V}'_2]$, i.e., $[\mathbf{V}'_1|\mathbf{V}'_2]\mathbf{A}_2 = \mathbf{V}_2$ for some $\mathbf{A}_2 \in \mathbb{R}^{r_2 \times (r_2-r_1)}$. Similarly, we can show $[\mathbf{V}'_1|\mathbf{V}'_2|\dots|\mathbf{V}'_k]\mathbf{A}_k = \mathbf{V}'\mathbf{A}_k = \mathbf{V}_k$ for some $\mathbf{A}_k \in \mathbb{R}^{r_k \times (r_k-r_{k-1})}$.

To prove the other direction of the statement, assume that there exist matrices $\mathbf{A}_1 \in \mathbb{R}^{r_1 \times r_1}$, $\mathbf{A}_2 \in \mathbb{R}^{r_2 \times (r_2-r_1)}$, … and $\mathbf{A}_k \in \mathbb{R}^{r_k \times (r_k-r_{k-1})}$ such that $\mathbf{V}'_1\mathbf{A}_1 = \mathbf{V}_1$, $[\mathbf{V}'_1|\mathbf{V}'_2]\mathbf{A}_2 = \mathbf{V}_2$, … and $[\mathbf{V}'_1|\mathbf{V}'_2|\dots|\mathbf{V}'_k]\mathbf{A}_k = \mathbf{V}'\mathbf{A}_k = \mathbf{V}_k$. Note that $\mathbf{V}$ is an appropriate basis for $\mathbf{U}$, and therefore the assumption $\mathbf{V}'_1\mathbf{A}_1 = \mathbf{V}_1$ results that $\mathbf{V}'_1$ and $\mathbf{V}_1$ span the same r_1-dimensional space. Hence, $\mathbf{V}'_1$ is basis for $\mathbf{U}_1$. The assumptions $\mathbf{V}'_1\mathbf{A}_1 = \mathbf{V}_1$ and

$[\mathbf{V}_1'|\mathbf{V}_2']\mathbf{A}_2 = \mathbf{V}_2$ together and the fact that $\mathbf{V}'$ is a full rank matrix results that $[\mathbf{V}_1'|\mathbf{V}_2']$ and $[\mathbf{V}_1|\mathbf{V}_2]$ span the same r_2-dimensional space. Therefore, $[\mathbf{V}_1'|\mathbf{V}_2']$ is a basis for $\mathbf{U}_2$. Similar reasoning results that $\mathbf{V}'$ is an appropriate basis for $\mathbf{U}$. □

Lemma 3 *There exists at most one appropriate decomposition* $\mathbf{U} = \mathbf{V}\mathbf{T}$ *such that* $\mathbf{V}$ *has the canonical structure.*

Proof By contradiction assume that there exist two different canonical bases $\mathbf{V}$ and $\mathbf{V}'$. Then, according to Lemma 2, we have $\mathbf{V}_1'\mathbf{A}_1 = \mathbf{V}_1$, $[\mathbf{V}_1'|\mathbf{V}_2']\mathbf{A}_2 = \mathbf{V}_2$, … and $[\mathbf{V}_1'|\mathbf{V}_2'|\dots|\mathbf{V}_k']\mathbf{A}_k = \mathbf{V}'\mathbf{A}_k = \mathbf{V}_k$ for some $\mathbf{A}_1 \in \mathbb{R}^{r_1\times r_1}$, $\mathbf{A}_2 \in \mathbb{R}^{r_2\times(r_2-r_1)}$, … and $\mathbf{A}_k \in \mathbb{R}^{r_k\times(r_k-r_{k-1})}$. Since both $\mathbf{V}$ and $\mathbf{V}'$ are canonical bases, $\mathbf{V}(1:r_1,:) = \mathbf{V}'(1:r_1,:) = \mathbf{I}_{r_1}$, and therefore the equation $\mathbf{V}'(1:r_1,:)\mathbf{A}_1 = \mathbf{V}(1:r_1,:)$ results that $\mathbf{A}_1 = \mathbf{I}_{r_1}$. As a result, $\mathbf{V}_1 = \mathbf{V}_1'$. Moreover, we have $[\mathbf{V}_1'|\mathbf{V}_2']\mathbf{A}_2 = \mathbf{V}_2$, which results $[\mathbf{V}_1'|\mathbf{V}_2'](1:r_1,:)\mathbf{A}_2 = \mathbf{V}_2(1:r_1,:) = \mathbf{0}_{r_1\times(r_2-r_1)}$. Since we have $\mathbf{V}_2'(1:r_1,:) = \mathbf{0}_{r_1\times(r_2-r_1)}$ and $\mathbf{V}_1'(1:r_1,:) = \mathbf{I}_{r_1}$, then $[\mathbf{V}_1'|\mathbf{V}_2'](1:r_1,:)\mathbf{A}_2 = \mathbf{0}_{r_1\times(r_2-r_1)}$ reduces to $\mathbf{I}_{r_1}\mathbf{A}_2 = \mathbf{0}_{r_1\times(r_2-r_1)}$, i.e., $\mathbf{A}_2(1:r_1,:) = \mathbf{0}_{r_1\times(r_2-r_1)}$. Therefore, $[\mathbf{V}_1'|\mathbf{V}_2']\mathbf{A}_2 = \mathbf{V}_2$ reduces to $\mathbf{V}_2'\mathbf{A}_2(r_1+1:r_2,:) = \mathbf{V}_2$. Now, with the similar approach that we showed $\mathbf{V}_1 = \mathbf{V}_1'$, we can show $\mathbf{V}_2' = \mathbf{V}_2$ since $\mathbf{V}_2'(r_1+1:r_2,:) = \mathbf{V}_2(r_1+1:r_2,:) = \mathbf{I}_{r_2}$. The similar approach results that $\mathbf{V}_3' = \mathbf{V}_3$, … and $\mathbf{V}_k' = \mathbf{V}_k$, and therefore $\mathbf{V}' = \mathbf{V}$, which contradicts the assumption. □

The following lemma shows the uniqueness of canonical structure in Definition 2.

Lemma 4 *With probability one, there exists a unique appropriate decomposition* $\mathbf{U} = \mathbf{V}\mathbf{T}$ *such that* $\mathbf{V}$ *has the canonical structure.*

Proof As in Lemma 3 we showed that there exist at most one appropriate canonical basis, it suffices to show the existence of one appropriate canonical basis for $\mathbf{U}$ with probability one. According to Lemma 1, there exists an appropriate basis $\mathbf{V}'$ for $\mathbf{U}$ and we will construct an appropriate canonical basis based on $\mathbf{V}'$ to complete the proof. The genericity assumption results that the submatrix consisting of any r_1 rows of $\mathbf{V}_1'$ is full rank as each column of $\mathbf{U}_1$ is chosen generically from the Grassmannian manifold of $\text{Gr}(r_1, \mathbb{R}^m)$. As a result, $\mathbf{V}_1'(1:r_1,:)$ is full rank, i.e., $\mathbf{V}_1'(1:r_1,:)$ is nonsingular, with probability one with respect to the probability measure $\mathbb{P}_{G_1}\mathbb{P}_{\theta_1}$. Define $\mathbf{A}_1 = \mathbf{V}_1'(1:r_1,:)^{-1} \in \mathbb{R}^{r_1\times r_1}$ and $\mathbf{V}_1 = \mathbf{V}_1'\mathbf{A}_1 \in \mathbb{R}^{m\times r_1}$. Note that $\mathbf{V}_1(1:r_1,:) = \mathbf{I}_{r_1}$.

Similarly, $[\mathbf{V}_1'|\mathbf{V}_2'](1:r_2,:)$ is full rank with probability one with respect to the probability measure $\Pi_{i=1}^2\mathbb{P}_{G_i}\mathbb{P}_{\theta_i}$. Define $\mathbf{A}_2' = [\mathbf{V}_1'|\mathbf{V}_2'](1:r_2,:)^{-1} \in \mathbb{R}^{r_2\times r_2}$, $\mathbf{A}_2 = \mathbf{A}_2'(:,r_1+1:r_2) \in \mathbb{R}^{r_2\times(r_2-r_1)}$ and $\mathbf{V}_2 = [\mathbf{V}_1'|\mathbf{V}_2']\mathbf{A}_2 \in \mathbb{R}^{m\times(r_2-r_1)}$. Therefore, $\mathbf{V}_2(1:r_2,:) = [\mathbf{0}_{(r_2-r_1)\times r_1}|\mathbf{I}_{(r_2-r_1)}]^\top$. By repeating this procedure we construct $\mathbf{V} = [\mathbf{V}_1|\dots|\mathbf{V}_k]$ such that $\mathbf{V}$ has the canonical structure with probability one with respect to the probability measure $\Pi_{i=1}^k\mathbb{P}_{G_i}\mathbb{P}_{\theta_i}$. Moreover, according to Lemma 2, $\mathbf{V}$ is an appropriate basis for $\mathbf{U}$. □

As a result of Lemma 4, for each $\mathbf{U}$ there exists a unique appropriate decomposition with the canonical basis and observe that an arbitrary appropriate decomposition with the canonical basis results in a certain matrix $\mathbf{U}$ that satisfies the given rank constraints. Hence, the canonical structure plays the role of a bijective mapping from a generic member of the manifold corresponding to $\mathbf{U}$ to the appropriate decomposition with canonical basis and generic entries (excluding the entries of the canonical pattern). Consequently, those entries excluding the canonical pattern entries are chosen with respect to the Lebesgue measure on $\mathbb{R}$, i.e., are chosen generically.

Remark 1 Similarly to the proof of Lemma 4, we can show the uniqueness of the bases having a structure of any permutation of the rows of the canonical structure given in Definition 2. Considering all these permutations of the canonical structure, we obtain some patterns that operate like an equivalence class such that with probability one, exactly one basis belongs to each class, i.e., exactly one basis satisfies a certain pattern, among all the bases for appropriate decompositions. This also leads to the fact that the dimension of all appropriate bases is equal to $mr_k - \sum_{i=1}^{k} r_i(r_i - r_{i-1})$, which is the number of unknown entries of the canonical structure.

3.2 Polynomials and finite completability

We consider an appropriate decomposition $\mathbf{U} = \mathbf{VT}$, where $\mathbf{V} \in \mathbb{R}^{m \times r_k}$ and $\mathbf{T} \in \mathbb{R}^{r_k \times n_k}$. We are interested in obtaining all entries of $\mathbf{V}$ and $\mathbf{T}$ using the sampled entries of $\mathbf{U}$. Assuming that the unknown entries of $\mathbf{V}$ and $\mathbf{T}$ are variables, each sampled entry of $\mathbf{U}$ results in a polynomial in terms of these variables as the following,

$$\mathbf{U}(i, j) = \sum_{l=1}^{r_k} \mathbf{V}(i, l)\mathbf{T}(l, j). \tag{1}$$

Here, we briefly mention the following two facts to highlight the fundamentals of our proposed analysis.

- **Fact 1**: As it can be observed from (1), any sampled entry $\mathbf{U}(i, j)$ results in a polynomial that involves the entries of the i-th row of $\mathbf{V}$ and the entries of the j-th column of $\mathbf{T}$. Moreover, for a sampled entry $\mathbf{U}(i, j)$, the values of i and j specify the location of the entries of $\mathbf{V}$ and $\mathbf{T}$ that are involved in the corresponding polynomial, respectively.
- **Fact 2**: It can be concluded from Bernstein's theorem [38] that in a system of n polynomials in n variables with each consisting of a given set of monomials such that the coefficients are chosen with respect to the Lebesgue measure on the manifold corresponding to the basis of the given rank, the n polynomials are algebraically independent with probability one, and therefore there exist only finitely many solutions. However, in the structure of the polynomials in our model, the set of involved monomials are different for different set of polynomials, and therefore to ensure algebraically independency we need to have for any selected subset of the original n polynomials, the number of involved variables should be more than the number of selected polynomials.

The following assumption will be used frequently in this paper.

Assumption 1 *Each column of* $\mathbf{U}_i$ *that does not belong to* $\mathbf{U}_{i-1}$ *includes at least* r_i *sampled entries,* $i = 1, \ldots, k$.

Lemma 5 *Given the basis* $\mathbf{V}$*, Assumption 1 holds if and only if* $\mathbf{T}$ *is uniquely solvable.*

Proof We prove that Assumption 1 is necessary and sufficient condition for unique solvability of each column of $\mathbf{T}$. We show that the first column of $\mathbf{U}_1$ has less than r_1 sampled entries if and only if the first column of $\mathbf{T}$ is infinitely many solvable, and the same reasoning works for other columns as well. According to Fact 1, only sampled entries of the first column of $\mathbf{U}_1$ result in a linear polynomial that involves the entries of the first column of $\mathbf{T}$ (since $\mathbf{V}$ is given the polynomials are linear). Note that as we consider appropriate decompositions, the first column of $\mathbf{T}$ includes r_1 unknown variables, and therefore exactly r_1 polynomials with generic coefficients results in a unique solution and less than r_1 polynomials results in infinitely many solutions. $\square$

Definition 3 For notational simplicity, define $M = \sum_{i=1}^{k} r_i(n_i - n_{i-1})$ (the number of non-zero entries of an appropriate $\mathbf{T}$, i.e., the number of sampled entries described in Assumption 1), $M' = r_k n_k - M$ (the number of zero entries of an appropriate $\mathbf{T}$), $N = \sum_{i=1}^{k} r_i(r_i - r_{i-1})$ (the number of fixed entries of a canonical basis) and $N' = mr_k - N$ (the number of entries of a canonical basis excluding the entries of the canonical pattern).

As a result of Lemma 4, we specify the M sampled entries described in Assumption 1 to obtain $\mathbf{T}$ uniquely based on $\mathbf{V}$. Hence, we want to obtain the condition on the sampling pattern for finite solvability of $\mathbf{V}$ given $\mathbf{T}$.

Definition 4 Let $\mathcal{P}(\mathbf{\Omega})$ denote the set of polynomials corresponding to the observed entries as in (1) excluding the M observed entries of Assumption 1. Note that since $\mathbf{T}$ is already solved in terms of $\mathbf{V}$, each polynomial in $\mathcal{P}(\mathbf{\Omega})$ is in terms of the entries of $\mathbf{V}$.

The following lemma provides the condition on $\mathcal{P}(\mathbf{\Omega})$ for finite completability of the sampled matrix $\mathbf{U}$.

Lemma 6 *Suppose that Assumption 1 holds. With probability one, there exist only finitely many completions of* $\mathbf{U}$ *if and only if there exist* N' *algebraically independent polynomials in* $\mathcal{P}(\mathbf{\Omega})$.

Proof The proof is omitted due to the similarity to the proof of Lemma 2 in [1]. The only minor difference is that here the dimension is N' instead of $\left(\Pi_{i=1}^{j} n_i\right)\left(\Pi_{i=j+1}^{d} r_i\right) - \left(\sum_{i=j+1}^{d} r_i^2\right)$ which is the dimension of the core for Tucker decomposition. $\square$

Having Lemma 6, we only need to obtain the maximum number of algebraically independent polynomials in $\mathcal{P}(\mathbf{\Omega})$ to determine if $\mathbf{U}$ is finitely many completable. In Section 3.3, we construct a binary matrix based on the sampling pattern $\mathbf{\Omega}$ to obtain this number.

3.3 Constraint matrix

In this section, we provide a procedure to construct a binary valued matrix based on the sampling pattern such that each column of it represents one polynomial, and therefore we can later obtain the maximum number of algebraically independent polynomials in $\mathcal{P}(\mathbf{\Omega})$ in terms of some combinatorial properties of the sampling pattern.

Let $l_i = N_{\mathbf{\Omega}}(\mathbf{U}_1(:, i))$ denote the number of observed entries in the i-th column of $\mathbf{U}_1$, where $i \in \{1, \dots, n_1\}$. Assumption 1 results that $l_i \geq r_1$. We construct a binary valued matrix $\breve{\mathbf{\Omega}}_1$ based on $\mathbf{\Omega}$ and r_1. Specifically, we construct $l_i - r_1$ columns with binary entries based on the locations of the observed entries in $\mathbf{U}_1(:, i)$ such that each column has exactly $r_1 + 1$ entries equal to one (if $l_i = r_1$ then $\breve{\mathbf{\Omega}}_1 = \emptyset$). Assume that $x_1, \dots, x_{l_i}$ are the row indices of all observed entries in this column. Let $\mathbf{\Omega}_1^i$ be the corresponding $m \times (l_i - r_1)$ matrix to this column which is defined as the following: for any $j \in \{1, \dots, l_i - r_1\}$, the j-th column has the value 1 in rows $\{x_1, \dots, x_{r_1}, x_{r_1+j}\}$ and zeros elsewhere. Define the binary constraint matrix as $\breve{\mathbf{\Omega}}_1 = \left[\mathbf{\Omega}_1^1 | \mathbf{\Omega}_1^2 \dots | \mathbf{\Omega}_1^{n_1}\right] \in \mathbb{R}^{m \times K_1}$ [34], where $K_1 = N_{\mathbf{\Omega}}(\mathbf{U}_1) - n_1 r_1$. Similarly, we construct $\breve{\mathbf{\Omega}}_i$ for the matrix consisting of the columns of $\mathbf{U}_i$ that do not belong to $\mathbf{U}_{i-1}$ based on the corresponding sampling pattern and r_i, $i = 2, \dots, k$. Then, we put together all these k binary matrices $\breve{\mathbf{\Omega}} = \left[\breve{\mathbf{\Omega}}_1 | \breve{\mathbf{\Omega}}_2 | \dots | \breve{\mathbf{\Omega}}_k\right] \in \mathbb{R}^{m \times K}$ and call it the **constraint matrix**, where $K = N_{\mathbf{\Omega}}(\mathbf{U}) - M$. We show this procedure on a simple example.

Example 1 Consider the sampled matrix $\mathbf{U} \in \mathbb{R}^{4\times 7}$, where $n_1 = 3$ and $n_2 = 7$, i.e., $\mathbf{U}_1 \in \mathbb{R}^{4\times 3}$ and $\mathbf{U}_2 \in \mathbb{R}^{4\times 7}$. Assume that $r_1 = 2$ and $r_2 = 3$. Moreover, assume that the sampled entries are $\mathcal{F} = \{(1,1), (2,1), (3,1), (1,2), (1,3), (2,3), (3,3), (4,3), (1,4), (2,4), (4,4)\}$ and those samples that are used to obtain $\mathbf{T}$ are $\mathcal{F}' = \{(1,1), (1,2), (1,3), (2,3), (1,4), (2,4)\}$. Then, the constraint matrix is

$$\breve{\mathbf{\Omega}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

where $\breve{\mathbf{\Omega}}_1 = [\breve{\mathbf{\Omega}}(:,1)|\breve{\mathbf{\Omega}}(:,2)]$ and $\breve{\mathbf{\Omega}}_2 = [\breve{\mathbf{\Omega}}(:,3)|\breve{\mathbf{\Omega}}(:,4)|\breve{\mathbf{\Omega}}(:,5)]$.

In Section 3.4, we characterize a relationship between the maximum number of algebraically independent polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}})$ and a combinatorial condition on the sampling pattern $\mathbf{\Omega}$. We next define the notion of proper submatrix of $\mathbf{C}(\mathbf{\Omega})$.

Definition 5 A submatrix $\breve{\Omega}'$ of the constraint matrix $\breve{\Omega}$ is called a proper submatrix if its columns correspond to different columns of the sampling pattern $\mathbf{\Omega}$.

3.4 Algebraic independence

In this subsection, we characterize the condition on the sampling pattern for finite completability of the sampled data given the rank constraints, i.e., the condition on the sampling pattern for having N' algebraically independent polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}) = \mathcal{P}(\mathbf{\Omega})$.

Definition 6 Let $\breve{\mathbf{\Omega}}'$ be a subset of columns of the constraint matrix $\breve{\mathbf{\Omega}}$. Let $g(\breve{\mathbf{\Omega}}')$ denote the number of nonzero rows of $\breve{\mathbf{\Omega}}'$ and $\mathcal{P}(\breve{\mathbf{\Omega}}')$ denote the set of polynomials that correspond to the columns of $\breve{\mathbf{\Omega}}'$. Moreover, let $\breve{\mathbf{\Omega}}_i'$ denote the columns of $\breve{\mathbf{\Omega}}'$ that include exactly $r_i + 1$ nonzero entries, i.e., correspond to the columns of $\mathbf{U}_i$ and not columns of $\mathbf{U}_{i-1}$.

The following lemma gives an upper bound on the maximum number of algebraically independent polynomials in any subset of columns of the constraint matrix $\breve{\mathbf{\Omega}}$. Simply put, for a set of polynomials with coefficients chosen generically, the total number of involved variables in the polynomials is an upper bound on the maximum number of algebraically independent polynomials.

Lemma 7 *Let $\breve{\mathbf{\Omega}}^{'} \in \mathbb{R}^{m \times t}$ be a proper subset of columns of the constraint matrix $\breve{\mathbf{\Omega}}$. Then, the maximum number of algebraically independent polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}^{'})$ is at most*

$$\sum_{i=1}^{k} (r_i - r_{i-1})(g(\breve{\mathbf{\Omega}}^{'}_i) - r_i)^+. \tag{2}$$

Proof Note that each observed entry of $\mathbf{U}_1$, i.e., each column of $\breve{\mathbf{\Omega}}^{'}_1$, results in a polynomial that involves all r_1 entries of a row of $\mathbf{V}_1$. As a result, the number of entries of $\mathbf{V}_1$ that are involved in the polynomials is exactly $(r_1 - r_0)g(\breve{\mathbf{\Omega}}^{'}_1)$. However, the rows of the canonical pattern in $\mathbf{V}_1$ can be permuted, and therefore in the case of $\breve{\mathbf{\Omega}}^{'}_1 \neq \emptyset$ the number of known entries of the pattern in $\mathbf{V}_1$ is r_1^2 for a pattern. Hence, the minimum number of variables (unknown entries) of $\mathbf{V}_1$ is $(r_1 - r_0)g(\breve{\mathbf{\Omega}}^{'}_1) - r_1^2 = (r_1 - r_0)(g(\breve{\mathbf{\Omega}}^{'}_1) - r_1)^+$ since $\breve{\mathbf{\Omega}}^{'}_1 \neq \emptyset$ implies $g(\breve{\mathbf{\Omega}}^{'}_1) \geq r_1 + 1$. Moreover, clearly in the case of $\breve{\mathbf{\Omega}}^{'}_1 = \emptyset$ the number of variables (unknown entries) of $\mathbf{V}_1$ is $(r_1 - r_0)(g(\breve{\mathbf{\Omega}}^{'}_1) - r_1)^+ = 0$. Similarly, we can show that the minimum number of variables (unknown entries) of $\mathbf{V}_1$ is $\sum_{i=1}^{k} (r_i - r_{i-1})(g(\breve{\mathbf{\Omega}}^{'}_i) - r_i)^+$. As a result, the maximum number of algebraically independent polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}^{'})$ is at most equal to the number of involved variables in the polynomials, i.e., $\sum_{i=1}^{k} (r_i - r_{i-1})(g(\breve{\mathbf{\Omega}}^{'}_i) - r_i)^+$. $\square$

A set of polynomials is called minimally algebraically dependent if the polynomials in that set are algebraically dependent but polynomials in every of its proper subset are algebraically independent. The next lemma which is Lemma 7 in [4], states an important property of a set of minimally algebraically dependent among polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}})$. This lemma is needed to derive the maximum number of algebraically independent polynomials in any subset of $\mathcal{P}(\breve{\mathbf{\Omega}})$.

Lemma 8 *Let $\breve{\mathbf{\Omega}}^{'} \in \mathbb{R}^{m \times t}$ be a proper subset of columns of the constraint matrix $\breve{\mathbf{\Omega}}$. Assume that polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}^{'})$ are minimally algebraically dependent. Then, the number of variables (unknown entries) of $\mathbf{V}$ that are involved in $\mathcal{P}(\breve{\mathbf{\Omega}}^{'})$ is equal to $t - 1$.*

Given a proper subset of columns $\breve{\mathbf{\Omega}}^{'}$ of the constraint matrix, the following lemma takes advantage of Lemmas 7 and 8 to characterize a relationship between the maximum number of algebraically independent polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}^{'})$ and the geometric structure of nonzero entries of $\breve{\mathbf{\Omega}}^{'}$.

Lemma 9 *Given a proper subset of columns $\breve{\mathbf{\Omega}}^{'} \in \mathbb{R}^{m \times t}$ of the constraint matrix, the polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}^{'})$ are algebraically independent if and only if for any $t' \in \{1, \ldots, t\}$ and any subset of columns $\breve{\mathbf{\Omega}}^{''} \in \mathbb{R}^{m \times t'}$ of $\breve{\mathbf{\Omega}}^{'}$ we have*

$$\sum_{i=1}^{k} (r_i - r_{i-1})(g(\breve{\mathbf{\Omega}}^{''}_i) - r_i)^+ \geq t'. \tag{3}$$

Proof Assume that the polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}^{'})$ are algebraically dependent. Then, there exists a subset of polynomials $\mathcal{P}(\breve{\mathbf{\Omega}}^{''})$ of the set $\mathcal{P}(\breve{\mathbf{\Omega}}^{'})$ such that the polynomials in

$\mathcal{P}(\breve{\boldsymbol{\Omega}}'')$ are minimally algebraically dependent. Let $\breve{\boldsymbol{\Omega}}'' \in \mathbb{R}^{m\times t'}$, where $t' \in \{1,\ldots,t\}$. According to Lemma 8 the number of involved variables in $\mathcal{P}(\breve{\boldsymbol{\Omega}}'')$ is $t'-1$. However, in Lemma 7 we showed that the number of involved variables in $\mathcal{P}(\breve{\boldsymbol{\Omega}}'')$ is at least $\sum_{i=1}^{k}(r_i - r_{i-1})(g(\breve{\boldsymbol{\Omega}}_i'') - r_i)^+$, and therefore $\sum_{i=1}^{k}(r_i - r_{i-1})(g(\breve{\boldsymbol{\Omega}}_i'') - r_i)^+ \leq t'-1 < t'$.

In order to show the other direction, assume that the polynomials in $\mathcal{P}(\breve{\boldsymbol{\Omega}}')$ are algebraically independent, and therefore any subset of polynomials of $\mathcal{P}(\breve{\boldsymbol{\Omega}}')$ are also algebraically independent. By contradiction assume that there exists a subset of columns $\breve{\boldsymbol{\Omega}}'' \in \mathbb{R}^{m\times t'}$ of $\breve{\boldsymbol{\Omega}}'$ such that (3) does not hold. Hence, $\sum_{i=1}^{k}(r_i - r_{i-1})(g(\breve{\boldsymbol{\Omega}}_i'') - r_i)^+$ is less than the number of polynomials in $\mathcal{P}(\breve{\boldsymbol{\Omega}}'')$. On the other hand, according to Lemma (7), the maximum number of algebraically independent polynomials in $\mathcal{P}(\breve{\boldsymbol{\Omega}}'')$ is at most $\sum_{i=1}^{k}(r_i - r_{i-1})(g(\breve{\boldsymbol{\Omega}}_i'') - r_i)^+$, which is less than the number of polynomials in $\mathcal{P}(\breve{\boldsymbol{\Omega}}'')$, and this contradicts the assumption. □

The next theorem which is the main result of this subsection characterizes the condition on the sampling pattern for finite completability of $\mathbf{U}$.

Theorem 1 *Suppose that Assumption 1 holds. With probability one, the sampled data* $\mathbf{U}$ *is finitely many completable if and only if there exists a proper subset of columns* $\breve{\boldsymbol{\Omega}}' \in \mathbb{R}^{m\times N'}$ *of the constraint matrix* $\breve{\boldsymbol{\Omega}}$ *such that for any* $t' \in \{1,\ldots,N'\}$ *and any subset of columns* $\breve{\boldsymbol{\Omega}}'' \in \mathbb{R}^{m\times t'}$ *of* $\breve{\boldsymbol{\Omega}}'$*, (3) holds.*

Proof First we assume that there exists a proper subset of columns $\breve{\boldsymbol{\Omega}}' \in \mathbb{R}^{m\times N'}$ of the constraint matrix $\breve{\boldsymbol{\Omega}}$ such that for any $t' \in \{1,\ldots,N'\}$ and any subset of columns $\breve{\boldsymbol{\Omega}}'' \in \mathbb{R}^{m\times t'}$, (3) holds and we need to show the finite completability of $\mathbf{U}$. According to Lemma 8, the N' polynomials corresponding to $\breve{\boldsymbol{\Omega}}'$ are algebraically independent, and therefore according to Lemma 6, $\mathbf{U}$ is finitely many completable.

In order to complete the proof, we assume that $\mathbf{U}$ is finitely many completable and show the existence of such $\breve{\boldsymbol{\Omega}}'$ described in the statement of theorem. According to Lemma 6, there exists N' algebraically independent polynomials in $\mathcal{P}(\breve{\Omega})$, and therefore according to Lemma 8, the submatrix corresponding to these N' polynomials satisfies the properties described in the statement of theorem. □

One challenge of applying Theorem 1 is the exhaustive enumeration that it takes to check if (3) holds for all the corresponding subsets of columns. In the next section, we provide a bound on the sampling probability in terms of $r_1, \ldots, r_k$ that ensures (3) holds with high probability for all the corresponding subsets of columns. Consequently, we do not need to check (3) but instead we can certify the above results with high probability and not deterministically anymore.

4 Probabilistic conditions for finite completability

We assume that the entries of $\mathbf{U}$ are sampled independently with probability p. In this section, we are interested in obtaining a condition in terms of the number of samples, i.e., the sampling probability, to ensure the combinatorial conditions on the sampling pattern given in Theorem 1 hold with high probability. Therefore, according to Theorem 1, the provided condition on the sampling probability ensures the finite completability of $\mathbf{U}$.

Lemma 10 below is Lemma 5 in [7], which will be used later as it connects a condition in terms of the number of samples to a combinatorial property on the sampling pattern.

Lemma 10 *Assume that* $r \leq \frac{m}{6}$ *and also each column of* $\mathbf{\Omega}$ *includes at least* l *nonzero entries, where*

$$l > \max\left\{9\ \log\left(\frac{m}{\epsilon}\right) + 3\ \log\left(\frac{q}{\epsilon}\right) + 6, 2r\right\}. \tag{4}$$

Let $\mathbf{\Omega}'$ *be an arbitrary set of* $m-r$ *columns of* $\mathbf{\Omega}$*. Then, with probability at least* $1-\frac{\epsilon}{q}$*, every subset* $\mathbf{\Omega}''$ *of columns of* $\mathbf{\Omega}'$ *satisfies*

$$g(\mathbf{\Omega}'') - r \geq t, \tag{5}$$

where t *is the number of columns of* $\mathbf{\Omega}''$*.*

Note that we are interested in obtaining a condition in terms of the number of samples to ensure finite completability, i.e., to certify that the conditions on the constraint matrix $\breve{\mathbf{\Omega}}$ (not the sampling pattern $\mathbf{\Omega}$) in Theorem 1 hold, with high probability. However, given the number of samples is large enough and using Lemma 10, we will be able to verify the mentioned combinatorial conditions on the sampling pattern. Then, the following lemma connects the conditions on the sampling pattern to the combinatorial conditions on the constraint matrix. In particular, the following lemma, which is Lemma 8 in [1], states that if the property in Lemma 10 holds for the sampling pattern, then it will be satisfied for the constraint matrix as well.

Lemma 11 *Let* r *be a given nonnegative integer. Assume that there exists a matrix* $\mathbf{\Omega}'$ *such that it consists of* $m-r$ *columns of* $\mathbf{\Omega}$ *and each column of* $\mathbf{\Omega}'$ *includes at least* $r+1$ *nonzero entries and satisfies the following property:*

- *Denote an arbitrary matrix obtained by choosing any subset of the columns of* $\mathbf{\Omega}'$ *by* $\mathbf{\Omega}''$*. Then,*

$$g(\mathbf{\Omega}'') - r \geq c(\mathbf{\Omega}''), \tag{6}$$

where $c(\mathbf{\Omega}'')$ *denotes the number of columns of* $\mathbf{\Omega}''$*. Then, there exists a matrix* $\breve{\mathbf{\Omega}}'$ *with the same size as* $\mathbf{\Omega}'$ *such that: each column has exactly* $r+1$ *entries equal to one, and if* $\breve{\mathbf{\Omega}}'(x, y) = 1$ *then we have* $\mathbf{\Omega}'(x, y) = 1$*. Moreover,* $\breve{\mathbf{\Omega}}'$ *satisfies the above-mentioned property.*

Definition 7 Let $\mathbf{\Omega}_i$ denote the subset of columns of $\mathbf{\Omega}$ that correspond those columns of $\mathbf{U}_i$ that do not belong to $\mathbf{U}_{i-1}$, i.e., the $(n_i - n_{i-1} + 1)$-th to the n_i-th columns of $\mathbf{\Omega}$.

The following lemma will be used to ensure that the condition on the constraint matrix $\breve{\mathbf{\Omega}}$ in Theorem 1 is satisfied.

Lemma 12 *Assume that* $i \in \{1, \ldots, k\}$, $r_i \leq \frac{m}{6}$, $(r_i - r_{i-1})(m - r_i) \leq n_i - n_{i-1}$ *and each column of* $\mathbf{\Omega}_i$ *includes at least* l_i *nonzero entries where*

$$l_i > \max\left\{9\ \log\left(\frac{m}{\epsilon}\right) + 3\ \log\left(\frac{(r_i - r_{i-1})k}{\epsilon}\right) + 6, 2r_i\right\}. \tag{7}$$

Then, with probability at least $1 - \frac{\epsilon}{k}$*, there exists a subset of columns* $\breve{\mathbf{\Omega}}_i^{'} \in \mathbb{R}^{m \times t_i}$ *of* $\breve{\mathbf{\Omega}}_i$*, where* $t_i = (r_i - r_{i-1})(m - r_i)$ *such that for any* $t_i^{'} \in \{1, 2, \ldots, t_i\}$ *and any subset of columns* $\breve{\mathbf{\Omega}}_i^{''} \in \mathbb{R}^{m \times t_i^{'}}$ *of* $\breve{\mathbf{\Omega}}_i^{'}$ *we have*

$$(r_i - r_{i-1})(g(\breve{\mathbf{\Omega}}_i^{''}) - r_i)^+ \geq t_i^{'}. \tag{8}$$

Proof Since $(r_i - r_{i-1})(m - r_i) \leq n_i - n_{i-1}$, we can randomly choose $(r_i - r_{i-1})$ disjoint matrices $\Omega_{i_s}^{'}$ (for $1 \leq s \leq r_i - r_{i-1}$), each consisting of $(m - r_i)$ columns of $\mathbf{\Omega}_i$. According to Lemma 10, each $\Omega_{i_s}^{'}$ satisfies the following property with probability at least $1 - \frac{\epsilon}{(r_i - r_{i-1})k}$, $1 \leq s \leq r_i - r_{i-1}$: for any $t_i^{''} \in \{1, 2, \ldots, m - r_i\}$ and any subset of columns $\Omega_{i_s}^{''} \in \mathbb{R}^{m \times t_i^{''}}$ of $\Omega_{i_s}^{'}$ we have

$$(g(\Omega_{i_s}^{''}) - r_i)^+ \geq t_i^{''}. \tag{9}$$

On the other hand, according to Lemma 11, there exist corresponding disjoint subsets of columns of the constraint matrix $\breve{\mathbf{\Omega}}_{i_s}^{'}$ (for $1 \leq s \leq r_i - r_{i-1}$), each consisting of $(m - r_i)$ columns of $\breve{\mathbf{\Omega}}_i$ such that they satisfy the above property as $\Omega_{i_s}^{'}$'s. As a result, all $\breve{\mathbf{\Omega}}_{i_s}^{'}$'s satisfy the mentioned property simultaneously with probability at least $1 - \frac{\epsilon}{k}$.

Define $\breve{\mathbf{\Omega}}_i^{'} = [\breve{\mathbf{\Omega}}_{i_1}^{'} | \breve{\mathbf{\Omega}}_{i_2}^{'} | \ldots | \breve{\mathbf{\Omega}}_{i_{r_i - r_{i-1}}}^{'}] \in \mathbb{R}^{m \times t_i}$. Consider any $t_i^{'} \in \{1, 2, \ldots, t_i\}$ and any subset of columns $\breve{\mathbf{\Omega}}_i^{''} \in \mathbb{R}^{m \times t_i^{'}}$ of $\breve{\mathbf{\Omega}}_i^{'}$. Let $\breve{\mathbf{\Omega}}_{i_s}^{''}$ denote those columns of $\breve{\mathbf{\Omega}}_i^{''}$ that belong to $\breve{\mathbf{\Omega}}_{i_s}^{'}$ and without loss of generality assume that $c(\breve{\mathbf{\Omega}}_{i_1}^{''}) = \max_{1 \leq s \leq r_i - r_{i-1}} \{c(\breve{\mathbf{\Omega}}_{i_s}^{''})\}$, where $c(\cdot)$ denotes the number of columns. Then, it is simply verified that

$$\begin{aligned}(r_i - r_{i-1})(g(\breve{\mathbf{\Omega}}_i^{''}) - r_i)^+ &\geq (r_i - r_{i-1})(g(\breve{\mathbf{\Omega}}_{i_1}^{''}) - r_i)^+ \geq (r_i - r_{i-1})c(\breve{\mathbf{\Omega}}_{i_1}^{''}) \\ &\geq \sum_{s=1}^{r_i - r_{i-1}} c(\breve{\mathbf{\Omega}}_{i_s}^{''}) \geq t_i^{'}. \end{aligned} \tag{10}$$

□

Finally, the following theorem gives the conditions on the number of samples to ensure that the conditions on the constraint matrix $\breve{\mathbf{\Omega}}$ in Theorem 1 hold with high probability, i.e., the sampled data is finitely many completable with high probability.

Theorem 2 *Assume that assumptions in the statement of Lemma 12 hold for any* $i \in \{1, 2, \ldots, k\}$. *Then, the sampled matrix* $\mathbf{U}$ *is finitely many completable with probability at least* $1 - \epsilon$.

Proof Consider the obtained subset of columns $\breve{\mathbf{\Omega}}_i^{'} \in \mathbb{R}^{m \times t_i}$ of $\breve{\mathbf{\Omega}}_i$ in Lemma 12, where $t_i = (r_i - r_{i-1})(m - r_i)$, for $i = 1, 2, \ldots, k$. Define $\breve{\mathbf{\Omega}}^{'} = [\breve{\mathbf{\Omega}}_1^{'} | \breve{\mathbf{\Omega}}_2^{'} | \ldots | \breve{\mathbf{\Omega}}_k^{'}] \in \mathbb{R}^{m \times t}$, where $t = \sum_{i=1}^{k} t_i = N'$. Each $\breve{\mathbf{\Omega}}_i^{'}$ satisfies the mentioned property in Lemma 12 with probability at least $1 - \frac{\epsilon}{k}$, and therefore all $\breve{\mathbf{\Omega}}_i^{'}$'s satisfy the corresponding properties simultaneously with probability at least $1 - \epsilon$.

Let $\breve{\mathbf{\Omega}}^{''} \in \mathbb{R}^{m \times t'}$ denote an arbitrary subset of columns of $\breve{\mathbf{\Omega}}^{'}$. Also, assume that $\breve{\mathbf{\Omega}}_i^{''}$ denote those columns of $\breve{\mathbf{\Omega}}^{''}$ that belong to $\breve{\mathbf{\Omega}}_i^{'}$, $i = 1, 2, \ldots, k$. Then, we can conclude that (8) holds, $i = 1, 2, \ldots, k$. Similarly to the last part of the proof of Lemma 12 we can show that (3) holds. Therefore, according to Theorem 1, $\mathbf{U}$ is finitely many completable with probability at least $1-\epsilon$. □

The following lemma is taken from [1] and ensures that with high probability the number of samples at each column is larger than a certain number given that the sampling probability is large enough.

Lemma 13 *Consider a vector with m entries where each entry is observed with probability p independently from the other entries. If $p > p' = \frac{z}{m} + \frac{1}{\sqrt[4]{m}}$, then with probability at least $\left(1 - \exp(-\frac{\sqrt{m}}{2})\right)$, more than z entries are observed.*

The following lemma makes use of Lemma 13 to derive a lower bound on the sampling probability that leads to the similar statement as Theorem 2 with high probability, i.e., finite completability of $\mathbf{U}$ with high probability, given that the sampling probability is larger than a certain number.

Lemma 14 *Assume that $r_i \leq \frac{m}{6}$, $(r_i - r_{i-1})(m - r_i) \leq n_i - n_{i-1}$, $1 \leq i \leq k$ and that the entries of $\mathbf{U}$ are sampled independently with probability p, where*

$$p > \frac{1}{m} \max \left\{ 9 \ \log \left(\frac{m}{\epsilon} \right) + 3 \ \log \left(\frac{qk}{\epsilon} \right) + 6, 2r_k \right\} + \frac{1}{\sqrt[4]{m}}, \tag{11}$$

where $q = \max_{1 \leq i \leq k} r_i - r_{i-1}$. Then, with probability at least $(1-\epsilon)\left(1 - \exp(-\frac{\sqrt{m}}{2})\right)^{n_k}$, $\mathbf{U}$ is finitely many completable.

Proof Since $q = \max_{1 \leq i \leq k} r_i - r_{i-1}$ and according to Lemma 13, the number of samples at each column of $\mathbf{\Omega}_i$ satisfies (7) with probability at least $\left(1 - \exp(-\frac{\sqrt{m}}{2})\right)$. The rest of the proof is easy to verify using Theorem 2. □

5 Deterministic and probabilistic conditions for unique completability

In Sections 3 and 4, we characterized the deterministic and probabilistic conditions on the sampling pattern for finite completability, respectively. In this section, we are interested in obtaining the deterministic and probabilistic conditions on the sampling pattern for unique completability. Note that for matrix completion problem (and therefore for our problem), finite completability does not necessarily imply unique completability [1]. Unique completability simply means that, any completion of the sampled data obtained by any algorithm is exactly the original sampled data. We show that adding a set of mild assumptions to those stated in Theorem 1 leads to unique completability.

Recall that there exists at least one completion of $\mathbf{U}$ since the original matrix that is sampled satisfies the rank constraints. The following lemma is a re-statement of Lemma 25 in [4].

Lemma 15 *Assume that Assumption 1 holds. Let $\breve{\mathbf{\Omega}}'$ be a proper subset of columns of the constraint matrix $\breve{\mathbf{\Omega}}$. Assume that polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}')$ are minimally algebraically* *dependent. Then, all variables (unknown entries) of* $\mathbf{V}$ *that are involved in* $\mathcal{P}(\breve{\mathbf{\Omega}}')$ *can be determined uniquely.*

The following theorem characterizes sufficient deterministic conditions on the sampling pattern for unique completability. In particular, condition (i) is the same condition as in Theorem 1, which results in finite completability, while by adding condition (ii) and using Lemma 15, we can ensure the unique completability with probability one.

Theorem 3 *Suppose that Assumption 1 holds. With probability one, the sampled data* $\mathbf{U}$ *is uniquely completable if there exists disjoint proper subsets of columns* $\breve{\mathbf{\Omega}}' \in \mathbb{R}^{m\times N'}$ *and* $\breve{\mathbf{\Omega}}'_i \in \mathbb{R}^{m\times(m-r_i)}$ $(1 \leq i \leq k)$ *of the constraint matrix* $\breve{\mathbf{\Omega}}$ *such that*

(i) *for any* $t' \in \{1, \dots, N'\}$ *and any subset of columns* $\breve{\mathbf{\Omega}}'' \in \mathbb{R}^{m\times t'}$ *of* $\breve{\mathbf{\Omega}}'$, *(3) holds.*
(ii) *for any* $t'_i \in \{1, \dots, m - r_i\}$ *and any subset of columns* $\breve{\mathbf{\Omega}}''_i \in \mathbb{R}^{m\times t'_i}$ *of* $\breve{\mathbf{\Omega}}'_i$ *we have*

$$(g(\breve{\mathbf{\Omega}}''_i) - r_i)^+ \geq t'_i, \tag{12}$$

$i = 1, 2, \dots, k.$

Proof According to Theorem 1, condition (i) results that there are at most finitely many completions of $\mathbf{U}$. As we showed in the proof of Theorem 1, there exist N' algebraically independent polynomials $\{p_1, p_2, \dots, p_{N'}\}$ in $\mathcal{P}(\breve{\mathbf{\Omega}}')$. Note that any set of $N' + 1$ polynomials are algebraically dependent. Consider a single polynomial p_0 from the set of polynomials $\cup_{i=1}^{k}\mathcal{P}(\breve{\mathbf{\Omega}}'_i)$. Hence, $\{p_0, p_1, \dots, p_{N'}\}$ are algebraically dependent and since $\{p_1, p_2, \dots, p_{N'}\}$ are algebraically independent, there exist a set of polynomials $\mathcal{P}(p_0) \subseteq \{p_0, p_1, \dots, p_{N'}\}$ that is minimally dependent.

According to Lemma 15, all variables involved in $\mathcal{P}(p_0)$ and therefore all variables involved in p_0 can be determined uniquely, or in other words, we obtain r_i linear polynomials in terms of the entries of $\mathbf{V}_i$ given that $p_0 \in \mathcal{P}(\breve{\mathbf{\Omega}}'_i)$. It is easily verified that given (ii) and substituting p_0 by all of the polynomials in $\mathcal{P}(\breve{\mathbf{\Omega}}'_i)$ one by one, $\mathbf{V}_i$ can be determined uniquely, $i = 1, 2, \dots, k$. □

Finally, using Theorems 2 and 3, we provide a bound on the number of samples to ensure unique completability with high probability. In particular, the next theorem gives a probabilistic guarantee for satisfying the conditions (i) and (ii) in the statement of Theorem 3.

Theorem 4 *Assume that* $r_i \leq \frac{m}{6}$, $(r_i - r_{i-1} + 1)(m - r_i) \leq n_i - n_{i-1}$ *and each column of* $\mathbf{\Omega}_i$ *includes at least* l_i *nonzero entries where*

$$l_i > \max\left\{9\ \log\left(\frac{m}{\epsilon}\right) + 3\ \log\left(\frac{(r_i - r_{i-1})2k}{\epsilon}\right) + 6, 2r_i\right\}, \tag{13}$$

for $i = 1, 2, \dots, k$. *Then, with probability at least* $1 - \epsilon$, $\mathbf{U}$ *is uniquely completable.*

Proof According to Theorem 2, condition (i) in the statement of Theorem 3 holds with probability at least $1-\frac{\epsilon}{2}$. According to Theorem 3, in order to complete the proof, it suffices to show that condition (ii) in the statement of Theorem 3 holds with probability at least $1 - \frac{\epsilon}{2}$. Note that according to Lemmas 10 and 11, condition (ii) for each value of i holds with probability at least $1 - \frac{\epsilon}{(r_i - r_{i-1})2k}$, and therefore condition (ii) holds with probability at least $1 - \frac{\epsilon}{(r_i - r_{i-1})2} \geq 1 - \frac{\epsilon}{2}$. □

The following lemma makes use of Lemma 13 to derive a lower bound on the sampling probability that leads to the similar statement as Theorem 4, i.e., unique completability with high probability.

Lemma 16 *Assume that $r_i \leq \frac{m}{6}$, $(r_i - r_{i-1} + 1)(m - r_i) \leq n_i - n_{i-1}$, $1 \leq i \leq k$ and that the entries of $\mathbf{U}$ are sampled independently with probability p, where*

$$p > \frac{1}{m} \max \left\{ 9 \log \left(\frac{m}{\epsilon} \right) + 3 \log \left(\frac{2qk}{\epsilon} \right) + 6, 2r_k \right\} + \frac{1}{\sqrt[4]{m}}, \tag{14}$$

where $q = \max_{1 \leq i \leq k} r_i - r_{i-1}$. Then, with probability at least $(1 - \epsilon) \left(1 - \exp(-\frac{\sqrt{m}}{2}) \right)^{n_k}$, $\mathbf{U}$ is uniquely completable.

Proof Using Theorem 4, the proof is similar to the proof of Lemma 14. □

6 Numerical results

As the first example, we compute the total number of samples that is required for finite completability based on Theorem 2 and compare with the number of samples required by simply using the conventional matrix analysis [34]. The mentioned numbers are $\sum_{i=1}^{k} (n_i - n_{i-1})$ $\max \left\{ 9 \log \left(\frac{m}{\epsilon} \right) + 3 \log \left(\frac{(r_i - r_{i-1})k}{\epsilon} \right) + 6, 2r_i \right\}$ and $n_k \max \left\{ 12 \log \left(\frac{m}{\epsilon} \right) + 12, 2r_k \right\}$, respectively. In this numerical example, we consider $m = 10000$, $k = 4$, $\epsilon = 0.01$ and also $r_i = r \times i^2$ and $n_i = 50 \times i^2 \times 10000$, $i = 1, 2, \ldots, k$, where we vary r from 1 to 50. The

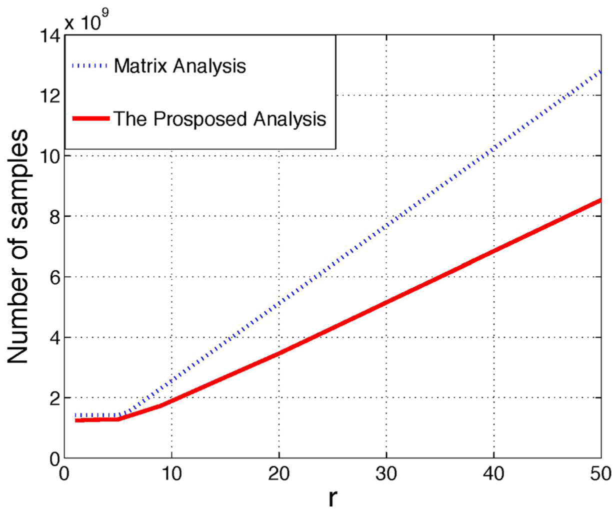


Fig. 5 Comparison of the number of samples for a union of 4 subspaces

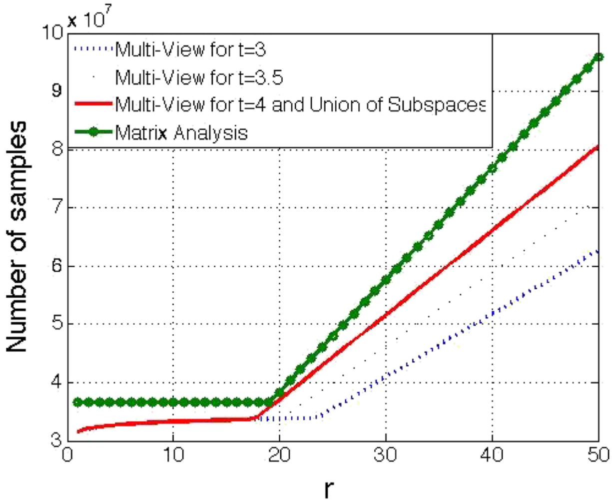


Fig. 6 Comparison of the union of two subspaces and multi-view matrix

corresponding curves are shown in Fig. 5. It is seen that the proposed analysis requires much less number of samples than the conventional matrix analysis for finite completability.

In another numerical example, we are interested in comparing our proposed method for the union of two subspaces with the multi-view scenario studied in [7]. First, we consider the union of two subspaces as $m = 1200$, $k = 2$, $\epsilon = 0.01$ and also $r_i = r \times i^2$ and $n_i = 50 \times i^2 \times 10000$, $i = 1, 2$, where we vary r from 1 to 50. Hence, for any value of $r \in \{1, \dots, 50\}$, we have $r_1 = \text{rank}(\mathbf{U}_1) = r$ and $r_2 = \text{rank}(\mathbf{U}) = 4r$. In the multi-view problem, an extra rank constraint is given, which is $R = \text{rank}\,(\mathbf{U}(:, n_1 + 1 : n_2))$. The number of samples that ensures finite completability for multi-view matrix is [7]

$$n_1 \ \max\left\{9 \log\left(\frac{m}{\epsilon}\right) + 3 \log\left(\frac{3 \max\{r_2 - r_1, r_2 - R, r_1 + R - r_2\}}{\epsilon}\right) + 6, 2r_1\right\}$$
$$+ (n_2 - n_1) \max\left\{9 \log\left(\frac{m}{\epsilon}\right) + 3 \log\left(\frac{3 \max\{r_2 - r_1, r_2 - R, r_1 + R - r_2\}}{\epsilon}\right) + 6, 2R\right\}.$$

It is easily verified that $r_1 \le r_2$, $R \le r_2$ and $r_2 \le r_1 + R$. Therefore, as $r_2 = 4r_1 = 4r$, we conclude that $3r \le R \le 4r$, i.e., $R = tr$ for some $3 \le t \le 4$. Note that in the union of subspaces scenario, the genericity assumption results that $R = r_2 = 4r$. Hence, in the multi-view scenario, $t = 4$ is basically almost the same as the union of subspaces scenario and for $3 \le t < 4$ we have more constraint in comparison with the union of subspaces scenario. The corresponding curves are shown in Fig. 6. It is seen that for $3 \le t < 4$ the multi-view matrix requires less number of samples than the union of two subspaces as we have one more rank constraint, and therefore more information about the data.

7 Conclusions

We consider the problem of union of low-rank subspaces completion. We analyze the manifold structure corresponding to the given rank constraints to characterize the deterministic conditions on the sampling pattern for finite completability of a matrix that represents the union of several subspaces satisfying the mentioned rank constraints. In order to obtain the deterministic conditions on the sampling pattern, we characterizes the algebraic independence of a set of polynomials defined based on the sampling pattern, which is closely related to finite completion. Moreover, assuming that the entries of the data are sampled independently with probability p and using the mentioned deterministic analysis, we propose a combinatorial method to derive a lower bound on the sampling probability p, or equivalently, the number of sampled entries that guarantees finite completability with high probability. Furthermore, using the proposed analysis for finite completability, we characterize deterministic and probabilistic conditions on the sampling pattern and the sampling probability that ensure there exists only one completion for the sampled data.

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References

1. Ashraphijuo, M., Aggarwal, V., Wang, X.: Deterministic and probabilistic conditions for finite completability of low-Tucker-rank tensor. Accepted to IEEE Transactions on Information Theory, arXiv:1612.01597 (2016)
2. Ashraphijuo, M., Aggarwal, V., Wang, X.: On deterministic sampling patterns for robust low-rank matrix completion. IEEE Signal Process. Lett. **25**(3), 343–347 (2018)
3. Ashraphijuo, M., Madani, R., Lavaei, J.: Characterization of rank-constrained feasibility problems via a finite number of convex programs. In: 2016 IEEE 55th Conference on Decision and Control (CDC), pp. 6544–6550 (2016)
4. Ashraphijuo, M., Wang, X.: Fundamental conditions for low-CP-rank tensor completion. J. Mach. Learn. Res. **18**(63), 1–29 (2017)
5. Ashraphijuo, M., Wang, X.: A characterization of sampling patterns for union of low-rank subspaces retrieval problem. In: International Symposium on Artificial Intelligence and Mathematics, pp. 1–8 (2018)
6. Ashraphijuo, M., Wang, X.: Clustering a union of low-rank subspaces of different dimensions with missing data. Pattern Recogn. Lett. **120**, 31–35 (2019)
7. Ashraphijuo, M., Wang, X., Aggarwal, V.: Deterministic and probabilistic conditions for finite completability of low-rank multi-view data. arXiv:1701.00737 (2017)
8. Ashraphijuo, M., Wang, X., Aggarwal, V.: Rank determination for low-rank data completion. J. Mach. Learn. Res. **18**(1), 3422–3450 (2017)
9. Ashraphijuo, M., Wang, X., Zhang, J.: Low-rank data completion with very low sampling rate using newton's method. IEEE Trans. Signal Process. **67**(7), 1849–1859 (2019)
10. Balzano, L., Eriksson, B., Nowak, R.: High rank matrix completion and subspace clustering with missing data. In: The Conference on Artificial Intelligence and Statistics (AIStats) (2012)
11. Cai, J.F., Candès, E.J., Shen, Z.: A singular value thresholding algorithm for matrix completion. SIAM J. Optim. **20**(4), 1956–1982 (2010)
12. Candès, E.J., Eldar, Y.C., Strohmer, T., Voroninski, V.: Phase retrieval via matrix completion. SIAM J. Imag. Sci. **6**(1), 199–225 (2013)
13. Candès, E.J., Recht, B.: Exact matrix completion via convex optimization. Found. Comput. Math. **9**(6), 717–772 (2009)
14. Candès, E.J., Tao, T.: The power of convex relaxation: Near-optimal matrix completion. IEEE Trans. Inf. Theory **56**(5), 2053–2080 (2010)
15. Eldén, L.: Matrix Methods in Data Mining and Pattern Recognition, vol. 4. Society for Industrial and Applied Mathematics (2007)
16. Elhamifar, E., Vidal, R.: Sparse subspace clustering. In: IEEE Conference on Computer Vision and Pattern Recognition, pp. 2790–2797. IEEE (2009)
17. Eriksson, B., Balzano, L., Nowak, R.: High-rank matrix completion. In: Artificial Intelligence and Statistics, pp 373–381 (2012)
18. Gandy, S., Recht, B., Yamada, I.: Tensor completion and low-n-rank tensor recovery via convex optimization. Inverse Probl. **27**(2), 1–19 (2011)

19. Gao, P., Wang, M., Chow, J.H., Berger, M., Seversky, L.M.: Missing data recovery for high-dimensional signals with nonlinear low-dimensional structures. IEEE Trans. Signal Process. **65**(20), 5421–5436 (2015)
20. Gao, P., Wang, M., Ghiocel, S.G., Chow, J.H., Fardanesh, B., Stefopoulos, G.: Missing data recovery by exploiting low-dimensionality in power system synchrophasor measurements. IEEE Trans. Power Syst. **31**(2), 1006–1013 (2016)
21. Gao, P., Wang, R., Wang, M., Chow, J.H.: Low-rank matrix recovery from quantized and erroneous measurements: Accuracy-preserved data privatization in power grids. In: 2016 50th Asilomar Conference on Signals, Systems and Computers, pp. 374–378. IEEE (2016)
22. Goldfarb, D., Qin, Z.: Robust low-rank tensor recovery: Models and algorithms. SIAM J. Matrix Anal Appl. **35**(1), 225–253 (2014)
23. Harvey, N.J.A., Karger, D.R., Murota, K.: Deterministic network coding by matrix completion. In: ACM-SIAM Symposium on Discrete algorithms, pp. 489–498 (2005)
24. Ji, H., Liu, C., Shen, Z., Xu, Y.: Robust video denoising using low rank matrix completion. In: IEEE Conference on Conference on Computer Vision and Pattern Recognition, pp. 1791–1798 (2010)
25. Kreimer, N., Stanton, A., Sacchi, M.D.: Tensor completion based on nuclear norm minimization for 5D seismic data reconstruction. Geophysics **78**(6), V273–V284 (2013)
26. Kressner, D., Steinlechner, M., Vandereycken, B.: Low-rank tensor completion by Riemannian optimization. BIT Numer. Math. **54**(2), 447–468 (2014)
27. Krishnamurthy, A., Singh, A.: Low-rank matrix and tensor completion via adaptive sampling. In: Advances in Neural Information Processing Systems, pp. 836–844 (2013)
28. Liu, X.Y., Aeron, S., Aggarwal, V., Wang, X., Wu, M.Y.: Adaptive sampling of RF fingerprints for fine-grained indoor localization. IEEE Trans. Mob. Comput. **15**(10), 2411–2423 (2016)
29. Liu, X.-Y., Aeron, S., Aggarwal, V., Wang, X.: Low-tubal-rank tensor completion using alternating minimization. In: International Society for Optics and Photonics, pp. 984809–984809 (2016)
30. Parsons, L., Haque, E., Liu, H.: Subspace clustering for high dimensional data: A review. ACM SIGKDD Explor. Newslett. **6**(1), 90–105 (2004)
31. Pimentel, D., Nowak, R., Balzano, L.: On the sample complexity of subspace clustering with missing data. In: IEEE Workshop on Statistical Signal Processing (SSP), pp. 280–283. IEEE (2014)
32. Pimentel-Alarcón, D., Boston, N., Nowak, R.D.: Deterministic conditions for subspace identifiability from incomplete sampling. In: IEEE International Symposium on Information Theory (ISIT), pp. 2191–2195 (2015)
33. Pimentel-Alarcón, D., Balzano, L., Nowak, R.: Necessary and sufficient conditions for sketched subspace clustering. In: 2016 54th Annual Allerton Conference on Communication, Control, and Computing (Allerton), pp. 1335–1343. IEEE (2016)
34. Pimentel-Alarcón, D., Boston, N., Nowak, R.: A characterization of deterministic sampling patterns for low-rank matrix completion. IEEE J. Selected Topics Signal Process. **10**(4), 623–636 (2016)
35. Pimentel-Alarcón, D., Nowak, R.: The information-theoretic requirements of subspace clustering with missing data (2016)
36. Romera-Paredes, B., Pontil, M.: A new convex relaxation for tensor completion. In: Advances in Neural Information Processing Systems, pp. 2967–2975 (2013)
37. Signoretto, M., Dinh, Q.T., De Lathauwer, L., Suykens, J.A.K.: Learning with tensors: A framework based on convex optimization and spectral regularization. Mach. Learn. **94**(3), 303–351 (2014)
38. Sturmfels, B.: Solving Systems of Polynomial Equations, Number 97, American Mathematical Society (2002)
39. Tomioka, R., Hayashi, K., Kashima, H.: Estimation of low-rank tensors via convex optimization. arXiv:1010.0789 (2010)
40. Wang, W., Aggarwal, V., Aeron, S.: Tensor completion by alternating minimization under the tensor train (TT) model. arXiv:1609.05587 (2016)

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