PACKING NON-ZERO A-PATHS IN GROUP-LABELED GRAPHS

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ABSTRACT. Let G=(V,E) be an oriented graph whose edges are labeled by the elements of a group Γ and let $A\subseteq V$. An A-path is a path whose ends are both in A. The weight of a path P in G is the sum of the group values on forward oriented arcs minus the sum of the backward oriented arcs in P. (If Γ is not abelian, we sum the labels in their order along the path.) We are interested in the maximum number of vertex-disjoint A-paths each of non-zero weight. When A=V this problem is equivalent to the maximum matching problem. The general case also includes Mader's S-paths problem. We prove that for any positive integer k, either there are k vertex-disjoint A-paths each of non-zero weight, or there is a set of at most 2k-2 vertices that meets each of the non-zero A-paths. This result is obtained as a consequence of an exact minmax theorem.

1. Introduction

Let Γ be a group, let G = (V, E) be an oriented graph where each edge e of G is assigned a weight $\gamma_e \in \Gamma$, and let $A \subseteq V$. (We will use additive notation for groups, although they need not be abelian.) An A-path is a path (with at least one edge) in the underlying graph whose ends are both in A. Let e be an edge of G oriented with tail u and head v. We let $\gamma(e, u) = -\gamma_e$ and $\gamma(e, v) = \gamma_e$. Now, if $P = (v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$ is a path in G, then the weight of P, denoted $\gamma(P)$, is defined to be $\sum_{i=1}^k \gamma(e_i, v_i)$. Note that, reversing the orientation on an edge e and replacing γ_e with $-\gamma_e$ does not change the weight of any path.

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We are interested in the maximum number of vertex-disjoint A-paths each of non-zero weight. We prove the following result.

Theorem 1.1. Let Γ be a group, let G = (V, E) be an oriented graph with edge-labels from Γ , and let $A \subseteq V$. Then, for any integer k, either

- (1) there are k vertex-disjoint A-paths each of non-zero weight, or
- (2) there is a set of at most 2k-2 vertices that meets each non-zero A-path.

Let $\nu(G, A, \gamma)$ denote the maximum number of vertex-disjoint A-paths each of non-zero weight. We prove Theorem 1.1 as a corollary to an exact min-max theorem for $\nu(G, A, \gamma)$. In fact we will give two different versions of the min-max theorem; the first provides a more intuitive upper-bound while the second is cleaner. Let $E(A, \gamma)$ denote the set of all edges $e \in E$ whose ends are both in A and that have $\gamma_e = 0$; note that deleting such edges does not affect ν . Let comp(G) denote the set of components of G. Finally, let $X, A' \subseteq V$ such that $A - X \subseteq A' \subseteq V - X$ and let $G' = G - X - E(A', \gamma)$. Then

$$\nu(G, A, \gamma) \leq |X| + \nu(G - X, A - X, \gamma)
\leq |X| + \nu(G - X, A', \gamma)
= |X| + \nu(G', A', \gamma)
= |X| + \sum_{H \in \text{comp}(G')} \nu(H, A' \cap V(H), \gamma)
\leq |X| + \sum_{H \in \text{comp}(G')} \left\lfloor \frac{|A' \cap V(H)|}{2} \right\rfloor.$$

We will see that after an appropriate change of edge-weights we can find X and A' such that the above inequalities hold with equality.

Let $x \in V$ and let $\delta \in \Gamma$. For each edge e of G with tail u and head v we define

$$\gamma_e' = \begin{cases} \gamma_e + \delta, & \text{if } v = x \\ -\delta + \gamma_e, & \text{if } u = x \\ \gamma_e, & \text{otherwise.} \end{cases}$$

We say that γ' is obtained by *shifting* γ by δ at x. Note that, if $x \notin A$ then this shift does not change the weight of any A-path (even when Γ is non-abelian). The main theorem is:

Theorem 1.2. Let Γ be a group, let G = (V, E) be an oriented graph with edge labels $(\gamma_e : e \in E)$ from Γ , and let $A \subseteq X$. Then there exist edge-labels $(\gamma'_e : e \in E)$ obtained by shifting γ at vertices in V - A and

there exist sets $X, A' \subseteq V$ such that $A - X \subseteq A' \subseteq V - X$ and

$$\nu(G, A, \gamma) = |X| + \sum_{H \in \text{comp}(G')} \left\lfloor \frac{|A' \cap V(H)|}{2} \right\rfloor,$$

where $G' = \text{comp}(G - X - E(A', \gamma'))$.

We now turn to an alternative min-max theorem. A set of edges $F \subseteq E$ is A-balanced if F contains no non-zero A-path and no non-zero circuit. We let V(F) denote the set of all vertices in G that are incident with an edge in F. It is straightforward to prove that $F \subseteq E$ is A-balanced if and only if there exist edge-labels $(\gamma'_e : e \in E)$ obtained by shifting γ at vertices in V - A such that $\gamma'_f = 0$ for all $f \in F$. With this in mind, the next result is an easy consequence of Theorem 1.2.

Corollary 1.3. Let Γ be a group, let G = (V, E) be an oriented graph with edge labels $(\gamma_e : e \in E)$ from Γ , and let $A \subseteq X$. Then

$$\nu(G, A, \gamma) = \min\left(|X| + \sum_{H \in \text{comp}(G - X - F)} \left\lfloor \frac{|(A \cup V(F)) \cap V(H)|}{2} \right\rfloor\right),$$

where the minimum is taken over all A-balanced sets $F \subseteq E$ and all sets $X \subseteq V$.

Note that, $\nu(G,V,\gamma)$ is the size of the largest matching in $G-E(V,\gamma)$. When A=V it is easy to see that Theorem 1.2 is equivalent to the Tutte-Berge Formula (Theorem 3.1) for the size of a maximum matching. Our proof of Theorem 1.2 is modeled on an easy proof of the Tutte-Berge Formula that we give in Section 3.

2. Some special cases

In this section we mention some path-packing problems that can be modeled via non-zero A-paths. In each of the applications we are given an undirected graph G = (V, E) and a set $A \subseteq V$. Then, we are interested in finding a maximum collection of "feasible" A-paths; where feasibility depends on the application. We then determine a group, a labeling of the edges, and an orientation of G so that the non-zero A-paths and feasible A-paths coincide. Unless explicitly defined, we assume that an arbitrary orientation of G has been prescribed.

A-paths. Here we consider any A-path to be feasible. We assign labels γ_e to edges $e \in E$ and let Γ be the free group generated by $\{\gamma_e : e \in E\}$. Thus, any non-trivial path has non-zero weight. Gallai [2] reduced this case to the maximum cardinality matching problem and proved the specialization of Theorem 1.1.

 $Odd\ A$ -paths. Here only the A-paths of odd length are feasible. We let $\Gamma = \mathbb{Z}_2$ and assign to each edge the label 1. Thus the non-zero paths are exactly those of odd length. The problem of finding a maximum collection of disjoint odd A-paths can be reduced to the maximum matching problem; see [1].

(The main result in [1] gives a structural characterization of signed-graphs with no odd- K_n minor. Signed-graphs can be considered as binary coextensions of graphic matroids. Theorem 1.1 allows us to extend those results to coextensions of graphic matroids over other finite fields.)

(S,T)-paths. Let (S,T) be a partition of A; an (S,T)-path is a path with one end in S and the other end in T. Let $\Gamma = \mathbb{Z}_2$. The edges with exactly one end in S are assigned a label of 1 and all other edges are labeled 0. Then, an A-path is an (S,T)-path if and only if it is non-zero. Now, $\nu(G,A,\gamma)$ is just the maximum number of vertex disjoint (S,T)-paths. It is an interesting exercise to deduce Menger's theorem from Theorem 1.2.

Composition of feasible sets. Suppose that we have groups Γ_1 and Γ_2 and two edge-labelings $(\alpha_e : e \in E)$ from Γ_1 and $(\beta_e : e \in E)$ from Γ_2 . We can define $\Gamma = \Gamma_1 \times \Gamma_2$ and define new edge-labels $\gamma_e = (\alpha_e, \beta_e)$. Now, a path P is non-zero with respect to γ if and only if P is non-zero with respect to either α or β .

Mader's S-paths. Let S be a partition of A and let l = |S|. A path is an S-path if its ends are in different parts of S. Thus, an A-path is an S-path if and only if it is an (S, A - S)-path for some set $S \in S$. Then, by composition, we can define a group $\Gamma = \mathbb{Z}_2^l$ and an edge-labeling γ from Γ such that the S-paths are precisely the non-zero A-paths. (There is a more direct formulation in which $\Gamma = \mathbb{Z}_l$.) Mader [3] proved a min-max theorem for the maximum number of disjoint S-paths; see Schrijver [5] for a shorter proof. Mader's Theorem is a direct specialization of Corollary 1.3.

(The problem of finding a maximum collection of vertex-disjoint S-paths is equivalent to the problem of finding a maximum collection of internally vertex-disjoint A-paths. It is natural then to consider the problem of finding a maximum collection of internally vertex-disjoint non-zero A-paths. This contains the problem of finding a maximum collection of internally vertex-disjoint odd paths between a given pair of vertices; we suspect that this latter problem is NP-hard.)

Paths on surfaces. Suppose that G = (V, E) is an oriented graph embedded on a surface S and that $A \subseteq V$ all lie on a common face F

in the embedding, where F is a closed disk. We fix a basepoint x in F; then, we associate to each A-path P a simple closed curve C(P) on S that is contained in $P \cup F$ and that has x as its basepoint. Now, we can designate an A-path P to be feasible, in different ways, according to the homotopy class of C(P).

Example 1: P is feasible if C(P) is non-contractible.

Example 2: P is feasible if C(P) is non-separating.

Example 3: P is feasible if C(P) is orientation reversing (that is, the neighbourhood of the curve C(P) is a Möbius band).

Let $\Gamma = \pi(S, x)$ be the fundamental group of S with respect to the basepoint x; see Munkres [4]. Recall that the elements of Γ are the equivalence classes of (x, x)-curves on S with respect to homotopy; thus, the identity of Γ consists of the set of contractible (x, x)-curves. Readers familiar with topology will see that:

Lemma 2.1. G can be assigned edge-labels $(\gamma_e : e \in E)$ from Γ such that, for any A-path P, $\gamma(P)$ is the homotopy class of C(P).

Thus, given the edge-labeling γ from Lemma 2.1, an A-path P is non-zero if and only if C(P) is non-contractible. This gives us a formulation for the first example. In each of the other two examples, the homotopy classes corresponding to non-feasible A-paths determine a normal subgroup of Γ . Therefore, formulations for these examples can be obtained, via Lemma 2.1, by applying appropriate homomorphisms to Γ .

3. Matching

Let G = (V, E) be a graph. The matching number of G, denoted $\nu(G)$, is the size of a maximum matching, and the deficiency of G is defined by $def(G) := |V| - 2\nu(G)$. We let odd(G) denote the number of components of G that have an odd number of vertices. Note that, for any $X \subseteq V$, we have

$$\operatorname{def}(G) \ge \operatorname{def}(G - X) - |X| \ge \operatorname{odd}(G - X) - |X|;$$

the following theorem shows that equality can be attained.

Theorem 3.1 (Tutte-Berge Formula). For any graph G,

$$def(G) = \max(odd(G - X) - |X| : X \subseteq V).$$

A set $S \subseteq V$ is matchable if there is a matching of G that covers every vertex in S (matchable sets need not have even cardinality). It is well-known that the matchable sets of G form the independent sets of a matroid on V; this is the $matching\ matroid$ of G.

We require some elementary matroid theory. Let M be a matroid with ground set V and let $u, v \in V$. Then, u is a coloop of M if u is in every basis of M. The elements u and v are in series if neither u nor v are coloops, but there is no basis that avoids both u and v. It is easy to show that series pairs are transitive. That is, if u is in series with v and v is in series with v and v is in series with v.

Lemma 3.2 (Gallai's Lemma). If G = (V, E) is a connected graph and $\nu(G - v) = \nu(G)$ for each vertex $v \in V$, then def(G) = 1 and |V| is odd.

Proof. The matching matroid M of G has no coloops since $\nu(G-v)=\nu(G)$ for each vertex $v\in V$. For each edge uv of G, we have $\nu(G-u-v)<\nu(G)$; that is, u is in series with v. Then, since G is connected, each pair of vertices is in series. Thus, no basis of M can avoid two or more vertices. Therefore, $\operatorname{def}(G)=1$ and, hence, |V| is odd.

Proof of the Tutte-Berge Formula. We have already seen that $def(G) \geq odd(G - X) - |X|$ for any set $X \subseteq V$, thus it suffices to prove that equality can be attained.

Choose $X\subseteq V$ maximal such that $\nu(G)=\nu(G-X)+|X|$. By our choice of X we have $\nu((G-X)-v)=\nu(G-X)$ for each $v\in V-X$. Then, applying Gallai's Lemma to each component H of G-X, we see that $\operatorname{def}(H)=1$ and |V(H)| is odd. Thus, $\operatorname{def}(G-X)=\operatorname{odd}(G-X)$. Therefore, $\operatorname{def}(G)=|V|-2\nu(G)=|V|-2(\nu(G-X)+|X|)=(|V-X|-2\nu(G-X))-|X|=\operatorname{def}(G-X)-|X|=\operatorname{odd}(G-X)-|X|$; as required.

4. A MATROID FROM NON-ZERO A-PATHS

Throughout this section we let Γ be a group, G = (V, E) be an oriented graph with edge-labels $(\gamma_e : e \in E)$ from Γ , and $A \subseteq V$. We let $def(G, A, \gamma) := |A| - 2\nu(G, A, \gamma)$.

A path is a sequence $P = (v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k)$ where v_0, \ldots, v_k are distinct vertices of G and e_i has ends v_{i-1} and v_i for each $i \in \{1, \ldots, k\}$. Thus, P is ordered in that it has distinguished start (v_0) and $end(v_k)$. However, P need not be "directed" in that an edge e_i of P may have v_{i-1} or v_i as its head. The path $(v_k, e_k, v_{k-1}, \ldots, v_1, e_1, v_0)$ is denoted by P. Also, for any $i, j \in \{1, \ldots, k\}$ with $i \leq j$, the path $(v_i, e_{i+1}, v_{i+1}, \ldots, e_j, v_j)$ is denoted by $P[v_i, v_j]$. We allow paths consisting of a single vertex; we refer to such paths as trivial.

An A-collection is a set \mathcal{P} of vertex disjoint paths such that:

- 1. each vertex in A is either the start or the end of a path in \mathcal{P} ,
- 2. the start of each path $P \in \mathcal{P}$ is in A, and

3. if $P \in \mathcal{P}$ is non-trivial and has its end in A, then $\gamma(P) \neq 0$.

A path $P \in \mathcal{P}$ is loose if it is trivial or its end is not in A; thus each path in \mathcal{P} is either an A-path or it is loose (not both). An A-collection is optimal if it contains $\nu(G, A, \gamma)$ A-paths; note that there are optimal A-collections.

Let $\Gamma' = \{\gamma(P) : P \text{ a path of } G\}$ (when Γ is finite we could just take $\Gamma' = \Gamma$). Now, let $S = \{(v, 0) : v \in A\} \cup \{(v, \gamma) : v \in V - A, \gamma \in \Gamma'\}$. We will define a matroid on the ground set S. Let \mathcal{P} be an A-collection. We let $B(\mathcal{P})$ denote the set of pairs $(v, \gamma(P))$ where v is the end of a loose path $P \in \mathcal{P}$. Note that, $B(\mathcal{P}) \subseteq S$. Now let \mathcal{B} denote the set of all $B(\mathcal{P})$ where \mathcal{P} is an optimal A-collection.

Note that $|B| = \text{def}(G, A, \gamma)$ for all $B \in \mathcal{B}$. Below we show that \mathcal{B} is the collection of bases of a matroid on S. (In the special case that our original A-path problem was just matching, this matroid is isomorphic to the dual of the matching matroid.)

Lemma 4.1. \mathcal{B} is the set of bases of a matroid on S.

Proof. As noted above, \mathcal{B} is nonempty and all sets in \mathcal{B} have the same cardinality. Suppose that \mathcal{B} is not the collection of bases of a matroid. Thus, there exist \mathcal{P} , \mathcal{P}' , and (u, α) satisfying:

4.1.1. \mathcal{P} and \mathcal{P}' are optimal A-collections and $(u, \alpha) \in B(\mathcal{P}) - B(\mathcal{P}')$ such that for each $(v, \beta) \in B(\mathcal{P}') - B(\mathcal{P})$ we have $(B(\mathcal{P}) - \{(u, \alpha)\}) \cup \{(v, \beta)\} \notin \mathcal{B}$.

Now:

4.1.2. we choose \mathcal{P} , \mathcal{P}' , and (u, α) satisfying 4.1.1 with $|E(\mathcal{P}) - E(\mathcal{P}')|$ as small as possible.

We use the following claim repeatedly.

4.1.3. There does not exist an optimal A-collection \mathcal{P}'' such that $B(\mathcal{P}) - B(\mathcal{P}'') = \{(u, \alpha)\}$ and $|E(\mathcal{P}'') - E(\mathcal{P}')| < |E(\mathcal{P}) - E(\mathcal{P}')|$.

Subproof. Suppose that there does exist such an A-collection \mathcal{P}'' . Since $|B(\mathcal{P}'')| = |B(\mathcal{P})|$ there is a unique element, say (u', α') , in $B(\mathcal{P}'') - B(\mathcal{P})$. Moreover, by 4.1.1, $(u', \alpha') \notin B(\mathcal{P}')$. However, $|E(\mathcal{P}'') - E(\mathcal{P}')| < |E(\mathcal{P}) - E(\mathcal{P}')|$. So, by 4.1.2, \mathcal{P}'' , \mathcal{P} , and (u', α') do not satisfy 4.1.1. That is, there exists an element $(v, \beta) \in B(\mathcal{P}') - B(\mathcal{P}'')$ such that $(B(\mathcal{P}') - \{(u', \alpha')\}) \cup \{(v, \beta)\} \in \mathcal{B}$. However, $(B(\mathcal{P}) - \{(u, \alpha)\}) \cup \{(v, \beta)\} = (B(\mathcal{P}') - \{(u', \alpha')\}) \cup \{(v, \beta)\} \in \mathcal{B}$, contradicting 4.1.1.

Let $P = (v_0, e_1, v_1, \dots, e_k, v_k)$ be the path in \mathcal{P} ending at u; thus, $u = v_k$. By possibly reversing the order, we may assume that there is a

path $P' = (v'_0, e'_1, v'_1, \dots, e'_l, v'_l)$ in \mathcal{P}' that starts at v_0 . Suppose that P is not contained in P' and let e_a be the first edge of P not in P'. Now let \mathcal{P}'' be the A-collection obtained from \mathcal{P} by replacing P with $P[v_0, v_{a-1}]$. Note that, \mathcal{P}'' is optimal. Moreover, $B(\mathcal{P}) - B(\mathcal{P}'') = \{(u, \alpha)\}$ and $|E(\mathcal{P}'') - E(\mathcal{P}')| \leq |E(\mathcal{P}) - E(\mathcal{P}')|$; contradicting 4.1.3. Hence, P is contained in P'.

Suppose that P' is disjoint from each path in \mathcal{P} other than the path P, and let \mathcal{P}'' be obtained from \mathcal{P} by replacing P with P'. Since \mathcal{P} is optimal, \mathcal{P}'' is also optimal and P' is loose. Note that, $(v'_l, \gamma(P')) \in B(\mathcal{P}') - B(\mathcal{P})$ and $(B(\mathcal{P}) - \{(u, \alpha)\}) \cup \{(v'_l, \gamma(P'))\} = B(\mathcal{P}'') \in \mathcal{B}$, contradicting 4.1.1. Therefore, there is some vertex that is both on P' and on a path in \mathcal{P} other than P; let v'_i be the first such vertex on P' and let $Q = (u_0, f_1, u_1, \ldots, f_m, u_m)$ be the path of \mathcal{P} containing v'_i . Suppose that $u_i = v'_i$. We consider two cases.

Case 1: Q is a loose path.

Let P_1 be the A-path contained in $P' \cup Q$ and let P_2 be the path in $P' \cup Q$ that starts at u and ends at u_m . Since \mathcal{P} is optimal, $\gamma(P_1) = 0$. Thus, $\gamma(P'[v'_0, v'_i]) = \gamma(Q[u_0, u_j])$ and, hence, $\gamma(P_2) = \gamma(Q)$. Now, let \mathcal{P}'' be the A-collection obtained from \mathcal{P} by replacing P and Q with P_2 and the trivial path (u_0) . Note that, $B(\mathcal{P}) - B(\mathcal{P}'') = \{(u, \alpha)\}$. Moreover, since $\gamma(P_1) = 0$, $P_1 \neq P'$. Thus, there is an edge of $Q[u_0, u_j]$ that is not in $E(\mathcal{P}')$. So, $|E(\mathcal{P}'') - E(\mathcal{P}')| < |E(\mathcal{P}) - E(\mathcal{P}')|$; contrading 4.1.3.

Case 2: Q is an A-path. Let P_1 and P_2 be the A-paths in $P' \cup Q$ that both start at u and that end with u_0 and u_m respectively. Note that $\gamma(P_1) + \gamma(Q) + \gamma(\bar{P_2}) = 0$ and $\gamma(Q) \neq 0$, so either $\gamma(P_1) \neq 0$ or $\gamma(P_2) \neq 0$. Moreover, either P' is loose (and hence different from P_1 and P_2) or $\gamma(P') \neq 0$. Thus, either $\gamma(P_1) \neq 0$ and $P_2 \neq P'$ or $\gamma(P_2) \neq 0$ and $P_1 \neq P'$. By possibly swapping P_1 and P_2 and reversing Q, we may assume that $\gamma(P_2) \neq 0$ and $P_1 \neq P'$. Let \mathcal{P}'' be the A-collection obtained from \mathcal{P} by replacing P and P0 with P2 and the trivial path P1. Note that, P2 and that is not in P3. Moreover, since P3 there is an edge of P4 and is not in P5. Thus, P4 there is an edge of P5 that is not in P6. Thus, P7 there is an edge of P8 that is not in P9. Thus, P9 there is an edge of P9 that is not in P9. Thus, P9 there is an edge of P9 that is not in P9. Thus, if P9 there is an edge of P9 that is not in P9. Thus, if P9 there is an edge of P9 that is not in P9. Thus, if P9 there is an edge of P9 that is not in P9 that is final contradiction completes the proof.

5. Proofs of the main results

Let Γ be a group, G = (V, E) be an oriented graph with edge-labels $(\gamma_e : e \in E)$ from Γ , and $A \subseteq V$. The triple (G, A, γ) is *critical* if

- (i) G is connected,
- (ii) $\nu(G \{v\}, A \{v\}, \gamma) = \nu(G, A, \gamma)$ for each $v \in V$,

(iii) for each $v \in V - A$ and for any edge-labeling γ' obtained from γ by shifting at v we have $\nu(G, A \cup \{v\}, \gamma') > \nu(G, A, \gamma)$, and (iv) $E(A, \gamma) = \emptyset$.

In Section 3 we defined "coloops" and "series pairs"; in this section we require the dual notions, "loops" and "parallel pairs". Let M be a matroid with ground set S and let $u, v \in S$. Then, u is a loop of M if u is not in any basis of M. The elements u and v are parallel if neither u nor v are loops, but there is no basis that contains both u and v. Parallel pairs are transitive; that is, if u is parallel with v and v is parallel with v is parallel with v and v is parallel with v is parallel with v in v is parallel with v in v in

Lemma 5.1. Let Γ be a group, let G = (V, E) be an oriented graph with edge labels $(\gamma_e : e \in E)$ from Γ , and let $A \subseteq X$. If (G, A, γ) is critical, then $def(G, A, \gamma) = 1$ and, hence, |A| is odd.

Proof. Suppose that (G, A, γ) is critical, and let $M = (S, \mathcal{B})$ be the matroid obtained from (G, A, γ) via Lemma 4.1. Let S' denote the set of all non-loop elements of M.

5.1.1. Let e be and edge of G with tail u and head v, and let $(u, \alpha), (v, \beta) \in S'$. If $\alpha + \gamma_e - \beta \neq 0$, then (u, α) and (v, β) are parallel.

Subproof. If (u, α) and (v, β) are not parallel, then there is a basis of M that contains them both. That is, there is an optimal A-collection \mathcal{P} with $(u, \alpha), (v, \beta) \in B(\mathcal{P})$. Now, let P_u and P_v be the paths in \mathcal{P} containing u and v respectively. Note that, $P = (P_u, e, \bar{P_v})$ is an A-path with $\gamma(P) = \alpha + \gamma_e - \beta$. Then, since \mathcal{P} is optimal, we have $\alpha + \gamma_e - \beta = 0$, as required. \square

5.1.2. For each $v \in A$, we have $(v, 0) \in S'$.

Subproof. Since (G, A, γ) is critical, $\nu(G - v, A - v, \gamma) = \nu(G, A, \gamma)$. Thus, there exists a set \mathcal{P} of $\nu(G, A, \gamma)$ non-zero A-paths each disjoint from v. Now, adding trivial A-paths to \mathcal{P} we obtain an optimal A-collection \mathcal{P}' with $(v, 0) \in B(\mathcal{P}')$. Thus, $(v, 0) \in S'$, as required. \square

5.1.3. For each $v \in V - A$, there exist two distinct elements $(v, \alpha), (v, \beta) \in S'$.

Subproof. Consider any element $\delta \in \Gamma$, and let γ' be the edge labels obtained from γ by shifting at v by δ . Since (G, A, γ) is critical, $\nu(G, A \cup \{v\}, \gamma') = \nu(G, A, \gamma) + 1$. Let \mathcal{P} be an optimal $A \cup \{v\}$ -collection with respect to γ' . Since $\nu(G, A \cup \{v\}, \gamma') > \nu(G, A, \gamma)$, v is the start or end of an $A \cup \{v\}$ -path P in \mathcal{P} ; by possibly reversing P we may assume that v is the end. Then, \mathcal{P} is an optimal A-path collection in G and $\gamma'(P) = \gamma(P) + \delta \neq 0$. Now, $(v, \gamma(P)) \in S'$ and

 $\gamma(P) \neq -\delta$. Since δ is any element of Γ , there must exist two distinct elements $(v, \alpha), (v, \beta) \in S'$.

5.1.4. Let e be an edge with tail u and head v. Then, there exist $(u, \alpha_u), (v, \alpha_v) \in S'$ that are parallel in M.

Subproof. First suppose that $u, v \in A$. Let $\alpha_v = \alpha_u = 0$. Since (G, A, γ) is critical, $0 \neq \gamma_e = \alpha_u + \gamma_e - \alpha_v$. Then, by 5.1.1, (u, α_u) and (v, α_v) are parallel. Now we may assume that $u \notin A$ or $v \notin A$; by symmetry we may assume that $v \notin A$. Now, by 5.1.2 and 5.1.3, there exists $\alpha_u \in \Gamma$ such that $(u, \alpha_u) \in S'$, and, by 5.1.3, there exists $\alpha_v \in \Gamma$ such that $(v, \alpha_v) \in S'$ and $\alpha_v \neq \alpha_u + \gamma_e$. Then, by 5.1.1, (u, α_u) and (v, α_v) are parallel.

For each $v \in V$, let $S'_v = \{(u, \alpha) \in S' : u = v\}$. Consider an optimal A-collection \mathcal{P} . Since there is at most one path in \mathcal{P} that ends at v, $|B(\mathcal{P}) \cap S'_v| \leq 1$. Thus, any two elements of S'_v are in parallel. Then, by 5.1.4 and since G is connected, each pair of elements in S' are parallel. Thus, if \mathcal{P} is an optimal A-collection, then $|B(\mathcal{P})| = 1$ and, hence, $def(G, A, \gamma) = 1$, as required.

Proof of Theorem 1.2. Choose $X \subseteq V$ maximal such that $\nu(G-X, A-X, \gamma) = \nu(G, A, \gamma) - |X|$. Now among all sets $A' \subseteq V - X$ with $A-X \subseteq A'$ and edge-labelings γ' obtained from γ by shifting on the vertices in A'-A such that $\nu(G-X,A',\gamma') = \nu(G-X,A-X,\gamma)$ we choose the pair (A',γ') with A' as large as possible. Now let H_1,\ldots,H_l be the components of $G-X-E(A',\gamma')$ and let A'_i denote $A' \cap V(H_i)$. Note that,

$$\nu(G, A, \gamma') = |X| + \sum_{i=1}^{l} \nu(H_i, A_i', \gamma').$$

By our choice of X and A', it is easy to check that each of the triples (H_i, A'_i, γ') is critical. Then, by Lemma 5.1, $\nu(H_i, A'_i, \gamma') = \left|\frac{|A'_i|}{2}\right|$. So,

$$\nu(G, A, \gamma') = |X| + \sum_{i=1}^{l} \left\lfloor \frac{|A'_i|}{2} \right\rfloor,$$

as required. \Box

Proof of Theorem 1.1. By Theorem 1.2 there exist edge-labels $(\gamma'_e:e\in E)$ obtained by shifting γ at vertices in V-A and there exist sets $X,A'\subseteq V$ such that $A-X\subseteq A'\subseteq V-X$ and

$$\nu(G, A, \gamma) = |X| + \sum_{i=1}^{l} \left\lfloor \frac{|V(H_i) \cap A'|}{2} \right\rfloor,$$

where H_1, \ldots, H_l are the components of $G - X - E(A', \gamma')$. Let $i \in \{1, \ldots, l\}$ and let A'_i denote $V(H_i) \cap A'$. Now, let $X_i \subseteq A'_i$ with $|X_i| = |A'_i| - 1$, and let $X^* = X \cup X_1 \cup \cdots \cup X_l$. Note that, $\nu(H_i - X_i, A'_i - X_i, \gamma') = 0$ since $|A'_i - X_i| = 1$. Now,

$$\nu(G - X^*, A - X^*, \gamma) = \nu(G - X^*, A - X^*, \gamma')
\leq \nu(G - X^*, A' - X^*, \gamma')
= \nu(G - X^* - E(A', \gamma'), A' - X^*, \gamma')
\leq \sum_{i=1}^{l} \nu(H_i - X_i, A'_i - X_i, \gamma')
= 0.$$

Thus, X^* meets every non-zero A-path in G. Suppose that $\nu(G, A, \gamma) < k$. Then,

$$2k - 2 \ge 2\nu(G, A, \gamma)$$

$$= 2|X| + \sum_{i=1}^{l} 2\left\lfloor \frac{|A'_i|}{2} \right\rfloor$$

$$\ge |X| + \sum_{i=1}^{l} (|A_i| - 1)$$

$$= |X| + \sum_{i=1}^{l} |X_i|$$

$$= |X^*|,$$

as required.

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