The Erdős-Hajnal Conjecture for Bull-free Graphs

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Abstract

The *bull* is a graph consisting of a triangle and two pendant edges. A graphs is called *bull-free* if no induced subgraph of it is a bull. In this paper we prove that every bull-free graph on n vertices contains either a clique or a stable set of size $n^{\frac{1}{4}}$, thus settling the Erdős-Hajnal conjecture [5] for the bull.

1 Introduction

All graphs in this paper are finite and simple. The *bull* is a graph with vertex set $\{x_1, x_2, x_3, y, z\}$ and edge set

$$\{x_1x_2, x_2x_3, x_1x_3, x_1y, x_2z\}.$$

Let G be a graph. We say that G is *bull-free* if no induced subgraph of G is isomorphic to the bull. The complement of G is the graph $\overline{G}$, on the same vertex set as G, and such that two vertices are adjacent in G if and only if they are non-adjacent in $\overline{G}$. We observe that G is bull-free if and only if $\overline{G}$

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is bull-free. A *clique* in G is a set of vertices, all pairwise adjacent. A *stable set* in G is a clique in the complement of G.

In [5] Erdős and Hajnal made the following conjecture:

1.1 *For every graph H, there exists $\delta(H) > 0$, such that if G is a graph and no induced subgraph of G is isomorphic to H, then G contains either a clique or a stable set of size at least $|V(G)|^{\delta(H)}$.*

This conjecture is known to be true for small graphs H (with $|V(H)| \leq 4$), and for graphs H obtained from them by certain operation [1, 5, 6]. Thus graphs H on at least five vertices, not obtained by the operations of [1, 5, 6], are the next interesting case. The bull is one of such graphs. In this paper we prove that the Erdős-Hajnal conjecture holds when H is the bull. Our main result is:

1.2 *Let G be a bull-free graph. Then G contains a stable set or a clique of size at least $|V(G)|^{\frac{1}{4}}$.*

In fact, we prove a stronger result. We say that a graph G is *narrow*, if $\sum_{v \in V(G)} g(v)^2 \leq 1$ for every function $g : V(G) \rightarrow \mathbb{R}^+$ such that $\sum_{v \in V(P)} g(v) \leq 1$ for every perfect induced subgraph P of G. We prove:

1.3 *Every bull-free graph is narrow.*

The connection between 1.2 and 1.3 is explained in the next section.

For a subset A of $V(G)$ and a vertex $b \in V(G) \setminus A$, we say that b is *complete* to A if b is adjacent to every vertex of A, and that b is *anticomplete* to A if b is not adjacent to any vertex of A. For two disjoint subsets A and B of $V(G)$, A is *complete* to B if every vertex of A is complete to B, and A is *anticomplete* to B if every vertex of A is anticomplete to B. For a subset X of $V(G)$, we denote by $G|X$ the subgraph induced by G on X, and by $G \setminus X$ the subgraph induced by G on $V(G) \setminus X$. We say that X is a *bull* if $G|X$ is a bull. A *hole* in a graph G is an induced cycle with at least four vertices. An *antihole* in G is a hole in $\overline{G}$. A hole (antihole) is *odd* if it has an odd number of vertices. A *path* in G is an induced connected subgraph P of G such that either P is a one-vertex graph, or two vertices of P have degree one and all the others have degree two. The *interior* of P is the set of all vertices that have degree two in P. An *antipath* in G is a path in $\overline{G}$. A *homogeneous set* in G is a proper subset X of $V(G)$ such that every vertex of $V(G) \setminus X$ is either complete or anticomplete to X. We say that G admits a *homogeneous set decomposition* if there exists a homogeneous set X in G with $1 < |X| < |V(G)|$.

We say that G is *composite* if G is bull-free and there exists an odd hole or antihole A in G, such that some vertex of $V(G) \setminus V(A)$ is complete to $V(A)$ and some vertex of $V(G) \setminus V(A)$ is anticomplete to $V(A)$. A graph is *basic* if it is bull-free and not composite.

In Section 3 we prove the following (this result, in a slightly greater generality, appears in [3], but we include the proof here for completeness):

1.4 *Every composite graph admits a homogeneous set decomposition.*

In Section 4 we prove that 1.3 holds for basic graphs. In Section 5 we use induction on the number of vertices of the graph, 1.4 and the fact that 1.3 is true for basic graphs, to prove 1.3 for composite graphs, thus completing the proof of 1.3.

Before we proceed with the proof, let us state two results about perfect graphs that we use:

the Weak Perfect Graph Theorem [7]

1.5 *A graph G is perfect if and only if $\overline{G}$ is perfect.*

and the Strong Perfect Graph Theorem [4]

1.6 *A graph is perfect if and only if it contains no odd hole and no odd antihole.*

2 Covering with perfect graphs

In this section we show how 1.3 implies 1.2. For a graph G, we denote by $\omega(G)$ the size of the largest clique of G, by $\alpha(G)$ the size of the largest stable set of G, and by $\chi(G)$ the chromatic number of G. G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G. We observe the following:

2.1 *Every perfect graph G contains a clique or stable set with at least $\sqrt{|V(G|}$ vertices.*

Proof. Since $\chi(G) = \omega(G)$, it follows that $|V(G)| \leq \alpha(G)\omega(G)$, and 2.1 follows. ∎

Next we prove the following lemma about narrow graphs:

2.2 *Let G be a narrow graph, let $w : V(G) \to \mathbb{R}^+$, and let $M = \sqrt{\sum_{v \in V(G)} w(v)^2}$. Let $\mathcal{P}$ be the family of all induced perfect subgraphs of G. Then there exists a function $f : \mathcal{P} \to \mathbb{R}^+$ such that*

- $\sum_{\{P \in \mathcal{P} \text{ s.t. } v \in P\}} f(P) \geq w(v)$ *for every $v \in V(G)$, and*
- $\sum_{P \in \mathcal{P}} f(P) \leq M$.

Proof. Consider the following linear program:

$$z = min \sum_{P \in \mathcal{P}} f(P)$$

subject to

$$\sum_{\{P \in \mathcal{P} \text{ s.t. } v \in P\}} f(P) \geq w(v) \text{ for every } v \in V(G)$$

$$f(P) \geq 0 \text{ for every } P \in \mathcal{P}.$$

By taking duals, using the LP-duality theorem (see, e.g. [2]), we get that

$$z = max \sum_{v \in V(G)} g(v)w(v)$$

subject to

$$\sum_{v \in V(P)} g(v) \leq 1 \text{ for every } P \in \mathcal{P}$$

$$g(v) \geq 0 \text{ for every } v \in V(G).$$

Let $g : V(G) \to \mathbb{R}^+$ be a function such that

$$\sum_{v \in V(P)} g(v) \leq 1 \text{ for every } P \in \mathcal{P}.$$

Then, by the Cauchy-Schwartz inequality, and since G is narrow, it follows that

$$\sum_{v \in V(G)} g(v)w(v) \leq \sqrt{\sum_{v \in V(G)} g(v)^2} \sqrt{\sum_{v \in V(G)} w(v)^2} \leq M.$$

This proves 2.2. ∎

Thus, 1.3 and 2.2, imply the following:

2.3 *Let G be a bull-free graph, let $w : V(G) \to \mathbb{R}^+$, and let $M = \sqrt{\sum_{v \in V(G)} w(v)^2}$. Let $\mathcal{P}$ be the family of all induced perfect subgraphs of G. Then there exists a function $f : \mathcal{P} \to \mathbb{R}^+$ such that*

- $\sum_{\{P \in \mathcal{P} \text{ s.t. } v \in P\}} f(P) \geq w(v)$ *for every* $v \in V(G)$, *and*
- $\sum_{P \in \mathcal{P}} f(P) \leq M$.

Next we show that in order to prove 1.2, it is enough to prove the following:

2.4 *Let G be a bull-free graph and let $\mathcal{P}$ be the family of all perfect induced subgraphs of G. Then there exists $f : \mathcal{P} \to \mathbb{R}^+$ such that*

- $\sum_{\{P \in \mathcal{P} \text{ s.t. } v \in P\}} f(P) \geq 1$ *for every* $v \in V(G)$, *and*
- $\sum_{P \in \mathcal{P}} f(P) \leq \sqrt{|V(G)|}$.

Proof of 1.2 assuming 2.4. Let G be a bull-free graph, and let f be as in 2.4. Let $K = \max_{P \in \mathcal{P}} |V(P)|$. Then

$$|V(G)| = \sum_{v \in V(G)} 1 \leq \sum_{v \in V(G)} \sum_{\{P \in \mathcal{P} \text{ s.t } v \in V(P)\}} f(P)$$

$$\leq \sum_{P \in \mathcal{P}} \sum_{v \in V(P)} f(P) \leq K \sum_{P \in \mathcal{P}} f(P) \leq K\sqrt{|V(G)|}.$$

Consequently,

$$K \geq \sqrt{|V(G)|}.$$

Let P be a perfect induced subgraph of G with $|V(P)| \geq \sqrt{|V(G)|}$. Now 2.1 implies that P, and therefore G, contains a clique or a stable set of size at least $\sqrt{|V(P)|} \geq |V(G)|^{\frac{1}{4}}$. This proves 1.2. ∎

Clearly, 2.4 is just a special case of 2.3 when $w(v) = 1$ for every $v \in V(G)$.

3 Homogeneous sets in composite graphs

The goal of this section is to prove 1.4. We start with some definitions. Let G be a graph, and let $S \subseteq V(G)$. We say that S is *split*, if $|S| > 1$, and for every vertex $x \in V(G) \setminus S$ that is not complete or anticomplete to S, there exist three distinct vertices $u, v, w \in S$ such that one of the following two alternatives holds:

1. u-v-w is a path, x is adjacent to u and v and non-adjacent to w, or

2. u-v-w is an antipath, x is adjacent to u and non-adjacent to v and w.

Please note that S is a split set in G if and only if S is a split set in $\overline{G}$.

We prove the following result, which, as we later show, implies 1.4:

3.1 *Let G be a bull-free graph, and let $S \subseteq V(G)$ be a split set. Assume that there exist vertices $a, c \in V(G) \setminus S$ such that c is complete to S and a is anticomplete to S. Then G admits a homogeneous set decomposition.*

Proof. We start with the following observation.

(1) If c is adjacent to a, then every vertex of $V(G) \setminus S$ is either complete to S, or anticomplete to S, or is adjacent to c. If c is non-adjacent to a, then every vertex of $V(G) \setminus S$ is either complete to S, or anticomplete to S, or is non-adjacent to a.

Let $x \in V(G) \setminus S$ and suppose that x violates (1). Let S_1 be the set of neighbors on x in S, and let $S_2 = S \setminus S_1$. Then $S_1, S_2 \neq \emptyset$. Since S is a split set, and by the definition of S_1, one of the following two cases holds.

Case 1. There exist vertices $u, v \in S_1$ and $w \in S_2$, such that u-v-w is a path.

In this case x is adjacent to u, v and non-adjacent to w. Since the set $\{a, x, u, v, w\}$ is not a bull in G, it follows that x is non-adjacent to a. Since x violates (1), it follows that c is adjacent to a, and x is non-adjacent to c. But now $\{x, v, w, c, a\}$ is a bull, a contradiction. This finishes Case 1.

Case 2. There exist vertices $u \in S_1$ and $v, w \in S_2$, such that u-v-w is an antipath.

In this case x is adjacent to u, and non-adjacent to v and w. Since $\{x, u, w, c, v\}$

is not a bull, it follows that x is adjacent to c. Since x violates (1), it follows that c is non-adjacent to a, and x is adjacent to a. But now $\{a, x, u, c, v\}$ is a bull, a contradiction. This finishes Case 2.

This proves (1).

Let C be the set of all vertices of G that are complete to S, and A be the set of all vertices of G that are anticomplete to S. Let $X = V(G) \setminus (A \cup C \cup S)$. We observe that either every vertex in C has a neighbor in A, or every vertex in A has a non-neighbor in C. From this, together with the fact that if G admits a homogeneous set decomposition then so does $\overline{G}$, we may assume, passing to the complement if necessary, that every vertex of C has a neighbor in A.

(2) C is complete to X.

Let $x \in X$ and $c' \in C$. Choose $a' \in A$ that is a neighbor of c'. Applying (1) to x, c' and a', we deduce that c' is adjacent to x. This proves (2).

Let A' be the set of vertices a' in A such that for some $x \in X$, there exists a path from a' to x with interior in A.

(3) A' is complete to C.

Let k be an integer, let $a_1, \ldots, a_k \in A'$ and $x \in X$ and let x-a_1-$\ldots$-a_k be a path. We prove by induction on k that a_k is complete to C. By (2) x is complete to C.

Suppose first that $k = 1$. Since S is a split set, x is not complete and not anticomplete to S, x is adjacent to a_1, and $a_1 \in A$, (1) implies that a_1 is complete to C.

So we may assume that $k > 1$, and $\{a_1, \ldots, a_{k-1}\}$ is complete to C. Let $a_0 = x$. Then a_{k-2} is defined, and there exists $s \in S$ anticomplete to $\{a_{k-2}, a_{k-1}, a_k\}$. But now, since $\{s, c', a_{k-2}, a_{k-1}, a_k\}$ is not a bull for any $c' \in C$, it follows that C is complete to a_k. This proves (3).

Let $Z = S \cup X \cup A'$. By the definition of A', every vertex of $A \setminus A'$ is anticomplete to Z, and by (2) and (3), C is complete to Z. Since $c \in C$, we deduce that $Z \neq V(G)$. But now Z is a homogeneous set in G; and since S is a split set, it follows that $|Z| > 1$. This proves 3.1. ∎

We can now prove 1.4.

Proof of 1.4. Since G is composite, we may assume, passing to the complement if necessary, that there is an odd hole A in G, such that some vertex of $V(G)\setminus V(A)$ is complete to $V(A)$, and some vertex of $V(G)\setminus V(A)$ is anticomplete to $V(A)$. By 3.1 it is enough to verify that $V(A)$ is a split set. Let $x \in V(G) \setminus V(A)$ and assume that x is not complete and not anticomplete to $V(A)$. We need to show that one of the two alternatives of the definition of a split set holds for x.

Let the vertices of A be a_1-a_2-$\ldots$-a_k-a_1 in order. From the symmetry, we may assume that x is adjacent to a_1 and non-adjacent to a_2. We may assume that x is non-adjacent to a_k, for otherwise the first alternative of the definition of a split set holds for the path a_k-a_1-a_2. If x is non-adjacent to a_{k-1}, then the second alternative of the definition holds for the antipath a_1-a_{k-1}-a_2. So we may assume that x is adjacent to a_{k-1}. But now the second alternative of the definition holds for the antipath a_{k-1}-a_2-a_k. This proves that $V(A)$ is a split set, and completes the proof of 1.4. ∎

4 Basic graphs

Let G be a graph. Let H be a hole in G with vertices $h_1, \ldots, h_k$ in order. We say that $v \in V(G)\setminus V(H)$ has *two consecutive neighbors* in H if for some $1 \leq i \leq k$, u is adjacent to both h_i and h_{i+1} (here the addition is *mod k*).

We start with two lemmas about basic graphs.

4.1 *Let G be a basic graph, let H be a hole in G with $|V(H)| \geq 5$, let $c \in V(G) \setminus V(H)$ be complete to $V(H)$, and let $u \in V(G)$ be non-adjacent to c. Then either*

1. *u is complete to $V(H)$, or*
2. *u has at least three neighbors in $V(H)$, and $|V(H)| = 5$.*

Proof. Let the vertices of H be h_1-h_2-$\ldots$-h_k-h_1 in order. Since G is basic, u has a neighbor in $V(H)$, and from the symmetry we may assume that u is adjacent to h_1.

Suppose first that u is adjacent to h_2. Since $\{h_k, h_1, u, h_2, h_3\}$ is not a bull in G, we may assume from the symmetry that u is adjacent to h_k. We may assume that u is not complete to $V(H)$, and that $|V(H)| > 5$, for otherwise one of the outcomes of 4.1 holds. Since $\{u, h_2, h_3, c, h_{k-1}\}$ is not a bull in G, it follows that u is adjacent to one of h_3, h_{k-1}, and again, from the symmetry we may assume that u is adjacent to h_3. Let i be minimum such that u is non-adjacent to h_i. Then $i > 3$. Since $\{h_i, h_{i-1}, h_{i-2}, u, h_k\}$ is not a bull in G, it follows that $i = k-1$, and therefore $i \geq 5$. But now $\{h_i, h_{i-1}, h_{i-2}, u, h_1\}$ is a bull in G, a contradiction. This proves that u is non-adjacent to h_2, and, from the symmetry, u does not have two consecutive neighbors in H.

Since $\{u, h_1, h_2, c, h_i\}$ is not a bull for $i \in \{4, \dots, k-1\}$, it follows that u is adjacent to every h_i with $i \in \{4, \dots, k-1\}$. From the reflectional symmetry about h_1, u is adjacent to every h_i with $i \in \{3, \dots, k-2\}$. But then u is adjacent to both h_3 and h_4, a contradiction. This proves 4.1. ■

4.2 *Let G be a basic graph, let H be a hole in G with $|V(H)| \geq 5$, let $a \in V(G) \setminus V(H)$ be anticomplete to $V(H)$, and let $u \in V(G)$ be adjacent to a. Then u does not have two consecutive neighbors in H, and, in particular, u has at least $\lceil \frac{|V(H)|}{2} \rceil$ non-neighbors in $V(H)$.*

Proof. Let the vertices of H be h_1-h_2-$\dots$-h_k-h_1 in order. Suppose u is adjacent to some two consecutive vertices of H, say h_1 and h_2. Since G is basic, u has a non-neighbor in $V(H)$. Let i be minimum such that u is non-adjacent to h_i. Then $\{a, u, h_{i-2}, h_{i-1}, h_i\}$ is a bull in G, a contradiction. This proves 4.2.

4.3 *Let G be a basic graph, and let $u \in V(G)$. Let N be the set of neighbors of u in G, and M the set of non-neighbors of u in G. Then one of the graphs $G|M$, $G|N$ is perfect.*

Proof. Let $G_N = G|N$ and $G_M = G|M$. Suppose neither of G_M, G_N is perfect.

(1) Not both G_M, G_N contain an odd hole.

Suppose (1) is false, and let H_M, H_N be odd holes in G_M, G_N, respectively. Then u is complete to $V(H_N)$ and anticomplete to $V(H_M)$. Let $m = |V(H_M)|$ and $n = |V(H_N)|$. Then $m, n \geq 5$. By 4.1 with $c = u$, every vertex of H_M has at least $n-2$ neighbors in $V(H_N)$, and hence there are at least $m(n-2)$ edges with one end in $V(H_M)$ and the other in $V(H_N)$. By 4.2 with $a = u$, every vertex of H_N has at least $\frac{m+1}{2}$ non-neighbors in $V(H_M)$, and hence there are at least $n\frac{m+1}{2}$ non-adjacent pairs of vertices with one vertex in $V(H_M)$ and the other in $V(H_N)$. Consequently,

$$m(n-2) + n\frac{m+1}{2} \leq nm,$$

and therefore

$$nm - 4m + n \leq 0.$$

But, since $n \geq 5$, it follows that

$$nm - 4m + n \geq m + 5,$$

a contradiction. This proves (1).

Since the complement of a basic graph is also basic, it follows that not both G_M, G_N contain an odd antihole.

(2) There is no odd hole in G_N.

Suppose there is an odd hole H_N in G_N. Since G_M is not perfect, it follows from (1) and 1.6 that there is an odd antihole H_M in G_M. Let $m = |V(H_M)|$ and $n = |V(H_N)|$. Since the complement of a hole of length five is a hole of length five, (1) implies that $m \geq 7$ and $n \geq 7$. Let $v \in V(H_M)$. By 4.1 with $c = u$, it follows that v is complete to $V(H_N)$. On the other hand, by 4.1 with $c = u$ applied in $\overline{G}$, every vertex of H_N is anticomplete to $V(H_M)$, a contradiction. This proves (2).

By (2) and 1.6, there is an odd antihole H_N in G_N. By (1) and 1.6, this implies that there is an odd hole H_M in G_M. Let $m = |V(H_M)|$ and $n = |V(H_N)$. By 4.2 with $a = u$, every vertex of H_N has at least $\frac{m+1}{2}$ non-neighbors in $V(H_M)$, and hence there are at least $n\frac{m+1}{2}$ non-adjacent pairs of vertices with non-empty intersection with each of $V(H_M), V(H_N)$. By 4.2 with $a = u$ applied in $\overline{G}$, every vertex of H_M has at least $\frac{n+1}{2}$ neighbors in $V(H_N)$, and hence there are at least $m\frac{n+1}{2}$ adjacent pairs of vertices with non-empty intersection with each of $V(H_M), V(H_N)$. But then

$$n\frac{m+1}{2} + m\frac{n+1}{2} \leq mn,$$

a contradiction. This proves 4.3. ∎

We can now prove the main result of this section, which is 1.3 for basic graphs:

4.4 *Let G be a basic graph. Then G is narrow.*

The proof is by induction on $|V(G)|$. Let $g : V(G) \to \mathbb{R}^+$ be such that $\sum_{v \in V(P)} g(v) \leq 1$ for every perfect induced subgraph P of G. We need to show that $\sum_{v \in V(G)} g(v)^2 \leq 1$. Since every two-vertex induced subgraph of G is perfect, we may assume that $g(u) < 1$ for every $u \in V(G)$. Choose $u \in V(G)$ with $g(u)$ maximum. Let N be the set of neighbors of u in G, and M the set of non-neighbors. Let $G_M = G|M$ and $G_N = G|N$. By 4.3, at least one of G_M, G_N is perfect. Since being basic, narrow and perfect are all properties that are invariant under taking complements (the first one is obvious, and the last two follow from 1.5), we may assume, passing to the complement if necessary, that G_N is perfect.

It follows from the inductive hypothesis that G_M is narrow. Define $f : M \to \mathbb{R}^+$ to be $f(v) = \frac{g(v)}{1-g(u)}$ for every $v \in M$. Let P be a perfect induced subgraph of G_M. Then $G|(V(P) \cup \{u\})$ is perfect, and therefore

$$\sum_{v \in V(P)} f(v) \leq 1.$$

Consequently,

$$\sum_{v \in M} f(v)^2 \leq 1,$$

and therefore

$$\sum_{v \in M} g(v)^2 \leq (1 - g(u))^2.$$

Since $G|N$ is perfect, it follows that $G|(N \cup \{u\})$ is perfect, and therefore $\sum_{v \in N \cup \{u\}} g(v) \leq 1$. Consequently, by the choice of u,

$$\sum_{v \in N} g(v)^2 \leq g(u) \sum_{v \in N} g(v) \leq g(u)(1 - g(u)).$$

Thus

$$\sum_{v \in V(G)} g(v)^2 = g(u)^2 + \sum_{v \in M} g(v)^2 + \sum_{v \in N} g(v)^2 \leq g(u)^2 + (1-g(u))^2 + g(u)(1-g(u))$$

$$= 1 - g(u) + g^2(u) \leq 1.$$

This proves 4.4. ∎

5 Composite graphs and the proof of 1.3.

Let H, F be graphs with $V(H) \cap V(F) = \emptyset$, and let $v \in V(H)$. Let $H(v, F)$ be the graph defined as follows:

- $V(H(v,F)) = V(H) \cup V(F) \setminus \{v\}$
- $u,w \in V(H)$ are adjacent in $H(v,F)$ if and only if they are adjacent in H
- $u,w \in V(F)$ are adjacent in $H(v,F)$ if and only if they are adjacent in F
- $u \in V(H)$ is adjacent to $w \in V(F)$ in $H(v,F)$ if and only if u is adjacent to v in H.

We say that $H(v,F)$ is obtained from H by *substituting F for v*.

In the proof of 1.3 we use the following result of [7]:

5.1 *Let H_1, H_2 be perfect graphs and let $v \in V(H_1)$. Then the graph obtained from H_1 by substituting H_2 for v is perfect.*

We can now prove 1.3.

Proof of 1.3. The proof is by induction on $|V(G)|$. Let $g : V(G) \rightarrow \mathbb{R}^+$ be such that $\sum_{v \in V(P)} g(v) \leq 1$ for every perfect induced subgraph P of G. We need to show that $\sum_{v \in V(G)} g(v)^2 \leq 1$. If G is basic, then 1.3 follows from 4.4, so we may assume that G is composite. By 1.4, there exists a homogeneous set X in G with $1 < |X| < |V(G)|$. Let N be the set of vertices of G that are complete to X, and M the set of vertices of G that are anticomplete to X. Then $V(G) = X \cup N \cup M$. Let H_1 be the graph obtained from $G \setminus X$ by adding a new vertex x and making x complete to N and anticomplete to M. Let $H_2 = G|X$. Then H_1, H_2 are both bull-free, and $|V(H_i)| < |V(G)|$ for $i = 1, 2$. Thus both H_1 and H_2 are narrow. For $i = 1, 2$, let $\mathcal{P}_i$ be the family of all perfect induced subgraphs of H_i. Let

$$K = \max_{P \in \mathcal{P}_2} \sum_{v \in P} g(v).$$

For $i = 1, 2$ define $g_i : V(H_i) \rightarrow \mathbb{R}^+$ as follows: for $v \in V(G) \cap V(H_1)$ let $g_1(v) = g(v)$, let $g_1(x) = K$, and for $v \in V(H_2)$ let $g_2(v) = \frac{g(v)}{K}$. Now it follows from 5.1 that $\sum_{v \in P} g_1(v) \leq 1$ for every $P \in \mathcal{P}_1$. Since H_1 is narrow, we deduce that

$$1 \geq \sum_{v \in V(H_1)} g_1^2(v) = \sum_{v \in N \cup M} g^2(v) + K^2.$$

Clearly, $\sum_{v\in P} g_2(v) \leq 1$ for every $P \in \mathcal{P}_2$. Since H_2 is narrow, it follows that

$$1 \geq \sum_{v\in V(H_2)} g_2^2(v) = \sum_{v\in X} \frac{g^2(v)}{K^2} = \frac{\sum_{v\in X} g^2(v)}{K^2},$$

and therefore $\sum_{v\in X} g^2(v) \leq K^2$. But now

$$\sum_{v\in V(G)} g^2(v) = \sum_{v\in N\cup M} g^2(v) + \sum_{v\in X} g^2(v) \leq \sum_{v\in N\cup M} g^2(v) + K^2 \leq 1.$$

This proves 1.3. ∎

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