Three-colourable perfect graphs without even pairs

Maria Chudnovsky[1]
Columbia University, New York, NY 10027

Paul Seymour[2]
Princeton University, Princeton, NJ 08544

May 16, 2008; revised October 13, 2011

[1]This research was partially conducted while the author served as a Clay Mathematics Institute Research Fellow, and partially supported by NSF grant DMS-0758364.
[2]Partially supported by ONR grant N00014-01-1-0608 and NSF grant DMS-0070912.

Abstract

We still do not know how to construct the "most general" perfect graph, not even the most general three-colourable perfect graph. But constructing all perfect graphs with no even pairs seems easier, and here we make a start on it; we construct all three-connected three-colourable perfect graphs without even pairs and without clique cutsets. They are all either line graphs of bipartite graphs, or complements of such graphs.

1 Introduction

A graph G is *perfect* if for every induced subgraph H, the chromatic number of H equals the size of the largest clique in H. A *hole* in a graph G is an induced subgraph that is a cycle of length at least four, and an *antihole* in G is an induced subgraph whose complement is a cycle of length at least four; and a hole or antihole is *odd* if it has an odd number of vertices. A graph is *Berge* if it has no odd hole and no odd antihole. Perfect graphs were introduced by Claude Berge in [1], where he proposed the "strong perfect graph conjecture", now a theorem [5], the following:

1.1 *A graph is perfect if and only if it is Berge.*

The recognition problem for Bergeness (and hence, by 1.1, for perfection) is also solved [2]:

1.2 *There is an algorithm with running time $O(|V(G)|^9)$, to test if an input graph G is Berge.*

But neither of these results gives us a way to build the most general perfect graph. Ideally we would like a theorem that a graph is Berge if and only if it can be built from some well-understood class of building blocks, by piecing them together in a way that preserves Bergeness. But we are far from such a theorem, and indeed we do not even know how to construct the most general Berge graph with no K_4 subgraph, which presumably should be an easier problem.

An *even pair* in G is a pair u, v of distinct vertices such that every induced path in G between u and v has even length (the *length* of a path or cycle is the number of edges in it), and consequently u, v are nonadjacent. As far as we know, finding an even pair does not give us a satisfactory way to construct our graph from a smaller graph; but still, an even pair u, v in a Berge graph G is quite a useful thing. For instance, if we identify u, v the graph remains Berge with the same clique number, which is helpful if we are trying to optimally colour G, or prove that G is perfect. (For a survey of recent work on even pairs see [6].) Thus, since finding a construction for all perfect graphs seems hopeless, what about finding a construction for all perfect graphs that have no even pairs? This problem, while still open, seems much more tractable.

In this paper we make a start on it; we construct all Berge graphs that have no K_4 subgraphs and have no even pairs. (Almost; we also assume that the graph admits no clique cutset, and is 3-connected. Graphs with a clique cutset can be constructed by overlapping two smaller graphs on the clique cutset, but this construction can introduce even pairs, and we have not been able to restrict the overlapping procedure to make it safe.) Let us say G is *K_4-free* if it has no K_4 subgraph. A *clique cutset* in G is a clique C of G such that $G \setminus C$ is disconnected. We denote the complement of the graph G by $\overline{G}$. Our main theorem is the following:

1.3 *Let G be a 3-connected K_4-free Berge graph with no even pair, and with no clique cutset. Then one of $G, \overline{G}$ is the line graph of a bipartite graph.*

The proof is lengthy, and similar to the proof of 1.1; for a sequence of different graphs H, we first assume that G contains H as an induced subgraph, and prove the theorem in this case, and thereafter we can assume that G does not contain H, and move on to the next graph of our sequence. (The sequence is shorter and the analysis easier than in the proof of 1.1, however.)

We used the fact that every K_4-free Berge graph is three-colourable (for instance, in the proof of 3.1), and so our work does not give an alternative proof of this fact, first proved by Tucker [9, 10].

It does, however, give a polynomial-time algorithm to three-colour K_4-free Berge graphs (first test if there is an even pair; to test if u, v is an even pair, just add an extra vertex adjacent to u, v and test for Bergeness.)

Here is a related question that has a surprisingly pretty answer: which K_4-free graphs have no odd hole and no even pair? In [4] (with Robertson and Thomas) we gave a construction for all K_4-free graphs with no odd hole, using as building blocks the K_4-free Berge graphs. Using this result, in [11], Zwols proved that there are only two K_4-free graphs without odd holes that are not perfect and do not admit a clique cutset, namely the complement of a seven-cycle, and a certain 11-vertex graph with cyclic symmetry.

Perhaps every Berge graph G such that G and its complement both have no even pair is "nice"; either G or its complement admits a clique cutset or a 2-join, or G or its complement is a line graph of a bipartite graph or a double split graph. Indeed, our work in this paper grew from an unpublished conjecture of Robin Thomas along these lines.

2 The Roussel-Rubio lemma

There was a result proved by Roussel and Rubio [8], that we used many times in the proof of 1.1, that will be important here. All graphs in this paper are finite, and without loops or parallel edges. Let us say a subset $X \subseteq V(G)$ is *connected* if the subgraph $G|X$ of G induced on X is connected, and *anticonnected* if $\overline{G}|X$ is connected. If $X, Y \subseteq V(G)$, we say X is *complete* to Y or *Y-complete* if every vertex in X is adjacent to every vertex in Y (and similarly, we say a vertex v is complete to Y or Y-complete if $\{v\}$ is complete to X, and an edge uv is Y-complete if both u, v are Y-complete); and X is *anticomplete* to Y if X is complete to Y in $\overline{G}$. If P is a path p_1-$\cdots$-p_k say, with $k > 1$, its *interior* is the set $\{p_2, \dots, p_{k-1}\}$, and we denote this by P^*.

If P is an induced path in G with vertices p_1-$\cdots$-p_k in order, with $k \geq 4$, a *leap* for P is a pair $\{x, y\}$ of nonadjacent vertices of $V(G) \setminus V(P)$ such that x is adjacent to p_1, p_2, p_k, and y is adjacent to p_1, p_{k-1}, p_k, and there are no other edges between $\{x, y\}$ and $V(P)$. The Roussel-Rubio lemma is the following:

2.1 *Let G be a Berge graph, and let P be an induced path in G of odd length, at least five. Let $X \subseteq V(G) \setminus V(P)$ be anticonnected, such that the ends of P are X-complete, and no edge of P is X-complete. Then X includes a leap for P.*

We also need a theorem of [5]:

2.2 *Let G be Berge, let X be an anticonnected subset of $V(G)$, and P be an induced path in $G \setminus X$ with odd length, such that both ends of P are X-complete, and no edge of P is X-complete. Then every X-complete vertex of G has a neighbour in P^*.*

Next, we need:

2.3 *Let G be a Berge graph, and let P be an induced path in G of odd length, with vertices p_1-$\cdots$-p_k in order. Let $X \subseteq V(G) \setminus V(P)$ be anticonnected, such that p_1, p_k are X-complete, and no edge of P is X-complete. Then $k \geq 4$ and every vertex in X is adjacent to one of p_2, p_{k-1}.*

Proof. Since no edge of P is X-complete it follows that $k \geq 4$. Suppose that $z \in X$ is nonadjacent to both p_2, p_{k-1}. If $k = 4$ then z-p_1-p_2-p_3-p_4-z is an odd hole, a contradiction, so $k > 4$. Choose an anticonnected subset $Z \subseteq X$, with $z \in Z$, maximal such that Z includes no leap for P. Thus $Z \neq X$ by 2.1; choose $x \in X \setminus Z$ such that $Z \cup \{x\}$ is anticonnected. From the maximality of Z, $Z \cup \{x\}$ includes a leap, and since Z includes no leap, it follows that there is a leap $\{x, y\}$ for some $y \in Z$. Consequently y is nonadjacent to $p_3, \ldots, p_{k-2}$. But from 2.1 applied to P and Z, since Z contains no leap, there is a Z-complete vertex p_i, where $2 \leq i \leq k-1$. Hence p_i is adjacent to both y, z. But z is nonadjacent to p_2, p_{k-1}, and y is nonadjacent to $p_3, \ldots, p_{k-2}$, a contradiction. This proves 2.3. ∎

In the three-colourable case we can say more:

2.4 *Let G be a Berge graph with a three-colouring $\phi : V(G) \to \{1, 2, 3\}$. Let P be an induced path in G of odd length, with vertices p_1-$\cdots$-p_k in order. Let $X \subseteq V(G) \setminus V(P)$ be anticonnected, such that p_1, p_k are X-complete and not all members of X have the same colour. Then*

- *$\phi(p_1) = \phi(p_k)$ ($= 3$ say), and in particular p_1, p_k are nonadjacent, so $k \geq 3$;*
- *no internal vertex of P is X-complete;*
- *$\{\phi(p_2), \phi(p_{k-1})\} = \{1, 2\}$, say $\phi(p_2) = 1$ and $\phi(p_{k-1}) = 2$;*
- *X is the union of two disjoint stable sets X_1, X_2, where the vertices in X_1 have colour 1 and are adjacent to p_{k-1}, and the vertices in X_2 have colour 2 and are adjacent to p_2; and*
- *there is a leap $\{x_1, x_2\}$ for P with $x_i \in X_i$ for $i = 1, 2$.*

Proof. Since not all members of X have the same colour, we may assume that some vertex in X has colour 1, and some vertex in X has colour 2; so every X-complete vertex has colour 3. In particular, p_1, p_k have colour 3 and therefore are nonadjacent, so $k \geq 4$. For the same reason no two X-complete vertices in P are adjacent. Choose a minimal subpath Q of P of odd length such that both its ends are X-complete. It follows that Q has length at least three; and none of its internal vertices are X-complete, from the minimality of Q. Since p_1, p_k are both X-complete, by 2.2 they both have neighbours in Q^*, and so $Q = P$. Consequently no internal vertex of P is X-complete.

Since p_1 has colour 3, it follows that $X = X_1 \cup X_2$ where for $i = 1, 2$, X_i is the set of vertices in X with colour i. Thus $X_1, X_2 \neq \emptyset$. Since p_2 is adjacent to p_1 and p_{k-1} to p_k, we deduce that p_2, p_{k-1} do not have colour 3, and from the symmetry we may assume that p_2 has colour 1. Thus p_2 is anticomplete to X_1, and so by 2.3, X_1 is complete to p_{k-1}. Since $X_1 \neq \emptyset$, it follows that p_{k-1} does not have colour 1; so it has colour 2. Thus X_2 is anticomplete to p_{k-1}, and therefore is complete to p_2, again by 2.3. We have shown then that every vertex in X is adjacent to one of p_2, p_{k-1} and nonadjacent to the other. Finally, we need to produce the leap. If P has length at least five, this follows from 2.1, so we may assume that P has length three, and therefore $k = 4$. Since p_2, p_3 are not X-complete, and X is anticonnected, there is an (induced) antipath p_2-q_1-$\cdots$-q_m-p_3 between p_2, p_3 with $q_1, \ldots, q_m \in X$. If $m \geq 3$ then q_2 is adjacent to both p_2, p_3, a contradiction; and if $m = 1$ then q_1 is nonadjacent to both p_2, p_3, again a contradiction; so $m = 2$ and $\{q_1, q_2\}$ is the desired leap. This proves 2.4. ∎

3 Complement line graphs

Let H be a graph with vertex set $\{v_1, \ldots, v_9\}$ and edges as follows:

- for $1 \le i \le 6$ v_i is adjacent to $v_{i+2}, v_{i+3}, v_{i+4}$ (reading subscripts modulo 6)
- v_7 is adjacent to v_3, v_4, v_5, v_6; v_8 is adjacent to v_5, v_6, v_1, v_2; and v_9 is adjacent to v_1, v_2, v_3, v_4, and there are no other edges.

We call such a graph H a *trampoline*. In this section we study K_4-free Berge graphs that contain trampolines. We prove the following:

3.1 *Let G be a K_4-free Berge graph with no even pair and no clique cutset. If G contains a trampoline as an induced subgraph, then G is the complement of the line graph of some bipartite graph.*

The proof needs several steps. Throughout this section, let G be a K_4-free Berge graph with no even pair and no clique cutset, that contains a trampoline. Consequently we may choose $t \ge 4$, and pairwise disjoint stable sets A_{ij} ($1 \le i \le 3, 1 \le j \le t$) with the following properties:

- for $1 \le i \le 3$, there is at most one value of $j \in \{1, \ldots, t\}$ such that $A_{ij} = \emptyset$
- for $1 \le j \le t$, there is at most one value of $i \in \{1, 2, 3\}$ such that $A_{ij} = \emptyset$
- for all distinct $i, i' \in \{1, 2, 3\}$ and all distinct $j, j' \in \{1, \ldots, t\}$, A_{ij} is complete to $A_{i'j'}$
- for $1 \le i \le 3$ and for all distinct $j, j' \in \{1, \ldots, t\}$, A_{ij} is anticomplete to $A_{ij'}$
- for $1 \le j \le t$, if A_{1j}, A_{2j}, A_{3j} are all nonempty then they are pairwise anticomplete
- for $1 \le j \le t$, and all distinct $i, i' \in \{1, 2, 3\}$, if $A_{i'j}$ is nonempty then every vertex in A_{ij} has a nonneighbour in $A_{i'j}$.

Choose these sets with maximal union W say. For $1 \le i \le 3$ let $Z_i = \cup_{1 \le j \le t} A_{ij}$, and for $1 \le j \le t$ let $A_j = A_{1j} \cup A_{2j} \cup A_{3j}$. Fix a 3-colouring ϕ of G. Since $t \ge 4$ it follows that the only partition of W into three stable sets is the partition Z_1, Z_2, Z_3; and we may therefore assume that for $1 \le i \le 3$, $\phi(v) = i$ for all $v \in Z_i$.

Let $v \in V(G) \setminus W$, and let N be the set of vertices in W that are adjacent to v. We say v is *major* if N is the union of two of Z_1, Z_2, Z_3; and v is *minor* if there exist $i, i' \in \{1, 2, 3\}$ and $j, j' \in \{1, \ldots, t\}$ such that $i \ne i'$ and $N \subseteq A_{ij} \cup A_{i'j'}$ and $N \cap A_{ij}$ is complete to $N \cap A_{i'j'}$.

3.2 *With notation as above, every vertex in $V(G) \setminus W$ is either major or minor.*

Proof. Let $v \in V(G) \setminus W$, and let N be the set of vertices in W that are adjacent to v. We may assume that $\phi(v) = 3$. Since v therefore has no neighbours in Z_3, it follows that

(1) $N \subseteq Z_1 \cup Z_2$.

(2) *For $1 \le j \le t$, if A_{1j}, A_{2j} are both nonempty and $A_{1j} \cup A_{2j}$ is neither a subset of N nor a*

subset of $V(G) \setminus N$, then there exist $a_{ij} \in A_{ij}$ for $i = 1,2$, nonadjacent, such that exactly one of them is in N.

For we may assume that $j = 1$; and suppose the claim is false. For $i = 1,2$, let $N_i = N \cap A_{i1}$ and let $M_i = A_{i1} \setminus N_i$. Since the claim is false, N_1 is complete to M_2, and N_2 is complete to M_1. If $x \in N_1$, then since x has a nonneighbour in A_{21}, it follows that x has a nonneighbour in N_2; and so, since by hypothesis one of N_1, N_2 is nonempty, it follows that there exist $n_i \in N_1$ for $i = 1,2$, nonadjacent. Similarly M_1, M_2 are both nonempty. Since A_{11} is not anticomplete to A_{21}, it follows that $A_{31} = \emptyset$. If $m_1 \in M_1$ is adjacent to $m_2 \in M_2$, then v-n_1-m_2-m_1-n_2-v is an odd hole, a contradiction; so M_1 is anticomplete to M_2. If say there exists $a_{12} \in A_{12} \setminus N$, then v-n_2-a_{12}-m_2-n_1-v is an odd hole, a contradiction; so $A_{12} \subseteq N$, and similarly $(Z_1 \cup Z_2) \setminus A_1 \subseteq N$. But then we can define $A'_{11} = N_1$, $A'_{21} = N_2$, $A'_{31} = \emptyset$, $A'_{1,t+1} = M_1$, $A'_{2,t+1} = M_2$, $A'_{3,t+1} = \{v\}$, and $A'_{ij} = A_{ij}$ for $1 \le i \le 3$ and $2 \le j \le t$, contrary to the maximality of W.

(3) *We may assume that there are at least two values of $j \in \{1, \dots, t\}$ such that $A_{1j} \cup A_{2j} \not\subseteq N$.*

For suppose not; say $Z_1 \cup Z_2 \subseteq N \cup A_1$. If $A_{11} = \emptyset$ and $A_{21} \subseteq N$, then $N = Z_1 \cup Z_2$ and v is major as required. If $A_{11} = \emptyset$ and $A_{21} \not\subseteq N$, then we can add v to A_{31}, contrary to the maximality of W. Thus we may assume that $A_{11} \neq \emptyset$, and similarly $A_{21} \neq \emptyset$. If N includes $A_{11} \cup A_{21}$ then again v is major, and if N is disjoint from $A_{11} \cup A_{21}$ then we can add v to A_{31}, again contradictory to the maximality of W. Thus we may assume that N includes some but not all of $A_{11} \cup A_{21}$; and so, from (2), we may assume that there exists $a_{11} \in A_{11} \setminus N$, and $a_{21} \in A_{21} \cap N$, nonadjacent. Since $t \ge 4$, there exists $j \in \{2, \dots, t\}$ such that $A_{2j}, A_{3j} \neq \emptyset$, say $j = 2$. Choose $a_{22} \in A_{22}$, and choose $a_{32} \in A_{32}$ nonadjacent to a_{22}. Then v-a_{21}-a_{32}-a_{11}-a_{22}-v is an odd hole, a contradiction. This proves (3).

(4) *For $1 \le j \le t$, $N \cap A_{1j}$ is complete to $N \cap A_{2j}$.*

For suppose that there exist $a_{i1} \in N \cap A_{i1}$ for $i = 1,2$, nonadjacent. By (3) we may assume that $A_{12} \cup A_{22} \not\subseteq N$. Suppose first that both A_{12}, A_{22} are nonempty, and $N \cap (A_{12} \cup A_{22}) \neq \emptyset$. From (2) we may assume that there exist $a_{12} \in A_{12} \setminus N$ and $a_{22} \in A_{22} \cap N$, nonadjacent. Since there is no odd hole of the form v-a_{21}-a_{12}-A_{31}-a_{22}-v, it follows that $A_{31} = \emptyset$; and so $A_{3k} \neq \emptyset$ for $2 \le k \le t$. Since $t \ge 4$, one of A_{23}, A_{24} is nonempty, say A_{23}; choose $a_{23} \in A_{23}$. If $a_{23} \in N$ then v-a_{23}-a_{12}-a_{33}-a_{22}-v is an odd hole (where $a_{33} \in A_{33}$ is nonadjacent to a_{23}), and if $a_{23} \notin N$ then v-a_{11}-a_{23}-a_{12}-a_{21}-v is an odd hole, in either case a contradiction. This proves that if both A_{12}, A_{22} are nonempty, then $N \cap (A_{12} \cup A_{22}) = \emptyset$. Since $A_{12} \cup A_{22} \not\subseteq N$, we may assume that there exists $a_{12} \in A_{12} \setminus N$. For $3 \le j \le t$, since there is no odd hole of the form v-a_{21}-a_{12}-A_{2j}-a_{11}-v, it follows that $A_{2j} \subseteq N$.

Suppose that $A_{22} \neq \emptyset$. By what we just proved, $N \cap (A_{12} \cup A_{22}) = \emptyset$, and from the symmetry between Z_1, Z_2 it follows that $A_{1j} \subseteq N$ for $3 \le j \le t$. By (3) it follows that $A_{11} \cup A_{21} \not\subseteq N$, and so by (2) and the symmetry between Z_1, Z_2, we may assume that $a'_{11} \in A_{11} \cap N$ and $a'_{21} \in A_{21} \setminus N$, nonadjacent. If both $A_{1j}, A_{2j} \neq \emptyset$ for some j with $3 \le j \le t$, then from the symmetry between A_1 and A_j it follows that $A_{11} \cup A_{21} \subseteq N$, a contradiction; so for all j with $3 \le j \le t$, one of $A_{1j}, A_{2j} = \emptyset$. Consequently $A_{3j} \neq \emptyset$, and since $t \ge 4$ we may assume that $A_{13}, A_{33} \neq \emptyset$. Choose $a_{i3} \in A_{i3}$ for $i = 1,3$; then v-a'_{11}-a_{33}-a'_{21}-a_{13}-v is an odd hole, a contradiction. This proves that $A_{22} = \emptyset$.

Consequently $A_{2j} \neq \emptyset$ for $3 \le j \le t$. For $3 \le j \le t$, exchanging A_2, A_j implies that $A_{1j} \subseteq N$. Since $t \ge 4$, at least one of $A_{13}, \ldots, A_{1t}$ is nonempty, say A_{13}; and so there exist vertices in $A_{13} \cap N, A_{23} \cap N$ that are nonadjacent. By exchanging A_1, A_3, it follows that $A_{11}, A_{21} \subseteq N$, contrary to (3). This proves (4).

(5) *There exist $j, j' \in \{1, \ldots, t\}$ such that $N \cap Z_1 \subseteq A_{1j}$ and $N \cap Z_2 \subseteq A_{2j'}$.*

For suppose that there exist $a_{1j} \in N \cap A_{1j}$ for $j = 1, 2$ say. Now either A_{31}, A_{22} are both nonempty, or A_{32}, A_{21} are both nonempty, and from the symmetry we may assume the former. Choose $a_{31} \in A_{31}$ nonadjacent to a_{11}, and choose $a_{22} \in A_{22}$ nonadjacent to a_{12}. By (1) and (4), $a_{31}, a_{22} \notin N$. Then v-a_{11}-a_{22}-a_{31}-a_{12}-v is an odd hole, a contradiction. This proves (5).

Let j, j' be as in (5). To show that v is minor, it remains to show that $N \cap A_{1j}$ is complete to $N \cap A_{2j'}$. This is true from the construction if $j \neq j'$, and by (4) if $j = j'$. Thus v is minor. This proves 3.2. ∎

3.3 *With notation as before, there is no major vertex.*

Proof. We begin with:

(1) *Every two major vertices are adjacent.*

For suppose that b_1, b_2 are nonadjacent major vertices. We may assume that b_1 is complete to $Z_2 \cup Z_3$ say. Suppose first that b_2 is not complete to $Z_2 \cup Z_3$; say b_2 is complete to $Z_3 \cup Z_1$. If there exists $j \in \{1, \ldots, t\}$ such that $A_{3j} = \emptyset$, we can add b_1 to A_{1j} and b_2 to A_{2j}, contrary to the maximality of W. Thus $A_{31}, \ldots, A_{3t}$ are all nonempty. But then we may define $A_{1,t+1} = \{b_1\}$, $A_{2,t+1} = \{b_2\}$, and $A_{3,t+1} = \emptyset$, contrary to the maximality of W. This proves that b_2 is complete to $Z_2 \cup Z_3$.

Since G has no even pair, there is an odd induced path $b_1 = p_1$-$\cdots$-$p_k = b_2$ in G. Since none of $p_2, \ldots, p_{k-1}$ is adjacent to both b_1, b_2, it follows that none of them is in $Z_2 \cup Z_3$. Moreover, $p_2, p_{k-1} \notin Z_1$, since b_1, b_2 are anticomplete to Z_1. Thus $p_2, p_{k-1} \in V(G) \setminus W$. Now p_2 is not complete to $Z_2 \cup Z_3$ since $Z_2 \cup Z_3$ is not stable and G is K_4-free; and since p_2, b_2 are nonadjacent, and we have already seen that every two nonadjacent major vertices have the same neighbours in W, it follows that p_2 is not major. Similarly p_{k-1} is not major. But by 2.4, one of p_2, p_{k-1} is complete to Z_2 and the other to Z_3, which is impossible since they are both minor. This proves (1).

Now to complete the proof of 3.3, suppose that b is a major vertex. Thus $b \notin W$, and we may assume that b is complete to $Z_2 \cup Z_3$ and anticomplete to Z_1. At least one of A_{11}, A_{12} is nonempty, say A_{11}; choose $a_{11} \in A_{11}$. Since G has no even pair, there is an odd induced path $b = p_1$-p_2-$\cdots$-$p_k = a_{11}$. Thus p_1, p_k are both complete to the anticonnected set $(Z_2 \cup Z_3) \setminus A_1$; and this anticonnected set is not stable since $t \ge 4$. Since k is even it follows that none of $p_1, \ldots, p_k$ belong to $(Z_2 \cup Z_3) \setminus A_1$; and so by 2.4, one of p_2, p_{k-1} is complete to $Z_2 \setminus A_1$, and the other to $Z_3 \setminus A_1$. Since p_{k-1} is adjacent to a_{11} and not to b, it follows that p_{k-1} is not in W; by (1) p_{k-1} is not major; and since p_{k-1} is complete to one of $Z_2 \setminus A_1, Z_3 \setminus A_1$ it follows that p_{k-1} is not minor, contrary to 3.2. This proves 3.3. ∎

3.4 *For $1 \leq i \leq 3$ and $1 \leq j \leq t$, $|A_{ij}| \leq 1$.*

Proof. Suppose that $u, v \in A_{11}$ say are distinct. Then u, v both have the same colour, and so are nonadjacent. Moreover, u, v are both complete to $(Z_2 \cup Z_3) \setminus A_1$, and there is an odd induced path $u = p_1\text{-}\cdots\text{-}p_k = v$ between u, v since they are not an even pair; so 2.4 implies that one of p_2, p_{k-1} has colour 3 and is complete to $Z_2 \setminus A_1$, and the other has colour 2 and is complete to $Z_3 \setminus A_1$; let the first be p_2, say. Consequently p_2 is not minor; by 3.3 it is not major; and so by 3.2 it belongs to W. Since it has colour 3 and has a neighbour and a nonneighbour in A_{11}, we deduce that $p_2 \in A_{31}$ and $A_{21} = \emptyset$. But similarly $p_{k-1} \in A_{21}$, a contradiction. This proves 3.4. ∎

Henceforth we denote the unique member of A_{ij} by a_{ij} (when it exists) without further explanation. Note that 3.4 implies that A_{ij} is anticomplete to $A_{i'j}$ for all distinct $i, i' \in \{1, 2, 3\}$ and all $j \in \{1, \ldots, t\}$.

3.5 *If X is a connected set of minor vertices and $u, v \in W$ both have neighbours in X, then u, v are adjacent.*

Proof. Suppose not, and choose nonadjacent $u, v \in W$ and a connected set X as in the claim, with $|X|$ minimum. It follows that X is the interior of an induced path $u\text{-}p_1\text{-}\cdots\text{-}p_k\text{-}v$ between u, v. Since the members of X are minor, 3.4 implies that $k \geq 2$.

(1) *For some $i \in \{1, 2, 3\}$ there are two members of Z_i with neighbours in X.*

For suppose not. We may therefore assume that $u = a_{11}$ and $v = a_{21}$, and $(Z_1 \cup Z_2) \setminus A_1$ is anticomplete to X. Suppose first that k is even. At most one vertex in $Z_3 \setminus A_1$ has a neighbour in X; choose $w \in Z_3 \setminus A_1$ with no neighbour in X, and then $w\text{-}u\text{-}p_1\text{-}\cdots\text{-}p_k\text{-}v\text{-}w$ is an odd hole. So k is odd. Now either a_{12}, a_{23} both exist, or a_{22}, a_{13} both exist, and from the symmetry we may assume the first; and then $u\text{-}p_1\text{-}\cdots\text{-}p_k\text{-}v\text{-}a_{12}\text{-}a_{23}\text{-}u$ is an odd hole, a contradiction. This proves (1).

In view of (1) we may assume that $u = a_{11}$ and $v = a_{12}$. From the minimality of X (and since $k \geq 2$) it follows that A_{1j} is anticomplete to X for $3 \leq j \leq t$.

(2) *It is impossible that k is even.*

For suppose k is even. We may assume that a_{23}, a_{34} exist. If a_{24} exists, then $\{a_{23}, a_{34}, a_{24}\}$ is anticonnected and not stable, and complete to u, v; so by 2.4 each of a_{23}, a_{34}, a_{24} is adjacent to one of p_1, p_k, contradicting that p_1, p_k are minor. So $A_{24} = \emptyset$, and similarly $A_{33} = \emptyset$. Hence a_{21}, a_{22}, a_{13} exist, and since

$$a_{13}\text{-}a_{22}\text{-}u\text{-}p_1\text{-}\cdots\text{-}p_k\text{-}v\text{-}a_{21}\text{-}a_{13}$$

is not an odd hole, one of a_{21}, a_{22} has a neighbour in X, say a_{21}. Since u is adjacent to p_1 and nonadjacent to a_{21}, we deduce that a_{21} is adjacent to p_k from the minimality of X. Since a_{23} also has a neighbour in X and a_{21}, a_{23} are nonadjacent, the minimality of X implies that a_{23} is adjacent to p_1. But similarly a_{34} is adjacent to one of p_1, p_k, contradicting that p_1, p_k are both minor. This proves (2).

(3) *It is impossible that k is odd.*

For suppose that k is odd. We may assume that a_{21}, a_{32} exist, and since

$$u\text{-}p_1\text{-}\cdots\text{-}p_k\text{-}v\text{-}a_{21}\text{-}a_{32}\text{-}u$$

is not an odd hole, we deduce that at least one of a_{21}, a_{32} has a neighbour in X, say a_{21}. Since u is adjacent to p_1, the minimality of X implies that p_k is the only neighbour of a_{21} in X. If also a_{32} has a neighbour in X, then similarly p_1 is its only neighbour, and then $a_{32}\text{-}p_1\text{-}\cdots\text{-}p_k\text{-}a_{21}\text{-}a_{32}$ is an odd hole, a contradiction. Thus a_{32} is anticomplete to X. We may assume that a_{13}, a_{24} exist, and we have seen that a_{13} is anticomplete to X. If also a_{24} is anticomplete to X, then

$$u\text{-}p_1\text{-}\cdots\text{-}p_k\text{-}a_{21}\text{-}a_{13}\text{-}a_{24}\text{-}u$$

is an odd hole. So a_{24} has a neighbour in X. From the minimality of X, its only neighbour in X is p_1; but then $v\text{-}a_{24}\text{-}p_1\text{-}\cdots\text{-}p_k\text{-}v$ is an odd hole. This proves (3).

From (2) and (3), we have a contradiction. This proves 3.5. ∎

Proof of 3.1.

Let G be a K_4-free Berge graph with no clique cutset and no even pair, that contains a trampoline. Define the sets A_{ij} as before. If there is a minor vertex, let X be a maximal connected set of minor vertices; then by 3.5 and 3.3, the set of vertices in W with a neighbour in X is a clique cutset, a contradiction. Thus there is no minor vertex, and by 3.3 and 3.4 it follows that G is the complement of the line graph of a bipartite graph. This proves 3.1. ∎

4 Trapezes and trestles

Let H be a graph, and let G be obtained from H by adding two more vertices, nonadjacent to each other and each adjacent to every vertex of H. We call G a *suspension* of H. We need to consider suspensions of several different small graphs. A *trapeze* is a suspension of a graph H that has four vertices and two edges, disjoint. A *trestle* is a suspension of a four-vertex path. An *extended 4-wheel* is a suspension of a graph with four vertices and two edges that share an end. An *octahedron* is a suspension of a cycle of length four. In this section we show that we can exclude these four kinds of subgraphs.

4.1 *Let G be a K_4-free Berge graph with no even pair, containing no trampoline. Then G does not contain a trapeze.*

Proof. Suppose that G contains a trapeze, with six vertices $a_1, b_1, a_2, b_2, c_1, c_2$, where c_1, c_2 are both complete to $\{a_1, b_1, a_2, b_2\}$, and a_ib_i is an edge for $i = 1, 2$. Fix a three-colouring ϕ of G; then $\phi(c_1) = \phi(c_2)$, and we may assume that $\phi(c_1) = 3$, and $\phi(a_i) = 1$ and $\phi(b_i) = 2$ for $i = 1, 2$.

There is an odd induced path between c_1, c_2, since G has no even pair. For $i = 1, 2$, let d_i be the neighbour of c_i in this path. For $i = 1, 2$, let X_i be the set of common neighbours of c_1, c_2 that have colour i. Then $X_1 \cup X_2$ is anticonnected and not stable (since $a_1, a_2 \in X_1$ and $b_1, b_2 \in X_2$).

Since c_1, c_2 are common neighbours of $X_1 \cup X_2$, we may assume by 2.4 that d_1 has colour 1 and is complete to X_2, and d_2 has colour 2 and is complete to X_1.

Now there is an odd induced path a_1-q_1-$\cdots$-q_k-a_2 between a_1, a_2. Since a_1, a_2 are common neighbours of $\{c_1, c_2, d_2\}$, we may assume by 2.4 (by exchanging a_1b_1 with a_2b_2 if necessary) that q_1 has colour 3 and is adjacent to d_2, and q_k has colour 2 and is complete to $\{c_1, c_2\}$. Moreover, $\{c_1, c_2, d_2\}$ includes a leap; and since the two vertices of the leap are nonadjacent and have different colours, it follows that the leap is $\{c_1, d_2\}$. Consequently c_1 is nonadjacent to $q_1, \ldots, q_{k-1}$, and d_2 is nonadjacent to $q_2, \ldots, q_k$. Since q_k is adjacent to a_2, c_1, c_2, it follows that $q_k \in X_2$, and so d_1 is adjacent to q_k.

Since b_1, q_k have the same colour, they are nonadjacent. Suppose that b_1 is nonadjacent to $q_1, \ldots, q_{k-1}$. Then b_1-a_1-q_1-$\cdots$-q_k is an odd path between common neighbours of $\{c_1, c_2, d_1\}$, and so by 2.4, it follows that d_1 is adjacent to q_{k-1} and not to $q_1, \ldots, q_{k-2}$. But then if d_1, d_2 are nonadjacent then

$$d_2\text{-}q_1\text{-}\cdots\text{-}q_{k-1}\text{-}d_1\text{-}c_1\text{-}a_2\text{-}d_2$$

is an odd hole, a contradiction; if d_1, d_2 are adjacent and $k \geq 4$ then d_2-q_1-$\cdots$-q_{k-1}-d_1-d_2 is an odd hole, a contradiction; and if d_1, d_2 are adjacent and $k = 2$ (and therefore d_1, q_1 are adjacent) then the subgraph induced on $\{a_1, a_2, b_1, q_k, c_1, c_2, d_1, d_2, q_1\}$ is a trampoline, a contradiction. This proves that b_1 is adjacent to q_i for some $i \in \{1, \ldots, k-1\}$. Choose i minimum. From the hole

$$d_2\text{-}q_1\text{-}\cdots\text{-}q_i\text{-}b_1\text{-}c_1\text{-}a_2\text{-}d_2$$

it follows that i is even, and since k is even, we deduce that q_i, q_k are nonadjacent. Suppose that d_1 is anticomplete to $\{q_1, \ldots, q_i\}$. If d_1, d_2 are nonadjacent then

$$d_2\text{-}q_1\text{-}\cdots\text{-}q_i\text{-}b_1\text{-}d_1\text{-}q_k\text{-}a_2\text{-}d_2$$

is an odd hole, and if d_1, d_2 are adjacent then d_2-q_1-$\cdots$-q_i-b_1-d_1-d_2 is an odd hole, a contradiction. Thus d_1 is adjacent to one of $q_1, \ldots, q_i$. Since d_2-q_1-$\cdots$-q_i-b_1-c_2-d_2 is not an odd hole, c_2 is also adjacent to one of $q_1, \ldots, q_i$. Consequently there is an induced path R between c_2 and d_1 with $R^* \subseteq \{q_1, \ldots, q_i\}$. But R can be completed to a hole via d_1-q_k-c_2 and via d_1-c_1-a_2-c_2, and one of these is an odd hole, a contradiction. This proves 4.1. ∎

4.2 *Let G be a K_4-free Berge graph with no even pair, containing no trampoline. Then G contains no trestle.*

Proof. (We remind the reader that all graphs in this paper are finite. This theorem in particular is false if we allow infinite graphs.) Let us say an *extended trestle* in G is a sequence $v_1, \ldots, v_n$ of distinct vertices, with $n \geq 8$, such that for $1 \leq i < j \leq n$, v_i and v_j are adjacent if $j - i \in \{1, 2, 4\}$, and they are nonadjacent if $j - i \notin \{1, 2, 4, 7\}$. Fix a three-colouring ϕ of G. By 4.1 it follows that G contains no trapeze. Suppose it contains a trestle.

(1) *G contains an extended trestle.*

For G contains a trestle, and so there are six vertices $v_2, \ldots, v_7$ in G such that v_2-v_4-v_5-v_7 is an induced path, and $\{v_3, v_6\}$ is complete to $\{v_2, v_4, v_5, v_7\}$, and there are no other edges among

$v_2, \ldots, v_7$. We may assume that v_2, v_5 have colour 1, and v_3, v_6 have colour 2, and v_4, v_7 have colour 3. There is an odd induced path between v_3, v_6, say v_3-p_1-$\cdots$-p_k-v_6. Since v_3, v_6 are both complete to $\{v_2, v_4, v_5, v_7\}$, and the latter is anticonnected and not stable, we may assume from 2.4 and the symmetry that p_1 has colour 3 and is complete to $\{v_2, v_5\}$, and p_k has colour 1 and is complete to $\{v_4, v_7\}$. But then the sequence $p_1, v_2, \ldots, v_7, p_k$ is an extended trestle. This proves (1).

In view of (1) and the finiteness of G, we may choose an extended trestle $v_1, \ldots, v_n$ with n maximum. We may assume that:

(2) *For* $1 \le i \le n$, $\phi(v_i) = n - i \mod 3$.

For v_i, v_{i+1}, v_{i+2} are pairwise adjacent (for $1 \le i \le n-2$), and so are $v_{i+1}, v_{i+2}, v_{i+3}$ (for $i \le n-3$), and so v_i, v_{i+3} have the same colour for $1 \le i \le n-3$. Thus for $1 \le i < j \le n$, if $j - i = 0 \mod 3$ then v_i, v_j have the same colour. Since we may assume that v_n has colour 3 and v_{n-1} has colour 1, the claim follows. This proves (2).

(3) *There is a vertex* $v_{n+1} \neq v_1, \ldots, v_n$, *with colour* 2, *adjacent to* v_n, v_{n-1}, v_{n-3} *and not to* $v_{n-2}, v_{n-4}, v_{n-5}$.

For there is an odd induced path v_{n-1}-p_1-$\cdots$-p_k-v_{n-4} between v_{n-1}, v_{n-4}. Since v_{n-1}, v_{n-4} are both complete to $\{v_n, v_{n-2}, v_{n-3}, v_{n-5}\}$, and the latter is anticonnected and not stable, it follows from 2.4 that one of p_1, p_k has colour 2 and is complete to $\{v_n, v_{n-3}\}$, and the other has colour 3 and is complete to $\{v_{n-2}, v_{n-5}\}$. Suppose that p_1 has colour 3. Then v_{n-2}, v_{n-5} are complete to $\{v_{n-1}, p_1, v_{n-4}, v_{n-6}\}$, and $v_{n-1}p_1$ and $v_{n-4}v_{n-6}$ are edges, and $\{v_{n-1}, p_1\}$ is anticomplete to $\{v_{n-4}, v_{n-6}\}$ (p_1 is not adjacent to v_{n-6} since they have the same colour). Thus G contains a trapeze, a contradiction. This proves that p_1 has colour 2, and is adjacent to v_n, v_{n-3}, and not to v_{n-4}.

Define $v_{n+1} = p_1$; we will show that v_{n+1} satisfies the claim. Since v_{n+1} has colour 2, it is nonadjacent to v_{n-2}, v_{n-5}. Thus, in summary, v_{n+1} is adjacent to v_n, v_{n-1}, v_{n-3} and not to $v_{n-2}, v_{n-4}, v_{n-5}$. Suppose that $v_{n+1} = v_i$ for some $i \in \{1, \ldots, n\}$. Then $n - i = 2 \mod 3$ by (2), since v_{n+1} has colour 2; and $i \neq n-5, n-2$ since v_{n+1} is nonadjacent to v_{n-4}. Thus $i \le n-8$. But the only neighbours of v_n in $\{v_1, \ldots, v_{n-1}\}$ are $v_{n-1}, v_{n-2}, v_{n-4}$ and possibly v_{n-7}, a contradiction. Thus v_{n+1} is different from $v_1, \ldots, v_n$. This proves (3).

(4) v_{n+1} *is nonadjacent to* v_{n-7}.

For suppose v_{n+1}, v_{n-7} are adjacent. Since v_{n+1}-v_{n-7}-v_{n-6}-v_{n-2}-v_{n-1}-v_{n+1} is not a hole of length five, it follows that v_{n+1} is adjacent to v_{n-6}. But then $v_{n+1}v_{n-7}$ and $v_{n-2}v_{n-4}$ are edges, and $\{v_{n+1}, v_{n-7}\}$ is anticomplete to $\{v_{n-2}, v_{n-4}\}$, and v_{n-3}, v_{n-6} are both complete to $\{v_{n-7}, v_{n-4}, v_{n-2}, v_{n+1}\}$, and hence G contains a trapeze, a contradiction. This proves (4).

(5) v_{n+1} *is nonadjacent to* v_i *for* $1 \le i \le n-8$.

For suppose that v_i is adjacent to v_{n+1} for some $i \in \{1, \ldots, n-8\}$, and choose i maximum. There are cases depending on the value of $n - i$ modulo 6. By (2), $n + 1 - i \neq 0, 3 \mod 6$ since v_{n+1}, v_i are adjacent and therefore have different colours; so $n - i$ is one of $0, 1, 3$ or $4 \mod 6$. If $n - i = 0$

mod 6, then $i \leq n-12$, and

$$v_{n+1}\text{-}v_i\text{-}v_{i+4}\text{-}v_{i+6}\text{-}v_{i+10}\text{-}\cdots\text{-}v_{n-12}\text{-}v_{n-8}\text{-}v_{n-4}\text{-}v_n\text{-}v_{n+1}$$

is an odd hole. If $n-i=1 \mod 6$, then $i \leq n-13$, and

$$v_{n+1}\text{-}v_i\text{-}v_{i+4}\text{-}v_{i+6}\text{-}v_{i+10}\text{-}\cdots\text{-}v_{n-13}\text{-}v_{n-9}\text{-}v_{n-5}\text{-}v_{n-1}\text{-}v_{n+1}$$

is an odd hole. If $n-i=3 \mod 6$ then $i \leq n-9$, and

$$v_{n+1}\text{-}v_i\text{-}v_{i+2}\text{-}v_{i+6}\text{-}v_{i+8}\text{-}\cdots\text{-}v_{n-13}\text{-}v_{n-9}\text{-}v_{n-5}\text{-}v_{n-4}\text{-}v_n\text{-}v_{n+1}$$

is an odd hole. If $n-i=4 \mod 6$, then $i \leq n-10$, and

$$v_{n+1}\text{-}v_i\text{-}v_{i+2}\text{-}v_{i+6}\text{-}v_{i+8}\text{-}\cdots\text{-}v_{n-14}\text{-}v_{n-10}\text{-}v_{n-8}\text{-}v_{n-4}\text{-}v_n\text{-}v_{n+1}$$

is an odd hole. This proves (5).

But from (5), $v_1, \ldots, v_{n+1}$ is an extended trestle, contrary to the maximality of n. This proves 4.2. ∎

4.3 *Let G be a K_4-free Berge graph with no even pair, containing no trampoline. Then G contains no extended 4-wheel.*

Proof. Suppose that G contains an extended 4-wheel, with vertex set $\{a_1, a_2, b_1, b_2, c_1, c_2\}$, where a_1-b_1-a_2 is a path and $\{c_1, c_2\}$ is complete to $\{a_1, a_2, b_1, b_2\}$. Fix a three-colouring of G; then we may assume that a_1, a_2 have colour 1, and b_1 has colour 2, and c_1, c_2 have colour 3 (and b_2 has colour 1 or 2). Since G has no even pair, there is an odd induced path c_1-p_1-$\cdots$-p_k-c_2, and since c_1, c_2 are complete to $\{a_1, b_1, a_2, b_2\}$, and the latter is anticonnected and not stable, we may assume from 2.4 and the symmetry between c_1, c_2 that p_1 has colour 2 and is adjacent to a_1, a_2. Since p_1 is not adjacent to c_2, it follows that $p_1 \neq b_1$, and c_2-b_1-c_1-p_1 is an induced path; but $\{a_1, a_2\}$ is complete to the vertex set of this path, and so G contains a trestle, contrary to 4.2. This proves 4.3. ∎

4.4 *Let G be a K_4-free Berge graph with no even pair, containing no trampoline. Then G contains no octahedron.*

Proof. Suppose it does; consequently we may choose three disjoint stable sets $A_1, A_2, A_3 \subseteq V(G)$, pairwise complete and each with cardinality at least two. Choose them with maximal union. Fix a three-colouring of G, and we may assume that the vertices in A_i have colour i for $i = 1, 2, 3$.

(1) *Every A_1-complete vertex belongs to $A_2 \cup A_3$.*

For suppose that v is A_1-complete and $v \notin A_2 \cup A_3$. Since G is K_4-free, v is anticomplete to at least one of A_2, A_3, say A_3. If v is A_2-complete then we may add v to A_3, contrary to the maximality of $A_1 \cup A_2 \cup A_3$. Thus v has a nonneighbour in A_2. Choose $a_1, a_1' \in A_1$. There is an odd induced path a_1-p_1-$\cdots$-p_k-a_1' between a_1, a_1'; and since $A_2 \cup A_3 \cup \{v\}$ is anticonnected and not stable, we may assume by 2.4 that p_1 is complete to A_3 and anticomplete to A_2. Choose distinct $a_3, a_3' \in A_3$, and choose $a_2 \in A_2$. Then p_1-a_1-a_2-a_1' is an induced path, and a_3, a_3' are complete to its vertex set, so G contains a trestle, contrary to 4.2. This proves (1).

Now since $|A_2| \geq 2$, there is an odd induced path with both ends in A_2; choose such a path with minimum length, say a_2-p_1-$\cdots$-p_k-a_2', where $a_2, a_2' \in A_2$. From the minimality of k, it follows that none of $p_1, \ldots, p_k$ is in A_2; and none of them is in $A_1 \cup A_3$ since none of them is adjacent to both a_2, a_2'. Consequently none of $p_1, \ldots, p_k$ is complete to A_1, by (1). By 2.3, every vertex in A_1 is adjacent to one of p_1, p_k; and similarly so is every vertex in A_3. Since p_1, p_k do not have colour 2 (because they have neighbours in A_2), we may assume that p_k has colour 1. Consequently p_k is anticomplete to A_1, and so p_1 is complete to A_1, contrary to (1). This proves 4.4. ∎

5 Jumps on a prism

In this section we present a collection of lemmas about attachments to a prism that we need later. We say a vertex v can be *linked* onto a triangle $\{a_1, a_2, a_3\}$ (via paths P_1, P_2, P_3) if:

- $v \neq a_1, a_2, a_3$
- the three paths P_1, P_2, P_3 are induced and mutually vertex-disjoint, and do not contain v
- for $i = 1, 2, 3$ a_i is an end of P_i
- for $1 \leq i < j \leq 3$, $a_i a_j$ is the unique edge of G between $V(P_i)$ and $V(P_j)$
- v has a neighbour in each of P_1, P_2 and P_3.

Our first lemma (theorem 2.4 of [5]) is well-known:

5.1 *Let G be Berge, and suppose v can be linked onto a triangle $\{a_1, a_2, a_3\}$. Then v is adjacent to at least two of a_1, a_2, a_3.*

A *prism* is a graph consisting of two vertex-disjoint triangles $\{a_1, a_2, a_3\}$, $\{b_1, b_2, b_3\}$, and three paths R_1, R_2, R_3, where each R_i has ends a_i, b_i, and for $1 \leq i < j \leq 3$ the only edges between $V(R_i)$ and $V(R_j)$ are $a_i a_j$ and $b_i b_j$. The three paths R_1, R_2, R_3 are said to *form* the prism. The prism is *long* if at least one of the three paths has length > 1. If G is a graph, a *prism in G* is an induced subgraph K that is a prism. If G is Berge, the three paths forming K are either all even or all odd, and we call the prism *even* or *odd* respectively. A vertex $w \in V(G) \setminus V(K)$ is said to be *major* with respect to K if it has at least two neighbours in each triangle of the prism.

If F, K are induced subgraphs of G, a vertex in $V(K)$ is said to be an *attachment* of F (or of $V(F)$) in K if either it belongs to $V(F)$ or it has a neighbour in $V(F)$. If K is a prism in G with R_1, R_2, R_3 as before, a subset $X \subseteq V(K)$ is *local* with respect to K if either $X \subseteq V(R_i)$ for some i, or X is a subset of one of the triangles of K. If $f_1, \ldots, f_n$ is an induced path disjoint from K, we say that f_1-$\cdots$-f_n is a *corner jump in position* a_1 with respect to K if f_1 is adjacent to a_2, a_3, and there is at least one edge between f_n and $V(R_1) \setminus \{a_1\}$, and every edge between $\{f_1, \ldots, f_n\}$ and $V(K) \setminus \{a_1\}$ is between f_1 and $\{a_2, a_3\}$ or between f_n and $V(R_1) \setminus \{a_1\}$. We define corner jumps in positions a_2, a_3, b_1, b_2, b_3 similarly. A *corner jump* means a path that is a corner jump in one of these six positions. Note that we are distinguishing between f_1-$\cdots$-f_n and f_n-$\cdots$-f_1 here.

We need theorem 10.1 of [5], specialized to K_4-free graphs, the following.

5.2 *Let R_1, R_2, R_3 form a prism K in a K_4-free Berge graph G, with triangles $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$, where each R_i has ends a_i and b_i. Let $F \subseteq V(G) \setminus V(K)$ be connected, such that its set of attachments in K is not local. Then there exist $n \geq 1$ and an induced path f_1-$\cdots$-f_n with $f_1, \ldots, f_n \in F$, such that either:*

- *$n = 1$ and f_1 is major, or*
- *for some distinct $i, j \in \{1, 2, 3\}$, f_1 has two adjacent neighbours in R_i, and f_n has two adjacent neighbours in R_j, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K)$, or*
- *$n \geq 2$, and for some distinct $i, j \in \{1, 2, 3\}$, f_1 is adjacent to a_i, a_j, and f_n is adjacent to b_i, b_j, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K)$, or*
- *f_1-$\cdots$-f_n is a corner jump.*

This has the following useful corollary.

5.3 *Let G be a K_4-free Berge graph containing no trestle, and let C be a hole of G. Let R_3 be an induced path of G, with $V(R_3) \cap V(C) = \emptyset$, and with ends a_3, b_3. (Possibly R_3 has length zero.) Let a_1a_2 and b_1b_2 be disjoint edges of C, such that the only edges between $V(R_3)$ and $V(C)$ are $a_1a_3, a_2a_3, b_1b_3, b_2b_3$. Let $w \in V(G) \setminus (V(R) \cup V(C))$, and let w be adjacent to a_1, a_2 and nonadjacent to at least two of b_1, b_2, b_3. Then w has no neighbours in C except a_1, a_2.*

Proof. We may assume that a_1, a_2, b_2, b_1 appear in this order in C. For $i = 1, 2$, let R_i be the path of C between a_i, b_i not using the edge a_1a_2, and let c_i be the neighbour of a_i in R_i. Suppose w has another neighbour in $V(C)$. Suppose first that R_3 has positive length, so R_1, R_2, R_3 form a prism K. The set of neighbours of w in K is not local, and so 5.2 implies that one of its outcomes holds if we set $n = 1$ and $f_1 = w$. Now the first outcome of 5.2 is false since w has at most one neighbour in $\{b_1, b_2, b_3\}$, and the third is false since $n = 1$. Suppose the second holds. Then w has exactly four neighbours in the hole C, namely c_1, a_1, a_2, c_2. Since C is even, and

$$w\text{-}c_1\text{-}R_1\text{-}b_1\text{-}b_2\text{-}R_2\text{-}c_2\text{-}w$$

is not an odd hole, it follows that C has length four; but then the prism is odd, so R_3 is odd, and

$$w\text{-}a_1\text{-}a_3\text{-}R_3\text{-}b_3\text{-}b_2\text{-}w$$

is an odd hole, a contradiction. Thus the fourth outcome holds. Since w is adjacent to a_1, a_2 it follows that w has neighbours in $V(R_3) \setminus \{a_3\}$, and has no other neighbours in $V(C)$, a contradiction.

We may therefore assume that R_3 has length zero, so $a_3 = b_3$. Suppose that R_2 has length one. Then since the subgraph induced on $(V(C) \setminus \{a_2, b_2\}) \cup \{w\}$ is not an odd hole, it follows that C has length four; and since w has more than two neighbours in C and is nonadjacent to one of b_1, b_2, it follows that G contains a trestle, a contradiction. Thus R_2 has length at least two, and similarly so does R_1. Since a_3-a_2-R_2-b_2-a_3 is a hole it follows that R_2 is even, and similarly R_1 is even.

Consequently R_1, R_2 are both even. Suppose that for $i = 1, 2$, w has a neighbour in R_i different from a_i. Since w cannot be linked onto $\{b_1, b_2, b_3\}$, we deduce that c_1, a_1, a_2, c_2 are the only neighbours of w in C, and then either C or the graph induced on $(V(C) \setminus \{a_1, a_2\}) \cup \{w\}$ is an odd hole. Thus we may assume that w has no neighbour in R_2 different from a_2; and so it does have a neighbour in R_1 different from a_1. Let Q be an induced path between b_1 and w with interior in $V(R_1)$. Since w-a_2-R_2-b_2-b_1-Q-w is not an odd hole, it follows that Q is even; but then w-a_2-a_3-b_1-Q-w is an odd hole, a contradiction. This proves 5.3. ∎

We use 5.2 to prove the following.

5.4 *Let G be a K_4-free Berge graph containing no trapeze, trestle, octahedron or extended 4-wheel. Let K be a prism in G, and let A, B and R_i, a_i, b_i $(i = 1, 2, 3)$ be as before. Let $w \in V(G) \setminus V(K)$ be major with respect to K. Let $F \subseteq V(G) \setminus V(K)$ be connected, such that its set of attachments in K is not local, and w is anticomplete to F. Then there is an induced path f_1-$\cdots$-f_n with $n \geq 1$ and $f_1, \ldots, f_n \in F$, such that either:*

- *$n \geq 3$ is odd, and for some distinct $i, j \in \{1, 2, 3\}$, f_1 has two adjacent neighbours c_i, d_i in R_i, and f_n has two adjacent neighbours c_j, d_j in R_j, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K)$, and w is adjacent to all of c_i, d_i, c_j, d_j, or*
- *K is even, and for some distinct $i, j \in \{1, 2, 3\}$, f_1 has two adjacent neighbours c_i, d_i in R_i, and f_n has two adjacent neighbours c_j, d_j in R_j, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K)$, and w is adjacent to a_i, b_i, a_j, b_j and nonadjacent to every internal vertex of R_i and of R_j, or*
- *$n \geq 2$, and for some distinct $i, j \in \{1, 2, 3\}$, f_1 is adjacent to a_i, a_j, and f_n is adjacent to b_i, b_j, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K)$, and w is adjacent to a_i, a_j, b_i, b_j, or*
- *f_1-$\cdots$-f_n is a corner jump in position a_i say (or b_i, similarly). Moreover, if w is adjacent to a_i, and therefore nonadjacent to a_j for some $j \in \{1, 2, 3\} \setminus \{i\}$, then R_i has length one, w is adjacent to b_i, b_j, and w has no neighbour in R_j except b_j.*

Proof. Let $f_1, \ldots, f_n$ be as in 5.2.

(1) *The first outcome of 5.2 does not hold.*

For suppose it does; thus f_1 is major. Let $\{u, v\} = \{w, f_1\}$; thus, u, v are nonadjacent major vertices, and there is symmetry between u, v. Let X be the set of vertices in K adjacent to both u, v. Thus $A \cap X, B \cap X \neq \emptyset$. If u, v have the same neighbours in $A \cup B$, then the subgraph induced on $X \cap (A \cup B)$ is either a 2-edge matching, or a 3-edge path, or a cycle of length four, and so H contains a trapeze, trestle or octahedron, contrary to the hypothesis. So we may assume that u, v have different neighbours in A; and since they both have exactly two neighbours in A (because G is K_4-free) we may assume that u is adjacent to a_1, a_3, and v is adjacent to a_2, a_3. Hence $a_3 \in X$. Since G contains no hole of length five, every vertex in X is adjacent to one of a_1, a_2. In particular $b_3 \notin X$, and for $i = 1, 2$, if $b_i \in X$ then R_i has length one.

If $b_1, b_2 \in X$, then R_1, R_2 both have length one; but then the subgraph induced on

$$\{a_1, a_2, a_3, b_1, b_2, u, v\}$$

is an odd antihole, a contradiction. Thus we may assume (exchanging u, v if necessary) that $b_2 \notin X$. Now also $b_3 \notin X$, so $b_1 \in X$ and therefore R_1 has length one. Moreover the subgraph induced on $\{u, v, b_2, b_3\}$ is a path of length three between u and v. Thus every vertex in X is adjacent to one of b_2, b_3; and since $a_3 \in X$, it follows that R_3 has length one. But then a_1, b_3, u, v are all adjacent to both a_3, b_1, and so G contains either a trapeze (if v is adjacent to b_3) or an extended 4-wheel (if u is adjacent to b_3). This proves (1).

(2) *If the second outcome of 5.2 holds then the theorem holds.*

For suppose, say, f_1 has two adjacent neighbours in R_1, and f_n has two adjacent neighbours in R_2, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K)$. Let c_1, d_1 be the two neighbours of $f_1 \in R_1$, where a_1, c_1, d_1, b_1 are in order in R_1, and choose $c_2, d_2 \in V(R_2)$ similarly. Suppose that w is adjacent to all of c_1, d_1, c_2, d_2. Thus n is odd, since w-c_1-f_1-$\cdots$-f_n-d_2-w is not an odd hole. If $n = 1$, then the subgraph induced on the set of common neighbours of f_1, w has two disjoint edges, and so G contains a trapeze, trestle or octahedron, a contradiction. Thus $n \geq 3$ and the theorem holds. Consequently we may assume that w is not adjacent to c_1 say, and so w cannot be linked onto the triangle $\{c_1, d_1, f_1\}$. Suppose that w is adjacent to both c_2, d_2. From 5.3 applied to the hole induced on $V(R_1) \cup V(R_2)$ and the path f_1-$\cdots$-f_n, it follows that w has no more neighbours in $V(R_1 \cup R_2)$, and since w is adjacent to at least one of a_1, a_2 and at least one of b_1, b_2, we deduce that R_2 has length one, and w is adjacent to a_3, b_3. But then R_1 is odd (since R_2 is odd), and so w-a_3-a_1-R_1-b_1-b_2-w is an odd hole, a contradiction. Thus w is adjacent to at most one of c_2, d_2, and therefore cannot be linked onto $\{c_2, d_2, f_n\}$.

For $i = 1, 2$, let C_i, D_i be the subpaths of R_i between a_i, c_i and between d_i, b_i respectively. Suppose that w has a neighbour in $V(C_1) \setminus \{a_1\}$. Since w cannot be linked onto $\{c_1, d_1, f_1\}$, it follows that w is nonadjacent to b_1, a_2. Since w is major, it is adjacent to b_2, b_3, and to a_1, a_3. Thus w can be linked onto $\{c_2, d_2, f_n\}$, a contradiction. It follows that w has no neighbour in R_1^*, and similarly none in R_2^*.

Suppose that w is nonadjacent to both a_1, b_1. Then w is adjacent to a_2, a_3, b_2, b_3, and K is even. From the symmetry we may assume that $a_2 \neq c_2$; but a_2 can be linked onto $\{c_2, d_2, f_n\}$, via paths with interiors in $V(C_1) \cup \{f_1, \ldots, f_n\}, V(C_2)$ and $\{w\} \cup V(D_2)$, a contradiction. Thus w is adjacent to at least one of a_1, b_1, and similarly to at least one of a_2, b_2. If w is adjacent to all of a_1, a_2, b_1, b_2 then the theorem holds, so we may assume that w is nonadjacent to b_1. Hence w is adjacent to a_1, b_2, b_3. From the hole w-a_1-R_1-b_1-b_3-w it follows that the prism is odd, and hence R_3 is odd. Let Q be the path

$$a_1\text{-}C_1\text{-}c_1\text{-}f_1\text{-}\cdots\text{-}f_n\text{-}d_2\text{-}D_2\text{-}b_2.$$

From the hole a_1-Q-b_2-b_3-R_3-a_3-a_1 it follows that Q is odd; but then w-a_1-Q-b_2-w is an odd hole, a contradiction. This proves (2).

(3) *If the third outcome of 5.2 holds then the theorem holds.*

Suppose the third outcome of 5.2 holds; so $n \geq 2$, and, say, f_1 is adjacent to a_1, a_2, and f_n is adjacent to b_1, b_2, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K)$. If w is adjacent to a_1, b_1, a_2, b_2, then the theorem holds, so we assume that w is nonadjacent to a_1 say. Hence w is adjacent to a_2, a_3. By 5.3 applied to the prism K' formed by R_1, R_2 and $f_1, \ldots, f_n$, it follows that w is nonadjacent to one of b_1, b_2, say b_i, and therefore adjacent to b_3. Since w-a_2-f_1-$\cdots$-f_n-b_i-b_3-w is not an odd hole, it follows that either n is even or a_2, b_2 are adjacent; and in either case K' is odd and therefore n is even.

For $j = 1, 2$, since w-a_2-f_1-$\cdots$-f_n-b_j-w is not an odd hole, it follows (from $j = 1$) that b_1 is nonadjacent to w, and (from $j = 2$) that R_2 has length one. Since w-a_3-a_1-R_1-b_1-b_2-w is not an odd hole, w has a neighbour $c_1 \in R_1^*$. Since w cannot be linked onto $\{a_1, a_2, f_1\}$, it follows that c_1 is adjacent to b_1. But similarly c_1 is adjacent to a_1, contradicting that R_1 has odd length. This proves (3).

(4) *If the fourth outcome of 5.2 holds then the theorem holds.*

Suppose that f_1-$\cdots$-f_n is a corner jump in position a_3, say. If w is adjacent to both a_1, a_2 then the theorem holds, so we may assume that w, a_1 are nonadjacent. Thus w is adjacent to a_2, a_3. Let R_3' be an induced path between f_1 and b_3 with interior in $\{f_2, \ldots, f_n\} \cup V(R_3)$. Then R_1, R_2, R_3' form a prism K', and by three applications of 5.3 applied to the three holes of this prism, we deduce that w is nonadjacent to one of b_1, b_2, and nonadjacent to one of b_2, b_3 (and hence adjacent to b_1, b_3), and has no neighbours in $R_1 \cup R_3'$ except b_1, b_3. Consequently K' (and therefore K) is odd. If a_3 has no neighbour in R_3', then w-a_3-a_1-f_1-R_3'-b_3-w is an odd hole, a contradiction; so a_3 has a neighbour in R_3', and hence there is an induced path Q between a_3, b_3 with interior in $V(R_3')$. In particular, w has no neighbour in Q^*, and Q is odd; and since w-a_3-Q-b_3-w is not an odd hole, we deduce that a_3, b_3 are adjacent. But then the theorem holds.

From 5.2 and (1)–(4), this proves 5.4. ∎

6 Prisms with balanced vertices

Let K be a prism in a graph G, formed by paths R_i with ends a_i, b_i $(1 \leq i \leq 3)$ as usual. We say a major vertex w is *balanced* if there are two values of $i \in \{1, 2, 3\}$ such that w is adjacent to both a_i, b_i; and w is *clear* if it is anticomplete to $V(R_i)$ for some $i \in \{1, 2, 3\}$. (Thus a clear major vertex is balanced.) In this section we prove that if G is a K_4-free Berge graph, containing no even pair and no trampoline, then no prism in G has a balanced major vertex. A 4-*wheel* is the graph obtained from a cycle of length four by adding one more vertex adjacent to every vertex of the cycle. We need:

6.1 *Let G be a K_4-free Berge graph containing no trapeze or trestle. Let K be a prism in G, and let A, B and R_i, a_i, b_i $(i = 1, 2, 3)$ be as before. Let $w \in V(G) \setminus V(K)$ be major with respect to K. Suppose that either w is balanced, or G does not contain a 4-wheel. Let w be nonadjacent to a_3, and let a_3-p_1-$\cdots$-p_k-w be an induced path from a_3 to w. Suppose that the set of attachments in K of $\{p_1, \ldots, p_{k-1}\}$ is local. Then k is odd.*

Proof. Suppose that k is even. Let X be the set of attachments in K of $\{p_1, \ldots, p_{k-1}\}$. For $i = 1, 2$, a_i is adjacent to both w, a_3. In particular, $a_i \notin \{p_1, \ldots, p_k\}$ since k is even. Moreover, since w-a_i-a_3-p_1-$\cdots$-p_k-w is not an odd hole, it follows that a_i has a neighbour in $\{p_1, \ldots, p_k\}$. Since G is K_4-free, not both a_1, a_2 are adjacent to p_k; say a_1 is not adjacent to p_k without loss of generality. Thus $a_1 \in X$; and since $a_3 \in X$ and X is local, we deduce that $X \subseteq A$. In particular, $p_1, \ldots, p_{k-1} \notin V(K)$. If $p_k \in V(K)$, then $p_k \in X$ since it is adjacent to p_{k-1}, and hence $p_k \in A$, which is impossible. Thus none of the vertices $p_1, \ldots, p_k, w$ belong to $V(K)$.

Now w is adjacent to at least one of b_2, b_3; let R be the induced path between w and a_3 with interior in $V(R_3) \cup \{b_2\}$. Since w-R-a_3-a_1-w is a hole, it follows that R is even. Consequently

$$w\text{-}R\text{-}a_3\text{-}p_1\text{-}\cdots\text{-}p_k\text{-}w$$

is not a hole (since it would be odd), and since no vertex in $V(R) \setminus \{a_3\}$ belongs to X, it follows that p_k has a neighbour in $V(R) \setminus \{a_3\}$. Let R' be the induced path between p_k and a_3 with interior in $V(R)$. Then a_3-p_1-$\cdots$-p_k-R'-a_3 is a hole, and so R' is even. Consequently w-p_k-R'-a_3-a_1-w is not a hole, and therefore w has a neighbour in the interior of R'. We deduce that the neighbour of w in R, and the neighbour of p_k in R', are the same vertex q say. Suppose that $q = b_2$. Then w, p_k are both anticomplete to $V(R_3)$, and therefore R_3 is even; and since

$$w\text{-}b_1\text{-}b_3\text{-}R_3\text{-}a_3\text{-}p_1\text{-}\cdots\text{-}p_k\text{-}w$$

is not a hole (because it would have odd length), and $b_1 \notin X$, we deduce that w, p_k are both adjacent to b_1, and so b_1, b_2, w, p_k are pairwise adjacent, a contradiction. Consequently $q \neq b_2$. Since we cannot link a_1 onto $\{w, p_k, q\}$, via a_1w and two paths with interiors in $V(R_3), \{p_1, \ldots, p_{k-1}\}$ respectively, it follows that p_1 is the only neighbour of a_1 in $\{p_1, \ldots, p_k\}$. Since G is K_4-free, a_2 is nonadjacent to p_1, and so we can link a_2 onto $\{w, p_k, q\}$; and so a_2, p_k are adjacent. Thus the set of attachments in K of $\{p_k\}$ is not local.

Let us apply 5.2 setting $F = \{p_k\}$. Now p_k is not major, since it has only one neighbour in A, and the third outcome of 5.2 does not hold since $|F| = 1$. Suppose that the second outcome of 5.2 holds; so p_k has two adjacent neighbours in R_2 (namely, a_2 and its neighbour in R_2) and two adjacent neighbours in R_3 (namely, q and its neighbour in R_3 between q and b_3; this is only possible if $q \neq b_3$), and p_k has no other neighbours in $V(K)$. But then we can link p_k onto $\{a_1, a_3, p_1\}$, via paths with interiors in $(V(R_2) \setminus \{a_2\}) \cup V(R_3)$, $V(R_1) \setminus \{b_1\}$, and $\{p_1, \ldots, p_{k-1}\}$, a contradiction. We deduce that the fourth outcome of 5.2 holds, and so the one-vertex path p_k is a corner jump. Since p_k has a neighbour in $V(R_3) \setminus \{a_3\}$, and is adjacent to a_2 and not to a_1, a_3, it follows that p_k is a corner jump in position b_2, and $q = b_3$. Since p_k, w, b_1, b_3 are not all pairwise adjacent, it follows that w is nonadjacent to b_1, and therefore adjacent to b_2. But then w is not balanced with respect to K, and yet the subgraph induced on $B \cup \{w, p_k\}$ is a 4-wheel, a contradiction. This proves 6.1. ∎

Next we show:

6.2 *Let G be a K_4-free Berge graph with no even pair and no trampoline. If K is a prism in G, then no major vertex is balanced with respect to K.*

Proof. Suppose that there is a prism with a balanced major vertex; and if possible choose one with a clear major vertex. Thus we have chosen a vertex w, and two paths R_1, R_2, with ends a_i, b_i for $i = 1, 2$, such that

- R_1, R_2 both have length at least one, and are disjoint, and $w \notin V(R_1 \cup R_2)$

- a_1a_2 and b_1b_2 are edges, and there are no other edges between $V(R_1)$ and $V(R_2)$
- w is adjacent to a_1, a_2, b_1, b_2
- there is a path R_3 with ends a_3, b_3, with $V(R_3)$ disjoint from $V(R_1 \cup R_2) \cup \{w\}$, such that a_3 is adjacent to a_1, a_2, and b_3 is adjacent to b_1, b_2, and there are no other edges between R_3 and $R_1 \cup R_2$
- if there is a prism in G with a clear major vertex, then w has no neighbour in R_3.

Consequently, we may choose three sets A, B, C, pairwise disjoint and each disjoint from $V(R_1 \cup R_2) \cup \{w\}$, such that

- every vertex in A is complete to $\{a_1, a_2\}$ and has no other neighbours in $R_1 \cup R_2$
- every vertex in B is complete to $\{b_1, b_2\}$ and has no other neighbours in $R_1 \cup R_2$
- no vertex in C has a neighbour in $R_1 \cup R_2$
- for every vertex $v \in A \cup B \cup C$, there is an induced path containing v with one end in A and the other end in B, and with interior in C
- $A, B \neq \emptyset$
- if there is a prism in G with a clear major vertex, then w has no neighbour in C.

Since G is K_4-free, it follows that A, B are stable, and w is anticomplete to $A \cup B$. Choose such a triple (A, B, C) with $A \cup B \cup C$ maximal. If R is an induced path with one end in A and the other end in B, and with interior in C, we call R a *rung*. Let $W = A \cup B \cup C \cup V(R_1) \cup V(R_2)$.

(1) *Let p_0-p_1-$\cdots$-p_k be an induced path such that $p_0 \in A$ and $p_1, \ldots, p_k \notin A \cup B$, and w is nonadjacent to $p_0, \ldots, p_k$. Let X be the set of vertices in W that either belong to $\{p_1, \ldots, p_k\}$ or are adjacent to some vertex in $\{p_1, \ldots, p_k\}$. Then either $X \subseteq A \cup B \cup C$, or $X \subseteq A \cup \{a_1, a_2\}$.*

For suppose not, and choose k minimum such that the claim is false. From the minimality of k it follows that $p_1, \ldots, p_k \notin V(R_1 \cup R_2)$, and from the hypothesis we have $p_1, \ldots, p_k \notin A \cup B$. (They might belong to C, however.) For $1 \leq i \leq k$ let X_i denote the set of vertices in W that either belong to $\{p_i, \ldots, p_k\}$ or are adjacent to some vertex in $\{p_i, \ldots, p_k\}$. Thus $X_1 = X$ and is not a subset of $A \cup B \cup C$, and not a subset of $A \cup \{a_1, a_2\}$. Since $p_0 \in A \cap X_1$, it follows that X_1 is not a subset of any of $A \cup B \cup C$, $A \cup \{a_1, a_2\}$, $B \cup \{b_1, b_2\}$, $V(R_1)$, $V(R_2)$. Choose $h \leq k$ maximum such that X_h is not a subset of any of these five sets.

Suppose that $p_j \in C$ for some j with $h \leq j \leq k$. Since $X_h \not\subseteq A \cup B \cup C$, there exists i with $h \leq i \leq k$ such that some vertex $y \in V(R_1 \cup R_2)$ is adjacent to p_i. Since one of $p_1, \ldots, p_{k-1}$ either belongs to C or has a neighbour in C, the minimality of k implies that $i = k$. Since $p_j \in C$ and therefore is nonadjacent to y, we deduce that $j < k$. But then $p_j, y \in X_{j+1}$, contrary to the maximality of h. This proves that $p_h, \ldots, p_k \notin C$, and therefore $p_h, \ldots, p_k \notin W$.

Choose a rung R_3 with ends $a_3 \in A$ and $b_3 \in B$, such that the set of attachments of $\{p_h, \ldots, p_k\}$ in the prism K formed by R_1, R_2, R_3 is not local. By 4.1, 4.2, 4.4 and 4.3, G contains no trapeze, trestle, octahedron or extended 4-wheel. From 5.4, we deduce that one of the five outcomes of 5.4 holds; and from the minimality of k and the maximality of h, the path f_1-$\cdots$-f_k of 5.4 is either the path p_h-$\cdots$-p_k or its reverse.

Suppose the first outcome holds; then $k - h + 1 \geq 3$ is odd, and for some distinct $i, j \in \{1, 2, 3\}$, p_h has two adjacent neighbours c_i, d_i in R_i, and p_k has two adjacent neighbours c_j, d_j in R_j, and there are no other edges between $\{p_h, \ldots, p_k\}$ and $V(K)$, and w is adjacent to all of c_i, d_i, c_j, d_j. The minimality of k implies that not both $i, j \in \{1, 2\}$; and so w has neighbours in R_3. Yet w is a clear major vertex with respect to the prism induced on $V(R_i \cup R_j) \cup \{p_h, \ldots, p_k\}$, contrary to the choice of R_1, R_2, w.

Suppose the second outcome of 5.4 holds; then K is an even prism, and for some distinct $s, t \in \{1, 2, 3\}$, p_h has two adjacent neighbours c_s, d_s in R_s, and p_k has two adjacent neighbours c_t, d_t in R_t, and there are no other edges between $\{p_h, \ldots, p_k\}$ and $V(K)$, and w is adjacent to a_s, b_s, a_t, b_t and nonadjacent to every internal vertex of R_s and of R_t. Since w is balanced it follows that $\{s, t\} = \{1, 2\}$, and since none of $p_1, \ldots, p_{k-1}$ has a neighbour in $V(R_1 \cup R_2) \setminus \{a_1, a_2\}$, it follows that $h = k$. We may assume that for $i = 1, 2$, a_i, c_i, d_i, b_i are in order in R_i. For $i = 1, 2$, let C_i, D_i be the subpaths of R_i between a_i, c_i and between d_i, b_i respectively. If $a_1 \neq c_1$, then we can link a_1 onto $\{p_k, c_1, d_1\}$ via paths with interiors in $\{w\} \cup V(D_1)$, $V(C_1)$, and $\{a_3\} \cup \{p_1, \ldots, p_k\}$, a contradiction. Thus $a_1 = c_1$ and similarly $a_2 = c_2$. Since K is even, it follows that

$$p_k\text{-}d_1\text{-}D_1\text{-}b_1\text{-}b_2\text{-}D_2\text{-}d_2\text{-}p_k$$

is an odd hole, a contradiction.

Suppose the third outcome of 5.4 holds; then $k > h$, and since w is nonadjacent to a_3, b_3, one of p_h, p_k is adjacent to a_1, a_2, and the other to b_1, b_2, and there are no other edges between $\{p_h, \ldots, p_k\}$ and $V(K)$. From the minimality of k, p_h is nonadjacent to both b_1, b_2; so p_k is adjacent to b_1, b_2, and p_h to a_1, a_2. But then we can add p_h to A and p_k to B and $p_{h+1}, \ldots, p_{k-1}$ to C, contrary to the maximality of $A \cup B \cup C$.

Suppose the fourth outcome of 5.4 holds; then one of p_h-$\cdots$-p_k, p_k-$\cdots$-p_h is a corner jump in one of the six positions, say position $x_i \in \{a_i, b_i\}$. There is no $j \in \{1, 2, 3\} \setminus \{i\}$ such that w is adjacent to just one end of R_j; and so from the fourth outcome of 5.4, it follows that w is nonadjacent to x_i, and so $i = 3$. But then we can add $p_h, \ldots, p_k$ to A, B or C (in the appropriate way, depending whether $x_3 = a_3$ or b_3, and depending whether the corner jump is p_h-$\cdots$-p_k or p_k-$\cdots$-p_h) contrary to the maximality of $A \cup B \cup C$.

We have shown then that none of the outcomes of 5.4 holds, which is impossible; and this proves (1).

(2) *If P is an induced path with both ends in $A \cup B$ such that w is anticomplete to $V(P)$, then P has even length.*

We proceed by induction on the length of P. If some internal vertex of P belongs to $A \cup B$, then the result follows from the inductive hypothesis, so we may assume that P is p_0-p_1-$\cdots$-p_{k+1} say, where $p_0 \in A$, and $p_{k+1} \in A \cup B$, and $p_1, \ldots, p_k \notin A \cup B$. Let X be the set of vertices in W that belong to $\{p_1, \ldots, p_k\}$ or have a neighbour in this set. By (1), either $X \subseteq A \cup \{a_1, a_2\}$, or $X \subseteq A \cup B \cup C$. Suppose first that $p_{k+1} \in B$. Since $p_{k+1} \in X$, it follows that $X \subseteq A \cup B \cup C$, and so w-a_1-p_0-P-p_{k+1}-b_2-w is a hole, and therefore P has even length. Thus we may assume that

$p_{k+1} \in A$. If a_1-p_0-P-p_{k+1}-a_1 is a hole then again P has even length, so we may assume that $a_1 \in X$; and so $X \not\subseteq A \cup B \cup C$, and therefore $X \subseteq A \cup \{a_1, a_2\}$. But there is an induced path Q joining p_0, p_{k+1} with interior in $B \cup C \cup \{b_2\}$, and it has even length since it can be completed to a hole via p_{k+1}-a_1-p_0. Since $P \cup Q$ is a hole, it follows that P has even length. This proves (2).

Since G has no even pair, there is an odd induced path between some vertex of $A \cup B$ and w. Choose such a path as short as possible. By (2), none of its internal vertices belong to $A \cup B$. Let this path be a_3-p_1-$\cdots$-p_k-w say, where $a_3 \in A_3$. Choose a rung R_3 with a_3 as one end, and let K be the prism formed by R_1, R_2, R_3. By 6.1 applied to a_3-p_1-$\cdots$-p_{k-1}, the set of attachments of $\{p_1, \ldots, p_{k-1}\}$ in K is not local. But this contradicts (1). This proves 6.2. ∎

A *square* in G is a hole of length four. We deduce:

6.3 *Let G be a K_4-free Berge graph with no even pair and no trampoline. Then G contains no 4-wheel.*

Proof. Suppose that G contains a 4-wheel, and let a_1-b_1-a_2-b_2-a_1 be a square in G, and let c be adjacent to a_1, a_2, b_1, b_2. Since a_1, a_2 is not an even pair, there is an odd induced path a_1-p_1-$\cdots$-p_k-a_1; and therefore $b_1, b_2, c \notin \{p_1, \ldots, p_k\}$. Suppose that there is an edge uv of the path a_1-p_1-$\cdots$-p_k-a_1 such that $\{u, v\}$ is complete to $\{b_1, b_2\}$. From the symmetry we may assume that $u, v \neq a_2$. Since $\{a_2, c, u, v\}$ is complete to $\{b_1, b_2\}$, and therefore includes no triangle, it follows that G contains a trapeze, trestle, or octahedron, a contradiction. Thus there is no such edge uv. We claim that $\{b_1, b_2\}$ is a leap for the path a_1-p_1-$\cdots$-p_k-a_1. This follows from 2.1 if $k \geq 3$, and so we may assume that $k = 2$, and therefore neither of p_1, p_2 is complete to $\{b_1, b_2\}$. But each of b_1, b_2 is adjacent to at least one of p_1, p_2 since G has no hole of length five; and so again $\{b_1, b_2\}$ is a leap. Thus we may assume that b_1 is adjacent to p_1, and b_2 to p_k, and there are no other edges between $\{b_1, b_2\}$ and $\{p_1, \ldots, p_k\}$. But then the paths p_1-$\cdots$-p_k, a_1b_2 and b_1a_2 form a prism and c is a balanced major vertex with respect to it, contrary to 6.2. This proves 6.3. ∎

7 Prisms with major-general vertices

Let K be a prism in a graph G, formed by paths R_i with ends a_i, b_i $(1 \leq i \leq 3)$ as usual. A vertex $w \in V(G) \setminus V(K)$ is said to be *major-general* with respect to K if it is major and there exists $i \in \{1, 2, 3\}$ such that R_i has length at least two and w is adjacent to both ends of R_i. Our next objective is to extend 6.2, proving the analogous theorem for major-general vertices, the following.

7.1 *Let G be a K_4-free Berge graph with no even pair and no trampoline. If K is a prism in G, then no vertex is major-general with respect to K.*

Proof. Suppose that there is a prism with a major-general vertex w. Then there is an induced path R_3 with length at least two, with ends a_3, b_3, and two other vertices a_2, b_1, and nine pairwise disjoint subsets A_i, C_i, B_i $(1 \leq i \leq 3)$ of $V(G) \setminus \{w\}$, satisfying

- a_2-a_3-R_3-b_3-b_1 is an induced path

- $A_i = \{a_i\}$ for $i = 2, 3$; $B_i = \{b_i\}$ for $i = 1, 3$; C_3 is the set of internal vertices of R_3
- for $1 \le i < j \le 3$, A_i is complete to A_j, and B_i is complete to B_j, and there are no other edges between $A_i \cup B_i \cup C_i$ and $A_i \cup B_j \cup C_j$
- for $1 \le i \le 3$ and every vertex $v \in A_i \cup B_i \cup C_i$, there is an i-rung containing v, where an i-rung means an induced path with one end in A_i and the other end in B_i, and with interior in C_i
- w is adjacent to a_2, a_3, b_1, b_3, and
- A_1, B_2 are nonempty.

(To see this, note that since w is major-general with respect to some prism, and not balanced, we may assume in the usual notation that w is adjacent to a_2, b_1, a_3, b_3, and R_3 has length at least two; and then the claim follows.) Let W be the union of the nine sets A_i, C_i, B_i ($1 \le i \le 3$), and choose A_1, C_1, C_2, B_2 such that W is maximal.

(1) *Let p_0-p_1-$\cdots$-p_k be an induced path such that $p_0 \in A_1$ and $p_1, \ldots, p_k \notin A_1 \cup B_1$, and w is nonadjacent to $p_0, \ldots, p_k$. Let X be the set of vertices in W that either belong to $\{p_1, \ldots, p_k\}$ or are adjacent to some vertex in $\{p_1, \ldots, p_k\}$. Then either $X \subseteq A_1 \cup B_1 \cup C_1$, or $X \subseteq A_1 \cup \{a_2, a_3\}$.*

For suppose not, and choose k minimum such that the claim is false, and choose $h \le k$ as in step (1) of the proof of 6.2. As in that proof, it follows that $p_h, \ldots, p_k \notin W$, and there is a prism K, formed by a 1-rung R_1, a 2-rung R_2, and the path R_3, such that the set of attachments of $\{p_h, \ldots, p_k\}$ in K is not local. Choose a_1, b_2 such that for $i = 1, 2, 3$, the ends of R_i are a_i, b_i. Again, one of the outcomes of 5.4 holds.

The first outcome does not hold since G contains no prism with respect to which w is balanced, by 6.2. The second and third outcomes do not hold since w is not balanced with respect to K. Thus the fourth outcome of 5.4 holds; so one of p_h-$\cdots$-p_k, p_k-$\cdots$-p_h is a corner jump in one of the six positions, say position x.

Suppose first that $x = a_1$, and so one of p_h-$\cdots$-p_k, p_k-$\cdots$-p_h is a corner jump in position a_1 with respect to K. If $\{p_h, \ldots, p_k\}$ is anticomplete to $B_2 \cup C_2$ then we can either add p_h to A_1 and $p_{h+1}, \ldots, p_k$ to C_1, or add p_k to A_1 and $p_h, \ldots, p_{k-1}$ to C_1 (depending whether p_h-$\cdots$-p_k or p_k-$\cdots$-p_h is the corner jump with respect to K), a contradiction to the maximality of W. Thus there is a 2-rung R_2' with ends a_2, b_2' say, such that one of $p_h, \ldots, p_k$ has a neighbour in $V(R_2') \setminus \{a_2\}$. From the minimality of k, no vertex in $\{p_1, \ldots, p_{k-1}\}$ has a neighbour in $V(R_2') \setminus \{a_2\}$; so p_k has such a neighbour. If p_k is adjacent to a_2, a_3, then the prism formed by R_1, R_2', R_3 does not satisfy 5.3, since p_k has at most one neighbour in $\{b_1, b_2', b_3\}$. Thus $h < k$, and p_h is adjacent to a_2, a_3, and p_k has a neighbour in $V(R_1) \setminus \{a_1\}$ and a neighbour in $V(R_2') \setminus \{a_2\}$. If p_k has a neighbour in R_2' different from b_2', we can link p_k onto $\{p_h, a_2, a_3\}$ via paths with interiors in $\{p_{h+1}, \ldots, p_{k-1}\}$, $V(R_2') \setminus \{b_2'\}$, and $(V(R_1) \setminus \{a_1\}) \cup V(R_3)$, a contradiction. So b_2' is the only neighbour of p_k in R_2'. But then we can link b_2' onto $\{p_h, a_2, a_3\}$, via paths with interiors in $V(R_2')$, $\{p_h, \ldots, p_k\}$ and $V(R_3)$, a contradiction. Thus $x \ne a_1$.

Suppose that $x = b_2$. From the minimality of k, no vertex in $\{p_1, \ldots, p_{k-1}\}$ is adjacent to b_3; so the corner jump is p_k-$\cdots$-p_h, and p_k is adjacent to b_1, b_3, and p_h has a neighbour in $V(R_2) \setminus \{b_2\}$. Then from the maximality of W, there is a 1-rung R_1' with ends a_1' and b_1, such that one of $p_h, \ldots, p_k$, say p_i, has a neighbour $v \in V(R_1')\setminus\{b_1\}$. If $i = k$ then the prism formed by R_1', R_2, R_3 does not satisfy 5.3 (since p_k has at most one neighbour in $\{a_1', a_2, a_3\}$). Thus $i < k$, and consequently $h < k$. From the minimality of k, p_h has no neighbour in R_2 except possibly a_2; and so p_h, a_2 are adjacent. From the minimality of k, since p_h is adjacent to a_2, it follows that none of $p_1, \ldots, p_{k-1}$ has a neighbour in $B_1 \cup C_1$; and in particular $v = a_1'$. But then the prism formed by R_1, R_3 and the path a_2-p_h-$\cdots$-p_k does not satisfy 5.3, since a_1' has at most one neighbour in $\{b_1, b_3, p_k\}$ (since we have shown that $p_i \neq p_k$). Thus $x \neq b_2$.

If $x = a_2$, there is a prism K' formed by R_1, R_3 and a path starting with one of p_h-$\cdots$-p_k, p_k-$\cdots$-p_h and with final vertex b_2, and with interior in $V(R_2) \setminus \{a_2\}$; and this prism does not satisfy 5.3, a contradiction. Similarly $x \neq b_1$; and so $x \in \{a_3, b_3\}$. By the fourth outcome of 5.4, since w is adjacent to x, it follows that R_3 has length one, a contradiction. This proves (1).

(2) *If P is an induced path with both ends in $A_1 \cup B_2$ such that w is anticomplete to $V(P)$, then P has even length.*

We proceed by induction on the length of P. If some internal vertex of P belongs to $A_1 \cup B_2$, then the result follows from the inductive hypothesis, so we may assume that P is p_0-p_1-$\cdots$-p_{k+1} say, where $p_0 \in A_1$, and $p_{k+1} \in A_1 \cup B_2$, and $p_1, \ldots, p_k \notin A_1 \cup B_2$. Let X be the set of vertices in W that belong to $\{p_1, \ldots, p_k\}$ or have a neighbour in this set. By (1), either $X \subseteq A_1 \cup \{a_2, a_3\}$, or $X \subseteq A_1 \cup B_1 \cup C_1$, and in particular, $p_{k+1} \notin B_2$. Thus $p_{k+1} \in A_1$. If a_2-p_0-P-p_{k+1}-a_2 is a hole then again P has even length, so we may assume that $a_2 \in X$; and so $X \not\subseteq A_1 \cup B_1 \cup C_1$, and therefore $X \subseteq A_1 \cup \{a_2, a_3\}$. But there is an induced path Q joining p_0, p_{k+1} with interior in $B_1 \cup C_1 \cup \{b_3\}$, and it has even length since it can be completed to a hole via p_{k+1}-a_2-p_0. Since $P \cup Q$ is a hole, it follows that P has even length. This proves (2).

Since G has no even pair, there is an odd induced path between some vertex of $A_1 \cup B_2$ and w. Choose such a path as short as possible. By (2), none of its internal vertices belong to $A_1 \cup B_2$. Let this path be a_1-p_1-$\cdots$-p_k-w, where $a_1 \in A_1$ say. Choose a 1-rung R_1 with a_1 as one end, and choose a 2-rung R_2; and let K be the prism formed by R_1, R_2, R_3. By 6.1 applied to a_1-p_1-$\cdots$-p_k, the set of attachments of $\{p_1, \ldots, p_{k-1}\}$ in K is not local. But this contradicts (1). This proves 7.1. ∎

8 Line graphs

A *cut* of a graph G is a partition (A_1, X, A_2) of $V(G)$ such that A_1, A_2 are nonempty and A_1 is anticomplete to A_2; and it is a *k-cut* if $|X| \leq k$. We say G is *k-connected* if $|V(G)| > k$ and there is no $(k-1)$-cut.

A *branch-vertex* of a graph H is a vertex with degree ≥ 3; and a *branch* of H means a maximal path P in H such that no internal vertex of P is a branch-vertex. Let J be a graph with minimum degree at least three. If H is a subdivision of J then $V(J)$ is the set of branch-vertices of H, and the branches of H are in 1-1 correspondence with the edges of J in the natural way.

If H is a graph, then $L(H)$ denotes its line graph; thus $E(H) = V(L(H))$. If J is 3-connected and H is a bipartite subdivision of J, and $L(H)$ is an induced subgraph of G, we call $L(H)$ an *appearance* of J in G. An appearance $L(H)$ of J in G is *degenerate* if $J = K_4$ and there is a cycle of H of length four containing all the vertices of J, or $H = J = K_{3,3}$, and *non-degenerate* otherwise. In this section we prove the following.

8.1 *Let G be a 3-connected K_4-free Berge graph, containing no even pair and no trampoline, and no clique cutset. Suppose that there is an appearance of a 3-connected graph J in G, nondegenerate if $J = K_4$. Then G is the line graph of a bipartite graph.*

If $L(H)$ is an appearance of J in G, a vertex $w \in V(G) \setminus V(L(H))$ is *major* with respect to $L(H)$ if for each $v \in V(J) \subseteq V(H)$, there is at most one edge x of H incidentwith v such that w is nonadjacent to x in G.

8.2 *Let G be a K_4-free Berge graph, containing no even pair and no trampoline. For every 3-connected graph J and every appearance $L(H)$ of J in G, no vertex is major with respect to $L(H)$.*

Proof. Suppose that w is major with respect to $L(H)$. There is a subgraph H' of H that is a bipartite subdivision of K_4, and w is major with respect to $L(H')$. Thus if the theorem holds when $J = K_4$ then it holds in general. We therefore may assume that $J = K_4$. Let the four vertices of J be $c_1, \ldots, c_4$. For all distinct $i, j \in \{1, \ldots, 4\}$, let $B_{ij} = B_{ji}$ be the branch of H with ends c_i, c_j, let $e(i, j)$ be the edge of $B_{ij} = B_{ji}$ incident with c_i, and let $H_{ij} = H_{ji}$ be the subgraph of H obtained by deleting the edges and interior vertices of B_{ij}. Let N be the set of neighbours of w in $V(L(H))$. Thus $N \subseteq E(H)$. For $1 \leq i \leq 4$, exactly two of the edges of H incident with c_i belong to N (for at least two are in N since w is major, and not all three since G is K_4-free).

(1) *For $1 \leq i < j \leq 4$, N contains at least one of $e(i, j), e(j, i)$.*

For let $(i, j) = (1, 2)$ say. Suppose that no end-edge of B_{12} is in N. Thus $e(1, 3), e(1, 4), e(2, 3), e(2, 4) \in N$. Suppose first that B_{12} has length one, and let x be its unique edge. Then $\{x, w\}$ is complete in G to $\{e(1, 3), e(1, 4), e(2, 3), e(2, 4)\}$, and so G contains a trapeze, trestle, or octahedron, a contradiction. Thus B_{12} has length at least two. Then $L(H_{34})$ is an induced subgraph of G, and it is a prism (since B_{12} has length at least two). Moreover, since w is nonadjacent to both end-edges of B_{12}, we deduce that w is a balanced major vertex with respect to this prism, contrary to 6.2. This proves (1).

We may assume that $e(1, 2), e(1, 3) \in N$, and therefore $e(1, 4) \notin N$. By (1), $e(4, 1) \in N$. From the symmetry between c_2 and c_3, we may assume that $e(4, 3) \notin N$, and hence $e(3, 4), e(4, 2) \in N$. Since $L(H_{12})$ is a prism (since B_{34} has at least two edges) and w is not major-general with respect to this prism, by 7.1, it follows that $e(3, 1) \notin N$, and $e(3, 2) \in N$, and B_{23}, B_{24} both have length one (and so $e(2, 1) \notin N$). But then w is major-general with respect to the prism $L(H_{24})$, a contradiction. This proves 8.2. ∎

An appearance $L(H)$ of J in G is *overshadowed* if there is a branch B of H with odd length ≥ 3, and a vertex $winV(G) \setminus V(L(H))$, such that for each end b of B in H, there is at most one edge of H that is incident with b in H and nonadjacent to w in G.

8.3 *Let G be a K_4-free Berge graph, containing no even pair and no trampoline. For every 3-connected graph J, there is no overshadowed appearance of J in G.*

Proof. Suppose $L(H)$ is an overshadowed appearance of J in G, and let B, w be as above. Let the ends of B in H be b_1, b_2. Since J is 3-connected, there are three paths P_1, P_2, P_3 of H between b_1, b_2, vertex-disjoint except for b_1, b_2, where $P_3 = B$. Let H' be the union of these paths; then $L(H')$ is an even prism (since B has odd length) and w is a major (and therefore major-general) vertex with respect to it, contrary to 7.1. This proves 8.3. ∎

Let J be a 3-connected graph. A *J-strip system* (S, N) in a graph G consists of a subset $S_{uv} = S_{vu} \subseteq V(G)$ for each edge uv of J, and a subset $N_v \subseteq V(G)$ for each vertex v of J, satisfying the following conditions:

- The sets S_{uv} $(uv \in E(J))$ are pairwise disjoint.
- For each $u \in V(J)$, $N_u \subseteq \bigcup(S_{uv} : v \in V(J)$ adjacent to $u)$.
- For each $uv \in E(J)$, every vertex of S_{uv} is in a uv-rung (a *uv-rung* is an induced path R of G with ends s, t say, where $V(R) \subseteq S_{uv}$, and s is the unique vertex of R in N_u, and t is the unique vertex of R in N_v).
- If $uv, wx \in E(J)$ with u, v, w, x all distinct, then there are no edges between S_{uv} and S_{wx}.
- If $uv, uw \in E(J)$ with $v \neq w$, then $N_u \cap S_{uv}$ is complete to $N_u \cap S_{uw}$, and there are no other edges between S_{uv} and S_{uw}.
- For each $uv \in E(J)$ there is a special uv-rung such that for every cycle C of J, the sum of the lengths of the special uv-rungs for $uv \in E(C)$ has the same parity as $|V(C)|$.

We define $V(S, N) = \bigcup(S_{uv} : uv \in E(J))$. If $u, v \in V(J)$ are adjacent, we define $N_{uv} = N_u \cap S_{uv}$. So every vertex of N_u belongs to N_{uv} for exactly one v. Note that N_{uv} is in general different from N_{vu}, but S_{uv} and S_{vu} mean the same thing.

If $L(H)$ is an appearance of J in G, then since $L(H)$ is an induced subgraph of G, there is a J-strip system (S, N) in G, defined by setting

- for each edge uv of J, S_{uv} is the set of edges of the branch of H with ends u, v
- for each $v \in V(J)$, N_v is the set of edges of H incident with v in H.

We call this the *strip system of H*.

A J-strip system (S', N') in G *extends* a J-strip system (S, N) in G if $V(S, N) \subset V(S', N')$, and $S'_{uv} \cap V(S, N) = S_{uv}$ for every $uv \in E(J)$, and $N'_v \cap V(S, N) = N_v$ for every $v \in V(J)$; and a J-strip system (S, N) in G is *maximal* if there is no J-strip system in G that extends (S, N).

Proof of 8.1.

Choose a 3-connected graph J maximal such that there is an appearance $L(H)$ of J in G, non-degenerate if $J = K_4$. (Thus $E(H) \subseteq V(G)$.) We will prove that $G = L(H)$. Since G is K_4-free it follows that J has maximum degree three. Since $L(H)$ is an appearance of J in G, we may choose a maximal J-strip system (S, N) that extends the strip system of H.

(1) *For all $uv \in E(J)$, all uv-rungs have lengths of the same parity.*

This follows from theorem 8.1 of [5].

(2) *For every edge uv of J, if some uv-rung has length zero then $|S_{uv}| = 1$.*

For by 8.3 and theorem 8.2 of [5] it follows that every uv-rung has length zero. Suppose that $x, y \in S_{uv}$ are distinct. Then x, y are both complete to $N_u \setminus N_{uv}$ and both complete to $N_v \setminus N_{vu}$; and so G contains a trapeze, trestle or octahedron, a contradiction. Thus $|S_{uv}| = 1$. This proves (2).

We say $X \subseteq V(S, N)$ is *local* (with respect to the strip system) if either $X \subseteq N_v$ for some $v \in V(J)$, or $X \subseteq S_{uv}$ for some edge $uv \in E(J)$. Let $\mathcal{F}$ be the set of all vertex sets of components of $G \setminus V(S, N)$.

(3) *For each $F \in \mathcal{F}$ the set of attachments of F in $V(S, N)$ is local.*

This follows from theorem 8.5 of [5], because of 8.2, 8.3, the choice of J, and the maximality of the strip system, using that $L(H)$ is nondegenerate if $J = K_4$, and that (S, N) extends the strip system of H.

(4) *For every edge $uv \in E(J)$, $|N_{uv}| = 1$.*

For we prove, by induction on the length of P, that if P is an induced path with both ends in N_{ax} for some edge ax of J then P is even. Let $a \in V(J)$, with neighbours x, y, z in J; and suppose that P is an induced path of G with both ends in N_{ax}. If some internal vertex of P belongs to N_{ax} the result follows from the inductive hypothesis, and if some vertex of P is in $N_{ay} \cup N_{az}$ then P has length two as required; so we may assume that $P^* \cap N_a = \emptyset$. Let the vertices of P be $p_1\text{-}\cdots\text{-}p_k$ in order. Since $p_1, p_k \in N_{ax} \subseteq S_{ax}$, (2) implies that every ax-rung has positive length, and so $N_{ax} \cap N_x = \emptyset$. Let F_1 be the union of all $F \in \mathcal{F}$ such that every attachment of F in $V(S, N)$ belongs to N_a, and let F_2 be the union of all $F \in \mathcal{F}$ such that every attachment of F is in S_{ax} and some attachment is not in N_a. From (3), every member of $\mathcal{F}$ with an attachment in $S_{ax} \setminus N_{xa}$ is a subset of one of F_1, F_2. Choose $c \in N_{ay}$.

Suppose first that some vertex of P belongs to F_1. Choose h, j with $1 \le h < j \le k$ and $j - h$ minimum such that $p_h, p_j \notin F_1$ and there exists i with $h < i < j$ and $p_i \in F_1$. It follows that $p_h, p_j \in N_a$, and therefore $i = 1$ and $j = k$. Let R, R' be ax-rungs containing p_1, p_k respectively, and let $b \in N_x \setminus N_{xa}$. Then there is an induced path Q between p_1, p_k with interior in $V(R) \cup V(R') \cup \{b\}$, and we claim it is even. For if $b \in V(Q)$ then Q is even since R, R' have the same parity by (1); and if $b \notin V(Q)$ then Q is even since Q can be complete to a hole via $p_k\text{-}c\text{-}p_1$. Thus in either case Q is even; but $P \cup Q$ is a hole, and so P is even as required.

Thus we may assume that no vertex of P belongs to F_1. If no vertex of P is in N_{xa}, then $P^* \subseteq F_2 \cup (S_{ax} \setminus (N_{ax} \cup N_{xa}))$, and therefore P can be completed to a hole via $p_k\text{-}c\text{-}p_1$, and so P is even as required. Thus we may assume that there exist $h, j \in \{2, \ldots, k-1\}$, minimum and maximum respectively such that $p_h, p_j \in N_{xa}$. (Possibly $h = j$.) From the maximality of $V(S, N)$, the internal vertices of $p_1\text{-}\cdots\text{-}p_h$ belong to S_{ax} (for otherwise they could be added to S_{ax}), and so $p_1\text{-}\cdots\text{-}p_h$ is an ax-rung, and so is $p_j\text{-}\cdots\text{-}p_k$. Consequently their lengths have the same parity, by (1); and from the inductive hypothesis the subpath $p_h\text{-}\cdots\text{-}p_j$ has even length; and so P has even length. This completes the proof that P has even length.

We deduce that for each edge uv of J, any two vertices in N_{uv} would be an even pair, and so $|N_{uv}| = 1$. This proves (4).

Thus each N_v is a clique. If there exists $F \in \mathcal{F}$ such that the set of attachments of F in $V(S, N)$ is contained in some N_v, then G admits a clique cutset, a contradiction. For each $uv \in E(J)$, let A_{uv} be the union of S_{uv} and all $F \in \mathcal{F}$ such that the set of attachments of F in $V(S, N)$ is a subset of S_{uv}. It follows that the sets A_{uv} ($uv \in E(J)$) are pairwise disjoint and have union $V(G)$.

(5) *For each edge uv of J, $|A_{uv}| \leq 2$.*

For every path in G between A_{uv} and $V(G) \setminus A_{uv}$ contains a member of $N_{uv} \cup N_{vu}$. But by (4), $|N_{uv} \cup N_{vu}| = 2$, and since G is 3-connected, it follows that $|A_{uv}| \leq 2$. This proves (5).

From (5) it follows that $G = L(H)$, and so G is a line graph. This proves 8.1. ∎

9 Degenerate K_4's

In this section we extend 8.1 to include the case when G contains an appearance of K_4, but all such appearances are degenerate. This case was excluded from 8.1 so that we could apply theorem 8.5 of [5], and we therefore need some workaround to replace that theorem. We begin with:

9.1 *Let G be a K_4-free Berge graph, containing no even pair or trampoline, and containing no appearance of $K_{3,3}$. Let $L(H)$ be a degenerate appearance of J in G, where J is isomorphic to K_4. Let $V(J) = \{c_1, \ldots, c_4\}$, and for $1 \leq i < j \leq 4$ let B_{ij} be the branch of H with ends c_i, c_j. Let c_1-c_2-c_3-c_4-c_1 be a cycle of H, and let b_1, b_2, b_3, b_4 be the unique edges of $B_{12}, B_{23}, B_{34}, B_{14}$ respectively. Then every path in G between $E(B_{13})$ and $E(B_{24})$ contains one of $b_1, \ldots, b_4$.*

Proof. First, we observe that b_1-b_2-b_3-b_4-b_1 is a square of G. Let the edges of B_{13} be $p_1, \ldots, p_m$ in order; thus, p_1-$\cdots$-p_m is an induced path P of G, and p_1 is adjacent to b_1, b_4, and p_m is adjacent to b_2, b_3. Similarly, let the edges of B_{24} form an induced path q_1-$\cdots$-q_n (which we call Q) in G, where q_1 is adjacent to b_1, b_2, and q_n to b_3, b_4. Since H is bipartite it follows that m, n are even. Suppose there is a path of G between $V(P)$ and $V(Q)$ containing none of $b_1, \ldots, b_4$, and choose a minimal such path. Thus we may assume that r_1-$\cdots$-r_k is an induced path R, where $r_1, \ldots, r_k \notin V(P \cup Q) \cup \{b_1, b_2, b_3, b_4\}$, and r_1 has neighbours in $V(P)$ and r_k has neighbours in $V(Q)$, and there are no other edges between $\{r_1, \ldots, r_k\}$ and $V(P \cup Q)$. Let us choose H and R such that R has minimum length.

(1) *If b_1, b_2 are anticomplete to $V(R)$, then r_1 has exactly two neighbours in $V(P)$ and they are adjacent.*

For suppose that b_1, b_2 are nonadjacent to $r_1, \ldots, r_k$. If r_1 has a unique neighbour $r_0 \in V(P)$, we can link r_0 onto $\{b_1, b_2, q_1\}$, a contradiction; and if r_1 has two nonadjacent neighbour in $V(P)$, we can link r_1 onto the same triangle, again a contradiction. This proves (1).

(2) *At least one of* $b_1,\dots,b_4$ *has a neighbour in* $V(R)$.

For suppose not. By (1), r_1 has exactly two neighbours in $V(P)$, and they are adjacent; and similarly r_k has exactly two neighbours in Q, and they are adjacent. But then the restriction of G to $V(P\cup Q\cup R)\cup\{b_1,\dots,b_4\}$ is the line graph of a bipartite subdivision of $K_{3,3}$, contrary to the hypothesis. This proves (2).

(3) *At least two of* $b_1,\dots,b_4$ *have a neighbour in* $V(R)$.

For suppose that b_1 has a neighbour in $V(R)$, and b_2,b_3,b_4 do not. By (1), r_1 is adjacent to one of $p_2,\dots,p_m$; and so we can link b_1 onto $\{b_2,b_3,p_m\}$, a contradiction. This proves (3).

(4) *Either* b_1,b_3 *both have neighbours in* $V(R)$, *or* b_2,b_4 *both have neighbours in* $V(R)$.

For suppose not; then by (3) we may assume that b_1,b_2 have neighbours in $V(R)$ and b_3,b_4 do not. By (1) and the symmetry, it follows that r_1 has exactly two neighbours in $V(P)$ and they are adjacent. Choose $i\in\{1,\dots,k\}$ minimum such that r_i is adjacent to one of b_1,b_2. If r_i is adjacent to b_1 and not to b_2, then we can link b_1 onto $\{b_2,b_3,p_m\}$, a contradiction; and similarly r_i is adjacent to both b_1,b_2. Let S be the induced path between b_1,b_3 with interior in $\{r_1,\dots,r_i,p_2,\dots,p_m\}$. Since b_4 is anticomplete to S^*, it follows that S is even; and so b_3-S-b_1-q_1-$\cdots$-q_n-b_3 is not a hole. Hence one of $r_1,\dots,r_i$ has a neighbour in Q, and therefore $i=k$. Since b_1,b_2,q_1,r_k are not all pairwise adjacent, it follows that r_k is nonadjacent to q_1, and therefore r_k is adjacent to one of $q_2,\dots,q_n$. Moreover, from the minimality of i, it follows that b_1,b_2 are nonadjacent to $r_1,\dots,r_{k-1}$. But then we can link r_k onto $\{b_3,b_4,q_n\}$, via r_k-b_1-b_4 and two paths with interiors in $\{r_1,\dots,r_{k-1},p_2,\dots,p_m\}$ and $\{q_2,\dots,q_{n-1}\}$, a contradiction. This proves (4).

From (4) there is a subpath S of R containing neighbours either of both b_1,b_3 or of both b_2,b_4. Choose such a path as short as possible. From the symmetry we may assume it contains neighbours of both b_1,b_3, and so $V(S)$ is the interior of an induced path between b_1,b_3.

(5) $S=R$, *and* S *has even length.*

Let S be s_1-$\cdots$-s_t say, where b_1-s_1-$\cdots$-s_t-b_3 is an induced path. Suppose first that S has odd length. It follows (since b_2-b_1-s_1-$\cdots$-s_t-b_3-b_2 is not an odd hole) that b_2, and similarly b_4, have neighbours in $V(S)$. From the minimality of S, and the symmetry between c_2,c_4, we may assume that s_1 is the unique vertex of S adjacent to b_2, and s_t is the unique vertex of S adjacent to b_4. If $r_1\notin V(S)$, then the subgraph induced on $V(P\cup S)\cup\{b_1,b_2,b_3,b_4\}$ is another degenerate appearance of K_4 in G, and there is a proper subpath of R with attachments in $V(P)$ and $V(S)$, contrary to our choice of H,R. Thus $r_1\in V(S)$, and so r_1 is one of s_1,s_t. Consequently r_1 is either complete to $\{b_1,b_2\}$ or to $\{b_3,b_4\}$, and we may assume the first from the symmetry. Since S is odd, it follows that r_1 is nonadjacent to b_3,b_4, and (since $k>1$, because S is odd) r_1 is anticomplete to $V(Q)$. Since b_2-b_3-b_4-p_1-r_1-b_2 is not an odd hole, it follows that r_1,p_1 are nonadjacent, and so r_1 has a neighbour in $\{p_2,\dots,p_n\}$; and hence we can link b_1 onto $\{b_3,b_4,q_m\}$ via b_1b_4, b_1-q_1-$\cdots$-q_n and a path between b_1,b_3 with interior in $\{r_1,p_2,\dots,p_m\}$, a contradiction.

Thus S is even. Since b_1-s_1-$\cdots$-s_t-b_3-p_m-P-p_1-b_1 is not an odd hole, there are edges between $V(S)$ and $V(P)$, and so $r_1 \in V(S)$, and similarly $r_k \in V(S)$, and so $R = S$. This proves (5).

From (5) and the symmetry between b_2, b_4, we may assume that b_1-r_1-$\cdots$-r_k-b_3 is an induced path.

(6) $k > 1$.

For suppose that $k = 1$. Thus r_1 is adjacent to both b_1, b_3, and has neighbours in both $V(P), V(Q)$. By 8.2, r_1 is not major with respect to $L(H)$, and so from the symmetry we may assume that r_1 has at most one neighbour in $\{b_3, b_4, q_n\}$. Hence r_1 is nonadjacent to b_4, q_n. By 5.3 applied to the prism induced on $V(Q) \cup \{b_1, \ldots, b_4\}$, r_1 is nonadjacent to b_2. Since r_1 has a neighbour in $\{q_1, \ldots, q_{n-1}\}$ (because it is nonadjacent to q_n), we can link r_1 onto $\{b_1, b_2, q_1\}$, and so r_1 is adjacent to q_1. By 5.3 applied to the same prism as before, r_1 has no neighbours in Q except q_1. Since r_1-q_1-$\cdots$-q_n-b_4-p_1-r_1 is not an odd hole, r_1 is nonadjacent to p_1. This restores the symmetry between p_1, q_n, and so from the symmetry r_1 is adjacent to p_m and has no other neighbour in P. But then b_1, q_1, p_m, b_3 are all common neighbours of r_1, b_2, and so G contains a trapeze, a contradiction. This proves (6).

(7) *Not both b_2, b_4 have neighbours in R.*

For suppose they do; then from the minimality of S and (5), it follows that S is the interior of an induced path between b_2, b_4. In particular, one of b_2, b_4 (say b_i) is adjacent to r_k and not to $r_1, \ldots, r_{k-1}$. But then b_1-r_1-$\cdots$-r_k-b_i-b_1 is an odd hole, by (5) and (6), a contradiction. This proves (7).

From (7) and the symmetry between b_2, b_4, we may assume that b_4 is anticomplete to $V(R)$. Since b_1-r_1-$\cdots$-r_k-q_1-b_1 is not an odd hole, and R is even of length at least two by (5) and (6), we deduce that r_k, q_1 are nonadjacent, and so r_k has a neighbour in $\{q_2, \ldots, q_n\}$. We can link r_k onto $\{q_n, b_3, b_4\}$, via r_k-b_3, r_k-R-r_1-b_1-b_4, and and a path between r_k, q_n with interior in $\{q_2, \ldots, q_{n-1}\}$; and so r_k, q_n are adjacent. Since r_k has at most one neighbour in $\{b_1, b_2, q_1\}$, 5.3 applied to the prism induced on $V(Q) \cup \{b_1, \ldots, b_4\}$ implies that r_k has no neighbours in Q except q_n. If b_2 has a neighbour in $V(R)$ we can link b_2 onto $\{b_3, q_n, r_k\}$, via b_2-b_3, b_2-q_1-$\cdots$-q_n and a path with interior in $V(R)$, and so b_2, r_k are adjacent; but then b_4-q_n-r_k-b_2-b_1-b_4 is an odd hole, a contradiction. Thus b_2 is anticomplete to $V(R)$. If r_1 has a neighbour in $\{p_2, \ldots, p_m\}$, then we can link r_1 onto $\{b_2, b_3, p_m\}$ via paths with interiors in $\{b_1\}, V(R)$ and $\{p_2, \ldots, p_m\}$, a contradiction. Thus p_1 is the only neighbour of r_1 in P. But then the subgraph induced on $V(P \cup Q \cup R) \cup \{b_1, b_3\}$ is an even prism, and b_4 is major-general with respect to this prism, contrary to 7.1. This proves 9.1. ∎

Now we prove the desired extension of 8.1, the following.

9.2 *Let G be a 3-connected K_4-free Berge graph, containing no even pair and no trampoline, and no clique cutset. Suppose that there is an appearance of a 3-connected graph J in G. Then G is the line graph of a bipartite graph.*

Proof. Choose a 3-connected graph J maximal such that there is an appearance $L(H)$ of J in G. By 8.1, we may assume that $J = K_4$ and $L(H)$ is degenerate. Let $V(J) = \{c_1, \ldots, c_4\}$, and for

$1 \leq i < j \leq 4$ let B_{ij} be the branch of H with ends c_i, c_j. Let $B_{12}, B_{23}, B_{34}, B_{14}$ all have length one, and let C be the cycle of H with vertices c_1-c_2-c_3-c_4-c_1 in order.

Let us choose a maximal J-strip system (S, N) that extends the strip system of H. For convenience we write N_i for N_{c_i} for $1 \leq i \leq 4$, and S_{ij} for $S_{c_i c_j}$ and N_{ij} for $N_{c_i c_j}$ for all distinct $i, j \in \{1, \ldots, 4\}$. As in the proof of 8.1, for each $uv \in E(J)$, every uv-rung has the same parity, and they either all have positive length zero or $|S_{uv}| = 1$. In particular, $S_{12}, S_{23}, S_{34}, S_{14}$ each have a unique member. Let b_{12} be the unique member of S_{12}, and define b_{23}, b_{34}, b_{14} similarly.

We say $X \subseteq V(S, N)$ is *local* (with respect to the strip system) if either $X \subseteq N_v$ for some $v \in V(J)$, or $X \subseteq S_{uv}$ for some edge $uv \in E(J)$.

(1) *If $F \subseteq V(G) \setminus V(S, N)$ is connected, then the set of attachments of F in $V(S, N)$ is local.*

For suppose not, and choose F minimal violating the claim. Let X be the set of attachments of F in $V(S, N)$. By 9.1, we may assume that $X \subseteq E(C) \cup S_{13}$. Since X is not local, $X \not\subseteq S_{13}$, and so we may assume that $b_{12} \in X$. Suppose that also $b_{34} \in X$. From the minimality of F, it follows that there is an induced path b_{12}-f_1-$\cdots$-f_k-b_{34}, where $F = \{f_1, \ldots, f_k\}$. Since the union of this path and a c_2c_4-rung induces a hole, and all c_2c_4-rungs are odd, it follows that k is even; and so b_{23}, b_{14} both have neighbours in F. From the minimality of F, f_1 is the unique neighbour in F of one of b_{23}, b_{14}, and f_k is the unique neighbour of the other. If b_{14} is adjacent to f_1 then we can add f_1 to N_1 and add f_k to N_3, and add F to S_{13}, contrary to the maximality of $V(S, N)$. Thus b_{23} is adjacent to f_1, and b_{14} to f_k, and $k > 1$. The minimality of F implies that no member of F has a neighbour in S_{13}; but then we can add f_1 to N_2, add f_k to N_4, and add F to S_{24}, again a contradiction.

This proves that $b_{34} \notin X$. Suppose that $b_{14} \in X$. Then similarly, $b_{23} \notin X$. Since X is not local, it follows that $X \cap S_{13} \not\subseteq N_{13}$. From the minimality of F, it follows that there is an induced path f_1-$\cdots$-f_k, where $F = \{f_1, \ldots, f_k\}$, and f_1 is adjacent to b_{12}, b_{14}, and f_k has neighbours in $S_{13} \setminus N_1$, and there are no other edges between $V(S, N) \setminus N_{13}$ and F. But then we can add f_1 to N_1 and F to S_{13}, contrary to the maximality of $V(S, N)$.

This proves that $b_{14} \notin X$. Suppose that $X \cap S_{13} \not\subseteq N_{13}$. Then there is an induced path between b_{12} and b_{34} with interior in $F \cup (S_{13} \setminus N_{13})$; this path is even since it can be completed to a hole via b_{34}-b_{14}-b_{12}, and yet it can also be completed to a hole via a path between b_{34}, b_{12} with interior a c_2c_4-rung, giving an odd hole, a contradiction. Thus $X \cap S_{13} \subseteq N_{13}$.

Since X is not local, and therefore $X \not\subseteq N_1$, it follows that $b_{23} \in X$. But then similarly $X \cap S_{13} \subseteq N_{31}$, and so $X \cap S_{13} = \emptyset$, contradicting that X is not local. This proves (1).

Now the proof is completed just like the proof of 8.1, using (1) above as a substitute for statement (3) in that proof. This proves 9.2. ∎

This has the following consequence.

9.3 *Let G be a 3-connected K_4-free Berge graph, containing no even pair, no trampoline, and no clique cutset. If G contains an even prism, then G is the line graph of a bipartite graph.*

Proof. By 9.2, we may assume that there is no appearance of K_4 in G. Since G contains an even prism, we can choose in G a collection of nine sets

$$\begin{matrix} A_1 & C_1 & B_1 \\ A_2 & C_2 & B_2 \\ A_3 & C_3 & B_3 \end{matrix}$$

with the following properties:

- all these sets are nonempty and pairwise disjoint
- for $1 \le i < j \le 3$, A_i is complete to A_j and B_i is complete to B_j, and there are no other edges between $A_i \cup B_i \cup C_i$ and $A_j \cup B_j \cup C_j$
- for $1 \le i \le 3$, every vertex of $A_i \cup B_i \cup C_i$ belongs to an induced path between A_i and B_i with interior in C_i
- some induced path between A_1 and B_1 with interior in C_1 is even.

Choose these nine sets with maximal union, and let H be the subgraph of G induced on their union. Let us write $S_i = A_i \cup B_i \cup C_i$ for $1 \le i \le 3$. Let us say a subset $X \subseteq V(H)$ is *local* if X is a subset of one of $S_1, S_2, S_3, A_1 \cup A_2 \cup A_3$ or $B_1 \cup B_2 \cup B_3$. By 7.1, there is no prism in G with a major-general vertex; so by the argument of step (2) of the proof of theorem 10.6 of [5], it follows that

(1) *For every connected subset F of $V(G) \setminus V(H)$, its set of attachments in H is local.*

Now since G is 3-connected, it follows from (1) that for $i = 1, 2, 3$, at least one of A_i, B_i has more than one member. Consequently we may assume that $|A_1|, |A_2| > 1$, from the symmetry. Since G is K_4-free, A_1, A_2 are both stable; but then the subgraph induced on $A_1 \cup A_2 \cup A_3$ contains a 4-wheel, contrary to 6.3. This proves 9.3. ∎

10 Long prisms

Our next goal is to eliminate all prisms. A prism is *long* if it has more than six vertices, and *short* otherwise. In this section we eliminate long prisms, and in the next we eliminate short prisms.

Let K be a short prism in G, and let w be a major vertex with respect to K. Let N be the set of vertices in K adjacent to w, and let x, y be the two vertices in $V(K) \setminus N$. We say w *separates* K if every path in G between x, y has a vertex in $N \cup \{w\}$. In this section we prove the following.

10.1 *Let G be a 3-connected K_4-free Berge graph, containing no even pair, no trampoline, and no clique cutset. Suppose that G contains no even prism, and no appearance of K_4, and that $|V(G)| > 6$. Then*

- *G contains no long prism,*
- *for every short prism K, every major vertex (with respect to K) separates K, and*
- *if there is a short prism then some short prism has a major vertex.*

Proof. Let K be a prism; we must show that K is short, and every major vertex separates K, and some short prism has a major vertex. We can choose a collection of nine subsets of $V(G)$

$$\begin{matrix} A_1 & C_1 & B_1 \\ A_2 & C_2 & B_2 \\ A_3 & C_3 & B_3 \end{matrix}$$

with the following properties:

- all these sets are pairwise disjoint, and $A_1, A_2, A_3, B_1, B_2, B_3$ are nonempty,
- for $1 \le i < j \le 3$, A_i is complete to A_j and B_i is complete to B_j, and there are no other edges between $A_i \cup B_i \cup C_i$ and $A_j \cup B_j \cup C_j$,
- for $1 \le i \le 3$, every vertex of $A_i \cup B_i \cup C_i$ belongs to an induced path between A_i and B_i with interior in C_i, and
- for $i = 1, 2, 3$ there is an induced path between A_i and B_i with interior in C_i, such that these three paths form the prism K.

Choose these nine sets with maximal union, and let H be the subgraph of G induced on their union. Let $A = A_1 \cup A_2 \cup A_3$, and define B, C similarly. For $1 \le i \le 3$, let $S_i = A_i \cup B_i \cup C_i$, and let us say an induced path between A_i and B_i with interior in C_i is an *i-rung*. Since G contains no even prism, it follows that for $1 \le i \le 3$, every i-rung is odd. Let us say a subset $X \subseteq V(H)$ is *local* if X is a subset of one of S_1, S_2, S_3, A or B. We say $v \in V(G) \setminus V(H)$ is *major* with respect to H if v has neighbours in at least two of A_1, A_2, A_3 and at least two of B_1, B_2, B_3.

(1) *Let $F \subseteq V(G) \setminus V(H)$ be connected, and contain no major vertex. Let X be the set of attachments of F in H. Then X is local.*

Suppose not, and choose F minimal with this property. Thus we may choose an i-rung R_i for $i = 1, 2, 3$, forming a prism K' say, such that $X \cap V(K')$ is not local with respect to K'. For $i = 1, 2, 3$, let R_i have ends $a_i \in A_i$ and $b_i \in B_i$. By 5.2, and the minimality of F, there is an induced path $f_1\text{-}\cdots\text{-}f_n$ in F with $n \ge 1$ and $F = \{f_1, \ldots, f_n\}$, such that (up to symmetry) either:

- $n = 1$ and f_1 is major with respect to K', or
- for some distinct $i, j \in \{1, 2, 3\}$, f_1 has two adjacent neighbours in R_i, and f_n has two adjacent neighbours in R_j, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K)$, or
- $n \ge 2$, and for some distinct $i, j \in \{1, 2, 3\}$, f_1 is adjacent to a_i, a_j, and f_n is adjacent to b_i, b_j, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K)$, or
- $f_1\text{-}\cdots\text{-}f_n$ is a corner jump.

The first is impossible since no vertex in F is major with respect to K' (since any such vertex would also be major with respect to H), and the second is impossible there is no appearance of K_4 in G. Suppose that the third holds, with $i = 1, j = 2$ say. It follows that n is even. Suppose that there exists $a_1' \in A_1 \setminus \{a_1\}$. If f_1 is adjacent to a_1', then the subgraph induced on $\{a_1, a_1', a_2, a_3, f_1\}$

is a 4-wheel, a contradiction. Thus f_1, a'_1 are nonadjacent. Let R'_1 be a 1-rung with ends a'_1 and $b'_1 \in B_1$. Since

$$f_1\text{-}\cdots\text{-}f_n\text{-}b_2\text{-}b'_1\text{-}R'_1\text{-}a'_1\text{-}a_3\text{-}a_1\text{-}f_1$$

is not an odd hole, it follows that F is not anticomplete to $V(R'_1)$. Consequently the set of attachments of F in the prism formed by R'_1, R_2, R_3 is not local with respect to this prism; and yet f_1 is nonadjacent to a'_1, and F is anticomplete to $V(R_3)$, contrary to 5.2. Thus there is no such vertex a'_1, and hence $A_1 = \{a_1\}$, and similarly $A_2 = \{a_2\}$, and $B_i = \{b_i\}$ for $i = 1, 2$. But then we can add f_1 to A_3, and f_n to B_3, and $f_2, \dots, f_{n-1}$ to C_3, contrary to the maximality of $V(H)$. This proves that the third outcome above does not hold.

We deduce that the fourth holds, and, say, f_1 is adjacent to a_1, a_2, and there is at least one edge between f_n and $V(R_3) \setminus \{a_3\}$, and there are no other edges between $\{f_1, \dots, f_n\}$ and $V(K') \setminus \{a_3\}$. Let R'_1 be a 1-rung, with ends $a'_1 \in A_1$ and $b'_1 \in B_1$. Thus the set of attachments of F in the prism formed by R'_1, R_2, R_3 is not local with respect to this prism, and so by 5.2 applied to this prism, there is a unique edge between F and $V(R'_1)$, and either f_1 is adjacent to a'_1, or f_n is adjacent to b'_1 and the only edges between $V(K') \cup V(R'_1)$ and F are $f_1a_1, f_1a_2, f_nb'_1, f_nb_3$. Suppose the latter. Then n is odd, since $f_1\text{-}\cdots\text{-}f_n\text{-}b_3\text{-}b_1\text{-}R_1\text{-}a_1\text{-}f_1$ is a hole. Since $b_1\text{-}R_1\text{-}a_1\text{-}f_1\text{-}\cdots\text{-}f_n\text{-}b'_1\text{-}b_2\text{-}b_1$ is not an odd hole, it follows that b'_1 is not anticomplete to $V(R_1)$, and so there is a 1-rung with ends a_1, b'_1; but this is impossible from what we showed above, since there are two edges between this 1-rung and F. This proves that for every choice of R'_1 (with ends a'_1, b'_1 as above) f_1 is adjacent to a'_1 and there are no other edges between F and $V(R'_1)$. Consequently, f_1 is complete to A_1, and there are no other edges between F and S_1. Similarly, the analogous statement holds for A_2, S_2; but then we can add f_1 to A_3 and $f_2, \dots, f_n$ to C_3, contary to the maximality of $V(H)$. Thus there is no such F. This proves (1).

Let W be the set of all major vertices with respect to H. From (1), we may partition $V(G) \setminus (V(H) \cup W)$ into five (possibly empty) sets A_0, B_0, D_1, D_2, D_3, pairwise anticomplete, such that

- every attachment of A_0 in $V(H)$ belongs to A, and every attachment of B_0 in $V(H)$ belongs to B
- for $i = 1, 2, 3$, every attachment of D_i in $V(H)$ belongs to S_i; and for every component X of D_i, some attachment of X in $V(H)$ does not belong to A, and some attachment does not belong to B.

(2) *For $i = 1, 2, 3$, if P is an induced path with both ends in A_i or both ends in B_i, and with no vertex in W, then P has even length.*

Suppose not, and choose i and P such that P is odd, with the length of P as small as possible. We may assume that both ends of P belong to A_1 say. If some internal vertex of P belongs to A_1, then it divides P into two subpaths, one of which is odd, contrary to the minimality of P. Thus no internal vertex of P is in A_1. Since A_2, A_3 are complete to A_1, it follows that no vertex of P is in $A_2 \cup A_3$. Let P have vertices $p_1\text{-}\cdots\text{-}p_k$ say. Now there is an induced path Q between p_1, p_k with interior in $C_1 \cup B_1 \cup B_2$, since p_1, p_k both belong to 1-rungs. Since Q can be completed to a hole via $p_k\text{-}a_3\text{-}p_1$ (where $a_3 \in A_3$) it follows that Q is even. Consequently the union of P and Q is not a hole, and so some internal vertex of P is equal to or adjacent to some internal vertex of Q.

Consequently P^* is not a subset of A_0, and (since no attachment of A_0 belongs to P^*) it follows that $V(P) \cap A_0 = \emptyset$. Thus $p_2, p_{k-1} \in B_1 \cup C_1 \cup D_1$. If $p_2, \dots, p_{k-1} \in C_1 \cup D_1$, then P can be completed to a hole via p_k-a_3-p_1, where $a_3 \in A_3$, which is impossible since P is odd. Thus there exist $i, j \in \{2, \dots, k-1\}$ such that $p_i, p_j \in B_1$, minimum and maximum respectively. The path p_1-$\cdots$-p_i is therefore a 1-rung, and so is p_j-$\cdots$-p_k; both these 1-rungs are odd, and so the path p_i-$\cdots$-p_j is also odd (and in particular $p_i \neq p_j$) contrary to the minimality of P. This proves (2).

(3) *$W \neq \emptyset$.*

For suppose that $W = \emptyset$. By (2), since there is no even pair, it follows that $|A_i| = |B_i| = 1$ for $i = 1, 2, 3$. Since G admits no clique cutset, and A is a clique, it follows that $A_0 = \emptyset$, and similarly $B_0 = \emptyset$; and since G is 3-connected, we deduce that $C_i \cup D_i = \emptyset$ for $1 \leq i \leq 3$. Hence G has only six vertices, a contradiction. This proves (3).

(4) *If $w \in V(G) \setminus V(H)$ is major with respect to H, then (up to symmetry) w is complete to $A_1 \cup B_2$, and has a unique neighbour $a_3 \in A_3$ and $b_3 \in B_3$, and a_3, b_3 are adjacent, and every 3-rung contains one of a_3, b_3, and $|A_1| = |B_2| = 1$.*

For let X be the set of neighbours of w in $V(H)$. We may assume that $X \cap A_1, X \cap A_3, X \cap B_3 \neq \emptyset$. Consequently $X \cap A_2 = \emptyset$, since G is K_4-free. For $1 \leq i \leq 3$ let R_i be an i-rung, with ends $a_i \in A_i$ and $b_i \in B_i$, such that $a_1, b_3 \in X$. Since w-a_1-a_2-R_2-b_2-b_3-w is not an odd hole, it follows that w has a neighbour in $V(R_2) \setminus \{a_2\}$. Thus w can be linked onto $\{b_1, b_2, b_3\}$, and so one of $b_1, b_2 \in X$. If $b_2 \notin X$, then similarly $a_3 \in X$, and so w is balanced with respect to the prism formed by R_1, R_2, R_3, a contradiction. Thus $b_2 \in X$, and so $X \cap B_1 = \emptyset$. Since this holds for all choices of R_2, we deduce that $B_2 \subseteq X$, and similarly $A_1 \subseteq X$. If there exist distinct $a_1, a_1' \in A_1$, then the subgraph induced on $\{a_1, a_1', w, a_2, a_3\}$ is a 4-wheel (where $a_2 \in A_2$ and $a_3 \in A_3 \cap X$), contrary to 6.3. Thus $|A_1| = 1$, and similarly $|B_2| = 1$. Let $A_1 = \{a_1\}$ and $B_2 = \{b_2\}$. If there exist distinct $a_3, a_3' \in A_3 \cap X$, then the subgraph induced on $\{a_3, a_3', w, a_1, a_2\}$ is a 4-wheel (where $a_2 \in A_2$), again a contradiction. Thus $|A_3 \cap X| = 1$, and similarly $|B_3 \cap X| = 1$. Let $A_3 \cap X = \{a_3\}$ and $B_3 \cap X = \{b_3\}$ say. Suppose there is a 3-rung R_3' containing neither of a_3, b_3; let its ends be $a_3' \in A_3$ and $b_3' \in B_3$ say. Since w-a_1-a_3'-R_3'-b_3'-b_2-w is not an odd hole, w has a neighbour in the interior of R_3'; but then w can be linked onto $\{a_1, a_2, a_3'\}$ (where $a_2 \in A_2$), a contradiction. Thus every 3-rung contains one of a_3, b_3. Next, suppose that a_3, b_3 are nonadjacent, and let R_3 be a 3-rung containing a_3. Let b_3' be its end in B_3. If $b_3 = b_3'$, then w is major-general with respect to the prism formed by R_3 and some 1-rung and 2-rung, contrary to 7.1. Thus $b_3 \neq b_3'$, and so $b_3' \notin X$. Moreover, we cannot choose a 3-rung with ends a_3, b_3, and so b_3 is anticomplete to $V(R_3)$. Since we cannot link w onto $\{b_1, b_2, b_3'\}$ (where $b_1 \in B_1$), it follows that $X \cap V(R_3) = \{a_3\}$. But then w-a_3-R_3-b_3'-b_1-b_3-w is an odd hole (where $b_1 \in B_1$), a contradiction. This proves that a_3, b_3 are adjacent, and so this proves (4).

(5) *$|W| = 1$.*

For suppose that u, v are distinct major vertices. By (4), we may assume that v is complete to $A_1 \cup B_2$, and has a unique neighbour $a_3 \in A_3$ and $b_3 \in B_3$, and a_3, b_3 are adjacent, and every 3-rung contains one of a_3, b_3, and $|A_1| = |B_2| = 1$. Let $A_1 = \{a_1\}$ and $B_2 = \{b_2\}$. Take a 3-colouring of G.

We may assume that every vertex in A_i has colour i, for $i = 1, 2, 3$. Since v has neighbours in A_1 and in A_3 it follows that v has colour 2; since b_3 is adjacent to v and to a_3, we deduce that b_3 has colour 1; since b_2 is adjacent to b_3 and to v, b_2 has colour 3; and therefore every vertex in B_1 has colour 2, and every vertex in B_3 has colour 1.

Suppose first that u has a neighbour in A_1 and one in B_1; thus u is adjacent to a_1 and to some $b_1 \in B_1$. Consequently a_1, b_1 are adjacent, by (4) applied to u. Moreover, u has colour 3, and therefore u is anticomplete to $A_3 \cup B_2$. Hence by (4) applied to u, u is adjacent to b_3, and so $\{a_1, b_3\}$ is complete to $\{u, v, a_3, b_1\}$, and G contains a trapeze or trestle, a contradiction. This proves that u is anticomplete to one of A_1, B_1, and similarly to one of A_2, B_2. By (4), u has neighbours in both A_3, B_3. It follows that u has colour 2, and therefore u, v are nonadjacent, and u is anticomplete to $A_2 \cup B_1$. By (4), u is adjacent to a_1, b_2. Now every 3-rung has a vertex adjacent to u, by (4), and since a_3-b_3 is a 3-rung, we may assume from the symmetry that a_3 is adjacent to u. Let b_3' be the unique neighbour of u in B_3. If $b_3 = b_3'$ then the subgraph induced on $\{u, v, a_1, a_3, b_2, b_3\}$ is a trestle, and if $b_3 \neq b_3'$ then the subgraph induced on $\{a_3, b_2, b_3, b_3', u, v\}$ is a trapeze, in either case a contradiction. Thus $|W| \leq 1$, and the result follows from (3). This proves (5).

Let $W = \{w\}$. By (4) we may assume that w is complete to $A_1 \cup B_2$, and has a unique neighbour $a_3 \in A_3$ and $b_3 \in B_3$, and a_3, b_3 are adjacent, and every 3-rung contains one of a_3, b_3, and $|A_1| = |B_2| = 1$. Let $A_1 = \{a_1\}$ and $B_2 = \{b_2\}$.

(6) *w is anticomplete to $C_3 \cup D_3$.*

For let X be a component of $C_3 \cup D_3$, and suppose that w has a neighbour in X. Let N be the set of all vertices not in X with a neighbour in X; thus, $w \in N \subseteq A_3 \cup B_3 \cup \{w\}$. Since $\{w, a_3, b_3\}$ are pairwise adjacent and G does not admit a clique cutset, it follows that $N \not\subseteq \{w, a_3, b_3\}$, and so we may assume that some $b_3' \in B_3 \setminus \{b_3\}$ belongs to N. Choose $b_1 \in B_1$; then we can link w onto $\{b_1, b_2, b_3'\}$, a contradiction. This proves (6).

(7) *For each $a_2 \in A_2$, every odd induced path between a_2 and w contains a vertex in $A_3 \setminus \{a_3\}$; and consequently $|A_3|, |B_3| \geq 2$ and $|A_2| = |B_1| = 1$.*

For let a_2-p_1-$\cdots$-p_k-w be an odd induced path, and suppose that $p_1, \ldots, p_k \notin A_3 \setminus \{a_3\}$. Choose a_2 and $p_1, \ldots, p_k$ with k minimum. If some $p_i \in A_2$, then none of $p_1, \ldots, p_i$ is in W, and so i is odd by (2); and so p_i-$\cdots$-p_k-w is an odd induced path, contrary to the minimality of k. Thus $p_1, \ldots, p_k \notin A_2$. Since p_1 is adjacent to a_2 and p_1 is nonadjacent to w, it follows that either $p_1 \in C_2 \cup D_2$ or $p_1 \in A_0$ (since $p_1 \notin A_3$ by hypothesis). If $p_1 \in C_2 \cup D_2$, then since none of $p_2, \ldots, p_{k-1}$ is adjacent to w, and therefore none of $p_2, \ldots, p_{k-1}$ belongs to $A_2 \cup B_2 \cup \{w\}$, it follows that $p_2, \ldots, p_{k-1} \in C_2 \cup D_2$. Consequently $p_k \in C_2 \cup D_2 \cup B_2$. But then a_2-p_1-$\cdots$-p_k-w-a_3-a_2 is an odd hole, a contradiction. Thus $p_1 \in A_0$. Since $p_2, \ldots, p_k$ are nonadjacent to a_2 and therefore not in A, it follows that $p_2, \ldots, p_k \in A_0$. But there is an induced path Q between a_2 and w with interior in $C_2 \cup B_2$, since a_2 belongs to a 2-rung with ends a_2, b_2; and Q is even since a_2-Q-w-a_3-a_2 is a hole; and so a_2-p_1-$\cdots$-p_k-w-Q-a_2 is an odd hole, a contradiction. This proves the first assertion of (7). Since w, a_2 is not an even pair, we deduce that $A_3 \setminus \{a_3\} \neq \emptyset$, and so $|A_3| \geq 2$; and similarly $|B_3| \geq 2$. Finally, note that that if also $|A_2| \geq 2$ then $G|A$ contains a 4-wheel, a contradiction. This proves (7).

Let $A_2 = \{a_2\}$ and $B_1 = \{b_1\}$.

(8) *$C_1, D_1, C_2, D_2 = \emptyset$.*

For let $a_3' \in A_3 \setminus \{a_3\}$. Since we cannot link w onto $\{a_1, a_2, a_3'\}$, it follows that w is anticomplete to $C_2 \cup D_2$; and since G is 3-connected, it follows that $C_2 \cup D_2 = \emptyset$, and similarly $C_1 \cup D_1 = \emptyset$. This proves (8).

(9) *If $|A_3| \geq 3$ then w is anticomplete to A_0.*

For by (8), a_2, b_2 are adjacent. Suppose that w has a neighbour in some component X of A_0. Since G admits no clique cutset, there is an attachment of X in one of $\{a_2\}, A_3 \setminus \{a_3\}$, and so there is an induced path w-p_1-$\cdots$-p_k-u between w and some $u \in \{a_2\} \cup A_3 \setminus \{a_3\}$, with $p_1, \ldots, p_k \in A_0$. Choose u and p_1-$\cdots$-p_k with k minimum. We claim that a_2, p_k are adjacent. For suppose not; then $u \in A_3 \setminus \{a_3\}$. Since there is a 3-rung R_3 with ends u and b_3, and w-p_1-$\cdots$-p_k-u-R_3-b_3-w is a hole by (6), it follows that k is odd; and since a_2 is anticomplete to $\{p_1, \ldots, p_k\}$ (by the minimality of k, and since a_2, p_k are nonadjacent) and a_2, b_2 are adjacent, it follows that w-p_1-$\cdots$-p_k-u-a_2-b_2-w is an odd hole, a contradiction. Thus a_2, p_k are adjacent, and so w-p_1-$\cdots$-p_k-a_2 is an induced path. By (7), k is odd.

Let $a_3' \in A_3 \setminus \{a_3\}$; we claim that a_3', p_k are adjacent. For suppose not; then from the minimality of k, it follows that a_3' is anticomplete to $\{p_1, \ldots, p_k\}$, and so w-p_1-$\cdots$-p_k-a_2-a_3'-R_3-b_3-w is an odd hole (where R_3 is a 3-rung with ends a_3', b_3), a contradiction. This proves that p_k is complete to $A_3 \setminus \{a_3\}$. Choose distinct $x, y \in A_3 \setminus \{a_3\}$ (this is possible since $|A_3| \geq 3$); then the subgraph induced on $\{a_1, p_k, a_2, x, y\}$ is a 4-wheel, a contradiction. This proves (9).

(10) *$|A_3|, |B_3| = 2$, and $C_3, D_3 = \emptyset$.*

Suppose first that either

- $|A_3| \geq 3$, or
- $|A_3| = 2$ and there is an induced path between the two members of A_3 with interior in $C_3 \cup D_3$.

Since G has no even pair, there is an odd induced path a_3-p_1-$\cdots$-p_k with $p_k \in A_3 \setminus \{a_3\}$; choose such a path with k minimum. By (2) w belongs to this path, and since a_3 is adjacent to w it follows that $w = p_1$, and $k \geq 3$. If $p_h \in A_3$ for some h with $1 \leq h < k$, then $h \geq 2$, and w does not belong to the path p_h-$\cdots$-p_k, and so this path is even by (2); and so a_3-p_1-$\cdots$-p_h is odd, contrary to the minimality of k. Thus $p_1, \ldots, p_{k-1} \notin A_3$. Now since the path is induced, none of $p_1, \ldots, p_{k-1} \in A$; and since $w = p_1$, none of $p_3, \ldots, p_k$ is adjacent to w, since $w = p_1$. Since p_{k-1} is adjacent to $p_k \in A_3$, it follows that $p_{k-1} \in A_0$ or $p_{k-1} \in B_3 \cup C_3 \cup D_3$.

Suppose that $p_{k-1} \in B_3 \cup C_3 \cup D_3$. Now $p_{k-1} \neq b_3$ since p_{k-1} is not adjacent to a_3, and $p_{k-1} \notin B_3 \setminus \{b_3\}$ since $p_{k-1}p_k$ is not a 3-rung (because every 3-rung contains a_3 or b_3). Thus $p_{k-1} \in C_3 \cup D_3$. Since w is anticomplete to $C_3 \cup D_3$, it follows that $p_2 \notin C_3 \cup D_3$, and so we may choose i with $2 \leq i \leq k-1$ maximum such that $p_i \notin C_3 \cup D_3$. It follows that $i \leq k-2$, and so $p_{i+1} \in C_3 \cup D_3$, and therefore $p_i \in A_3 \cup B_3$. Since $2 \leq i < k$ it follows that $p_i \notin A_3$, so $p_i \in B_3$. Since $p_j \in C_3 \cup D_3$ for $i < j < k$, it follows that p_i-p_{i+1}-$\cdots$-p_k is an induced path with ends in B_3

and A_3, and with interior in $C_3 \cup D_3$. From the maximality of $V(H)$, this path belongs to H and therefore is a 3-rung; so $p_i = b_3$, a contradiction since none of $p_2, \ldots, p_k$ are adjacent to a_3.

Thus $p_{k-1} \in A_0$. Since none of $p_2, \ldots, p_{k-1}$ belongs to $A \cup \{w\}$, it follows that $p_2, \ldots, p_{k-1} \in A_0$. Since p_2 is adjacent to $p_1 = w$, it follows that w has a neighbour in A_0, and so $|A_3| = 2$ by (9); and therefore $A_3 = \{a_3, p_k\}$. By hypothesis there is an induced path Q between a_3, p_k with interior in $C_3 \cup D_3$. Since Q can be completed to a hole via p_k-a_1-a_3, it follows that Q is even; and so a_3-p_1-$\cdots$-p_k-Q-a_3 is an odd hole, a contradiction.

Thus the bulletted statements above are both false. In particular, $|A_3| = 2$, and similarly $|B_3| = 2$. If $C_3 \cup D_3 = \emptyset$ then the claim holds; so we may assume (for a contradiction) that X is a component of $C_3 \cup D_3$. Let N be the set of vertices not in X with a neighbour in X. Now $N \subseteq A_3 \cup B_3$ since w is anticomplete to $C_3 \cup D_3$ by (6), and $N \not\subseteq \{a_3, b_3\}$ since G does not admit a clique cutset. Thus from the symmetry we may assume that $a'_3 \in N$, for some $a'_3 \in A_3 \setminus \{a_3\}$. Consequently $A_3 = \{a_3, a'_3\}$. Since the second bulletted statement above is false, it follows that $a_3 \notin N$. Since G is 3-connected, we deduce that $N \not\subseteq \{a'_3, b_3\}$; choose $b'_3 \in N \setminus \{a'_3, b_3\}$. Thus $b'_3 \in B_3 \setminus \{b_3\}$. There is an induced path between a'_3, b'_3 with interior in X, and hence this is a 3-rung (from the maximality of $V(H)$), contradicting that every 3-rung contains either a_3 or b_3. This proves (10).

Now (8) and (10) imply that C_1, C_2, C_3 are all empty; and so the prism K is short. This proves the first assertion of the theorem. Let K' be the subgraph induced on $\{a_1, a_2, a_3, b_1, b_2, b_3\}$. Now suppose that x is a major vertex with respect to K. Then x is major with respect to H, and so $x = w$, and therefore $K = K'$ (since w is not major with respect to any other prism contained in H) and so w separates K. Thus the second assertion of the theorem holds. Finally the third assertion holds since K' is a short prism and w is a major vertex with respect to it. This proves 10.1. ∎

11 Short prisms

In this section we complete the elimination of prisms, and hence complete the proof of 1.3. We first prove the following.

11.1 *Let G be a 3-connected K_4-free Berge graph, containing no even pair, no trampoline, and no clique cutset. Suppose that G contains a prism. Then G is the line graph of a bipartite graph.*

Proof. If $|V(G)| = 6$ then since G contains a prism, it follows that G is a short prism and therefore the theorem holds; so we may assume that $|V(G)| > 6$. Suppose that G contains no appearance of K_4 and no even prism. By 10.1, G contains no long prism; and G contains a short prism with a major vertex w say. Let the short prism have vertex set $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ where $\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}$ are triangles, and for $1 \leq i, j \leq 3$ a_i is adjacent to b_j if and only if $i = j$. Since w is not balanced by 6.2, we may assume that w is adjacent to a_1, b_2, a_3, b_3.

Let us say a *prism-sequence* in G is a sequence $v_1, \ldots, v_n$ of distinct vertices of G such that $n \geq 7$ and for $1 \leq i < j \leq n$, v_i, v_j are adjacent if and only if $j - i \in \{1, 2, 5\}$. We observe that the sequence

$$a_2, a_1, a_3, w, b_3, b_2, b_1$$

is a prism-sequence. Let us choose a prism-sequence $v_1, \ldots, v_n$ in G with n maximum. Choose a 3-colouring of G. We may assume that v_n has colour 1, and v_{n-1} has colour 2; and so v_{n-2} has colour 3, v_{n-3} has colour 1, and so on.

Since v_{n-3}, v_n is not an even pair, there is an odd induced path P in G between v_{n-3}, v_n. Now $v_{n-1}, v_{n-2}, v_{n-5}$ are all complete to $\{v_{n-3}, v_n\}$. By 2.4, $\{v_{n-1}, v_{n-2}, v_{n-5}\}$ contains a leap for P, and since G contains no long prism it follows that P has length three. Let P have vertices v_{n-3}-x-y-v_n in order. Thus one of x, y has colour 2 and is adjacent to v_{n-2}, v_{n-5} and not to v_{n-1}, and the other has colour 3 and is adjacent to v_{n-1} and not to v_{n-2}, v_{n-5}. Suppose that x has colour 3. Then $\{v_{n-2}, v_{n-5}\}$ is complete to y, v_n, v_{n-3}, v_{n-4}; and $y \neq v_{n-4}$ since y is adjacent to v_n, so y, v_n, v_{n-3}, v_{n-4} are all different, and the subgraph induced on $\{v_{n-2}, v_{n-5}, y, v_n, v_{n-3}, v_{n-4}\}$ is a trapeze, a contradiction. Thus x has colour 2. Hence x is adjacent to v_{n-2}, v_{n-5} and not to v_{n-1}, and y is adjacent to v_{n-1} and not to v_{n-2}, v_{n-5}. The subgraph induced on $\{x, v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}\}$ is not a 4-wheel, and so $x \in \{v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}\}$; and since x has colour 2 it follows that $x = v_{n-4}$. We deduce that y is adjacent to v_n, v_{n-1}, v_{n-4}, and not to $v_{n-2}, v_{n-3}, v_{n-5}$. Consequently $y \neq v_{n-1}, v_{n-2}, v_{n-5}$, and since y is adjacent to v_n it follows that y is different from $v_1, \dots, v_n$. Now the subgraph induced on $\{v_{n-6}, v_{n-5}, v_{n-4}, v_{n-2}, v_{n-1}, v_n\}$ is a short prism K say, and v_{n-3} is a major vertex with respect to K. By 10.1 it follows that v_{n-3} separates K, and so y is anticomplete to $\{v_1, \dots, v_{n-6}\}$. Hence the sequence $v_1, \dots, v_n, y$ is a prism-sequence, contrary to the maximality of n.

This proves that G contains either an appearance of K_4 and or an even prism. From 9.2 and 9.3 it follows that G is the line graph of a bipartite graph. This proves 11.1. ∎

We deduce:

11.2 *Let G be a 3-connected K_4-free Berge graph, containing no even pair, no trampoline, and no clique cutset. Suppose that G contains a square. Then G is the line graph of a bipartite graph.*

Proof. Fix a three-colouring of G. Choose a square a_1-b_1-a_2-b_2-a_1, such that if possible a_1, a_2 have different colours. Since b_1, b_2 is not an even pair, there is an odd induced path P between b_1, b_2. Let the vertices of P be p_1-$\cdots$-p_k, where $p_1 = b_1$ and $p_k = b_2$. If $\{a_1, a_2\}$ contains a leap for this path, then G contains a prism and the result follows from 11.1. Thus we suppose that $\{a_1, a_2\}$ contains no leap. By 2.4 it follows that a_1, a_2 have the same colour, say colour 1. From the choice of the square a_1-b_1-a_2-b_2-a_1, it follows that there is no square in which some two nonadjacent vertices have different colours. In particular, b_1, b_2 have the same colour, say colour 2.

We claim that some edge of P is complete to $\{a_1, a_2\}$. For if $k > 4$ this follows from 2.1, so we assume $k = 4$. Since a_1-p_1-$\cdots$-p_4-a_1 is not an odd hole, a_1 is adjacent to one of p_2, p_3, and similarly so is a_2. If neither of p_2, p_3 is complete to $\{a_1, a_2\}$, then $\{a_1, a_2\}$ is a leap, a contradiction; so from the symmetry we may assume that p_2 is complete to $\{a_1, a_2\}$, and so the edge p_1p_2 is complete to $\{a_1, a_2\}$. This proves that some edge of P is complete to $\{a_1, a_2\}$, and consequently there exists i with $1 < i < k$ such that p_i is complete to $\{a_1, a_2\}$ and p_i has colour different from 2. Now p_i is nonadjacent to one of b_1, b_2, say b_j; and so p_i-a_1-b_j-a_2-p_i is a square, and p_i, b_j have different colours, a contradiction. This proves 11.2. ∎

Next we use a theorem of Linhares Sales and Maffray [7], the following (thanks to the referee for pointing out this result, which eliminates the hard part of our original argument):

11.3 *Let G be a Berge graph with no prism and no square, and with no even pair. Then G is complete.*

Now we can complete the proof of the main theorem.

Proof of 1.3. Let G be a 3-connected K_4-free Berge graph with no even pair and no clique cutset. If G contains a trampoline, then $\overline{G}$ is a line graph by 3.1. Thus, we assume that G contains no trampoline. If G contains a prism or square, then G is the line graph of a bipartite graph, by 11.1 and 11.2. Thus we may assume that G contains no prism or square. By 11.3, G is complete, and hence is the line graph of a bipartite graph. This proves 1.3. ∎

References

[1] C. Berge, "Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind", *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe* 10 (1961), 114.

[2] M. Chudnovsky, G. Cornuéjols, X. Liu, P. Seymour and K. Vušković, "Recognizing Berge graphs", *Combinatorica*, 25 (2005), 143–186.

[3] M. Chudnovsky and P. Seymour, "Even pairs in Berge graphs", *J. Combinatorial Theory, Ser. B*, 99 (2009), 370–377.

[4] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas, "K_4-free graphs with no odd holes", *J. Combinatorial Theory, Ser. B*, 100 (2010), 313–331.

[5] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas, "The strong perfect graph theorem", *Annals of Mathematics* 164 (2006), 51–229.

[6] H. Everett, C.M.H. de Figueiredo, C. Linhares Sales, F. Maffray, O. Porto and B.A. Reed, "Even pairs", in *Perfect Graphs* (J. L. Ramírez-Alfonsín and B.A. Reed, eds.), Wiley Interscience, New York, 2001, 67–92.

[7] C. Linhares Sales and F. Maffray, "Even pairs in square-free Berge graphs", *Matemática Contemporânea*, 25 (2003), 161–176.

[8] F. Roussel and P. Rubio, "About skew partitions in minimal imperfect graphs", *J. Combinatorial Theory, Ser. B* 83 (2001), 171–190.

[9] A. Tucker, "Critical perfect graphs and perfect 3-chromatic graphs", *J. Combinatorial Theory, Ser. B* 23 (1977), 143–149.

[10] A. Tucker, "The validity of the perfect graph conjecture for K_4-free graphs", in *Topics On Perfect Graphs* (C. Berge and V. Chvátal, eds.), North-Holland, Amsterdam, *Annals of Discrete Math.* 21 (1984), 225–252.

[11] Y. Zwols, "K_4-free graphs with no odd hole: even pairs and the circular chromatic number", *Journal of Graph Theory* 65 (2010), 303–322.