Three-colourable perfect graphs without even pairs

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Abstract

We still do not know how to construct the "most general" perfect graph, not even the most general three-colourable perfect graph. But constructing all perfect graphs with no even pairs seems easier, and here we make a start on it; we construct all three-colourable perfect graphs without even pairs. They are all built from line graphs and some other basic classes by composition operations.

1 Introduction

A graph G is *perfect* if for every induced subgraph H, the chromatic number of H equals the size of the largest clique in H. A *hole* in a graph G is an induced subgraph that is a cycle of length at least four, and an *antihole* in G is an induced subgraph whose complement is a cycle of length at least four; and a hole or antihole is *odd* if it has an odd number of vertices. A graph is *Berge* if it has no odd hole and no odd antihole. Perfect graphs were introduced by Claude Berge in [1], where he proposed the "strong perfect graph conjecture", now a theorem [5], the following:

1.1 *A graph is perfect if and only if it is Berge.*

The recognition problem for Bergeness (and hence, by 1.1, for perfection) is also solved [2];

1.2 *There is an algorithm with running time $O(|V(G)|^9)$, to test if an input graph G is Berge.*

But neither of these results gives us a way to build the most general perfect graph. Ideally we would like a theorem that a graph is Berge if and only if it can be built from some well-understood class of building blocks, by piecieng them together in a way that preserves Bergeness. But we are far from such a theorem, and indeed we do not even know how to construct the most general Berge graph with no K_4 subgraph, which presumably should be an easier problem.

An *even pair* in G is a pair u, v of distinct vertices such that every induced path in G between u and v has even length (the *length* of a path or cycle is the number of edges in it), and consequently u, v are nonadjacent. As far as we know, finding an even pair does not give us a satisfactory way to construct our graph from a smaller graph; but still, an even pair u, v in a Berge graph G is quite a useful thing. For instance, if we identify u, v the graph remains Berge with the same clique number, which is helpful if we are trying to optimally colour G, or prove that G is perfect. (For a survey of recent work on even pairs see [6].) Thus, since finding a construction for all perfect graphs seems hopeless, what about finding a construction for all perfect graphs that have no even pairs? This problem, while still open, seems much more tractable.

In this paper we make a start on it; we construct all Berge graphs that have no K_4 subgraphs and have no even pairs. Let us say G is *K_4-free* if it has no K_4 subgraph. A *clique cutset* in G is a clique C of G such that $G \setminus V(C)$ is disconnected. We denote the complement of the graph G by $\overline{G}$. Our main theorem is the following:

1.3 *Let G be a K_4-free Berge graph with no even pair, and with no clique cutset. Then one of $G, \overline{G}$ is a line graph.*

The proof is lengthy, and similar to the proof of 1.1; for a sequence of different graphs H, we first assume that G contains H as an induced subgraph, and prove the theorem in this case, and thereafter we can assume that G does not contain H, and move on to the next graph of our sequence. (The sequence is shorter and the analysis easier than in the proof of 1.1, however.)

Here is a related question that has a surprisingly pretty answer: which K_4-free graphs have no odd hole and no even pair? In [4] we gave a construction for all K_4-free graphs with no odd hole, using as building blocks the K_4-free Berge graphs. Using this result, in [3], Chudnovsky and Zwols proved that there are only two K_4-free graphs without odd holes that are not perfect and do not admit a 3-vertex clique cutset, namely the complement of a seven-cycle, and a certain 11-vertex graph with cyclic symmetry.

2 The Roussel-Rubio theorem

There was a result proved by Roussel and Rubio [7], that we used many times in the proof of 1.1, that will be important here. All graphs in this paper are finite, and without loops or parallel edges. Let us say a subset $X \subseteq V(G)$ is *connected* if the subgraph $G|X$ of G induced on X is connected, and *anticonnected* if $\overline{G}|X$ is connected. If $X, Y \subseteq V(G)$, we say X is *complete* to Y or Y-complete if every vertex in X is adjacent to every vertex in Y (and similarly, we say a vertex v is complete to Y or Y-complete if $\{v\}$ is complete to X, and an edge uv is Y-complete if both u, v are Y-complete); and X is *anticomplete* to Y if X is complete to Y in $\overline{G}$.

If P is an induced path in G with vertices p_1-$\cdots$-p_k in order, with $k \geq 4$, a *leap* for P is a pair $\{x, y\}$ of nonadjacent vertices of $V(G) \setminus V(P)$ such that x is adjacent to p_1, p_2, p_k, and y is adjacent to p_1, p_{k-1}, p_k, and there are no other edges between $\{x, y\}$ and $V(P)$. The Roussel-Rubio theorem is the following:

2.1 *Let G be a Berge graph, and let P be an induced path in G of odd length, at least five. Let $X \subseteq V(G) \setminus V(P)$ be anticonnected, such that the ends of P are X-complete, and no edge of P is X-complete. Then X includes a leap for P.*

We also need a theorem of [5]:

2.2 *Let G be Berge, let X be an anticonnected subset of $V(G)$, and P be an induced path in $G \setminus X$ with odd length, such that both ends of P are X-complete, and no edge of P is X-complete. Then every X-complete vertex of G has a neighbour in P^*.*

Next, we need:

2.3 *Let G be a Berge graph, and let P be an induced path in G of odd length, with vertices p_1-$\cdots$-p_k in order. Let $X \subseteq V(G) \setminus V(P)$ be anticonnected, such that p_1, p_k are X-complete, and no edge of P is X-complete. Then $k \geq 4$ and every vertex in X is adjacent to one of p_2, p_{k-1}.*

Proof. Since no edge of P is X-complete it follows that $k \geq 4$. Suppose that $z \in X$ is nonadjacent to both p_2, p_{k-1}. If $k = 4$ then z-p_1-p_2-p_3-p_4-z is an odd hole, a contradiction, so $k > 4$. Choose an anticonnected subset $Z \subseteq X$, with $z \in Z$, maximal such that Z includes no leap for P. Thus $Z \neq X$; choose $x \in X \setminus Z$ such that $Z \cup \{x\}$ is anticonnected. From the maximality of Z, $Z \cup \{x\}$ includes a leap, and since Z includes no leap, it follows that there is a leap $\{x, y\}$ for some $x \in Z$. Consequently y is nonadjacent to $p_3, \ldots, p_{k-2}$. But from 2.1 applied to P and Z, since Z contains no leap, there is a Z-complete vertex p_i, where $2 \leq i \leq k-1$. Hence p_i is adjacent to both y, z. But z is nonadjacent to p_2, p_{k-1}, and y is nonadjacent to $p_3, \ldots, p_{k-2}$, a contradiction. This proves 2.3. ∎

In the three-colourable case we can say more:

2.4 *Let G be a Berge graph with a three-colouring $\phi : V(G) \to \{1, 2, 3\}$. Let P be an induced path in G of odd length, with vertices p_1-$\cdots$-p_k in order. Let $X \subseteq V(G) \setminus V(P)$ be anticonnected and not stable, such that p_1, p_k are X-complete. Then*

- *$\phi(p_1) = \phi(p_k)$ ($= 3$ say), and in particular p_1, p_k are nonadjacent, so $k \geq 3$;*

- *no internal vertex of P is X-complete;*
- *$\{\phi(p_2), \phi(p_{k-1}\} = \{1, 2\}$, say $\phi(p_2) = 1$ and $p_{k-1} = 2$;*
- *X is the union of two disjoint stable sets X_1, X_2, where the vertices in X_1 have colour 1 and are adjacent to p_{k-1}, and the vertices in X_2 have colour 2 and are adjacent to p_2; and*
- *there is a leap $\{x_1, x_2\}$ for P with $x_i \in X_i$ for $i = 1, 2$.*

Proof. Since X is not stable, we may assume that some vertex in X has colour 1, and some vertex in X has colour 2; so every X-complete vertex has colour 3. In particular, p_1, p_k have colour 3 and therefore are nonadjacent, so $k \geq 3$. For the same reason no two X-complete vertices in P are adjacent. Choose a minimal subpath Q of P of odd length such that both its ends are X-complete. It follows that Q has length at least three; and none of its internal vertices are X-complete, from the minimality of Q. Since p_1, p_k are both X-complete, by 2.2 they both have neighbours in the interior of Q, and so $Q = P$. Consequently no internal vertex of P is X-complete.

Since c_1 has colour 3, it follows that $X = X_1 \cup X_2$ where for $i = 1, 2$, X_i is the set of vertices in X with colour i. Thus $X_1, X_2 \neq \emptyset$. Since p_2 is adjacent to p_1 and p_{k-1} to p_k, we deduce that p_2, p_{k-1} do not have colour 3, and from the symmetry we may assume that p_2 has colour 1. Thus p_2 is anticomplete to X_1, and so by 2.3, X_1 is complete to p_{k-1}. Since $X_1 \neq \emptyset$, it follows that p_{k-1} does not have colour 1; so it has colour 2. Thus X_2 is anticomplete to p_{k-1}, and therefore is complete to p_2. We have shown then that every vertex in X is adjacent to one of p_2, p_{k-1} and nonadjacent to the other. Finally, we need to produce the leap. If P has length at least five, this follows from 2.1, so we may assume that P has length three, and therefore $k = 4$. Since p_2, p_3 are not X-complete, and X is anticonnected, there is an (induced) antipath p_2-q_1-$\cdots$-q_m-p_3 between p_2, p_3 with $q_1, \ldots, q_m \in X$. If $m \geq 3$ then q_m is adjacent to both p_2, p_3, a contradiction; and if $m = 1$ then q_1 is nonadjacent to both p_2, p_3, again a contradiction; so $m = 2$ and $\{q_1, q_2\}$ is the desired leap. This proves 2.4. ∎

3 Complement line graphs

Let H be a graph with vertex set $\{v_1, \ldots, v_9\}$ and edges as follows:

- for $1 \leq i \leq 6$ v_i is adjacent to $v_{i+2}, v_{i+3}, v_{i+4}$ (reading subscripts modulo 6)
- v_7 is adjacent to v_3, v_4, v_5, v_6; v_8 is adjacent to v_5, v_6, v_1, v_2; and v_9 is adjacent to v_1, v_2, v_3, v_4, and there are no other edges.

We call such a graph H a *trampoline*. In this section we study K_4-free Berge graphs that contain trampolines. We prove the following:

3.1 *Let G be a K_4-free Berge graph with no even pair and no clique cutset. If G contains a trampoline as an induced subgraph, then G is the complement of the line graph of some bipartite graph.*

The proof needs several steps. Throughout this section, let G be a K_4-free Berge graph with no even pair and no clique cutset, that contains a trampoline. Consequently we may choose $t \geq 4$, and pairwise disjoint stable sets $A_{ij}, \ldots, ; (1 \leq i \leq 3, 1 \leq j \leq t)$ with the following properties:

- for $1 \le i \le 3$, there is at most one value of $j \in \{1, \dots, t\}$ such that $A_{ij} = \emptyset$
- for $1 \le j \le t$, there is at most one value of $i \in \{1, 2, 3\}$ such that $A_{ij} = \emptyset$
- for all distinct $i, i' \in \{1, 2, 3\}$ and all distinct $j, j' \in \{1, \dots, t\}$, A_{ij} is complete to $A_{i'j'}$
- for $1 \le i \le 3$ and for all distinct $j, j' \in \{1, \dots, t\}$, A_{ij} is anticomplete to $A_{ij'}$
- for $1 \le j \le t$, if A_{1j}, A_{2j}, A_{3j} are all nonempty then they are pairwise anticomplete
- for $1 \le j \le t$, and all distinct $i, i' \in \{1, 2, 3\}$, if $A_{i'j}$ is nonempty then every vertex in A_{ij} has a nonneighbour in $A_{i'j}$.

Choose these sets with maximal union W say. For $1 \le i \le 3$ let $Z_i = \cup_{1 \le j \le t} A_{ij}$, and for $1 \le j \le t$ let $A_j = A_{1j} \cup A_{2j} \cup A_{3j}$. Fix a 3-colouring ϕ of G. Since $t \ge 4$ it follows that the only partition of W into three stable sets is the partition Z_1, Z_2, Z_3; and we may therefore assume that for $1 \le i \le 3$, $\phi(v) = i$ for all $v \in Z_i$.

Let $v \in V(G) \setminus W$, and let N be the set of vertices in W that are adjacent to v. We say v is *big* if N is the union of two of Z_1, Z_2, Z_3; and v is *small* if there exist $i, i' \in \{1, 2, 3\}$ and $j, j' \in \{1, \dots, t\}$ such that $i \ne i'$ and $N \subseteq A_{ij} \cup A_{i'j'}$ and $N \cap A_{ij}$ is complete to $N \cap A_{i'j'}$.

3.2 *With notation as above, every vertex in $V(G) \setminus W$ is either big or small.*

Proof. Let $v \in V(G) \setminus W$, and let N be the set of vertices in W that are adjacent to v. Since v has no neighbours in Z_i where $i = \phi(v)$, it follows that

(1) *N is a subset of the union of two of Z_1, Z_2, Z_3.*

(2) *For $1 \le j \le t$, if A_{1j}, A_{2j} are both nonempty and $A_{1j} \cup A_{2j}$ is neither a subset of N nor a subset of $V(G) \setminus N$, then there exist $a_{ij} \in A_{ij}$ for $i = 1, 2$, nonadjacent, such that exactly one of them is in N.*

For we may assume that $j = 1$; and suppose the claim is false. For $i = 1, 2$, let $N_i = N \cap A_{i1}$ and let $M_i = A_{i1} \setminus N_i$. Since the claim is false, N_1 is complete to M_2, and N_2 is complete to M_1. If $x \in N_1$, then since x has a nonneighbour in A_{21}, it follows that x has a nonneighbour in N_2; and so, since by hypothesis one of N_1, N_2 is nonempty, it follows that there exist $n_i \in N_1$ for $i = 1, 2$, nonadjacent. Similarly M_1, M_2 are both nonempty. Since A_{11} is not anticomplete to A_{21}, it follows that $A_{31} = \emptyset$. If $m_1 \in M_1$ is adjacent to $m_2 \in M_2$, then v-n_1-m_2-m_1-n_2-v is an odd hole, a contradiction; so M_1 is anticomplete to M_2. If say there exists $a_{12} \in A_{12} \setminus N$, then v-n_2-a_{12}-m_2-n_1-v is an odd hole, a contradiction; so $A_{12} \subseteq N$, and similarly $(Z_1 \cup Z_2) \setminus N \subseteq A_1$. But then we can define $A'_{11} = N_1$, $A'_{21} = N_2$, $A'_{31} = \emptyset$, $A'_{1,t+1} = M_1$, $A'_{2,t+1} = M_2$, $A'_{3,t+1} = \{v\}$, and $A'_{ij} = A_{ij}$ for $1 \le i \le 3$ and $2 \le j \le t$, contrary to the maximality of W.

(3) *We may assume that there are at least two values of $j \in \{1, \dots, t\}$ such that $A_{1j} \cup A_{2j} \not\subseteq N$.*

For suppose not; say $N \cup A_1 = Z_1 \cup Z_2$. If $A_{11} = \emptyset$ and $A_{21} \subseteq N$, then $N = Z_1 \cup Z_2$ and v is big as required. If $A_{11} = \emptyset$ and $A_{21} \not\subseteq N$, then we can add v to $_{31}$, contrary to the maximality of

W. Thus we may assume that $A_{11} \neq \emptyset$, and similarly $A_{21} \neq \emptyset$. If N includes $A_{11} \cup A_{21}$ then again v is big, and if N is disjoint from $A_{11} \cup A_{21}$ then we can add v to A_{31}, again contradictory to the maximality of W. Thus we may assume that N includes some but not all of $A_{11} \cup A_{21}$; and so, from (1), we may assume that there exists $a_{11} \in A_{11} \setminus N$, and $a_{21} \in A_{21} \cap N$, nonadjacent. Since $t \geq 4$, there exists $j \in \{2, \ldots, t\}$ such that $A_{2j}, A_{3j} \neq \emptyset$, say $j = 2$. Choose $a_{22} \in A_{22}$, and choose $a_{32} \in A_{32}$ nonadjacent to a_{22}. Then v-a_{21}-a_{32}-a_{11}-a_{22}-v is an odd hole, a contradiction. This proves (3).

(4) *For $1 \leq j \leq t$, $N \cap A_{1j}$ is complete to $N \cap A_{2j}$.*

For suppose that there exist $a_{i1} \in N \cap A_{i1}$ for $i = 1, 2$, nonadjacent. By (3) we may assume that $A_{12} \cup A_{22} \not\subseteq N$. Suppose first that both A_{12}, A_{22} are nonempty, and $N \cap (A_{12} \cup A_{22}) \neq \emptyset$. From (1) we may assume that there exist $a_{12} \in A_{12} \setminus N$ and $a_{22} \in A_{22} \cap N$, nonadjacent. Since there is no odd hole of the form v-a_{21}-a_{12}-A_{31}-a_{22}-v, it follows that $A_{31} = \emptyset$; and so $A_{3k} \neq \emptyset$ for $2 \leq k \leq t$. Since $t \geq 4$, one of A_{23}, A_{24} is nonempty, say A_{23}; choose $a_{23} \in A_{23}$. If $a_{23} \in N$ then v-a_{23}-a_{12}-a_{33}-a_{22}-v is an odd hole (where $a_{33} \in A_{33}$), and if $a_{23} \notin N$ then v-a_{11}-a_{23}-a_{12}-a_{21}-v is an odd hole, in either case a contradiction. This proves that if both A_{12}, A_{22} are nonempty, then $N \cap (A_{12} \cup A_{22}) = \emptyset$.

Since $A_{12} \cup A_{22} \not\subseteq N$, we may assume that there exists $a_{12} \in A_{12} \setminus N$. For $3 \leq j \leq t$, since there is no odd hole of the form v-a_{21}-a_{12}-A_{2j}-a_{11}-v, it follows that $A_{2j} \subseteq N$. Suppose that $A_{22} \neq \emptyset$. By what we just proved, $N \cap (A_{12} \cup A_{22}) = \emptyset$, and from the symmetry between Z_1, Z_2 it follows that $A_{1j} \subseteq N$ for $3 \leq j \leq t$. Since $t \geq 4$, one of A_{33}, A_{34} is nonempty, say A_{33}, and therefore A_{13} is anticomplete to A_{23}; but then from the symmetry between A_1 and A_3, we deduce that $A_{11}, A_{21} \subseteq N$. Then v can be added to A_{32} contrary to the maximality of W. This proves that $A_{22} = \emptyset$.

Consequently $A_{2j} \neq \emptyset$ for $3 \leq j \leq t$. For $3 \leq j \leq t$, exchanging A_2, A_j implies that $A_{1j} \subseteq N$. Since $t \geq 4$, at least one of $A_{13}, \ldots, A_{1t}$ is nonempty, say A_{13}; and so there exist vertices in $A_{13} \cap N, A_{23} \cap N$ that are nonadjacent. By exchanging A_1, A_3, it follows that $A_{11}, A_{21} \subseteq N$. But then v can be added to A_{32}, a contradiction. This proves (4).

(5) *There exist $j, j' \in \{1, \ldots, t\}$ such that $N \cap Z_1 \subseteq A_{1j}$ and $N \cap Z_2 \subseteq A_{2j'}$.*

For suppose that there exist $a_{1j} \in N \cap A_{1j}$ for $j = 1, 2$ say. Now either A_{31}, A_{22} are both nonempty, or A_{32}, A_{21} are both nonempty, and from the symmetry we may assume the former. Choose $a_{31} \in A_{31}$ nonadjacent to a_{11}, and choose $a_{22} \in A_{22}$ nonadjacent to a_{12}. By (4), $a_{31}, a_{22} \notin N$. Then v-a_{11}-a_{22}-a_{31}-a_{12}-v is an odd hole, a contradiction. This proves (5).

Let j, j' be as in (5). To show that v is small, it remains to show that $N \cap A_{1j}$ is complete to $N \cap A_{2j'}$. This is true from the construction if $j \neq j'$, and by (4) if $j = j'$. Thus v is small. This proves 3.2. ∎

3.3 *With notation as before, there is no big vertex.*

Proof. We begin with:

(1) *Every two big vertices are adjacent.*

For suppose that b_1, b_2 are nonadjacent big vertices. We may assume that b_1 is complete to $Z_2 \cup Z_3$ say. Suppose first that b_2 is not complete to $Z_1 \cup Z_2$; say b_2 is complete to $Z_3 \cup Z_1$. If there exists $j \in \{1, \dots, t\}$ such that $A_{3j} = \emptyset$, we can add b_1 to A_{1j} and b_2 to A_{2j}, contrary to the maximality of W. Thus $A_{31}, \dots, A_{3t}$ are all nonempty. But then we may define $A_{1,t+1} = \{b_1\}$, $A_{2,t+1} = \{b_2\}$, and $A_{3,t+1} = \emptyset$, contrary to the maximality of W. This proves that b_2 is complete to $Z_2 \cup Z_3$.

Since G has no even pair, there is an odd induced path $b_1 = p_1\text{-}\cdots\text{-}p_k = b_2$ in G. Since none of $p_2, \dots, p_{k-1}$ is adjacent to both b_1, b_2, it follows that none of them is in $Z_2 \cup Z_3$. Moreover, $p_2, p_{k-1} \notin Z_1$, since b_1, b_2 are anticomplete to Z_1. Thus $p_2, p_{k-1} \in V(G) \setminus W$. Now p_2 is not complete to $Z_2 \cup Z_3$ since $Z_2 \cup Z_3$ is not stable and G is K_4-free; and since p_2, b_2 are nonadjacent, and we have already seen that every two nonadjacent big vertices have the same neighbours in W, it follows that p_2 is not big. Similarly p_{k-1} is not big. But by 2.4, one of p_2, p_{k-1} is complete to Z_2 and the other to Z_3, which is impossible since they are both small. This proves (1).

Now to complete the proof of 3.3, suppose that b is a big vertex. Thus $b \notin W$, and we may assume that b is complete to $Z_2 \cup Z_3$ and anticomplete to Z_1. At least one of A_{11}, A_{12} is nonempty, say A_{11}; choose $a_{11} \in A_{11}$. Since G has no even pair, there is an odd induced path $b = p_1\text{-}p_2\text{-}\cdots\text{-}p_k = a_{11}$. Thus p_1, p_k are both complete to the anticonnected set $(Z_2 \cup Z_3) \setminus A_1$; and this anticonnected set is not stable since $t \geq 4$. Since k is even it follows that none of $p_1, \dots, p_k$ belong to $(Z_2 \cup Z_3) \setminus A_1$; and so by 2.4, one of p_2, p_{k-1} is complete to $Z_2 \setminus A_1$, and the other to $Z_3 \setminus A_1$. Since p_k is adjacent to a_{11} and not to b, it follows that p_k is not in W; by (1) p_k is not big; and since p_k is complete to one of $Z_2 \setminus A_1, Z_3 \setminus A_1$ it follows that p_k is not small, contrary to 3.2. This proves 3.3. ∎

3.4 *For $1 \leq i \leq 3$ and $1 \leq j \leq t$, $|A_{ij}| \leq 1$.*

Proof. Suppose that $u, v \in A_{11}$ say are distinct. Then u, v both have the same colour, and so are nonadjacent. Moreover, u, v are both complete to $(Z_2 \cup Z_3) \setminus A_1$, and there is an odd induced path $u = p_1\text{-}\cdots\text{-}p_k = v$ between u, v; so 2.4 implies that one of p_2, p_{k-1} has colour 3 and is complete to $Z_2 \setminus A_1$, and the other has colour 3 and is complete to $Z_3 \setminus A_1$; let the first be p_2, say. Consequently p_2 is not small; by 3.3 it is not big; and so by 3.2 it belongs to W. Since it has colour 3 and has a neighbour and a nonneighbour in A_{11}, we deduce that $p_2 \in A_{31}$ and $A_{21} = \emptyset$. But similarly $p_{k-1} \in A_{21}$, a contradiction. This proves 3.4. ∎

Henceforth we denote the unique member of A_{ij} by a_{ij} (when it exists) without further explanation. Note that 3.4 implies that A_{ij} is anticomplete to $A_{i'j}$ for all distinct $i, i' \in \{1, 2, 3\}$ and all $j \in \{1, \dots, t\}$.

3.5 *If X is a connected set of small vertices and $u, v \in W$ both have neighbours in X, then u, v are adjacent.*

Proof. Suppose not, and choose nonadjacent $u, v \in W$ and a connected set X as in the claim, with $|X|$ minimum. It follows that X is the interior of an induced path $u\text{-}p_1\text{-}\cdots\text{-}p_k\text{-}v$ between u, v. Since the members of X are small, 3.4 implies that $k \geq 2$.

(1) *For some $i \in \{1, 2, 3\}$ there are two members of Z_i with neighbours in X.*

For suppose not. We may therefore assume that $u = a_{11}$ and $v = a_{21}$, and $(Z_1 \cup Z_2) \setminus A_1$ is anticomplete to X. Suppose first that k is even. At most one vertex in $Z_3 \setminus A_1$ has a neighbour in X; choose $w \in Z_3 \setminus A_1$ with no neighbour in X, and then w-u-p_1-$\cdots$-p_k-v-w is an odd hole. So k is odd. Now either a_{12}, a_{23} both exist, or a_{22}, a_{13} both exist, and from the symmetry we may assume the first; and then u-p_1-$\cdots$-p_k-v-a_{12}-a_{23}-u is an odd hole, a contradiction. This proves (1).

In view of (1) we may assume that $u = a_{11}$ and $v = a_{12}$. From the minimality of X (and since $k \geq 2$) it follows that A_{1j} is anticomplete to X for $3 \leq j \leq t$.

(2) *It is impossible that k is even.*

For suppose k is even. We may assume that $a_{23}, a34$ exist. If a_{24} exists, then $\{a_{23}, a_{34}, a_{24}\}$ is anticonnected and not stable, and complete to u, v; so by 2.4 each of a_{23}, a_{34}, a_{24} is adjacent to one of p_1, p_k, contradicting that p_1, p_k are small. So $A_{24} = \emptyset$, amd similarly $A_{33} = \emptyset$. Hence a_{21}, a_{22}, a_{13} exist, and since

$$a_{13}\text{-}a_{22}\text{-}u\text{-}p_1\text{-}\cdots\text{-}p_k\text{-}v\text{-}a_{21}\text{-}a_{13}$$

is not an odd hole, one of a_{21}, a_{22} has a neighbour in X, say a_{21}. Since u is adjacent to p_1 and nonadjacent to a_{21}, we deduce that a_{21} is adjacent to p_k from the minimality of X. Since a_{23} also has a neighbour in X and a_{21}, a_{23} are nonadjacent, the minimality of X implies that a_{23} is adjacent to p_1. But similarly a_{34} is adjacent to one of p_1, p_k, contradicting that p_1, p_k are both small. This proves (2).

(3) *It is impossible that k is odd.*

For suppose that k is odd. We may assume that a_{21}, a_{32} exist, and since

$$u\text{-}p_1\text{-}\cdots\text{-}p_k\text{-}v\text{-}a_{21}\text{-}a_{32}\text{-}u$$

is not an odd hole, we deduce that at least one of a_{21}, a_{32} has a neighbour in X, say a_{21}. Since u is adjacent to p_1, the minimality of X implies that p_k is the only neighbour of a_{21} in X. If also a_{32} has a neighbour in X, then similarly p_1 is its only neighbour, and then a_{21}-p_1-$\cdots$-p_k-a_{32}-a_{21} is an odd hole, a contradiction. Thus a_{32} is anticomplete to X. We may assume that a_{13}, a_{24} exist. From the minimality of X, a_{13} is anticomplete to X. If also a_{24} is anticomplete to X, then

$$u\text{-}p_1\text{-}\cdots\text{-}p_k\text{-}a_{21}\text{-}a_{13}\text{-}a_{24}\text{-}u$$

is an odd hole. So a_{24} has a neighbour in X. From the minimality of X, its only neighbour in X is p_1; but then v-a_{24}-p_1-$\cdots$-p_k-v is an odd hole. This proves (3).

From (2) and (3), we have a contradiction. This proves 3.5. ∎

Proof of 3.1.

Let G be a K_4-free Berge graph with no clique cutset and no even pair, that contains a trampoline. Define the sets A_{ij} as before. If there is a small vertex, let X be a maximal connected set of small vertices; then by 3.5 and 3.3, the set of vertices in W with a neighbour in X is a clique cutset, a contradiction. Thus there is no small vertex, and by 3.3 and 3.4 it follows that G is the complement of the line graph of a bipartite graph. This proves 3.1. ∎

4 Trapezes and trestles

We need four special graphs. Each has six vertices $a_1, a_2, b_1, b_2, c_1, c_2$, and in each the pairs a_1a_2, b_1b_2 and c_1c_2 are nonadjacent, and c_1, c_2 are both complete to $\{a_1, a_2, b_1, b_2\}$. Let F be the set of edges between $\{a_1, a_2\}$ and $\{b_1, b_2\}$; the four graphs differ only on the choice of F. We say such a graph is

- a *trapeze* if $F = \{a_1b_1, a_2b_2\}$.
- a *trestle* if $F = \{a_1b_1, a_2b_1, a_2b_2\}$
- an *extended 4-wheel* if $F = \{a_1b_1, a_2b_1\}$, and
- an *octahedron* if $F = \{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}$.

In this section we show that we can exclude these four kinds of subgraphs.

4.1 *Let G be a K_4-free Berge graph with no even pair, containing no trampoline. Then G does not contain a trapeze.*

Proof. Suppose that G contains a trapeze, with six vertices $a_1, b_1, a_2, b_2, c_1, c_2$ as above. Fix a three-colouring ϕ of G; then $\phi(c_1) = \phi(c_2)$, and we may assume that $\phi(c_1) = 3$, and $\phi(a_i) = 1$ and $\phi(b_i) = 2$ for $i = 1, 2$.

There is an odd induced path between c_1, c_2, since G has no even pair. For $i = 1, 2$, let d_i be the neighbour of c_i in this path. For $i = 1, 2$, let X_i be the set of common neighbours of c_1, c_2 that have colour i. Then $X_1 \cup X_2$ is anticonnected and not stable (since $a_1, a_2 \in X_1$ and $b_1, b_2 \in X_2$). Since c_1, c_2 are common neighbours of $X_1 \cup X_2$, we may assume by 2.4 that d_1 has colour 1 and is complete to X_2, and d_2 has colour 2 and is complete to X_1.

Now there is an odd induced path a_1-q_1-$\cdots$-q_k-a_2 between a_1, a_2. Since a_1, a_2 are common neighbours of $\{c_1, c_2, d_2\}$, we may assume by 2.4 (by exchanging a_1b_1 with a_2b_2 if necessary) that q_1 has colour 3 and is adjacent to d_2, and q_k has colour 2 and is complete to c_1, c_2. Moreover, $\{c_1, c_2, d_2\}$ includes a leap; and since the two vertices of the leap are nonadjacent and have different colours, it follows that the leap is $\{c_1, d_2\}$. Consequently c_1 is nonadjacent to $q_1, \ldots, q_{k-1}$, and d_2 is nonadjacent to $q_2, \ldots, q_k$. Since q_k is adjacent to a_2, c_1, c_2, it follows that $q_k \in X_2$, and so d_1 is adjacent to q_k.

Since b_1, q_k have the same colour, they are nonadjacent. Suppose that b_1 is nonadjacent to $q_1, \ldots, q_{k-1}$. Then b_1-a_1-q_1-$\cdots$-q_k is an odd path between common neighbours of $\{c_1, c_2, d_1\}$, and so by 2.4, it follows that d_1 is adjacent to q_{k-1} and not to $q_1, \ldots, q_{k-2}$. But then if d_1, d_2 are nonadjacent then

$$d_2\text{-}q_1\text{-}\cdots\text{-}q_{k-1}\text{-}d_1\text{-}c_1\text{-}a_2\text{-}d_2$$

is an odd hole, a contradiction; if d_1, d_2 are adjacent and $k \geq 4$ then d_2-q_1-$\cdots$-q_{k-1}-d_1-d_2 is an odd hole, a contradiction; and if d_1, d_2 are adjacent and $k = 2$ then the subgraph induced on $\{a_1, a_2, b_1, q_k, c_1, c_2, d_1, d_2, q_1\}$ is a trampoline, a contradiction. This proves that b_1 is adjacent to q_i for some $i \in \{1, \ldots, k-1\}$. Choose i minimum. From the hole

$$d_2\text{-}q_1\text{-}\cdots\text{-}q_i\text{-}b_1\text{-}c_1\text{-}a_2\text{-}d_2$$

it follows that i is even, and since k is even, we deduce that q_i, q_k are nonadjacent. Suppose that d_1 is anticomplete to $\{q_1, \ldots, q_i\}$. If d_1, d_2 are nonadjacent then

$$d_2\text{-}q_1\text{-}\cdots\text{-}q_i\text{-}b_1\text{-}d_1\text{-}q_k\text{-}a_2\text{-}d_2$$

is an odd hole, and if d_1, d_2 are adjacent then d_2-q_1-$\cdots$-q_i-b_1-d_1-d_2 is an odd hole, a contradiction. Thus d_1 is adjacent to one of $q_1, \ldots, q_i$. Since d_2-q_1-$\cdots$-q_i-b_1-c_2-d_2 is not an odd hole, c_2 is also adjacent to one of $q_1, \ldots, q_i$. Consequently there is an induced path between c_2 and d_1 with interior in $\{q_1, \ldots, q_i\}$, say R. But R can be completed to a hole via d_1-q_k-c_2 and via d_1-c_1-a_2-c_2, and one of these is an odd hole, a contradiction. This proves 4.1. ∎

4.2 *Let G be a K_4-free Berge graph with no even pair, containing no trampoline. Then G contains no trestle.*

Proof. (We remind the reader that all graphs in this paper are finite. This theorem in particular is false if we allow infinite graphs.) Let us say an *extended trestle* in G is a sequence $v_1, \ldots, v_n$ of distinct vertices, with $n \geq 8$, such that for $1 \leq i < j \leq n$, v_i and v_j are adjacent if $j - i \in \{1, 2, 4\}$, and they are nonadjacent if $j - i \notin \{1, 2, 4, 7\}$. Fix a three-colouring ϕ of G. By 4.1 it follows that G contains no trapeze. Suppose it contains a trestle.

(1) *G contains an extended trestle.*

For G contains a trestle, and so there are six vertices $v_2, \ldots, v_7$ in G such that v_2-v_4-v_5-v_7 is an induced path, and $\{v_3, v_6\}$ are complete to $\{v_2, v_4, v_5, v_7\}$, and there are no other edges among $v_2, \ldots, v_7$. We may assume that v_2, v_5 have colour 1, and v_3, v_6 have colour 2, and v_4, v_7 have colour 3. There is an odd induced path between v_3, v_6, say v_3-p_1-$\cdots$-p_k-v_6. Since v_3, v_6 are both complete to $\{v_2, v_4, v_5, v_7\}$, and the latter is anticonnected and not stable, we may assume from 2.4 and the symmetry that p_1 has colour 1 and is complete to $\{v_4, v_7\}$, and p_k has colour 3 and is complete to $\{v_2, v_5\}$. But then the sequence $p_k, v_2, \ldots, v_7, p_1$ is an extended trestle. This proves (1).

In view of (1) and the finiteness of G, we may choose an extended trestle $v_1, \ldots, v_n$ with n maximum. We may assume that:

(2) *For $1 \leq i \leq n$, $\phi(v_i) = n - i \mod 3$.*

For v_i, v_{i+1}, v_{i+2} are pairwise adjacent (for $1 \leq i \leq n-2$), and so are $v_{i+1}, v_{i+2}, v_{i+3}$ (for $i \leq n-3$), and so v_i, v_{i+3} have the same colour for $1 \leq i \leq n-3$. Thus for $1 \leq i < j \leq n$, if $j - i = 0 \mod 3$ then v_i, v_j have the same colour. Since we may assume that v_n has colour 3 and v_{n-1} has colour 1, the claim follows. This proves (2).

(3) *There is a vertex $v_{n+1} \neq v_1, \ldots, v_n$, with colour 2, adjacent to v_n, v_{n-1}, v_{n-3} and not to $v_{n-2}, v_{n-4}, v_{n-5}$.*

For there is an odd induced path v_{n-1}-p_1-$\cdots$-p_k-p_{n-4} between v_{n-1}, v_{n-4}. Since v_{n-1}, v_{n-4} are both complete to $\{v_n, v_{n-2}, v_{n-3}, v_{n-5}\}$, and the latter is anticonnected and not stable, it follows from 2.4 that one of p_1, p_k has colour 2 and is complete to $\{v_n, v_{n-3}\}$, and the other has colour

3 and is complete to $\{v_{n-2}, v_{n-5}\}$. Suppose that p_1 has colour 3. Then v_{n-2}, v_{n-5} are complete to $\{v_{n-1}, p_1, v_{n-4}, v_{n-6}\}$, and $v_{n-1}p_1$ and $v_{n-4}v_{n-6}$ are edges, and $\{v_{n-1}, p_1\}$ is anticomplete to $\{v_{n-4}, v_{n-6}\}$ (p_1 is not adjacent to v_{n-6} since they have the same colour). Thus G contains a trapeze, a contradiction. This proves that p_1 has colour 2, and is adjacent to v_n, v_{n-3}, and not to v_{n-4}.

Define $v_{n+1} = p_1$; we will show that v_{n+1} satisfies the claim. Since v_{n+1} has colour 2, it is nonadjacent to v_{n-2}, v_{n-5}. Thus, in summary, v_{n+1} is adjacent to v_n, v_{n-1}, v_{n-3} and not to $v_{n-2}, v_{n-4}, v_{n-5}$. Suppose that $v_{n+1} = v_i$ for some $i \in \{1, \ldots, n\}$. Then $n - i = 2 \mod 3$ by (2), since v_{n+1} has colour 2; and $i \neq n-5, n-2$ since v_{n+1} does not belong to the carriage above. Thus $i \leq n-8$. But the only neighbours of v_n in $\{v_1, \ldots, v_{n-1}\}$ are $v_{n-1}, v_{n-2}, v_{n-4}$ and possibly v_{n-7}, a contradiction. Thus v_{n+1} is different from $v_1, \ldots, v_n$. This proves (3).

(4) *v_{n+1} is nonadjacent to v_{n-7}.*

For suppose v_{n+1}, v_{n-7} are adjacent. Since v_{n+1}-v_{n-7}-v_{n-6}-v_{n-2}-v_{n-1}-v_{n+1} is not a 5-hole, it follows that v_{n+1} is adjacent to v_{n-6}. But then $v_{n+1}v_{n-7}$ and $v_{n-2}v_{n-4}$ are edges, and $\{v_{n+1}, v_{n-7}\}$ is anticomplete to $\{v_{n-2}, v_{n-4}\}$, and v_{n-3}, v_{n-6} are both complete to $\{v_{n-7}, v_{n-4}, v_{n-2}, v_{n+1}\}$, and hence G contains a trapeze, a contradiction. This proves (4).

(5) *v_{n+1} is nonadjacent to v_i for $1 \leq i \leq n-8$.*

For suppose that some $i \in \{1, \ldots, n-8\}$ is adjacent to v_{n+1}, and choose i maximum. There are cases depending on the value of $n - i$ modulo 6. By (2), $n + 1 - i \neq 0, 3 \mod 6$ since v_{n+1}, v_i are adjacent and therefore have different colours; so $n - i$ is one of $0, 1, 3$ or $4 \mod 6$. If $n - i = 0 \mod 6$, then $i \leq n - 12$, and

$$v_{n+1}\text{-}v_i\text{-}v_{i+4}\text{-}v_{i+6}\text{-}v_{i+10}\text{-}\cdots\text{-}v_{n-12}\text{-}v_{n-8}\text{-}v_{n-4}\text{-}v_n\text{-}v_{n+1}$$

is an odd hole. If $n - i = 1 \mod 6$, then $i \leq n - 13$, and

$$v_{n+1}\text{-}v_i\text{-}v_{i+4}\text{-}v_{i+6}\text{-}v_{i+10}\text{-}\cdots\text{-}v_{n-13}\text{-}v_{n-9}\text{-}v_{n-5}\text{-}v_{n-1}\text{-}v_{n+1}$$

is an odd hole. If $n - i = 3 \mod 6$ then $i \leq n - 9$, and

$$v_{n+1}\text{-}v_i\text{-}v_{i+2}\text{-}v_{i+6}\text{-}v_{i+8}\text{-}\cdots\text{-}v_{n-13}\text{-}v_{n-9}\text{-}v_{n-5}\text{-}v_{n-4}\text{-}v_n\text{-}v_{n+1}$$

is an odd hole. If $n - i = 4 \mod 6$, then $i \leq n - 10$, and

$$v_{n+1}\text{-}v_i\text{-}v_{i+2}\text{-}v_{i+6}\text{-}v_{i+8}\text{-}\cdots\text{-}v_{n-14}\text{-}v_{n-10}\text{-}v_{n-8}\text{-}v_{n-4}\text{-}v_n\text{-}v_{n+1}$$

is an odd hole. This proves (5).

But from (5), $v_1, \ldots, v_{n+1}$ is an extended trestle, contrary to the maximality of n. This proves 4.2. ∎

4.3 *Let G be a K_4-free Berge graph with no even pair, containing no trampoline. Then G contains no extended 4-wheel.*

Proof. Suppose that G contains an extended 4-wheel labelled as above. Fix a three-colouring of G; then we may assume that a_1, a_2 have colour 1, and b_1 has colour 2, and c_1, c_2 have colour 3 (and b_2 has colour 1 or 2). There is an odd induced path c_1-p_1-$\cdots$-p_k-c_2, and since c_1, c_2 are complete to $\{a_1, b_1, a_2, b_2\}$, and the latter is anticonnected and not stable, we may assume from 2.4 and the symmetry between c_1, c_2 that p_1 has colour 2 and is adjacent to a_1, a_2. Since p_1 is not adjacent to c_2, it follows that $p_k \neq b_1$, and c_1-b_1-c_2-p_2 is an induced path; but a_1, a_2 are both complete to the vertex set of this path, and so G contains a trestle, contrary to 4.2. This proves 4.3. ∎

4.4 *Let G be a K_4-free Berge graph with no even pair, containing no trampoline. Then G contains no octahedron.*

Proof. Suppose it does; consequently we may choose three disjoint stable sets $A_1, A_2, A_3 \subseteq V(G)$, pairwise complete and each with cardinality at least two. Choose them with maximal union. Fix a three-colouring of G, and we may assume that the vertices in A_i have colour i for $i = 1, 2, 3$.

(1) *Every A_1-complete vertex belongs to $A_2 \cup A_3$.*

For suppose that v is A_1-complete and $v \notin A_2 \cup A_3$. Since G is K_4-free, v is anticomplete to at least one of A_2, A_3, say A_3. If v is A_2-complete then we may add v to A_3, contrary to the maximality of $A_1 \cup A_2 \cup A_3$. Thus v has a nonneighbour in A_2. Choose $a_1, a_1' \in A_1$. There is an odd induced path a_1-p_1-$\cdots$-p_k-a_1' between a_1, a_1'; and since $A_2 \cup A_3 \cup \{v\}$ is anticonnected and not stable, we may assume by 2.4 that p_1 is complete to A_3 and anticomplete to A_2. Choose distinct $a_3, a_3' \in A_3$, and choose $a_2 \in A_2$. Then p_1-a_1-a_2-a_1' is an induced path, and a_3, a_3' are complete to its vertex set, so G contains a trestle, contrary to 4.2. This proves (1).

Now since $|A_2| \geq 2$, there is an odd induced path with both ends in A_2; choose such a path with minimum length, say a_2-p_1-$\cdots$-p_k-a_2', where $a_2, a_2' \in A_2$. From the minimality of k, it follows that none of $p_1, \ldots, p_k$ is in A_2; and none of them is in $A_1 \cup A_3$ since none of them is adjacent to both a_1, a_1'. Consequently none of $p_1, \ldots, p_k$ is complete to A_1, by (1). By 2.3, every vertex in A_1 is adjacent to one of p_1, p_k; and similarly so is every vertex in A_3. Since p_1, p_k do not have colour 2 (because they have neighbours in A_2), we may assume that p_k has colour 1. Consequently p_k is anticomplete to A_1, and so p_1 is complete to A_1, contrary to (1). This proves 4.4. ∎

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