K_4-free graphs with no odd holes

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Abstract

All K_4-free graphs with no odd hole and no odd antihole are three-colourable, but what about K_4-free graphs with no odd hole? They are not necessarily three-colourable, but we prove a conjecture of Ding that they are all four-colourable. This is a consequence of a decomposition theorem for such graphs; we prove that every such graph either has no odd antihole, or belongs to one of two explicitly-constructed classes, or admits a decomposition.

1 Introduction

All graphs in this paper are finite and have no loops or multiple edges. A *hole* in a graph is an induced cycle of length at least four, and an *antihole* is an induced subgraph isomorphic to the complement of a cycle of length at least four. As usual we denote by $\chi(G)$ the chromatic number of G and by $\omega(G)$ the clique number. Recently [2] we were able to prove the "strong perfect graph conjecture" of Berge [1], the following:

1.1 *If a graph G has no odd holes and no odd antiholes, then $\chi(G) = \omega(G)$.*

A graph is said to be *perfect* if every induced subgraph has chromatic number equal to clique number; and so 1.1 implies that graphs with no odd holes or antiholes are perfect. Since odd holes and odd antiholes do not satisfy the conclusion of 1.1, none of them can be left out from the hypothesis of the theorem. However, it is possible that the hypotheses can be relaxed and we could still deduce that $\chi(G)$ is bounded by some function of $\omega(G)$, where the function does not depend on G, of course. Gyarfás [4] conjectured:

1.2 Conjecture. *For each integer $k \geq 0$ there is a least integer $g(k)$ such that every graph G with no odd hole and with $\omega(G) = k$ satisfies $\chi(G) \leq g(k)$.*

Clearly $g(i) = i$ for $i \leq 2$, but $g(3) \geq 4$ since the complement of a cycle of length seven is not 3-colourable, and Ding [3] conjectured that $g(3) = 4$. We prove Ding's conjecture. For a graph F we say that a graph is F-free if it has no induced subgraph isomorphic to F, and for a family $\mathcal{F}$ we say that a graph is $\mathcal{F}$-free if it has no subgraph isomorphic to a member of $\mathcal{F}$. Our main result is:

1.3 *Every K_4-free graph with no odd hole is 4-colourable.*

We deduce 1.3 from a decomposition theorem 3.1 for K_4-free graphs with no odd holes. The decomposition theorem requires a number of definitions before it can be formulated, and so we postpone its statement until Section 3. Let us remark that our decomposition theorem is not completely satisfactory in that it only applies to non-perfect graphs. It would be nice to have an analogous result for K_4-free perfect graphs, but that remains open.

There is a conjectured strengthening of 1.2 due to C. T. Hoàng and C. McDiarmid [5], the following.

1.4 Conjecture. *For every graph G with no odd hole and with at least two vertices, there is a partition (V_1, V_2) of $V(G)$ such that every maximum clique of G meets both V_1 and V_2.*

Our result 1.3 shows that 1.4 is true for all K_4-free graphs.

2 Harmonious cutsets

The *length* of a path or cycle is the number of edges in it, and we say a path or cycle is *even* or *odd* depending whether its length is even or odd. If $A, B \subseteq V(G)$ are disjoint, we say that A is *complete* to B if every vertex in A is adjacent to every vertex in B, and A is *anticomplete* to B if no vertex in A is adjacent to a vertex in B. (We say a vertex v is complete to a set B if $\{v\}$ is complete to

B, and the same for anticomplete.) If $X \subseteq V(G)$, $G|X$ denotes the subgraph of G induced on X, and $G \setminus X$ denotes the graph obtained by deleting X, that is, the subgraph induced on $V(G) \setminus X$. A *cutset* in a graph G is a set $X \subseteq V(G)$ such that $G \setminus X$ has at least two components. A cutset X is *harmonious* if X can be partitioned into disjoint sets $X_1, X_2, \ldots, X_k$ such that:

- for all $i, j \in \{1, 2, \ldots, k\}$, if P is an induced path with one end in X_i and the other end in X_j, then P is even if $i = j$ and odd otherwise, and
- if $k \geq 3$, then $X_1, \ldots, X_k$ are pairwise complete to each other.

Thus the first condition implies that each X_i is a stable set.

2.1 *Let X be a harmonious cutset in a graph G, let C_1, C_2 be a partition of $V(G) \setminus X$ into two nonempty sets that are anticomplete to each other, and for $t = 1, 2$ let G_t be $G|(C_t \cup X)$. If G_1, G_2 have no odd hole then G has no odd hole.*

We omit the (easy) proof since we do not need the result, which is included just to motivate the concept of harmonious cutset. 2.1 implies that if we understand all graphs with no odd hole and no harmonious cutset, then by repeatedly piecing them together on harmonious cutsets we can "construct" all graphs with no odd hole. However, this does not really count as a construction. If G, X, G_1, G_2 are as above, and we wish to view this as a construction of G from things that we already understand, we need to know not only that G_1, G_2 have no odd hole, but that the cutset X of G will be harmonious. This can be stated as a property of the pairs (G_1, X) and (G_2, X); but we need to have constructions for the pairs (G_1, X) and (G_2, X), not just for G_1, G_2, before we can claim to have a construction for G. We have not yet resolved this issue.

Let us return to the colouring problem.

2.2 *Let X be a harmonious cutset in a graph G, let C_1, C_2 be a partition of $V(G) \setminus X$ into two nonempty sets that are anticomplete to each other, and for $t = 1, 2$ let G_t be $G|(C_t \cup X)$. If G_1, G_2 are 4-colourable then G is 4-colourable.*

Proof. Let $X_1, X_2, \ldots, X_k$ be as in the definition of a harmonious cutset. By hypothesis both G_1 and G_2 are 4-colourable. Let $t \in \{1, 2\}$, and let c be a 4-colouring of G_t (using colours $1, 2, 3, 4$, and so c is a map into $\{1, 2, 3, 4\}$). We say that a vertex $v \in X$ is *c-compliant* if $c(v) = i$, where i is the index such that $v \in X_i$. We claim

(1) *G_t has a 4-colouring c_t such that every vertex of X is c_t-compliant.*

To prove this claim let c be a 4-colouring of G_t that maximizes the number of c-compliant vertices. We will show that c is as desired. To this end, suppose for a contradiction that $v \in X$ is not c-compliant, say $v \in X_i$ and $c(v) = j$, where $i \neq j$. Let H be the component containing v of the subgraph of G_t induced by vertices coloured i or j. We claim that no vertex of H in X is c-compliant. For let $u \in V(H) \cap X$, and let P be an induced path of H joining u, v. Now $c(u) = c(v)$ (that is, $c(u) = j$) if and only if P has even length, from the definition of H; but P has even length if and only if u, v belong to the same member of $\{X_1, \ldots, X_k\}$ (that is, $u \in X_i$), since X is harmonious. Consequently $c(u) = j$ if and only if $u \in X_i$, and so u is not c-compliant. This proves that no vertex of H in X is c-compliant.

Let c' be the colouring obtained from c by swapping the colours i and j for every vertex of H. Then v is c'-compliant. Since no vertex of H is c-compliant, it follows that more vertices in X are c'-compliant than are c-compliant, contrary to our choice of c. This proves (1).

Now the colourings c_1 and c_2 can be combined to produce a 4-colouring of G, as desired. ∎

It is easy to see that a graph with a harmonious cutset has either what is called an even pair, an odd pair, or a clique cutset (we omit the definitions of these standard terms, which we do not need any more), and one could eliminate the use of 2.2 by using these three things instead, and three corresponding theorems from the literature. The interested reader can easily work this out.

What follows is a lemma to make it easier to prove that a given cutset is harmonious.

2.3 *Let G be a graph with no odd hole, let X be a cutset in G, and let $X_1, \dots, X_k$ be a partition of X into stable sets, such that if $k \geq 3$ then the sets $X_1, \dots, X_k$ are pairwise complete. Suppose that for all nonadjacent $a, b \in X$, there is an induced path P joining a, b, with interior in $V(G) \setminus X$, such that P is even if some X_i contains both a, b, and odd otherwise. Then G admits a harmonious cutset.*

Proof. If some proper subset X' of X is a cutset, then X' and the sets $X' \cap X_i$ $(1 \leq i \leq k)$ satisfy the hypotheses of the theorem and we may replace X by X'. We may therefore assume that X is a minimal cutset. Let $C_1, \dots, C_t$ be the vertex sets of the components of $G \setminus X$; thus every member of X has a neighbour in C_i for all i with $1 \leq i \leq t$.

(1) *Let $a, b \in X$. Every induced path between a, b with no internal vertex in X is even if some X_i contains both a, b, and odd otherwise.*

For we may assume that a, b are nonadjacent, since $X_1, \dots, X_k$ are stable. By hypothesis, there is an induced path P joining a, b, with interior in $V(G) \setminus X$, such that P is even if some X_i contains both a, b, and odd otherwise. Since no internal vertex of P is in X, the interior of P is contained in one of $C_1, \dots, C_t$, say C_1. Now $t > 1$, so a, b both have neighbours in C_2 from the minimality of X, and hence there is an induced path Q joining a, b with interior in C_2. Since the union of P, Q is an even hole, it follows that Q, P have the same parity. Now let R be any path with ends a, b and with interior disjoint from X. Then there exists $j \in \{1, \dots, t\}$ such that the interior of R is a subset of C_j. Consequently one of $P \cup R, Q \cup R$ is a hole, and since P, Q have the same parity, it follows that R also has the same parity. This proves (1).

Let P be an induced path with both ends in X, and let its ends be v, v' say, where $v \in X_i$ and $v' \in X_{i'}$. We must show that P is even if and only if $i = i'$. We proceed by induction on the length of P. If no internal vertex of P is in X, the claim follows from (1), so we may assume that there is an internal vertex u of P in X_j say. Let Q, Q' be the subpaths of P between v, u and between u, v' respectively. From the inductive hypothesis, Q is even if and only if $i = j$, and Q' is even if and only if $i' = j$. Now P is odd if and only if exactly one of Q, Q' is odd, that is, if exactly one of i, i' is equal to j. It follows that if P is odd then $i \neq i'$. For the converse, suppose that P is even; then either both i, i' are equal to j or both i, i' are different from j. In the first case $i = i'$ as required. In the second case, if $k \leq 2$ then $i = i'$ as required, and if $k \geq 3$ then $i = i'$ since v, v' are nonadjacent. This proves that P is even if and only if $i = i'$, and so proves 2.3. ∎

For X as in 2.3, we call $(X_1, \dots, X_k)$ the "corresponding colouring".

3 The main theorem

In this section we state the main result. If $A, B \subseteq V(G)$ are disjoint, we say that A, B are *linked* if every member of A has a neighbour in B, and every member of B has a neighbour in A. We need to define two kinds of graphs.

We say a graph G is of T_{11} *type* if there is a partition of $V(G)$ into eleven nonempty stable subsets $W_1, \ldots, W_{11}$, such that (with index arithmetic modulo 11) for $1 \leq i \leq 11$, W_i is anticomplete to $W_{i+1} \cup W_{i+2}$ and complete to $W_{i+3} \cup W_{i+4} \cup W_{i+5}$.

We say that G is of *heptagram type* if there is a partition of $V(G)$ into fourteen stable subsets $W_1, \ldots, W_7, Y_1, \ldots, Y_7$, where $W_1, \ldots, W_7$ are nonempty but $Y_1, \ldots, Y_7$ may be empty, satisfying the following (with index arithmetic modulo 7).

1. For $1 \leq i \leq 7$, W_i is anticomplete to W_{i+3}.

2. For $2 \leq i \leq 7$, W_i is complete to W_{i+2}, and W_1, W_3 are linked.

3. For $i \in \{3, 4, 6, 7\}$, W_i is complete to W_{i+1}; for $i = 1, 2, 5$, W_i, W_{i+1} are linked.

4. If $v_i \in W_i$ for $i = 1, 2, 3$, and v_2 is adjacent to v_1, v_3, then v_1 is adjacent to v_3.

5. If $v_i \in W_i$ for $i = 1, 2, 3$, and v_2 is nonadjacent to v_1, v_3, then v_1 is nonadjacent to v_3.

6. For $1 \leq i \leq 7$, every vertex in Y_i has a neighbour in each of W_i, W_{i+3}, W_{i-3} and has no neighbour in $W_{i+1}, W_{i+2}, W_{i-1}, W_{i-2}$.

7. For $1 \leq i \leq 7$ and each $y \in Y_i$, let N_j be the set of neighbours of y in W_j for $j = i, i+3, i-3$; then N_{i+3} is complete to N_{i-3}, and N_{i+3} is anticomplete to $W_{i-3} \setminus N_{i-3}$, and N_{i-3} is anticomplete to $W_{i+3} \setminus N_{i+3}$, and N_i is complete to $W_{i+1} \cup W_{i-1}$.

8. For $1 \leq i \leq 7$, Y_i is complete to Y_{i+1} and anticomplete to $Y_{i+2} \cup Y_{i+3}$.

9. For $1 \leq i \leq 7$, if Y_i is not complete to $W_{i+3} \cup W_{i-3}$ then $W_{i-3} \cup W_{i+3}$ is complete to $W_{i-2} \cup W_{i+2}$, and $Y_{i-1}, Y_{i+1}, Y_{i-3}, Y_{i+3}$ are all empty.

10. For $1 \leq i \leq 7$, at least one of Y_i, Y_{i+1}, Y_{i+2} is empty.

It is questionable whether the description given above of graphs of heptagram type really counts as an explicit construction. We return to this in the final section, where we give a more complicated but more explicit construction of the same class of graphs. We leave the reader to check that graphs of T_{11} type and graphs of heptagram type have no odd hole, are K_4-free, do not admit a harmonious cutset, and contain an antihole of length seven. (To check that graphs of heptagram type have no odd hole, we suggest the use of theorem 5.2 below.) Our main result is the converse, the following.

3.1 *Let G be a K_4-free graph with no odd hole, and with no harmonious cutset, containing an antihole of length seven. Then G is either of heptagram type or of T_{11} type.*

This has the corollary mentioned earlier:

3.2 *Every K_4-free graph with no odd hole is four-colourable.*

Proof. Let G be a K_4-free graph with no odd hole; we prove by induction on $|V(G)|$ that G is four-colourable. If G admits a harmonious cutset, the result follows from 2.2 and the inductive hypothesis. If G contains no antihole of length seven, then it contains no odd hole or antihole, and therefore is perfect by 1.1 (or Tucker's earlier result [6]), and so is three-colourable. We may therefore assume that G satisfies the hypotheses of 3.1; but then, by 3.1, G is of one of the two types listed. It is easy to check that graphs of these two types are four-colourable. This proves 3.2. ∎

4 Graphs of T_{11} type

Let $X_1, \dots, X_n$ be disjoint subsets of $V(G)$; by an *induced path of the form* $X_1\text{-}\cdots\text{-}X_n$ we mean an induced path $x_1\text{-}\cdots\text{-}x_n$ where $x_i \in X_i$ for $1 \le i \le n$ (and when some X_i is a singleton, say $\{x\}$, we sometimes write x instead of X_i). We use analogous terminology for holes. Let T_{11} be the graph with vertex set $w_1, \dots, w_{11}$, in which for $1 \le i \le 11$, w_i is nonadjacent to w_{i+1}, w_{i+2} and adjacent to $w_{i+3}, w_{i+4}, w_{i+5}$. (Throughout this section, index arithmetic is modulo 11.) In this section we show the following.

4.1 *Let G be a K_4-free graph with no odd holes and no harmonious cutset. If G contains T_{11} as an induced subgraph then G is of T_{11} type.*

Proof. Since G contains T_{11} as an induced subgraph, we may choose eleven nonempty stable sets $W_1, \dots, W_{11}$, pairwise disjoint, such that for $1 \le i \le 11$, W_i is anticomplete to W_{i+1}, W_{i+2} and complete to $W_{i+3}, W_{i+4}, W_{i+5}$. Choose them with maximal union, and let their union be W.

(1) *If $v \in V(G) \setminus W$, and $a, b \in W$ are adjacent to v, then either a, b are adjacent or $a, b \in W_i$ for some $i \in \{1, \dots, 11\}$.*

For suppose not; then from the symmetry we may assume that $a \in W_1$ and $b \in W_2 \cup W_3$. Let N be the set of neighbours of v in W. By a *v-path* we mean an induced path in $G|W$ with both ends in N and with no internal vertices in N. Since G has no odd hole, every odd v-path has length one. For $1 \le i \le 11$ choose $w_i \in W_i$. Suppose first that $b \in W_2$. Since there is no v-path of the form $a\text{-}W_4\text{-}W_{10}\text{-}b$, it follows that N includes one of W_4, W_{10}; and from the symmetry we may assume that $W_4 \subseteq N$. Since no three members of N are pairwise adjacent (since G is K_4-free) it follows that N is disjoint from W_7, W_8, W_9. Since there is no v-path of the form $b\text{-}(W_5 \cup W_6)\text{-}(W_1 \cup W_{11})\text{-}w_4$ it follows that N includes one of $W_5 \cup W_6, W_1 \cup W_{11}$, and we claim we may assume the second. For if $w_5 \notin N$ then the second statement holds anyway; and if $w_5 \in N$ then $W_2 \subseteq N$ (since there is no v-path of the form $w_4\text{-}W_7\text{-}W_2\text{-}w_5$), and so there is symmetry between the pairs (W_1, W_2) and (W_5, W_4), and we may assume that $W_1 \cup W_{11} \subseteq N$ because of this symmetry. Thus, we may assume that $W_1 \cup W_{11} \subseteq N$. Since there is no v-path of the form $a\text{-}w_9\text{-}W_3\text{-}w_{11}$ it follows that $W_3 \subseteq N$; and since N includes no triangle within $W_3 \cup W_6 \cup W_{11}$, it follows that $N \cap W_6 = \emptyset$. There is no v-path of the form $a\text{-}W_5\text{-}W_{10}\text{-}w_3$, so N includes one of W_5, W_{10}, and from the symmetry exchanging W_1, W_3 we may assume that $W_5 \subseteq N$. Since N includes no triangle within $W_2 \cup W_5 \cup W_{10}$, it follows that N is disjoint from W_{10}. Since there is no v-path of the form $w_4\text{-}w_7\text{-}W_2\text{-}w_5$, we deduce that $W_2 \subseteq N$, and so N is the union of W_i for $i = 11, 1, 2, 3, 4, 5$. But then v can be added to W_8, contradicting the maximality of W.

This proves that $b \notin W_2$, and more generally for $1 \leq i \leq 11$, N is disjoint from one of W_i, W_{i+1}. Now $b \in W_3$, and so N is disjoint from W_{11}, W_2, W_4. But then there is a v-path a-w_4-w_{11}-b, a contradiction. This proves (1).

(2) *Let $X \subseteq V(G) \setminus W$ such that $G|X$ is connected. If $a, b \in W$ have neighbours in X then either a, b are adjacent or $a, b \in W_i$ for some $i \in \{1, \ldots, 11\}$.*

For suppose not, and choose X minimal such that some such pair a, b violates (2). It follows that there is an induced path a-x_1-$\cdots$-x_k-b where $X = \{x_1, \ldots, x_k\}$. By (1), a, b have no common neighbour in X, and so $k \geq 2$. From the symmetry we may assume that $a \in W_1$ and $b \in W_2 \cup W_3$. For $1 \leq i \leq 11$ choose $w_i \in W_i$, choosing $w_i \in \{a, b\}$ if possible. For $1 \leq i \leq 11$, the minimality of X implies that not all of w_i, w_{i+1}, w_{i+2} have neighbours in X, since then some two of them would be joined by a proper subpath of x_1-$\cdots$-x_k. In particular, not all of w_6, w_7, w_8 have neighbours in X; say w_j does not, where $6 \leq j \leq 8$. Consequently w_j-a-x_1-$\cdots$-x_k-b-w_j is a hole, and therefore k is odd.

Suppose first that $b \in W_2$. Since a-x_1-$\cdots$-x_k-b-w_{10}-w_4-a is not an odd hole, we may assume from the symmetry that w_4 has a neighbour in X. From the minimality of X, w_4 is adjacent to x_1 and to no other member of X. Since not all w_{11}, w_1, w_2 have neighbours in X, it follows that w_{11} has no neighbour in X. Since not all w_4, w_5, w_6 have neighbours in X, there exists $i \in \{5, 6\}$ such that w_i has no neighbour in X. But then w_4-x_1-$\cdots$-x_k-b-w_i-w_{11}-w_4 is an odd hole, a contradiction.

Thus $b \notin W_2$, so $b \in W_3$, and more generally for $1 \leq i \leq 11$ at least one of W_i, W_{i+1} is anticomplete to X. In particular, w_{11}, w_2, w_4 have no neighbour in X. Thus a-x_1-$\cdots$-x_k-b-w_{11}-w_4-a is an odd hole, a contradiction. This proves (2).

Suppose that $W \neq V(G)$; we shall prove that G admits a harmonious cutset. Choose $C \subseteq V(G) \setminus W$ maximal such that $G|C$ is connected. Let N be the set of vertices in W with neighbours in C. By (2) (and since $11/4 < 3$), $N \cap W_i$ is nonempty for at most three values of $i \in \{1, \ldots, 11\}$, and $N \cap W_i$ is complete to $N \cap W_j$ for all distinct $i, j \in \{1, \ldots, 11\}$. Thus by 2.3 it suffices to show that if $a, b \in N \cap W_1$ then there is an even path joining a, b with interior in $W \setminus N$. But a, b have a common neighbour in W_j for $j = 4, 5$, and not both these belong to N by (2). This completes the proof of 4.1. ∎

5 Heptagrams

In view of 4.1, to prove 3.1 it suffices to prove it for $\{K_4, T_{11}\}$-free graphs, and that is the main goal of the remainder of the paper.

If a graph G contains an antihole of length seven, then the vertices of that antihole can be numbered $w_1, w_2, \ldots, w_7$ in such a way that w_i is adjacent to w_j if and only if $|i - j| \in \{1, 2, 5, 6\}$. This motivates the following definition. We say that $W = (W_1, W_2, \ldots, W_7)$ is a *heptagram* in G if (here and later index arithmetic is modulo 7)

(S1) the sets $W_1, W_2, \ldots, W_7 \subseteq V(G)$ are disjoint, nonempty, and stable,

(S2) for $1 \leq i \leq 7$, W_i is anticomplete to $W_{i+3} \cup W_{i+4}$

(S3) for $1 \le i \le 7$, the sets W_i, W_{i+1}, W_{i+2} are pairwise linked

(S4) if $u \in W_{i-1}$, $v \in W_i$, $w \in W_{i+1}$ and v is adjacent to both u and w, then u is adjacent to w,

(S5) if $u \in W_{i-1}$, $v \in W_i$, $w \in W_{i+1}$ and v is non-adjacent to both u and w, then u is non-adjacent to w, and

(S6) if $u \in W_{i-1}$, $v \in W_i$, $w \in W_{i+1}$, $x \in W_{i+2}$, u is adjacent to w and v is adjacent to x, then either u is adjacent to v or w is adjacent to x.

If $W = (W_1, \dots, W_7)$ is a heptagram in G, we also use W to denote the set $W_1 \cup \dots \cup W_7$. This mild abuse of notation should cause no confusion.

Let us explain briefly where these conditions came from. It is clear that (S1)–(S3) are designed to mimic the edge-structure of the antihole on seven vertices, but (S4)–(S6) are less natural. They arose from the following consideration. Let $(W_1, \dots, W_7)$ satisfy (S1)–(S3), in a graph G. One can check that if (S4)–(S6) are also satisfied, then $G|W$ has no odd hole (to prove this, use 5.3 below); and also the converse holds, that is, if $G|W$ has no odd hole then (S4)–(S6) hold, provided all the graphs $G|W_i \cup W_{i+1}$ are connected.

Our strategy to prove 3.1 is to choose a heptagram W in G with W maximal, and to analyze how the remainder of G attaches to W. But first, in this section we study the internal structure of a heptagram. We begin with:

5.1 *Let $(W_1, W_2, \dots, W_7)$ be a heptagram in a graph G. For $1 \le i \le 7$, if W_i is complete to W_{i+1}, then W_i is complete to W_{i+2} and W_{i-1} is complete to W_{i+1}.*

Proof. Let $u \in W_i$ and $w \in W_{i+2}$, and let $v \in W_{i+1}$ be a neighbour of w. (This exists by (S3).) Since W_i is complete to W_{i+1}, it follows that v is adjacent to both u, w; and so u is adjacent to w by (S4). This proves that W_i is complete to W_{i+2}. The second assertion follows by symmetry. This proves 5.1. ▮

5.2 *Let $(W_1, W_2, \dots, W_7)$ be a heptagram in a graph G. For $1 \le i \le 7$ either W_i is complete to W_{i+1} or W_{i+2} is complete to W_{i+3}.*

Proof. From the symmetry we may assume $i = 1$.

(1) *Let $w_i \in W_i$ for $i = 1, 3, 4$. Then w_3 is adjacent to one of w_1, w_4.*

For suppose not. By (S3), w_1 has a neighbour $w_2 \in W_2$; by (S4), w_2, w_3 are nonadjacent, and so by (S5), w_2, w_4 are nonadjacent. By (S3) again, w_2 has a neighbour $n_3 \in W_3$; by (S4), w_1, n_3 are adjacent, and by (S4) again, n_3, w_4 are nonadjacent. Again by (S3), w_4 has a neighbour $n_2 \in W_2$; by (S5), n_2, w_3 are adjacent, and so by (S4), n_2, w_1 are nonadjacent. But then w_1, n_2, n_3, w_4 violate (S6). This proves (1).

To prove the theorem, suppose that $w_i \in W_i$ for $1 \le i \le 4$, say, and w_1, w_2 are nonadjacent, and w_3, w_4 are nonadjacent. By (1), w_1, w_3 are adjacent, and similarly so are w_2, w_4; but then (S6) is violated. This proves 5.2. ▮

5.3 *Let $(W_1, W_2, \dots, W_7)$ be a heptagram in a graph G. Then there exists $t \in \{1, \dots, 7\}$ such that W_{j-1} is complete to W_{j+1} for all $j \in \{1, \dots, 7\} \setminus \{t\}$, and W_j is complete to W_{j+1} for all $j \in \{t-3, t-2, t+1, t+2\}$. Consequently, for all $i \in \{1, \dots, 7\}$, if $u \in W_{i-2}$ and $v \in W_{i+2}$, then*

- *u, v have common neighbours in W_{i-3}, in W_i and in W_{i+3}, and*
- *there is a path of the form u-W_{i-1}-W_{i+1}-v.*

Proof. The first assertion follows from 5.2 and 5.1, and the others follow from this and (S3). This proves 5.3. ∎

6 Y-vertices

Until the end of section 8, where we complete the proof of 3.1, G is a $\{K_4, T_{11}\}$-free graph with no odd hole, containing an antihole of length seven. Consequently there is a heptagram in G, say $W = (W_1, \dots, W_7)$; and let us choose the heptagram with $W_1 \cup \dots \cup W_7$ maximal. (We call this the "maximality" of W.) Again, W is fixed until the end of section 8.

We say that $y \in V(G) \setminus W$ is a *Y-vertex* or a *Y-vertex of type t* if the following hold, where N_i denotes the set of neighbours of y in W_i for $1 \le i \le 7$:

- N_t, N_{t+3}, N_{t-3} are nonempty, and $N_i = \emptyset$ for $i = t-2, t-1, t+1, t+2$
- N_{t-3} is complete to N_{t+3}, and N_{t-3} is anticomplete to $W_{t+3} \setminus N_{t+3}$, and N_{t+3} is anticomplete to $W_{t-3} \setminus N_{t-3}$
- N_t is complete to $W_{t+1} \cup W_{t+2} \cup W_{t-1} \cup W_{t-2}$

The main result of this section is the following:

6.1 *Let $v \in V(G) \setminus W$. Then one of the following holds:*

- *v is a Y-vertex, or*
- *let N be the set of neighbours of v in W; then $N \cap W_i$ is nonempty for at most two values of $i \in \{1, \dots, 7\}$, and if there are two such values, i and j say, then $j \in \{i-2, i-1, i+1, i+2\}$ and $N \cap W_i$ is complete to $N \cap W_j$.*

Proof. Let $N_i = N \cap W_i$ and $M_i = W_i \setminus N_i$ for $1 \le i \le 7$. Let

$$I = \{i \in \{1, \dots, 7\} : N_i \neq \emptyset\}.$$

By a *v-path* we mean an induced path of $G|W$ such that its ends are in N and its internal vertices are not in N. Since G has no odd hole, every odd v-path has length one. Since G is K_4-free, no three members of N are pairwise adjacent (briefly, N is triangle-free).

(1) *For $1 \le i \le 7$, not all $i, i+1, i+2, i+3$ belong to I.*

For suppose that $1, 2, 3, 4 \in I$ say, and choose $n_i \in N_i$ for $1 \le i \le 4$. By 5.2, either n_1, n_2 are adjacent or n_3, n_4 are adjacent, and we may assume the first by the symmetry. Since N is triangle-free, $\{n_1, n_2, n_3\}$ is not a triangle, and so (S4) implies that n_2, n_3 are nonadjacent. By 5.2 W_1 is complete to W_7, so by 5.1 W_2 is complete to W_7; and so $N_7 = \emptyset$ since N is triangle-free; and by 5.2 again, W_4 is complete to W_5. Choose $w_7 \in W_7$ adjacent to n_2; and choose $n_5 \in W_5$ and $w_6 \in W_6$, both adjacent to w_7. By 5.3, n_3, n_5 are adjacent, and since n_3-n_5-w_7-n_2 is not a v-path, it follows that $n_5 \in N_5$. Since $N_2, N_3, N_4, N_5 \neq \emptyset$, the argument earlier in this paragraph implies that n_3, n_4 are nonadjacent, and $N_6 = \emptyset$. Now n_3 is nonadjacent to both n_2, n_4, and so (S5) implies that n_2, n_4 are nonadjacent. By 5.3, n_4-w_6-w_7-n_2 is a v-path, a contradiction. This proves (1).

(2) *$|I| \leq 4$.*

For (1) implies that $|I| \leq 5$; suppose that $|I| = 5$. From (1) again we may assume that $I = \{1, 2, 4, 5, 7\}$. Choose $n_1 \in N_1$. If n_1 has a neighbour in N_2 and one in N_7, then by (S4) there is a triangle in N, a contradiction. Thus we may assume that n_1 is anticomplete to N_2. By 5.2, W_3 is complete to W_4, and W_6 to W_7. Choose $n_2 \in N_2$. If n_2 has a neighbour $w_1 \in M_1$, then since W_1 is complete to W_6 by 5.1, there is a v-path of the form n_2-w_1-W_6-n_1, a contradiction. This proves that n_2 is anticomplete to M_1. Choose $n_1' \in W_1$ adjacent to n_2; it follows that $n_1' \in N_1$. Since $n_1' \in N_1$ and has a neighbour in N_2, it follows from our previous argument that n_1' is anticomplete to N_7. By 5.2, W_2 is complete to W_3, and W_5 to W_6. Choose $n_7 \in N_7$. Now n_1 has a neighbour in W_2, necessarily in M_2; let w_2 be such a neighbour. Similarly let $w_7 \in M_7$ be adjacent to n_1'. Choose $n_4 \in N_4$. If n_4 is anticomplete to N_5, then since W_5 is complete to W_7 by 5.1, and n_4 has a neighbour (say w_5) in W_5, n_4-w_5-w_7-n_5 is a v-path (where $n_5 \in N_5$), a contradiction. Thus we may choose $n_5 \in N_5$ adjacent to n_4. Choose $w_3 \in W_3$ and $w_6 \in W_6$. Now n_2, w_7 are adjacent by (S4). If n_2, n_7 are nonadjacent, then n_2-w_7-w_6-n_7 is a v-path, a contradiction. Thus n_2, n_7 are adjacent, and so by (S5), n_1, n_7 are adjacent. By (S4), n_7, w_2 are adjacent. By (S5), n_1', w_2 are adjacent, and similarly n_1, w_7 are adjacent. By (S4), w_7, w_2 are adjacent. But then the subgraph induced on

$$\{v, w_3, w_7, n_7, n_4, n_1', n_1, n_5, n_2, w_2, w_6\}$$

is isomorphic to T_{11} (and these eleven vertices are written in the appropriate order), a contradiction. This proves (2).

(3) *$|I| \leq 3$.*

For suppose not; then $|I| = 4$ by (2), and we may assume that $1, 4 \in I$. By 5.3, there is a path of the form N_1-W_7-W_5-N_4. Since this is not a v-path, it follows that one of $N_5, N_7 \neq \emptyset$, and from the symmetry we may assume that $5 \in I$. Suppose that $6 \in I$, and so $I = \{1, 4, 5, 6\}$. If N_4 is not complete to N_5 there is a v-path of the form N_5-W_7-W_2-N_4, a contradiction, so N_4 is complete to N_5. Choose $n_6 \in N_6$. Since N_4 is complete to N_5 and N is triangle-free, it follows from (S4) that n_6 has no neighbour in N_5; and consequently n_6 is adjacent to some $w_5 \in M_5$. But then by 5.3 there is a v-path of the form N_5-W_3-w_5-n_6, a contradiction. This proves that $6 \notin I$, and similarly $3 \notin I$, and so from the symmetry we may assume that $2 \in I$, and therefore $I = \{1, 2, 4, 5\}$.

In this case we will show that we can add v to W_3, forming a heptagram W', contrary to the maximality of W. Define $W_i' = W_i$ for $1 \leq i \leq 7$ with $i \neq 3$, and $W_3' = W_3 \cup \{v\}$; and let $W' = (W_1', \ldots, W_7')$. We must check that W' satisfies (S1)–(S6). The first three are clear. Since W

satisfies (S4)–(S6), in order to check that W' satisfies (S4)–(S6), it suffices from the symmetry to show that:

1. N_2 is complete to N_4
2. N_4 is anticomplete to M_5
3. M_2 is anticomplete to M_4
4. M_4 is complete to N_5
5. if $M_2 \neq \emptyset$ then N_4 is complete to N_5
6. every vertex in W_6 is either anticomplete to M_4 or complete to N_5.

Let us prove these statements. For the first, if $n_2 \in N_2$ and $n_4 \in N_4$ are nonadjacent, choose $w_i \in W_i$ for $i = 6, 7$, adjacent; then by 5.2, n_4, w_6 are adjacent and so are n_2, w_7, and therefore n_4-w_6-w_7-n_4 is a v-path, a contradiction.

For the second, suppose that $n_4 \in N_4$ is adjacent to $w_5 \in M_5$. Choose $n_1 \in N_1$ and $w_7 \in W_7$ adjacent to both n_1, w_5 (this is possible by 5.3); then n_4-w_5-w_7-n_1 is a v-path, a contradiction.

For the third statement, suppose that $w_2 \in M_2$ and $w_4 \in M_4$ are adjacent. Choose $n_1 \in N_1$ and $n_5 \in N_5$. Since n_1-w_2-w_4-n_5 is not a v-path, we may assume that n_1, w_2 are nonadjacent, and indeed w_2 has no neighbour in N_1. Choose $w_1 \in W_1$ adjacent to w_2 (necessarily in M_1), and choose $w_7 \in W_7$ adjacent to w_1. By (S4), w_2, w_7 are adjacent, and by (S5), n_1, w_7 are adjacent. Choose $n_4 \in N_4$; by 5.3, n_4, w_2 are adjacent, since w_2, n_1 are not adjacent. But then n_1-w_7-w_2-n_4 is a v-path, a contradiction.

For the fourth statement, suppose that $w_4 \in M_4$ and $n_5 \in N_5$ are nonadjacent. Choose $w_6 \in W_6$ adjacent to w_4; then (S5) implies that n_5, w_6 are adjacent. Choose $n_2 \in N_2$; by 5.3, n_2, w_4 are adjacent. But then n_2-w_4-w_6-n_5 is a v-path, a contradiction.

For the fifth statement, suppose that $w_2 \in M_2$, $n_4 \in N_4$ and $n_5 \in N_5$, where n_4, n_5 are nonadjacent. By 5.3, w_2, n_4 are adjacent. By 5.3, there exists $w_7 \in W_7$ adjacent to both w_2, n_5; but then n_4-w_2-w_7-n_5 is a v-path, a contradiction.

Finally, for the last statement, suppose that $w_6 \in W_6$ is adjacent to $w_4 \in M_4$ and nonadjacent to $n_5 \in N_5$. Choose $n_1 \in N_1$. By (S5), n_5, w_4 are adjacent, and by 5.3, w_6, n_1 are adjacent; but then n_1-w_6-w_4-n_5 is a v-path, a contradiction.

This proves that W' is a heptagram, contrary to the maximality of W. This completes the proof of (3).

(4) *If $|I| = 3$ then the first outcome of the theorem holds.*

For suppose first that $I = \{1, 2, 3\}$, and choose $n_i \in N_i$ for $i = 1, 2, 3$. Since N is triangle-free, we may assume from (S4) that n_1, n_2 are nonadjacent. Choose $w_4 \in W_4$ and $w_6 \in W_6$, adjacent; then by 5.3, n_2-w_4-w_6-n_1 is a v-path, a contradiction.

Thus I does not consist of three consecutive integers (modulo seven), and so we may assume that $1, 4 \in I$. Since there is no v-path of the form N_4-W_5-W_7-N_1, 5.3 implies one of N_5, N_7 is nonempty, and from the symmetry we may assume that the former. Thus $I = \{1, 4, 5\}$. By the same argument, N_4 is anticomplete to M_5, and N_5 is anticomplete to M_4. If N_4 is not complete to N_5, 5.3 implies that there is a v-path of the form N_5-W_7-W_2-N_4, a contradiction. Thus N_4 is complete to N_5. Suppose that N_1 is not complete to W_2, and choose $n_1 \in N_1$ and $w_2 \in W_2$, nonadjacent. Choose $w_7 \in W_7$ adjacent to w_2; then (S5) implies that n_1, w_7 are adjacent. But by 5.3, w_2, n_4 are adjacent, and so n_1-w_7-w_2-n_4 is a v-path, a contradiction. Thus N_1 is complete to W_2 and therefore to W_3, by (S4). Similarly N_1 is complete to W_7, W_6. But then v is a Y-vertex of type 1, and the first statement of the theorem holds. This proves (4).

(5) *If $|I| = 2$ then the second outcome of the theorem holds.*

For then we may assume that $I = \{1, t\}$ where $t \in \{2, 3, 4\}$. If $t = 4$, there is a v-path of the form N_4-W_5-W_7-N_1, a contradiction. Thus $t \in \{2, 3\}$. Suppose there exist $n_1 \in N_1$ and $n_t \in N_t$, nonadjacent. Choose $w_6 \in W_6$ adjacent to n_1. By 5.3, there exists $w_4 \in W_4$ adjacent to both n_t, w_6; but then n_1-w_6-w_4-n_7 is a v-path, a contradiction. Thus N_1 is complete to N_t and the second outcome of the theorem holds. This proves (5).

From (2)–(5), we may assume that $|I| \leq 1$; but then the second outcome of the theorem holds. This proves 6.1. ∎

7 V-vertices

Let $1 \leq t \leq 7$. A *tail*, or *tail of type t*, is an induced path v_1-$\cdots$-v_k with the following properties:

- $k \geq 1$ is odd, and $v_1, \ldots, v_k \in V(G) \setminus W$
- v_1 has a neighbour in W_{t-3} and a neighbour in W_{t+3}, and W_{t-3}, W_{t+3} are anticomplete to $\{v_2, \ldots, v_k\}$
- W_{t-1}, W_{t+1} and at least one of W_{t-2}, W_{t+2} are anticomplete to $\{v_1, \ldots, v_k\}$
- v_k has a neighbour in W_t, and W_t is anticomplete to $\{v_1, \ldots, v_{k-1}\}$
- for $j = t-3, t+3$ let N_j be the set of neighbours of v_1 in W_j; then N_{t-3} is complete to N_{t+3}, N_{t-3} is anticomplete to $W_{t+3} \setminus N_{t+3}$, and N_{t+3} is anticomplete to $W_{t-3} \setminus N_{t-3}$
- every neighbour of v_k in W_t is complete to each of $W_{t-2}, W_{t-1}, W_{t+1}, W_{t+2}$.

We see that every Y-vertex forms a 1-vertex path that is a tail of length zero, and for every tail of length zero, its unique vertex is a Y-vertex, by 6.1, and so we may regard tails as a generalization of Y-vertices. If v_1-$\cdots$-v_k is a tail, we say it is a *tail for* v_1. If $1 \leq t \leq 7$, a vertex $v \in V(G) \setminus W$ with neighbours in W_{t-3} and in W_{t+3}, and anticomplete to W_j for $j = t-2, t-1, t, t+1, t+2$, is called a *hat of type t*. If $v_1, \ldots, v_k$ is a tail of type t, and has length greater than zero, then v_1 is a hat of type t. We say a vertex $v \in V(G) \setminus W$ is a *V-vertex of type t* if there is a tail of type t for v. Thus, every V-vertex of type t is either a Y-vertex of type t or a hat of type t.

Before we go on, let us give some idea where we are going. If every vertex in $V(G) \setminus W$ is a V-vertex, then since every tail only contains one V-vertex it follows that every tail has length zero, and so every vertex in $V(G) \setminus W$ is a Y-vertex, and we shall deduce that the graph is of heptagram type. On the other hand, if some vertex in $V(G)\setminus W$ is not a V-vertex, we shall prove that G admits a harmonious cutset.

If $X \subseteq V(G)$, we define $N(X)$ to be the set of vertices in $V(G)\setminus X$ with a neighbour in X. Here is a nice property of tails:

7.1 *Let $X \subseteq V(G)\setminus W$, such that $G|X$ is connected and contains no tail of G. Then there exists $i \in \{1,\dots,7\}$ such that $N(X)\cap W \subseteq W_{i-1}\cup W_i \cup W_{i+1}$.*

Proof. Suppose this is false, and choose a minimal counterexample X. Consequently there exists $i \in \{1,\dots,7\}$ such that N_i, N_{i+3} are both not anticomplete to X, and we may therefore assume that $N(X)\cap W_1, N(X)\cap W_4 \neq \emptyset$. Choose a minimal path from W_4 to W_1 with interior in X, say n_4-v_1-$\cdots$-v_k-n_1. From the minimality of X, it follows that $X=\{v_1,\dots,v_k\}$, and from 6.1 it follows that $k>1$. From the minimality of X, W_1 is anticomplete to $\{v_1,\dots,v_{k-1}\}$, and W_4 is anticomplete to $\{v_2,\dots,v_k\}$. Suppose first that k is even. Then by 5.3, n_1, n_4 have a common neighbour $w_j \in W_j$ for $j=2,3,6$, and since G has no odd hole, it follows that w_2, w_3, w_6 each are adjacent to one of $v_1,\dots,v_k$. But each of v_1, v_k is nonadjacent to one of w_2, w_3, by 6.1, and so one of w_2, w_3 is joined to w_6 by a path with interior a proper subpath of $v_1,\dots,v_k$, contrary to the minimality of X. This proves that k is odd. Since there is no odd hole of the form

$$n_4\text{-}v_1\text{-}\cdots\text{-}v_k\text{-}n_1\text{-}W_7\text{-}W_5\text{-}n_4,$$

it follows that some vertex of $W_5\cup W_7$ is adjacent to one of $v_1,\dots,v_k$, and from the symmetry we may assume this vertex is in W_5. From the minimality of X, $\{v_2,\dots,v_k\}$ is anticomplete to W_5, and so v_1 a has neighbour in W_5. By 6.1, and since $G|X$ contains no tail of G and hence X contains no Y-vertex, it follows that v_1 is a hat of type 1. We will prove that $v_1,\dots,v_k$ is a tail.

From the minimality of $|X|$, W_2 and W_7 are both anticomplete to $\{v_1,\dots,v_{k-1}\}$. Suppose that v_k has a neighbour $n_2 \in W_2$ say. Then by 6.1, v_k is a hat of type 5, and so W_7 is anticomplete to X, and the minimality of X implies that W_6 is anticomplete to X. If n_2, n_4 are adjacent then n_4-v_1-$\cdots$-v_k-n_2-n_4 is an odd hole, and if n_2, n_4 are nonadjacent then there is an odd hole of the form

$$n_4\text{-}v_1\text{-}\cdots\text{-}v_k\text{-}n_2\text{-}W_7\text{-}W_6\text{-}n_4,$$

in either case a contradiction. This proves that v_k has no neighbour in W_2, and so X is anticomplete to W_2, and similarly to W_7. Now v_1 is anticomplete to both W_3, W_6, and from the minimality of X, at least one of W_3, W_6 is anticomplete to $X\setminus\{v_1\}$, and so at least one of W_3, W_6 is anticomplete to X. We have therefore verified that $v_1,\dots,v_k$ satisfies the first four conditions in the definition of a tail.

To verify the fifth condition, let N_i be the set of neighbours of v_1 in W_i for $i=4,5$. By 6.1, N_4 is complete to N_5. If $w_4 \in N_4$ is adjacent to some $w_5 \in W_5\setminus N_5$, then there is an odd hole of the form

$$w_4\text{-}v_1\text{-}\cdots\text{-}v_k\text{-}n_1\text{-}W_7\text{-}w_5\text{-}w_4,$$

a contradiction. Similarly N_5 is anticomplete to $W_4\setminus N_4$, and this verifies the fifth condition.

To verify the sixth and last condition, let $w_1 \in W_1$ be adjacent to v_k. If w_1 is nonadjacent to some $w_2 \in W_2$, choose $w_7 \in W_7$ adjacent to w_2; then (S5) implies that w_1, w_7 are adjacent, and so by 5.3 there is an odd hole

$$n_4\text{-}v_1\text{-}\cdots\text{-}v_k\text{-}w_1\text{-}w_7\text{-}w_2\text{-}n_4,$$

a contradiction. Thus w_1 is complete to W_2, and therefore to W_3 by (S4), and similarly to W_7, W_6. This verifies the sixth condition.

Consequently $v_1, \dots, v_k$ is a tail in $G|X$, a contradiction. Thus there is no such X. This proves 7.1. ∎

7.2 *Let U be the set of all vertices in $V(G) \setminus W$ that are not V-vertices. For $1 \le t \le 7$, there is no path x_1-$\cdots$-x_k in G satisfying the following:*

- *x_1 is either a hat or Y-vertex of type t*
- *$x_2, \dots, x_{k-1} \in U$*
- *$x_k \in V(G) \setminus W$ has a neighbour in $W_{t+1} \cup W_{t-1}$, and*
- *x_k is not a Y-vertex of type $t+1$ or $t-1$.*

Proof. For suppose there is, and choose k minimum such that for some t there is such a path. We may assume that $t = 1$, and x_1 is either a hat or a Y-vertex of type 1, and $x_k \in V(G) \setminus W$ has a neighbour in W_2, and $x_2, \dots, x_{k-1} \in U$, and x_k is not a Y-vertex of type 2 or 7. Let $X = \{x_1, \dots, x_k\}$. From the minimality of k, W_2, W_7 are both anticomplete to $X \setminus \{x_k\}$. Choose $w_2 \in W_2$ adjacent to x_k. Choose $w_4 \in W_4$ adjacent to x_1, and also adjacent to w_2 if possible. We claim that if x_1 is a V-vertex, then w_2, w_4 are adjacent; for if W_4 is complete to W_5 then x_1 is complete to W_4 (since x_1 is a V-vertex), and if W_4 is not complete to W_5 then W_4 is complete to W_2 by 5.3. In either case it follows that w_2, w_4 are adjacent.

(1) *$G|X$ contains a tail for x_k and a tail for x_1, and in particular x_1 and x_k are V-vertices.*

For suppose it contains no tail for x_k. By 7.1 applied to $X \setminus \{x_1\}$ we deduce that W_5, W_6 are anticomplete to $X \setminus \{x_1\}$. From 7.1, $G|X$ contains a tail of G, and since X contains no V-vertex except possibly x_1 and x_k, we may assume that $G|X$ contains a tail for x_1. Thus x_1 is a V-vertex, and so w_2, w_4 are adjacent. Moreover, there exists $j \le k$ such that x_1-$\cdots$-x_j is a tail for x_1. In particular, W_2 is anticomplete to $\{x_1, \dots, x_j\}$, and so $j < k$.

Suppose that k is even. Since there is no odd hole of the form

$$x_1\text{-}\cdots\text{-}x_k\text{-}w_2\text{-}W_7\text{-}W_5\text{-}x_1,$$

it follows that x_k has a neighbour $w_7 \in W_7$. But then W_4 is anticomplete to $X \setminus \{x_1\}$ by 7.1, and so there is an odd hole of the form

$$x_1\text{-}\cdots\text{-}x_k\text{-}w_7\text{-}W_6\text{-}W_4\text{-}x_1,$$

a contradiction.

Thus k is odd. Since x_1-$\cdots$-x_k-w_2-w_4-x_1 is not an odd hole, we deduce that w_4 has a neighbour in $X \setminus \{x_1\}$. From 7.1 applied to $X \setminus \{x_1\}$, we deduce that W_1, W_7 are anticomplete to $X \setminus \{x_1\}$, and therefore $j = 1$, and so x_1 is a Y-vertex. Choose $w_1 \in W_1$ adjacent to x_1. Then w_1 is complete to W_2 from the definition of a Y-vertex, and in particular w_1, w_2 are adjacent. But then x_1-$\cdots$-x_k-w_2-w_1-x_1 is an odd hole, a contradiction.

This proves that $G|X$ contains a tail for x_k. In particular, x_k is either a hat or Y-vertex of type s say, where $s = 5$ or 6, and x_1 has a neighbour in W_{s-1}. Thus there is symmetry between x_1 and x_k, and since we have shown that $G|X$ contains a tail for x_k, it follows that it also contains a tail for x_1. This proves (1).

(2) *x_k is not a V-vertex of type* 6.

For suppose it is; then it has neighbours in W_3. From the minimality of k, W_7 is anticomplete to X, and W_2 is anticomplete to $X \setminus \{x_k\}$, and W_5 is anticomplete to $X \setminus \{x_1\}$. Since there is no odd hole of the form

$$x_1\text{-}\cdots\text{-}x_k\text{-}w_2\text{-}W_7\text{-}W_5\text{-}x_1,$$

it follows that k is odd. Since $w_4\text{-}x_1\text{-}\cdots\text{-}x_k\text{-}w_2\text{-}w_4$ is not an odd hole, it follows that w_4 has a neighbour in $X \setminus \{x_1, x_k\}$. By 7.1 applied to $X \setminus \{x_1, x_k\}$, it follows that W_1 is anticomplete to $X \setminus \{x_1, x_k\}$. But by (1), some vertex $w_1 \in W_1$ has a neighbour in a tail for x_1 contained in $x_1\text{-}\cdots\text{-}x_k$; w_1 is not adjacent to x_k since x_k is a V-vertex of type 6; and so w_1 is adjacent to x_1 and to none of $x_2, \ldots, x_k$. Since x_1 is a V-vertex, w_1 is complete to W_2 and in particular adjacent to w_2. But then $w_1\text{-}x_1\text{-}\cdots\text{-}x_k\text{-}w_2\text{-}w_1$ is an odd hole, a contradiction. This proves (2).

(3) *x_k is not a V-vertex of type* 5.

For suppose it is, and so it has neighbours in W_1. By the minimality of k, W_4, W_6 are both anticomplete to $X \setminus \{x_1\}$. From the hole $x_1\text{-}\cdots\text{-}x_k\text{-}w_2\text{-}w_4\text{-}x_1$ we deduce that k is even. Choose $w_5 \in W_5$ adjacent to x_1, and $w_1 \in W_1$ adjacent to x_k. There is no odd hole of the form

$$x_1\text{-}\cdots\text{-}x_k\text{-}w_2\text{-}W_7\text{-}w_5\text{-}x_1,$$

and so w_5 is not anticomplete to $X \setminus \{x_1\}$. Similarly w_1 is not anticomplete to $X \setminus \{x_k\}$. By 7.1 applied to $X \setminus \{x_1, x_k\}$, not both w_1, w_5 have neighbours in $X \setminus \{x_1, x_k\}$; so from the symmetry we may assume that w_1 is adjacent to x_1 and not to $x_2, \ldots, x_{k-1}$. In particular x_1 is a Y-vertex. Since $x_1\text{-}\cdots\text{-}x_k\text{-}w_1\text{-}x_1$ is not an odd hole, it follows that $k = 2$, and so w_5 is adjacent to x_2; and therefore x_2 is also a Y-vertex.

Since x_1 is a Y-vertex, it has a neighbour in W_1 that is complete to W_2, and therefore $G|(W_1 \cup W_2)$ is connected. Since x_2 is a Y-vertex of type 5, its set of neighbours in $W_1 \cup W_2$ is the vertex set of a component of $G|(W_1 \cup W_2)$; and consequently x_2 is complete to $W_1 \cup W_2$, and W_1 is complete to W_2. Similarly x_1 is complete to $W_4 \cup W_5$ and W_4 is complete to W_5. We claim that x_1 is complete to W_1. For suppose that x_1 is nonadjacent to some $w_1 \in W_1$. Then there is an odd hole of the form

$$x_1\text{-}x_2\text{-}w_1\text{-}W_3\text{-}w_4\text{-}x_1,$$

a contradiction. This proves that x_1 is complete to W_1, and similarly x_2 is complete to W_5.

Define $W'_6 = W_6 \cup \{x_1\}$, and $W'_7 = W_7 \cup \{x_2\}$, and let $W' = (W_1, \ldots, W_5, W'_6, W'_7)$. We claim that W' is a heptagram. We must check (S1)–(S6), but they are all obvious and we leave this to the reader. Thus W' is a heptagram, contrary to the maximality of W. This proves (3).

Since x_k is a V-vertex with a neighbour in W_2, and is not a Y-vertex of type 2, (1)–(3) are contradictory. Consequently there is no such path $x_1, \ldots, x_k$. This proves 7.2. ∎

We conclude this section with some more lemmas about V-vertices.

7.3 *For $1 \leq i \leq 7$, no two V-vertices of type i are adjacent.*

Proof. Suppose that a, b are adjacent V-vertices of type 5 say. For $j = 1, 2$, let A_j, B_j be the set of neighbours in W_j of a, b respectively. Since G is K_4-free, and A_1 is complete to A_2, it follows that $A_1 \cup A_2 \neq B_1 \cup B_2$. Since $A_1 \cup A_2$ and $B_1 \cup B_2$ are both vertex sets of components of $G|(W_1 \cup W_2)$, we deduce that $A_j \cap B_j = \emptyset$ for $j = 1, 2$. Since G is K_4-free, and A_1 is complete to A_2, some vertex of $A_1 \cup A_2$ is not adjacent to b, and so $A_j \cap B_j = \emptyset$ for $j = 1, 2$. In particular, W_1 is not complete to W_2, and so W_1 is complete to W_6 by 5.3. Choose $a_1 \in A_1$, $b_1 \in B_1$, and $w_6 \in W_6$. Then w_6-a_1-a-b-b_1-w_6 is a hole of length five, a contradiction. This proves 7.3. ∎

7.4 *For $1 \leq i \leq 7$, if a is a V-vertex of type i, and a is not complete to $W_{i-3} \cup W_{i+3}$, then $W_{i-2} \cup W_{i+2}$ is complete to $W_{i-3} \cup W_{i+3}$.*

Proof. We may assume that $i = 5$ say. For $j = 1, 2$, let N_j be the set of neighbours of a in W_j, and let $M_j = W_i \setminus N_j$. Thus N_1 is complete to N_2, and N_1 is anticomplete to M_2, and M_1 is anticomplete to N_2. By hypothesis $M_1 \cup M_2 \neq \emptyset$, and since each member of M_1 has a neighbour in W_2 (and therefore in M_2), and vice versa, it follows that $M_1, M_2 \neq \emptyset$. Let $w_3 \in W_3$; we will show that w_3 is complete to $W_1 \cup W_2$. Suppose first that w_3 is anticomplete to M_1. Then w_3 has a neighbour in N_1, and so by (S5), w_3 is complete to M_2. Yet w_3 is anticomplete to M_1, and every vertex in M_2 has a neighbour in M_1, contrary to (S4). This proves that w_3 has a neighbour in M_1, say m_1. By (S5), since m_1 is anticomplete to N_2, it follows that w_3 is complete to N_2, and consequently complete to N_1, by (S4). Choose $n_1 \in N_1$; then since n_1 is anticomplete to M_2, (S5) implies that w_3 is complete to M_2, and hence to M_1, by (S4). This proves our claim that w_3 is complete to $W_1 \cup W_2$. We deduce that W_3 is complete to $W_1 \cup W_2$, and similarly so is W_7. This proves 7.4. ∎

7.5 *For $1 \leq i \leq 7$, if a is a V-vertex of type i, and b is a V-vertex of type $i+1$, then a, b are adjacent, and both are complete to W_{i-3}.*

Proof. We may assume that $i = 5$, say. Let a, b be V-vertices of types 5 and 6 respectively, and let their tails be S, T respectively. For $j = 1, 2$, let A_j be the set of neighbours of a in W_j, and for $j = 2, 3$, let B_j be the set of neighbours of b in W_j. By 7.4, at least one of a, b is complete to W_2.

(1) *a, b are adjacent.*

For suppose a, b are nonadjacent. Since at least one of a, b is complete to W_2, they have a common neighbour $w_2 \in W_2$. Suppose first that S, T are disjoint and there is no edge between them. Then there is an induced path Q of odd length between a, b of the form

$$a\text{-}S\text{-}W_5\text{-}W_6\text{-}T\text{-}b,$$

and we can complete it to an odd hole via b-w_2-a (note that w_2 has no neighbours in $S \cup T$ except a, b), a contradiction. Thus $V(S) \cup V(T)$ induces a connected subgraph of G.

Now by 7.2, a is anticomplete to $V(T) \setminus \{b\}$ and hence to $V(T)$, and similarly b is anticomplete to $V(S)$. Let $X = V(S) \cup V(T) \setminus \{a, b\}$. Since $V(S) \cup V(T)$ induces a connected subgraph of G, it follows that S, T both have positive length and $G|X$ is connected. Since X contains no V-vertex, and $N(X)$ has nonempty intersection with W_5, W_6, 7.2 implies that W_1, W_3 have no neighbours in X. Choose $a_1 \in A_1$, and $b_3 \in B_3$. Since w_2 is adjacent to a_1, b_3, (S4) implies that a_1, b_3 are adjacent. But there is an induced path Q between a, b with interior in X, and it can be completed to holes via b-w_2-a and via b-b_3-a_1-a, and one of these is odd, a contradiction. This proves (1).

Suppose there exists $a_2 \in W_2 \setminus B_2$, say. Thus b is not complete to W_2, and so by 7.4, a is complete to $W_1 \cup W_2$, and in particular $a_2 \in A_2$. Choose $b_3 \in B_3$; then a_2, b_3 are nonadjacent since b is a V-vertex. Choose $w_4 \in W_4$ adjacent to a_2 and therefore to b_3, by (S5). Then a-b-b_3-w_4-a_2-a is a hole of length five, a contradiction. This proves that $B_2 = W_2$, and similarly $A_2 = W_2$, and hence proves 7.5. ∎

7.6 *For $1 \le i \le 7$, if a is a V-vertex of type i, and a is not complete to $W_{i-3} \cup W_{i+3}$, then there is no V-vertex of type j for $j \in \{i-3, i-1, i+1, i+3\}$.*

Proof. We may assume that $i = 5$. By 7.5, there is no V-vertex of type 6, since a is not complete to W_2. Similarly there is none of type 4. Since no vertex in W_1 is complete to W_2, there is no V-vertex of type 1, and similarly there is none of type 2. This proves 7.6. ∎

8 Attachments of the remaining vertices

In this section we complete the proof of 3.1. The main part of this proof is the next result.

8.1 *Let U be the set of all vertices in $V(G) \setminus W$ that are not V-vertices. If $U \neq \emptyset$ then G admits a harmonious cutset.*

Proof. Suppose that $U \neq \emptyset$, and let $X \subseteq U$ be maximal such that $G|X$ is connected. Thus $X \neq \emptyset$, and $N(X) \subseteq V(G) \setminus U$. For $1 \le i \le 7$, let $N_i = N(X) \cap W_i$, let V_i be the set of all V-vertices of type i, and let $P_i = N(X) \cap V_i$. Let $I = \{i \in \{1, \dots, 7\} : N_i \neq \emptyset\}$ and $J = \{i \in \{1, \dots, 7\} : P_i \neq \emptyset\}$. By 7.1 there exists t such that $I \subseteq \{t-1, t, t+1\}$ and by 7.2 there exists t such that $J \subseteq \{t, t+1\}$.

(1) *If $1 \le i \le 7$ and $a, b \in N_i$ then there is an induced even path joining a, b with interior in X.*

Let Q be an induced path between a, b with interior in X. We will prove that Q is even. Let $a, b \in W_3$ say; thus $6, 7 \notin I$ and not both $1, 5 \in I$. From the symmetry we may assume that $1 \notin I$. If a, b have a common neighbour $w_1 \in W_1$ then the claim holds, since w_1-a-Q-b-w_1 is an even hole, so we assume not; and therefore W_1 is complete to W_7, by 5.3. Choose $a', b' \in W_1$ adjacent to a, b respectively. Thus a, b' are nonadjacent, and a', b are nonadjacent. Choose $w_7 \in W_7$; then w_7-b'-b-Q-a-a'-w_7 is a hole, and so Q is even. This proves (1).

(2) *For $1 \le i \le 7$, N_i is complete to N_{i+1}.*

For suppose that $i = 1$ say, and $n_1 \in N_1$ and $n_2 \in N_2$ are nonadjacent. Let Q be an induced path between n_1, n_2 with interior in X. By 7.1, $4, 6 \notin I$, and not both $3, 7 \in I$ and we may assume that $3 \notin I$. Choose $w_3 \in W_3$ adjacent to n_1; then (S5) implies that n_2, w_3 are adjacent. From the hole w_3-n_1-Q-n_2-w_3 we deduce that Q is even. But there is a hole of the form

$$n_1\text{-}Q\text{-}n_2\text{-}W_4\text{-}W_6\text{-}n_1,$$

and it is odd, a contradiction. This proves (2).

(3) *For $1 \leq i \leq 7$, every two members of P_i have the same neighbours in $W_{i-3} \cup W_{i+3}$, and P_i is complete to $N_{i-3} \cup N_{i+3}$.*

For we may assume that $i = 5$, say, and we may assume that $P_5 \neq \emptyset$. For $j = 1, 2$ let R_j be the set of vertices in W_j with a neighbour in $X \cup P_5$. We claim first that R_1 is complete to R_2. For suppose that $r_1 \in R_1$ and $r_2 \in R_2$ are nonadjacent, and let Q be a path joining r_1, r_2 with interior in $X \cup P_5$. It follows from 7.2 (since $P_5 \neq \emptyset$) that $X \cup P_5$ is anticomplete to W_4, W_6, and (by 7.1) anticomplete to at least one of W_3, W_7, say W_7. Consequently Q can be completed to a hole via r_2-W_7-r_1 and via r_2-W_4-W_6-r_1, and one of these is odd, a contradiction. This proves that R_1 is complete to R_2. Since each $p_5 \in P_5$ is a V-vertex, and therefore its neighbour set in $W_1 \cup W_2$ is the vertex set of a component of $G|(W_1 \cup W_2)$, it follows that each $p_5 \in P_5$ is complete to $R_1 \cup R_2$. This proves (3).

We wish to prove that G admits a harmonious cutset, and henceforth we assume (for a contradiction) that it does not.

(4) *$J \neq \emptyset$.*

For suppose that $J = \emptyset$; and we may assume that $I \subseteq \{1, 2, 3\}$. By (2), N_1 is complete to N_2, and N_2 to N_3, so if $N_2 \neq \emptyset$ then N_1 is complete to N_3 by (S4), and by (1) and 2.3 applied to the cutset $N_1 \cup N_2 \cup N_3$, we deduce that G admits a harmonious cutset, a contradiction. We may therefore assume that $N_2 = \emptyset$. Let $n_1 \in N_1$ and $n_3 \in N_3$ be nonadjacent; and let Q be a path between them with interior in X. By 5.3 there is a hole of the form n_1-Q-n_3-W_4-W_6-n_1, so Q is odd. Thus it again follows from (1) and 2.3 that G admits a harmonious cutset, a contradiction. This proves (4).

(5) *$I \cap J = \emptyset$.*

For suppose that $5 \in I \cap J$ say. By 7.1, $1, 2 \notin I$. Since $5 \in J$, 7.2 implies that $4, 6 \notin I$ and $1, 2, 3, 7 \notin J$. Since $5 \in I$, 7.2 implies that $4, 6 \notin J$. Consequently $I \subseteq \{3, 5, 7\}$ and $J = \{5\}$. By 7.1 not both $3, 7 \in I$, so we may assume that $I \subseteq \{3, 5\}$. We claim that $P_5 \cup N_3 \cup N_5$ is a harmonious cutset (where $(P_5 \cup N_3, N_5)$ is the corresponding colouring). We must check:

- if $a, b \in P_5 \cup N_3$ then there is an induced even path joining them with interior disjoint from $P_5 \cup N_3 \cup N_5$
- if $a, b \in N_5$ then there is an induced even path joining them with interior disjoint from $P_5 \cup N_3 \cup N_5$

- if $a \in P_5 \cup N_3$ and $b \in N_5$ then there is an induced odd path joining them with interior disjoint from $P_5 \cup N_3 \cup N_5$.

For the first, if $a, b \in N_3$ this follows from (1), so we may assume that $a \in P_5$. But then a, b have a common neighbour in W_2 by 7.4 and (3), and so the claim follows since $2 \notin I$. The second follows from (1). For the third, let $a \in P_5 \cup N_3$ and $b \in N_5$, and we may assume that a, b are nonadjacent; then there is an induced path of the form a-W_1-W_6-b satisfying the claim. Consequently, 2.3 implies that G admits a harmonious cutset, a contradiction. This proves (5).

In view of (5), since the same conclusion holds for every choice of X, we may therefore assume that every tail has length zero, and therefore every V-vertex is a Y-vertex.

(6) *There exists* $t \in \{1, \ldots, 7\}$ *such that* $I \subseteq \{t-1, t, t+1\}$ *and* $J \subseteq \{t-3, t+3\}$.

For we may assume that $5 \in J$ say. By (5), $5 \notin I$; and by 7.2, $4, 6 \notin I$; and not both $3, 7 \in I$, say $7 \notin I$. But 7.2 implies that $7, 1, 2, 3 \notin J$, and not both $4, 6 \in J$. If $4 \notin J$ then the claim holds with $t = 2$, so we may assume that $4 \in J$. By 7.2, $3 \notin I$, and now the claim holds with $t = 1$. This proves (6).

In view of (6) we henceforth assume that $I \subseteq \{1, 2, 3\}$ and $J \subseteq \{5, 6\}$. We claim that $N(X)$ is a cutset satisfying the hypotheses of 2.3, with corresponding colouring $(N_2, N_1 \cup P_6, N_3 \cup P_5)$. Certainly it is a cutset, and the three sets $N_2, N_1 \cup P_6, N_3 \cup P_5$ are pairwise complete, by (1), (3) and 7.5. It suffices therefore (by the symmetry) to show that

- if $a, b \in N_2$ then they are joined by an even induced path with interior disjoint from $N(X)$, and
- if $a, b \in N_1 \cup P_6$ then they are joined by an even induced path with interior disjoint from $N(X)$.

The first is proved in (1). For the second, if $a, b \in N_1$, then again the claim follows from (1). If $a, b \in P_6$, then since they both have neighbours in W_6 that are complete to W_5, there is an induced path between a, b of length two or four with interior in $W_5 \cup W_6$, satisfying the claim. If $a \in N_1$ and $b \in P_6$, then b has a neighbour $w_6 \in W_6$ that is complete to W_1, and so the path a-w_6-b satisfies the claim. This completes the proof of the two displayed statements above. Consequently, by 2.3, we deduce that G admits a harmonious cutset, a contradiction. This proves 8.1. ∎

Finally we can prove our main decomposition theorem.

Proof of 3.1. Let G be a K_4-free graph with no odd hole, and with no harmonious cutset, containing an antihole of length seven. By 4.1 we may assume that G is T_{11}-free. Choose a maximal heptagram $W = (W_1, \ldots, W_7)$. By 8.1, every vertex of G either belongs to W or is a V-vertex; and, since a tail contains only one V-vertex, it follows that every tail has length zero and so every V-vertex is a Y-vertex. For $1 \leq i \leq 7$ let Y_i be the set of all Y-vertices of type i. We need to check the ten conditions in the definition of heptagram type. The first is clear; and by 5.3 we may assume that the second and third hold by renumbering $W_1, \ldots, W_7$. Conditions 4–7 are clear. For the eighth condition, we see from 7.2 that Y_i is anticomplete to Y_{i+2}, Y_{i+3}, and from 7.5 that Y_i is complete to Y_{i+1}. The ninth condition follows from 7.4 and 7.6. For the tenth condition, suppose that $y_{i-1} \in Y_{i-1}$, and

$y_i \in Y_i$, and $y_{i+1} \in Y_{i+1}$. Thus y_i is adjacent to y_{i-1}, y_{i+1}, and y_{i-1}, y_{i+1} are nonadjacent. But then there is an odd hole of the form

$$y_i\text{-}y_{i+1}\text{-}W_{i+1}\text{-}W_{i-1}\text{-}y_{i-1}\text{-}y_i,$$

a contradiction. This proves 3.1. ∎

9 A more explicit construction

We hesitate to claim that our current definition of graphs of heptagram type is an "explicit construction"; it is certainly a helpful description, but the way the various hypotheses interact is not transparent. In this section we make it more explicit.

Let us say that G is of the *first heptagram type* if there exist $t \geq 1$ and a partition of $V(G)$ into ten stable sets

$$W_1, \ldots, W_7, Y_2, Y_4, Y_7$$

where Y_4, Y_7 may be empty but the other sets are nonempty, such that, with index arithmetic modulo seven:

- for $1 \leq i \leq 7$, W_i is complete to W_{i+2} and anticomplete to W_{i+3}
- for $i \in \{3, 4, 6, 7\}$, W_i is complete to W_{i+1}, and for $i = 1, 2$, W_i, W_{i+1} are linked; and every vertex in W_2 is complete to one of W_1, W_3
- for $i = 4, 7$, every vertex in Y_i is complete to $W_{i+3} \cup W_{i-3}$, has a neighbour in W_i, and has no neighbour in $W_{i+1}, W_{i+2}, W_{i-1}, W_{i-2}$
- Y_2, Y_4, Y_7 are pairwise anticomplete
- there is a nonempty subset $C \subseteq W_2$ such that C is complete to $W_1 \cup W_3$, and Y_2, C are linked, and Y_2 is anticomplete to $W_2 \setminus C$
- there exist partitions $M_0, \ldots, M_t$ of W_5 and $N_0, \ldots, N_t$ of W_6 where M_0, N_0 may be empty but the other sets are nonempty, such that for $1 \leq i \leq t$, M_i is complete to N_i, M_i is anticomplete to $W_6 \setminus N_i$, $W_5 \setminus M_i$ is anticomplete to N_i, and M_0, N_0 are linked (and consequently W_5, W_6 are linked)
- there is a partition $X_1, \ldots, X_t$ of Y_2 where $X_1, \ldots, X_t$ are all nonempty, such that for $1 \leq i \leq t$, X_i is complete to $M_i \cup N_i$, and anticomplete to each of

$$W_5 \setminus M_i, W_6 \setminus N_i, W_7, W_1, W_3, W_4.$$

That completes the definition of the first heptagram type. Before we define the second, we need another definition. Let us say a triple (W_1, W_2, W_3) of disjoint stable subsets of $V(G)$ is a *crescent* in G if the following hold:

- if $v_i \in W_i$ for $i = 1, 2, 3$, and v_2 is adjacent to v_1, v_3, then v_1 is adjacent to v_3
- if $v_i \in W_i$ for $i = 1, 2, 3$, and v_2 is nonadjacent to v_1, v_3, then v_1 is nonadjacent to v_3.

We say that G is of the *second heptagram type* if there is a partition of $V(G)$ into fourteen stable subsets $W_1, \ldots, W_7, Y_1, \ldots, Y_7$, where $W_1, \ldots, W_7$ are nonempty but $Y_1, \ldots, Y_7$ may be empty, such that (with index arithmetic modulo 7)

- for $1 \leq i \leq 7$, W_i is anticomplete to W_{i+3}
- for $2 \leq i \leq 7$, W_i is complete to W_{i+2}, and the sets W_1, W_2, W_3 are pairwise linked
- (W_1, W_2, W_3) is a crescent, and if W_1 is not complete to W_3 then $Y_2, Y_5, Y_6 = \emptyset$
- for $i \in \{3, 4, 6, 7\}$, W_i is complete to W_{i+1}; W_5, W_6 are linked
- for $1 \leq i \leq 7$, every vertex in Y_i is complete to $W_{i+3} \cup W_{i-3}$, has a neighbour in W_i, and has no neighbour in $W_{i+1}, W_{i+2}, W_{i-1}, W_{i-2}$
- for $1 \leq i \leq 7$, every vertex in W_i with a neighbour in Y_i is complete to $W_{i+1} \cup W_{i-1}$
- for $1 \leq i \leq 7$, Y_i is complete to Y_{i+1} and anticomplete to $Y_{i+2} \cup Y_{i+3}$
- for $1 \leq i \leq 7$, at least one of Y_i, Y_{i+1}, Y_{i+2} is empty.

Then we have:

9.1 *A graph is of heptagram type if and only if it is of either the first or second heptagram type.*

Proof. (A sketch, we leave the details to the reader.) Let G be of heptagram type, with notation as usual. Suppose first that some Y_i is not complete to $W_{i-3} \cup W_{i+3}$. Then we may assume that $i = 2$; by 7.6 Y_1, Y_3, Y_5, Y_6 are empty; and if C denotes the set of vertices in W_2 with neighbours in Y_2, then C is complete to $W_1 \cup W_3$ and so (S4) implies that W_1 is complete to W_3. By (S5), every vertex in W_2 is complete to one of W_1, W_3. By 7.4 and 5.3, W_j is complete to W_{j+1} for $j = 3, 7$, and so 5.1 implies that W_j is complete to W_{j+2} for all j. Every two vertices in Y_2 either have the same neighbours in $W_5 \cup W_6$ or disjoint neighbour sets in $W_5 \cup W_6$. It follows that G is of the first heptagram type. On the other hand, if each Y_i is complete to $W_{i-3} \cup W_{i+3}$, then G is of the second type. ∎

The two descriptions are more explicit than before, and the first heptagram type description is explicit and satisfactory; but there is still some degree of opacity in the description of the second type, due principally to the use of "crescents". We need to transform the definition of a crescent into something transparent.

Let W_1, W_2, W_3 be disjoint sets, and let f be a function from their union to the set of all integers, such that there do not exist $w_i \in W_i$ $(i = 1, 2, 3)$ with $f(w_1) = f(w_2) = f(w_3)$. We define a graph H_f with vertex set $W_1 \cup W_2 \cup W_3$ as follows. W_1, W_2, W_3 are stable in H_f. For $1 \leq i < j \leq 3$, and all $u \in W_i$ and $v \in W_j$, let u, v be adjacent if $f(u) < f(v)$, and nonadjacent if $f(u) > f(v)$; if $f(u) = f(v)$ then the adjacency between u and v is arbitrary. It is easy to check that (W_1, W_2, W_3) is a crescent in H_f. We prove in the next section that the converse is also true; if (W_1, W_2, W_3) is a crescent in G, then there is a function f as above such that $H_f = G|(W_1 \cup W_2 \cup W_3)$. This gives an explicit construction of all crescents, and hence can be used to convert our definition of the second heptagram type to an explicit construction.

10 Constructing a crescent

Let (W_1, W_2, W_3) be a partition of the vertex set of a graph G. We say the quadruple (G, W_1, W_2, W_3) is a *trident* if W_1, W_2, W_3 are stable, and for all choices of $w_i \in W_i$ for $1 \le i \le 3$, w_1, w_2, w_3 are not all pairwise adjacent and not all pairwise nonadjacent. How do we construct the most general trident? This will answer the crescent problem of the previous section, because if (W_1, W_2, W_3) is a partition of $V(G)$, and H is obtained from G by reversing all adjacencies between W_1 and W_3, then (G, W_1, W_2, W_3) is a trident if and only if (W_1, W_2, W_3) is a crescent in H.

Let W_1, W_2, W_3 be three disjoint sets with union W say, and let f be a function from W to the set of integers, such that there do not exist $w_i \in W_i$ $(1 \le i \le 3)$ satisfying $f(w_1) = f(w_2) = f(w_3)$. Let G be a graph with vertex set W defined as follows. For $1 \le i \le 3$, let $j = i + 1$ if $i < 3$ and $j = 1$ if $i = 3$; then for all $u \in W_i$ and $v \in W_j$, let u, v be adjacent if $f(u) < f(v)$, and nonadjacent if $f(u) > f(v)$, and either adjacent or nonadjacent if $f(u) = f(v)$. It is easy to check that (G, W_1, W_2, W_3) is a trident.

The result of this section is the converse: that every trident arises in this way from some appropriate function f. More precisely, let (G, W_1, W_2, W_3) be a trident. We say a function f from $V(G)$ to the set of integers is a *certificate* for this trident if it satisfies the following:

- there do not exist $w_1 \in W_1, w_2 \in W_2$ and $w_3 \in W_3$ such that $f(w_1) = f(w_2) = f(w_3)$, and
- for all $i, j \in \{1, 2, 3\}$ such that $j - i = 1$ modulo 3, and all $u \in W_i$ and $v \in W_j$, if $f(u) < f(v)$ then u, v are adjacent, and if $f(u) > f(v)$ then u, v are nonadjacent.

We shall prove:

10.1 *Every trident admits a certificate.*

Proof. Let (G, W_1, W_2, W_3) be a trident. We prove by induction on $|V(G)|$ that (G, W_1, W_2, W_3) admits a certificate. If $V(G) = \emptyset$ then the claim is true, so we may assume that $V(G) \neq \emptyset$. Below, all index arithmetic is modulo three.

(1) *There exists $i \in \{1, 2, 3\}$ and $v \in W_i$ such that v is adjacent to every member of W_{i+1}.*

For we may assume that $W_1 \neq \emptyset$. Choose $w_1 \in W_1$ with as many neighbours in W_2 as possible, and let N_2 be the set of vertices in W_2 adjacent to w_1. We may assume that some vertex w_2 is nonadjacent to w_1. Similarly we may assume that some vertex $w_3 \in W_3$ is nonadjacent to w_2. Since $\{w_1, w_2, w_3\}$ is not a stable set it follows that w_1, w_3 are adjacent. For $n_2 \in N_2$, since $\{w_1, n_2, w_3\}$ is not a clique, it follows that n_2, w_3 are nonadjacent, and so w_3 is anticomplete to N_2. We may assume that there exists $w_1' \in W_1$ nonadjacent to w_3. For $n_2 \in N_2 \cup \{w_2\}$, since $\{w_1', n_2, w_3\}$ is not a stable set, w_1' is adjacent to n_2, and so w_1' is complete to $N_2 \cup \{w_2\}$. But then w_1' has more neighbours in W_2 than w_1, contrary to the choice of w_1. This proves (1).

In view of (1), we may assume that some vertex in W_1 is complete to W_2. Let A_1 be the set of all vertices in W_1 that are complete to W_2, and let A_3 be the set of all vertices in W_3 with a neighbour in A_1. For each $a_3 \in A_3$, since a_3 is adjacent to some $a_1 \in A_1$, and a_1 is adjacent to each $w_2 \in W_2$, and $\{a_1, w_2, a_3\}$ is not a clique, it follows that a_3, w_2 are nonadjacent, and so A_3 is anticomplete to W_2. Also, for each $w_1 \in W_1 \setminus A_1$, since w_1 has a non-neighbour $w_2 \in W_2$, and each $a_3 \in A_3$ is nonadjacent to w_2, and $\{w_1, w_2, a_3\}$ is not a stable set, it follows that w_1, a_3 are adjacent, and so A_3 is complete to $W_1 \setminus A_1$. Let $W' = V(G) \setminus (A_1 \cup A_3)$; then

$$(G|W', W_1 \setminus A_1, W_2, W_3 \setminus A_3)$$

is a trident, and since $A_1 \neq \emptyset$, it follows from the inductive hypothesis that there is a certificate, f' say, for this trident. Choose an integer n such that $n < f'(v)$ for all $v \in W'$. Define a map f from W to the set of integers by setting $f(v) = n$ if $v \in A_1 \cup A_3$, and $f(v) = f'(v)$ otherwise. Then f is a certificate for (G, W_1, W_2, W_3) as required. This proves 10.1. ∎

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