Three-colourable perfect graphs without even pairs

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Abstract

We still do not know how to construct the “most general” perfect graph, not even the most general three-colourable perfect graph. But constructing all perfect graphs with no even pairs seems easier, and here we make a start on it; we construct all three-connected three-colourable perfect graphs without even pairs and without clique cutsets. They are all either line graphs of bipartite graphs, or complements of such graphs.
1 Introduction

A graph $G$ is perfect if for every induced subgraph $H$, the chromatic number of $H$ equals the size of the largest clique in $H$. A hole in a graph $G$ is an induced subgraph that is a cycle of length at least four, and an antihole in $G$ is an induced subgraph whose complement is a cycle of length at least four; and a hole or antihole is odd if it has an odd number of vertices. A graph is Berge if it has no odd hole and no odd antihole. Perfect graphs were introduced by Claude Berge in [1], where he proposed the “strong perfect graph conjecture”, now a theorem [5], the following:

1.1 A graph is perfect if and only if it is Berge.

The recognition problem for Berge-ness (and hence, by 1.1, for perfection) is also solved [2]:

1.2 There is an algorithm with running time $O(|V(G)|^9)$, to test if an input graph $G$ is Berge.

But neither of these results gives us a way to build the most general perfect graph. Ideally we would like a theorem that a graph is Berge if and only if it can be built from some well-understood class of building blocks, by piecing them together in a way that preserves Berge-ness. But we are far from such a theorem, and indeed we do not even know how to construct the most general Berge graph with no $K_4$ subgraph, which presumably should be an easier problem.

An even pair in $G$ is a pair $u, v$ of distinct vertices such that every induced path in $G$ between $u$ and $v$ has even length (the length of a path or cycle is the number of edges in it), and consequently $u, v$ are nonadjacent. As far as we know, finding an even pair does not give us a satisfactory way to construct our graph from a smaller graph; but still, an even pair $u, v$ in a Berge graph $G$ is quite a useful thing. For instance, if we identify $u, v$ the graph remains Berge with the same clique number, which is helpful if we are trying to optimally colour $G$, or prove that $G$ is perfect. (For a survey of recent work on even pairs see [6].) Thus, since finding a construction for all perfect graphs seems hopeless, what about finding a construction for all perfect graphs that have no even pairs? This problem, while still open, seems much more tractable.

In this paper we make a start on it; we construct all Berge graphs that have no $K_4$ subgraphs and have no even pairs. (Almost; it is helpful also to assume that the graph admits no clique cutset, and is 3-connected.) Let us say $G$ is $K_4$-free if it has no $K_4$ subgraph. A clique cutset in $G$ is a clique $C$ of $G$ such that $G \setminus V(C)$ is disconnected. We denote the complement of the graph $G$ by $\overline{G}$. Our main theorem is the following:

1.3 Let $G$ be a 3-connected $K_4$-free Berge graph with no even pair, and with no clique cutset. Then one of $G, \overline{G}$ is the line graph of a bipartite graph.
2 The Roussel-Rubio lemma

There was a result proved by Roussel and Rubio [7], that we used many times in the proof of 1.1, that will be important here. All graphs in this paper are finite, and without loops or parallel edges.

Let us say a subset \( X \subseteq V(G) \) is connected if the subgraph \( G[X] \) of \( G \) induced on \( X \) is connected, and anticonnected if \( \overline{G}[X] \) is connected. If \( X, Y \subseteq V(G) \), we say \( X \) is complete to \( Y \) or \( Y \)-complete if every vertex in \( X \) is adjacent to every vertex in \( Y \) (and similarly, we say a vertex \( v \) is complete to \( Y \) or \( Y \)-complete if \( \{v\} \) is complete to \( X \), and an edge \( uv \) is \( Y \)-complete if both \( u, v \) are \( Y \)-complete); and \( X \) is anticonnected to \( Y \) if \( X \) is complete to \( Y \) in \( \overline{G} \). If \( P \) is a path \( p_1 \cdots p_k \) say, with \( k > 1 \), its interior is the set \( \{p_2, \ldots, p_{k-1}\} \), and we denote this by \( P^* \).

If \( P \) is an induced path in \( G \) with vertices \( p_1 \cdots p_k \) in order, with \( k \geq 4 \), a leap for \( P \) is a pair \( \{x, y\} \) of nonadjacent vertices of \( G \setminus V(P) \) such that \( x \) is adjacent to \( p_1, p_2, p_k \), and \( y \) is adjacent to \( p_1, p_{k-1}, p_k \), and there are no other edges between \( \{x, y\} \) and \( V(P) \). The Roussel-Rubio lemma is the following:

2.1 Let \( G \) be a Berge graph, and let \( P \) be an induced path in \( G \) of odd length, at least five. Let \( X \subseteq V(G) \setminus V(P) \) be anticonnected, such that the ends of \( P \) are \( X \)-complete, and no edge of \( P \) is \( X \)-complete. Then \( X \) includes a leap for \( P \).

We also need a theorem of [5]:

2.2 Let \( G \) be Berge, let \( X \) be an anticonnected subset of \( V(G) \), and \( P \) be an induced path in \( G \setminus X \) with odd length, such that both ends of \( P \) are \( X \)-complete, and no edge of \( P \) is \( X \)-complete. Then every \( X \)-complete vertex of \( G \) has a neighbour in \( P^* \).

Next, we need:

2.3 Let \( G \) be a Berge graph, and let \( P \) be an induced path in \( G \) of odd length, with vertices \( p_1 \cdots p_k \) in order. Let \( X \subseteq V(G) \setminus V(P) \) be anticonnected, such that \( p_1, p_k \) are \( X \)-complete, and no edge of \( P \) is \( X \)-complete. Then \( k \geq 4 \) and every vertex in \( X \) is adjacent to one of \( p_2, p_{k-1} \).

**Proof.** Since no edge of \( P \) is \( X \)-complete it follows that \( k \geq 4 \). Suppose that \( z \in X \) is nonadjacent to both \( p_2, p_{k-1} \). If \( k = 4 \) then \( z-p_1-p_2-p_3-p_4-z \) is an odd hole, a contradiction, so \( k > 4 \). Choose an anticonnected subset \( Z \subseteq X \), with \( z \in Z \), maximal such that \( Z \) includes no leap for \( P \). Thus \( Z \neq X \) by 2.1; choose \( x \in X \setminus Z \) such that \( Z \cup \{x\} \) is anticonnected. From the maximality of \( Z \), \( Z \cup \{x\} \) includes a leap, and since \( Z \) includes no leap, it follows that there is a leap \( \{x, y\} \) for some \( y \in Z \). Consequently \( y \) is nonadjacent to \( p_3, \ldots, p_{k-2} \). But from 2.1 applied to \( P \) and \( Z \), since \( Z \) contains no leap, there is a \( Z \)-complete vertex \( p_i \), where \( 2 \leq i \leq k-1 \). Hence \( p_i \) is adjacent to both \( y, z \). But \( z \) is nonadjacent to \( p_2, p_{k-1} \), and \( y \) is nonadjacent to \( p_3, \ldots, p_{k-2} \), a contradiction. This proves 2.3.

In the three-colourable case we can say more:

2.4 Let \( G \) be a Berge graph with a three-colouring \( \phi : V(G) \rightarrow \{1, 2, 3\} \). Let \( P \) be an induced path in \( G \) of odd length, with vertices \( p_1 \cdots p_k \) in order. Let \( X \subseteq V(G) \setminus V(P) \) be anticonnected, such that \( p_1, p_k \) are \( X \)-complete and not all members of \( X \) have the same colour. Then
• \( \phi(p_1) = \phi(p_k) \) (= 3 say), and in particular \( p_1, p_k \) are nonadjacent, so \( k \geq 3 \);
• no internal vertex of \( P \) is \( X \)-complete;
• \( \{\phi(p_2), \phi(p_{k-1})\} = \{1, 2\} \), say \( \phi(p_2) = 1 \) and \( \phi(p_{k-1}) = 2 \);
• \( X \) is the union of two disjoint stable sets \( X_1, X_2 \), where the vertices in \( X_1 \) have colour 1 and are adjacent to \( p_{k-1} \), and the vertices in \( X_2 \) have colour 2 and are adjacent to \( p_2 \); and
• there is a leap \( \{x_1, x_2\} \) for \( P \) with \( x_i \in X_i \) for \( i = 1, 2 \).

**Proof.** Since not all members of \( X \) have the same colour, we may assume that some vertex in \( X \) has colour 1, and some vertex in \( X \) has colour 2; so every \( X \)-complete vertex has colour 3. In particular, \( p_1, p_k \) have colour 3 and therefore are nonadjacent, so \( k \geq 4 \). For the same reason no two \( X \)-complete vertices in \( P \) are adjacent. Choose a minimal subpath \( Q \) of \( P \) of odd length such that both its ends are \( X \)-complete. It follows that \( Q \) has length at least three; and none of its internal vertices are \( X \)-complete, from the minimality of \( Q \). Since \( p_1, p_k \) are both \( X \)-complete, by 2.2 they both have neighbours in \( Q^{*} \), and so \( Q = P \). Consequently no internal vertex of \( P \) is \( X \)-complete.

Since \( p_1 \) has colour 3, it follows that \( X = X_1 \cup X_2 \) where for \( i = 1, 2 \), \( X_i \) is the set of vertices in \( X \) with colour \( i \). Thus \( X_1, X_2 \neq \emptyset \). Since \( p_2 \) is adjacent to \( p_1 \) and \( p_{k-1} \) to \( p_k \), we deduce that \( p_2, p_{k-1} \) do not have colour 3, and from the symmetry we may assume that \( p_2 \) has colour 1. Thus \( p_2 \) is anticomplete to \( X_1 \), and so by 2.3, \( X_1 \) is complete to \( p_{k-1} \). Since \( X_1 \neq \emptyset \), it follows that \( p_{k-1} \) does not have colour 1; so it has colour 2. Thus \( X_2 \) is anticomplete to \( p_{k-1} \), and therefore is complete to \( p_2 \), again by 2.3. We have shown then that every vertex in \( X \) is adjacent to one of \( p_2, p_{k-1} \) and nonadjacent to the other. Finally, we need to produce the leap. If \( P \) has length at least five, this follows from 2.1, so we may assume that \( P \) has length three, and therefore \( k = 4 \). Since \( p_2, p_3 \) are not \( X \)-complete, and \( X \) is anticonnected, there is an (induced) antipath \( p_2, q_1, \ldots, q_m, p_3 \) between \( p_2, p_3 \) with \( q_1, \ldots, q_m \in X \). If \( m \geq 3 \) then \( q_2 \) is adjacent to both \( p_2, p_3 \), a contradiction; and if \( m = 1 \) then \( q_1 \) is nonadjacent to both \( p_2, p_3 \), again a contradiction; so \( m = 2 \) and \( \{q_1, q_2\} \) is the desired leap. This proves 2.4.

### 3 Complement line graphs

Let \( H \) be a graph with vertex set \( \{v_1, \ldots, v_9\} \) and edges as follows:

- for \( 1 \leq i \leq 6 \) \( v_i \) is adjacent to \( v_{i+2}, v_{i+3}, v_{i+4} \) (reading subscripts modulo 6)
- \( v_7 \) is adjacent to \( v_3, v_4, v_5, v_6 \); \( v_8 \) is adjacent to \( v_5, v_6, v_1, v_2 \); and \( v_9 \) is adjacent to \( v_1, v_2, v_3, v_4 \), and there are no other edges.

We call such a graph \( H \) a **trampoline**. In this section we study \( K_4 \)-free Berge graphs that contain trampolines. We prove the following:

**3.1 Let \( G \) be a \( K_4 \)-free Berge graph with no even pair and no clique cutset. If \( G \) contains a trampoline as an induced subgraph, then \( G \) is the complement of the line graph of some bipartite graph.**
The proof needs several steps. Throughout this section, let \( G \) be a \( K_4 \)-free Berge graph with no even pair and no clique cutset, that contains a trampoline. Consequently we may choose \( t \geq 4 \), and pairwise disjoint stable sets \( A_{ij} \) (\( 1 \leq i \leq 3, 1 \leq j \leq t \)) with the following properties:

- for \( 1 \leq i \leq 3 \), there is at most one value of \( j \in \{1, \ldots, t\} \) such that \( A_{ij} = \emptyset \)
- for \( 1 \leq j \leq t \), there is at most one value of \( i \in \{1, 2, 3\} \) such that \( A_{ij} = \emptyset \)
- for all distinct \( i, i' \in \{1, 2, 3\} \) and all distinct \( j, j' \in \{1, \ldots, t\} \), \( A_{ij} \) is complete to \( A_{i'j'} \)
- for \( 1 \leq i \leq 3 \) and for all distinct \( j, j' \in \{1, \ldots, t\} \), \( A_{ij} \) is anticomplete to \( A_{i'j} \)
- for \( 1 \leq j \leq t \), if \( A_{1j}, A_{2j}, A_{3j} \) are all nonempty then they are pairwise anticomplete
- for \( 1 \leq j \leq t \), and all distinct \( i, i' \in \{1, 2, 3\} \), if \( A_{i'j} \) is nonempty then every vertex in \( A_{ij} \) has a nonneighbour in \( A_{i'j} \).

Choose these sets with maximal union \( W \) say. For \( 1 \leq i \leq 3 \) let \( Z_i = \bigcup_{1 \leq j \leq t} A_{ij} \), and for \( 1 \leq j \leq t \) let \( A_j = A_{1j} \cup A_{2j} \cup A_{3j} \). Fix a 3-colouring \( \phi \) of \( G \). Since \( t \geq 4 \) it follows that the only partition of \( W \) into three stable sets is the partition \( Z_1, Z_2, Z_3 \); and we may therefore assume that for \( 1 \leq i \leq 3 \), \( \phi(v) = i \) for all \( v \in Z_i \).

Let \( v \in V(G) \setminus W \), and let \( N \) be the set of vertices in \( W \) that are adjacent to \( v \). We say \( v \) is major if \( N \) is the union of two of \( Z_1, Z_2, Z_3 \); and \( v \) is minor if there exist \( i, i' \in \{1, 2, 3\} \) and \( j, j' \in \{1, \ldots, t\} \) such that \( i \neq i' \) and \( N \subseteq A_{ij} \cup A_{i'j'} \) and \( N \cap A_{ij} \) is complete to \( N \cap A_{i'j'} \).

3.2 With notation as above, every vertex in \( V(G) \setminus W \) is either major or minor.

**Proof.** Let \( v \in V(G) \setminus W \), and let \( N \) be the set of vertices in \( W \) that are adjacent to \( v \). We may assume that \( \phi(v) = 3 \). Since \( v \) therefore has no neighbours in \( Z_3 \), it follows that

1. \( N \subseteq Z_1 \cup Z_2 \).

2. For \( 1 \leq j \leq t \), if \( A_{1j}, A_{2j} \) are both nonempty and \( A_{1j} \cup A_{2j} \) is neither a subset of \( N \) nor a subset of \( V(G) \setminus N \), then there exist \( a_{ij} \in A_{ij} \) for \( i = 1, 2 \), nonadjacent, such that exactly one of them is in \( N \).

For we may assume that \( j = 1 \); and suppose the claim is false. For \( i = 1, 2 \), let \( N_i = N \cap A_{i1} \) and let \( M_i = A_{i1} \setminus N_i \). Since the claim is false, \( N_1 \) is complete to \( M_2 \), and \( N_2 \) is complete to \( M_1 \).

If \( x \in N_1 \), then since \( x \) has a nonneighbour in \( A_{21} \), it follows that \( x \) has a nonneighbour in \( N_2 \); and so, since by hypothesis one of \( N_1, N_2 \) is nonempty, it follows that there exist \( n_i \in N_i \) for \( i = 1, 2 \), nonadjacent. Similarly \( M_1, M_2 \) are both nonempty. Since \( A_{11} \) is not anticomplete to \( A_{21} \), it follows that \( A_{31} = \emptyset \). If \( m_1 \in M_1 \) is adjacent to \( m_2 \in M_2 \), then \( v-n_1-m_2-m_1-n_2-v \) is an odd hole, a contradiction; so \( M_1 \) is anticomplete to \( M_2 \). If say there exists \( a_{12} \in A_{12} \setminus N \), then \( v-n_2-a_{12}-m_2-m_1-v \) is an odd hole, a contradiction; so \( A_{12} \subseteq N \), and similarly \( (Z_1 \cup Z_2) \setminus A_1 \subseteq N \). But then we can define \( A'_{11} = N_1, A'_{21} = N_2, A'_{31} = \emptyset, A'_{1,t+1} = M_1, A'_{2,t+1} = M_2, A'_{3,t+1} = \{v\} \), and \( A'_{ij} = A_{ij} \) for \( 1 \leq i \leq 3 \) and \( 2 \leq j \leq t \), contrary to the maximality of \( W \).
(3) We may assume that there are at least two values of $j \in \{1, \ldots, t\}$ such that $A_{1j} \cup A_{2j} \not\subseteq N$.

For suppose not; say $Z_1 \cup Z_2 \subseteq N \cap A_1$. If $A_{11} = \emptyset$ and $A_{21} \subseteq N$, then $N = Z_1 \cup Z_2$ and $v$ is major as required. If $A_{11} = \emptyset$ and $A_{21} \not\subseteq N$, then we can add $v$ to $A_{31}$, contrary to the maximality of $W$. Thus we may assume that $A_{11} \neq \emptyset$, and similarly $A_{21} \neq \emptyset$. If $N$ includes $A_{11} \cup A_{21}$ then again $v$ is major, and if $N$ is disjoint from $A_{11} \cup A_{21}$ then we can add $v$ to $A_{31}$, again contradictory to the maximality of $W$. Thus we may assume that $N$ includes some but not all of $A_{11} \cup A_{21}$; and so, from (2), we may assume that there exists $a_{11} \in A_{11} \cap N$, and $a_{21} \in A_{21} \cap N$, nonadjacent. Since $t \geq 4$, there exists $j \in \{2, \ldots, t\}$ such that $A_{2j}, A_{3j} \neq \emptyset$, say $j = 2$. Choose $a_{22} \in A_{22}$, and choose $a_{32} \in A_{32}$ nonadjacent to $a_{22}$. Then $v-a_{21}-a_{32}-a_{11}-a_{22}-v$ is an odd hole, a contradiction. This proves (3).

(4) For $1 \leq j \leq t$, $N \cap A_{1j}$ is complete to $N \cap A_{2j}$.

For suppose that there exist $a_{i1} \in N \cap A_{1i}$ for $i = 1, 2$, nonadjacent. By (3) we may assume that $A_{12} \cup A_{22} \not\subseteq N$. Suppose first that both $A_{12}, A_{22}$ are nonempty, and $N \cap (A_{12} \cup A_{22}) \neq \emptyset$. From (2) we may assume that there exist $a_{12} \in A_{12} \setminus N$ and $a_{22} \in A_{22} \cap N$, nonadjacent. Since there is no odd hole of the form $v-a_{21}-a_{12}-a_{31}-a_{22}-v$, it follows that $A_{31} = \emptyset$; and so $A_{3k} \neq \emptyset$ for $2 \leq k \leq t$. Since $t \geq 4$, one of $A_{23}$, $A_{43}$ is nonempty, say $A_{23}$; choose $a_{23} \in A_{23}$. If $A_{23} \cap N$ then $v-a_{23}-a_{21}-a_{23}$$-a_{22}-v$ is an odd hole (where $a_{33} \in A_{33}$ is nonadjacent to $a_{23}$), and if $a_{23} \not\in N$ then $v-a_{11}-a_{23}$$-a_{12}-a_{21}-v$ is an odd hole, in either case a contradiction. This proves that if both $A_{12}, A_{22}$ are nonempty, then $N \cap (A_{12} \cup A_{22}) = \emptyset$. Since $A_{12} \cup A_{22} \not\subseteq N$, we may assume that there exists $a_{12} \in A_{12} \cap N$. For $3 \leq j \leq t$, since there is no odd hole of the form $v-a_{21}-a_{12}$$-A_{2j}-a_{11}-v$, it follows that $A_{2j} \subseteq N$.

Suppose that $A_{22} \neq \emptyset$. By what we just proved, $N \cap (A_{12} \cup A_{22}) = \emptyset$, and from the symmetry between $Z_1, Z_2$ it follows that $A_{1j} \subseteq N$ for $3 \leq j \leq t$. By (3) it follows that $A_{11} \cup A_{21} \not\subseteq N$, and so by (2) and the symmetry between $Z_1, Z_2$, we may assume that $a'_{11} \in A_{11} \cap N$ and $a'_{21} \cap A_{21} \cap N$, nonadjacent. If both $A_{1j}, A_{2j} \neq \emptyset$ for some $j$ with $3 \leq j \leq t$, then from the symmetry between $A_{1j}$ and $A_{2j}$ it follows that $A_{11} \cup A_{21} \subseteq N$, a contradiction; so for all $j$ with $3 \leq j \leq t$, one of $A_{1j}, A_{2j} = \emptyset$. Consequently $A_{3j} \neq \emptyset$, and since $t \geq 4$ we may assume that $A_{13}, A_{33} \neq \emptyset$. Choose $a_{i3} \in A_{i3}$ for $i = 1, 3$; then $v-a'_{11}$$-a_{33}$$-a'_{21}$$-a_{13}$$-v$ is an odd hole, a contradiction. This proves that $A_{22} = \emptyset$.

Consequently $A_{2j} \neq \emptyset$ for $3 \leq j \leq t$. For $3 \leq j \leq t$, exchanging $A_{2j}, A_{j}$ implies that $A_{1j} \subseteq N$. Since $t \geq 4$, at least one of $A_{12}, \ldots, A_{1t}$ is nonempty, say $A_{13}$; and so there exist vertices in $A_{13} \cap N, A_{23} \cap N$ that are nonadjacent. By exchanging $A_{13}, A_{3}$, it follows that $A_{11}, A_{21} \subseteq N$, contrary to (3). This proves (4).

(5) There exist $j, j' \in \{1, \ldots, t\}$ such that $N \cap Z_1 \subseteq A_{1j}$ and $N \cap Z_2 \subseteq A_{2j'}$.

For suppose that there exist $a_{1j} \in N \cap A_{1j}$ for $j = 1, 2$ say. Now either $A_{31}, A_{22}$ are both nonempty, or $A_{32}, A_{21}$ are both nonempty, and from the symmetry we may assume the former. Choose $a_{31} \in A_{31}$ nonadjacent to $a_{11}$, and choose $a_{22} \in A_{22}$ nonadjacent to $a_{12}$. By (1) and (4), $a_{31}, a_{22} \not\in N$. Then $v-a_{11}-a_{22}$$-a_{31}$$-a_{12}$$-v$ is an odd hole, a contradiction. This proves (5).

Let $j, j'$ be as in (5). To show that $v$ is minor, it remains to show that $N \cap A_{1j}$ is complete to $N \cap A_{2j'}$. This is true from the construction if $j \neq j'$, and by (4) if $j = j'$. Thus $v$ is minor. This proves 3.2.
3.3 With notation as before, there is no major vertex.

Proof. We begin with:

(1) Every two major vertices are adjacent.

For suppose that \( b_1, b_2 \) are nonadjacent major vertices. We may assume that \( b_1 \) is complete to \( Z_2 \cup Z_3 \) say. Suppose first that \( b_2 \) is not complete to \( Z_2 \cup Z_3 \); say \( b_2 \) is complete to \( Z_3 \cup Z_1 \). If there exists \( j \in \{1, \ldots, t\} \) such that \( A_{3j} = \emptyset \), we can add \( b_1 \) to \( A_{1j} \) and \( b_2 \) to \( A_{2j} \), contrary to the maximality of \( W \). Thus \( A_{31}, \ldots, A_{3t} \) are all nonempty. But then we may define \( A_{1,t+1} = \{b_1\}, A_{2,t+1} = \{b_2\} \), and \( A_{3,t+1} = \emptyset \), contrary to the maximality of \( W \). This proves that \( b_2 \) is complete to \( Z_2 \cup Z_3 \).

Since \( G \) has no even pair, there is an odd induced path \( b_1 = p_1 \cdots p_k = b_2 \) in \( G \). Since none of \( p_2, \ldots, p_{k-1} \) is adjacent to both \( b_1, b_2 \), it follows that none of them is in \( Z_2 \cup Z_3 \). Moreover, \( p_2, p_{k-1} \notin Z_1 \), since \( b_1, b_2 \) are anticomplete to \( Z_1 \). Thus \( p_2, p_{k-1} \in V(G) \setminus W \). Now \( p_2 \) is not complete to \( Z_2 \cup Z_3 \) since \( Z_2 \cup Z_3 \) is not stable and \( G \) is \( K_4 \)-free; and since \( p_2, b_2 \) are nonadjacent, and we have already seen that every two nonadjacent major vertices have the same neighbours in \( W \), it follows that \( p_2 \) is not major. Similarly \( p_{k-1} \) is not major. But by 2.4, one of \( p_2, p_{k-1} \) is complete to \( Z_2 \) and the other to \( Z_3 \), which is impossible since they are both minor. This proves (1).

Now to complete the proof of 3.3, suppose that \( b \) is a major vertex. Thus \( b \notin W \), and we may assume that \( b \) is complete to \( Z_2 \cup Z_3 \) and anticomplete to \( Z_1 \). At least one of \( A_{11}, A_{12} \) is nonempty, say \( A_{11} \); choose \( a_{11} \in A_{11} \). Since \( G \) has no even pair, there is an odd induced path \( b = p_1 \cdots p_k = a_{11} \). Thus \( p_1, p_k \) are both complete to the anticonnected set \( (Z_2 \cup Z_3) \setminus A_1 \); and this anticonnected set is not stable since \( t \geq 4 \). Since \( k \) is even it follows that none of \( p_1, \ldots, p_k \) belong to \( (Z_2 \cup Z_3) \setminus A_1 \), and so by 2.4, one of \( p_2, p_{k-1} \) is complete to \( Z_2 \setminus A_1 \), and the other to \( Z_3 \setminus A_1 \). Since \( p_{k-1} \) is adjacent to \( a_{11} \) and not to \( b \), it follows that \( p_{k-1} \) is not in \( W \); by (1) \( p_{k-1} \) is not major; and since \( p_{k-1} \) is complete to one of \( Z_2 \setminus A_1, Z_3 \setminus A_1 \) it follows that \( p_{k-1} \) is not minor, contrary to 3.2. This proves 3.3.

3.4 For \( 1 \leq i \leq 3 \) and \( 1 \leq j \leq t \), \( |A_{ij}| \leq 1 \).

Proof. Suppose that \( u, v \in A_{11} \) say are distinct. Then \( u, v \) both have the same colour, and so are nonadjacent. Moreover, \( u, v \) are both complete to \( (Z_2 \cup Z_3) \setminus A_1 \), and there is an odd induced path \( u = p_1 \cdots p_k = v \) between \( u, v \) since they are not an even pair; so 2.4 implies that one of \( p_2, p_{k-1} \) has colour 3 and is complete to \( Z_2 \setminus A_1 \), and the other has colour 2 and is complete to \( Z_3 \setminus A_1 \); let the first be \( p_2 \), say. Consequently \( p_2 \) is not minor; by 3.3 it is not major; and so by 3.2 it belongs to \( W \). Since it has colour 3 and has a neighbour and a nonneighbour in \( A_{11} \), we deduce that \( p_2 \in A_{31} \) and \( A_{21} = \emptyset \). But similarly \( p_{k-1} \in A_{21} \), a contradiction. This proves 3.4.

Henceforth we denote the unique member of \( A_{ij} \) by \( a_{ij} \) (when it exists) without further explanation. Note that 3.4 implies that \( A_{ij} \) is anticomplete to \( A_{i'j} \) for all distinct \( i, i' \in \{1, 2, 3\} \) and all \( j \in \{1, \ldots, t\} \).

3.5 If \( X \) is a connected set of minor vertices and \( u, v \in W \) both have neighbours in \( X \), then \( u, v \) are adjacent.
Proof. Suppose not, and choose nonadjacent $u, v \in W$ and a connected set $X$ as in the claim, with $|X|$ minimum. It follows that $X$ is the interior of an induced path $u-p_1\cdots-p_k-v$ between $u, v$. Since the members of $X$ are minor, 3.4 implies that $k \geq 2$.

(1) For some $i \in \{1, 2, 3\}$ there are two members of $Z_i$ with neighbours in $X$.

For suppose not. We may therefore assume that $u = a_{11}$ and $v = a_{21}$, and $(Z_1 \cup Z_2) \setminus A_1$ is anticomplete to $X$. Suppose first that $k$ is even. At most one vertex in $Z_3 \setminus A_1$ has a neighbour in $X$; choose $w \in Z_3 \setminus A_1$ with no neighbour in $X$, and then $w-u-p_1\cdots-p_k-v-w$ is an odd hole. So $k$ is odd. Now either $a_{12}, a_{22}$ both exist, or $a_{22}, a_{13}$ both exist, and from the symmetry we may assume the first; and then $u-p_1\cdots-p_k-v-a_{12}-a_{23}-u$ is an odd hole, a contradiction. This proves (1).

In view of (1) we may assume that $u = a_{11}$ and $v = a_{12}$. From the minimality of $X$ (and since $k \geq 2$) it follows that $A_{1j}$ is anticomplete to $X$ for $3 \leq j \leq t$.

(2) It is impossible that $k$ is even.

For suppose $k$ is even. We may assume that $a_{23}, a_{34}$ exist. If $a_{24}$ exists, then $\{a_{23}, a_{34}, a_{24}\}$ is anticonnected and not stable, and complete to $u, v$; so by 2.4 each of $a_{23}, a_{34}, a_{24}$ is adjacent to one of $p_1, p_k$, contradicting that $p_1, p_k$ are minor. So $A_{24} = \emptyset$, and similarly $A_{33} = \emptyset$. Hence $a_{21}, a_{22}, a_{13}$ exist, and since $a_{13}-a_{22}-u-p_1\cdots-p_k-v-a_{21}-a_{13}$ is not an odd hole, one of $a_{21}, a_{22}$ has a neighbour in $X$, say $a_{21}$. Since $u$ is adjacent to $p_1$ and nonadjacent to $a_{21}$, we deduce that $a_{21}$ is adjacent to $p_k$ from the minimality of $X$. Since $a_{23}$ also has a neighbour in $X$ and $a_{21}, a_{23}$ are nonadjacent, the minimality of $X$ implies that $a_{23}$ is adjacent to $p_1$. But similarly $a_{34}$ is adjacent to one of $p_1, p_k$, contradicting that $p_1, p_k$ are both minor. This proves (2).

(3) It is impossible that $k$ is odd.

For suppose that $k$ is odd. We may assume that $a_{21}, a_{32}$ exist, and since $u-p_1\cdots-p_k-v-a_{21}-a_{32}-u$ is not an odd hole, we deduce that at least one of $a_{21}, a_{32}$ has a neighbour in $X$, say $a_{21}$. Since $u$ is adjacent to $p_1$, the minimality of $X$ implies that $p_k$ is the only neighbour of $a_{21}$ in $X$. If also $a_{32}$ has a neighbour in $X$, then similarly $p_1$ is its only neighbour, and then $a_{32}-p_1\cdots-p_k-a_{21}-a_{32}$ is an odd hole, a contradiction. Thus $a_{32}$ is anticomplete to $X$. We may assume that $a_{13}, a_{24}$ exist, and we have seen that $a_{13}$ is anticomplete to $X$. If also $a_{24}$ is anticomplete to $X$, then $u-p_1\cdots-p_k-a_{21}-a_{13}-a_{24}-u$ is an odd hole. So $a_{24}$ has a neighbour in $X$. From the minimality of $X$, its only neighbour in $X$ is $p_1$; but then $v-a_{24}-p_1\cdots-p_k-v$ is an odd hole. This proves (3).

From (2) and (3), we have a contradiction. This proves 3.5.
Proof of 3.1.

Let $G$ be a $K_4$-free Berge graph with no clique cutset and no even pair, that contains a trampoline. Define the sets $A_{ij}$ as before. If there is a minor vertex, let $X$ be a maximal connected set of minor vertices; then by 3.5 and 3.3, the set of vertices in $W$ with a neighbour in $X$ is a clique cutset, a contradiction. Thus there is no minor vertex, and by 3.3 and 3.4 it follows that $G$ is the complement of the line graph of a bipartite graph. This proves 3.1. \hfill \Box

4 Trapezes and trestles

Let $H$ be a graph, and let $G$ be obtained from $H$ by adding two more vertices, nonadjacent to each other and each adjacent to every vertex of $H$. We call $G$ a suspension of $H$. We need to consider suspensions of several different small graphs. A trapeze is a suspension of a graph $H$ that has four vertices and two edges, disjoint. A trestle is a suspension of a four-vertex path. An extended 4-wheel is a suspension of a graph with four vertices and two edges that share an end. An octahedron is a suspension of a cycle of length four. In this section we show that we can exclude these four kinds of subgraphs.

4.1 Let $G$ be a $K_4$-free Berge graph with no even pair, containing no trampoline. Then $G$ does not contain a trapeze.

Proof. Suppose that $G$ contains a trapeze, with six vertices $a_1, b_1, a_2, b_2, c_1, c_2$, where $c_1, c_2$ are both complete to $\{a_1, b_1, a_2, b_2\}$, and $aibi$ is an edge for $i = 1, 2$. Fix a three-colouring $\phi$ of $G$; then $\phi(c_1) = \phi(c_2)$, and we may assume that $\phi(c_1) = 3$, and $\phi(a_i) = 1$ and $\phi(b_i) = 2$ for $i = 1, 2$.

There is an odd induced path between $c_1, c_2$, since $G$ has no even pair. For $i = 1, 2$, let $d_i$ be the neighbour of $c_i$ in this path. For $i = 1, 2$, let $X_i$ be the set of common neighbours of $c_1, c_2$ that have colour $i$. Then $X_1 \cup X_2$ is anticonnected and not stable (since $a_1, a_2 \in X_1$ and $b_1, b_2 \in X_2$).

Since $c_1, c_2$ are common neighbours of $X_1 \cup X_2$, we may assume by 2.4 that $d_1$ has colour 1 and is complete to $X_2$, and $d_2$ has colour 2 and is complete to $X_1$.

Now there is an odd induced path $a_1-q_1-\cdots-q_k-a_2$ between $a_1, a_2$. Since $a_1, a_2$ are common neighbours of $\{c_1, c_2, d_2\}$, we may assume by 2.4 (by exchanging $a_1b_1$ with $a_2b_2$ if necessary) that $q_1$ has colour 3 and is adjacent to $d_2$, and $q_k$ has colour 2 and is complete to $\{c_1, c_2\}$. Moreover, $\{c_1, c_2, d_2\}$ includes a leap; and since the two vertices of the leap are nonadjacent and have different colours, it follows that the leap is $\{c_1, d_2\}$. Consequently $c_1$ is nonadjacent to $q_1, \ldots, q_{k-1}$, and $d_2$ is nonadjacent to $q_2, \ldots, q_k$. Since $q_k$ is adjacent to $a_2, c_1, c_2$, it follows that $q_k \in X_2$, and so $d_1$ is adjacent to $q_k$.

Since $b_1, q_k$ have the same colour, they are nonadjacent. Suppose that $b_1$ is nonadjacent to $q_1, \ldots, q_{k-1}$. Then $b_1-a_1q_1-\cdots-q_k$ is an odd path between common neighbours of $\{c_1, c_2, d_1\}$, and so by 2.4, it follows that $d_1$ is adjacent to $q_{k-1}$ and not to $q_1, \ldots, q_{k-2}$. But then if $d_1, d_2$ are nonadjacent then

$$d_2-q_1-\cdots-q_{k-1}-d_1-c_1-a_2-d_2$$

is an odd hole, a contradiction; if $d_1, d_2$ are adjacent and $k \geq 4$ then $d_2-q_1-\cdots-q_{k-1}-d_1-d_2$ is an odd hole, a contradiction; and if $d_1, d_2$ are adjacent and $k = 2$ (and therefore $d_1, q_1$ are adjacent) then
the subgraph induced on \( \{a_1, a_2, b_1, q_k, c_1, c_2, d_1, d_2, q_1\} \) is a trampoline, a contradiction. This proves that \( b_1 \) is adjacent to \( q_i \) for some \( i \in \{1, \ldots, k - 1\} \). Choose \( i \) minimum. From the hole

\[
d_2 - q_1 - \cdots - q_i - b_1 - c_1 - a_2 - d_2
\]

it follows that \( i \) is even, and since \( k \) is even, we deduce that \( q_i, q_k \) are nonadjacent. Suppose that \( d_1 \) is anticomplete to \( \{q_1, \ldots, q_i\} \). If \( d_1, d_2 \) are nonadjacent then

\[
d_2 - q_1 - \cdots - q_i - b_1 - d_1 - q_k - a_2 - d_2
\]

is an odd hole, and if \( d_1, d_2 \) are adjacent then \( d_2 - q_1 - \cdots - q_i - b_1 - d_1 - d_2 \) is an odd hole, a contradiction. Thus \( d_1 \) is adjacent to one of \( q_1, \ldots, q_i \). Since \( d_2 - q_1 - \cdots - q_i - b_1 - c_2 - d_2 \) is not an odd hole, \( c_2 \) is also adjacent to one of \( q_1, \ldots, q_i \). Consequently there is an induced path \( R \) between \( c_2 \) and \( d_1 \) with \( R^* \subseteq \{q_1, \ldots, q_i\} \). But \( R \) can be completed to a hole via \( d_1 - q_k - c_2 \) and via \( d_1 - c_1 - a_2 - c_2 \), and one of these is an odd hole, a contradiction. This proves 4.1.

4.2 Let \( G \) be a \( K_4 \)-free Berge graph with no even pair, containing no trampoline. Then \( G \) contains no trestle.

Proof. (We remind the reader that all graphs in this paper are finite. This theorem in particular is false if we allow infinite graphs.) Let us say an extended trestle in \( G \) is a sequence \( v_1, \ldots, v_n \) of distinct vertices, with \( n \geq 8 \), such that for \( 1 \leq i < j \leq n \), \( v_i \) and \( v_j \) are adjacent if \( j - i \in \{1, 2, 4\} \), and they are nonadjacent if \( j - i \notin \{1, 2, 4, 7\} \). Fix a three-colouring \( \phi \) of \( G \). By 4.1 it follows that \( G \) contains no trapeze. Suppose it contains a trestle.

(1) \( G \) contains an extended trestle.

For \( G \) contains a trestle, and so there are six vertices \( v_2, \ldots, v_7 \) in \( G \) such that \( v_2 - v_4 - v_5 - v_7 \) is an induced path, and \( \{v_3, v_6\} \) is complete to \( \{v_2, v_4, v_5, v_7\} \), and there are no other edges among \( v_2, \ldots, v_7 \). We may assume that \( v_2, v_5 \) have colour 1, and \( v_3, v_6 \) have colour 2, and \( v_4, v_7 \) have colour 3. There is an odd induced path between \( v_3, v_6 \), say \( v_3 - p_1 - \cdots - p_k - v_6 \). Since \( v_3, v_6 \) are both complete to \( \{v_2, v_4, v_5, v_7\} \), and the latter is anticonnected and not stable, we may assume from 2.4 and the symmetry that \( p_1 \) has colour 3 and is complete to \( \{v_2, v_5\} \), and \( p_k \) has colour 1 and is complete to \( \{v_4, v_7\} \). But then the sequence \( p_1, v_2, \ldots, v_7, p_k \) is an extended trestle. This proves (1).

In view of (1) and the finiteness of \( G \), we may choose an extended trestle \( v_1, \ldots, v_n \) with \( n \) maximum. We may assume that:

(2) For \( 1 \leq i \leq n \), \( \phi(v_i) = n - i \mod 3 \).

For \( v_i, v_{i+1}, v_{i+2} \) are pairwise adjacent (for \( 1 \leq i \leq n - 2 \)), and so are \( v_{i+1}, v_{i+2}, v_{i+3} \) (for \( i \leq n - 3 \)), and so \( v_i, v_{i+3} \) have the same colour for \( 1 \leq i \leq n - 3 \). Thus for \( 1 \leq i < j \leq n \), if \( j - i = 0 \mod 3 \) then \( v_i, v_j \) have the same colour. Since we may assume that \( v_n \) has colour 3 and \( v_{n-1} \) has colour 1, the claim follows. This proves (2).
For there is an odd induced path $v_{n-1}p_1\cdots p_kv_{n-4}$ between $v_{n-1}, v_{n-4}$. Since $v_{n-1}, v_{n-4}$ are both complete to $\{v_n, v_{n-2}, v_{n-3}, v_{n-5}\}$, and the latter is anticonnected and not stable, it follows from 2.4 that one of $p_1, p_k$ has colour 2 and is complete to $\{v_n, v_{n-3}\}$, and the other has colour 3 and is complete to $\{v_{n-2}, v_{n-5}\}$. Suppose that $p_1$ has colour 3. Then $v_{n-2}, v_{n-5}$ are complete to $\{v_{n-1}, p_1, v_{n-4}, v_{n-6}\}$, and $v_{n-1}p_1$ and $v_{n-4}v_{n-6}$ are edges, and $\{v_{n-1}, p_1\}$ is anticomplete to $\{v_{n-4}, v_{n-6}\}$ ($p_1$ is not adjacent to $v_{n-6}$ since they have the same colour). Thus $G$ contains a trapeze, a contradiction. This proves that $p_1$ has colour 2, and is adjacent to $v_n, v_{n-3}$, and not to $v_{n-4}$.

Define $v_{n+1} = p_1$; we will show that $v_{n+1}$ satisfies the claim. Since $v_{n+1}$ has colour 2, it is nonadjacent to $v_{n-2}, v_{n-5}$. Thus, in summary, $v_{n+1}$ is adjacent to $v_n, v_{n-1}, v_{n-3}$ and not to $v_{n-2}, v_{n-4}, v_{n-5}$.

Suppose that $v_{n+1} = v_i$ for some $i \in \{1, \ldots, n\}$. Then $n - i = 2 \mod 6$ by (2), since $v_{n+1}$ has colour 2; and $i \neq n - 5, n - 2$ since $v_{n+1}$ is nonadjacent to $v_{n-4}$. Thus $i \leq n - 8$. But the only neighbours of $v_n$ in $\{v_1, \ldots, v_{n-1}\}$ are $v_{n-1}, v_{n-2}, v_{n-4}$ and possibly $v_{n-7}$, a contradiction. Thus $v_{n+1}$ is different from $v_1, \ldots, v_n$. This proves (3).

(4) $v_{n+1}$ is nonadjacent to $v_{n-7}$.

For suppose $v_{n+1}, v_{n-7}$ are adjacent. Since $v_{n+1}v_{n-7}v_{n-6}v_{n-2}v_{n-1}v_{n+1}$ is not a hole of length five, it follows that $v_{n+1}$ is adjacent to $v_{n-6}$. But then $v_{n+1}v_{n-7}$ and $v_{n-2}v_{n-4}$ are edges, and $\{v_{n+1}, v_{n-7}\}$ is anticomplete to $\{v_{n-2}, v_{n-4}\}$, and $v_{n-3}, v_{n-6}$ are both complete to $\{v_{n-7}, v_{n-4}, v_{n-2}, v_{n+1}\}$, and hence $G$ contains a trapeze, a contradiction. This proves (4).

(5) $v_{n+1}$ is nonadjacent to $v_i$ for $1 \leq i \leq n - 8$.

For suppose that $v_i$ is adjacent to $v_{n+1}$ for some $i \in \{1, \ldots, n - 8\}$, and choose $i$ maximum. There are cases depending on the value of $n - i$ modulo 6. By (2), $n + 1 - i \neq 0, 3 \mod 6$ since $v_{n+1}, v_i$ are adjacent and therefore have different colours; so $n - i$ is one of 0, 1, 3 or 4 mod 6. If $n - i = 0$ mod 6, then $i \leq n - 12$, and

$v_{n+1}v_i\cdots v_{i+6}v_{i+10}\cdots v_{n-12}v_{n-8}v_{n-4}v_nv_{n+1}$

is an odd hole. If $n - i = 1 \mod 6$, then $i \leq n - 13$, and

$v_{n+1}v_i\cdots v_{i+4}v_{i+6}v_{i+10}\cdots v_{n-13}v_{n-9}v_{n-5}v_{n-1}v_{n+1}$

is an odd hole. If $n - i = 3 \mod 6$ then $i \leq n - 9$, and

$v_{n+1}v_i\cdots v_{i+2}v_{i+6}v_{i+8}\cdots v_{n-13}v_{n-9}v_{n-5}v_{n-4}v_{n+1}$

is an odd hole. If $n - i = 4 \mod 6$, then $i \leq n - 10$, and

$v_{n+1}v_i\cdots v_{i+2}v_{i+6}v_{i+8}\cdots v_{n-14}v_{n-10}v_{n-8}v_{n-4}v_{n+1}$

is an odd hole. This proves (5).

But from (5), $v_1, \ldots, v_{n+1}$ is an extended trestle, contrary to the maximality of $n$. This proves 4.2.
4.3 Let $G$ be a $K_4$-free Berge graph with no even pair, containing no trampoline. Then $G$ contains no extended 4-wheel.

**Proof.** Suppose that $G$ contains an extended 4-wheel, with vertex set $\{a_1, a_2, b_1, b_2, c_1, c_2\}$, where $a_1-b_1-a_2$ is a path and $\{c_1, c_2\}$ is complete to $\{a_1, a_2, b_1, b_2\}$. Fix a three-colouring of $G$; then we may assume that $a_1, a_2$ have colour 1, and $b_1$ has colour 2, and $c_1, c_2$ have colour 3 (and $b_2$ has colour 1 or 2). Since $G$ has no even pair, there is an odd induced path $c_1-p_1-\cdots-p_k-c_2$, and since $c_1, c_2$ are complete to $\{a_1, a_2, b_1, b_2\}$, and the latter is anticonnected and not stable, we may assume from 2.4 and the symmetry between $c_1, c_2$ that $p_1$ has colour 2 and is adjacent to $a_1, a_2$. Since $p_1$ is not adjacent to $c_2$, it follows that $p_1 \neq b_1$, and $c_2-b_1-c_1-p_1$ is an induced path; but $\{a_1, a_2\}$ is complete to the vertex set of this path, and so $G$ contains a trestle, contrary to 4.2. This proves 4.3. $lacksquare$

4.4 Let $G$ be a $K_4$-free Berge graph with no even pair, containing no trampoline. Then $G$ contains no octahedron.

**Proof.** Suppose it does; consequently we may choose three disjoint stable sets $A_1, A_2, A_3 \subseteq V(G)$, pairwise complete and each with cardinality at least two. Choose them with maximal union. Fix a three-colouring of $G$, and we may assume that the vertices in $A_i$ have colour $i$ for $i = 1, 2, 3$.

(1) Every $A_1$-complete vertex belongs to $A_2 \cup A_3$.

For suppose that $v$ is $A_1$-complete and $v \notin A_2 \cup A_3$. Since $G$ is $K_4$-free, $v$ is anticomplete to at least one of $A_2, A_3$, say $A_3$. If $v$ is $A_2$-complete then we may add $v$ to $A_3$, contrary to the maximality of $A_1 \cup A_2 \cup A_3$. Thus $v$ has a nonneighbour in $A_2$. Choose $a_1, a'_1 \in A_1$. There is an odd induced path $a_1-p_1-\cdots-p_k-a'_1$ between $a_1, a'_1$; and since $A_2 \cup A_3 \cup \{v\}$ is anticonnected and not stable, we may assume by 2.4 that $p_1$ is complete to $A_3$ and anticomplete to $A_2$. Choose distinct $a_3, a'_3 \in A_3$, and choose $a_2 \in A_2$. Then $p_1-a_1-a_2-a'_1$ is an induced path, and $a_3, a'_3$ are complete to its vertex set, so $G$ contains a trestle, contrary to 4.2. This proves (1).

Now since $|A_2| \geq 2$, there is an odd induced path with both ends in $A_2$; choose such a path with minimum length, say $a_2-p_1-\cdots-p_k-a'_2$, where $a_2, a'_2 \in A_2$. From the minimality of $k$, it follows that none of $p_1, \ldots, p_k$ is in $A_2$; and none of them is in $A_1 \cup A_3$ since none of them is adjacent to both $a_1, a'_1$. Consequently none of $p_1, \ldots, p_k$ is complete to $A_1$, by (1). By 2.3, every vertex in $A_1$ is adjacent to one of $p_1, p_k$; and similarly so is every vertex in $A_3$. Since $p_1, p_k$ do not have colour 2 (because they have neighbours in $A_2$), we may assume that $p_k$ has colour 1. Consequently $p_k$ is anticomplete to $A_1$, and so $p_1$ is complete to $A_1$, contrary to (1). This proves 4.4. $lacksquare$

5 Jumps on a prism

In this section we present a collection of lemmas about attachments to a prism that we need later. We say a vertex $v$ can be linked onto a triangle $\{a_1, a_2, a_3\}$ (via paths $P_1, P_2, P_3$) if:

• $v \neq a_1, a_2, a_3$
the three paths $P_1, P_2, P_3$ are induced and mutually vertex-disjoint, and do not contain $v$

- for $i = 1, 2, 3$ $a_i$ is an end of $P_i$

- for $1 \leq i < j \leq 3$, $a_ia_j$ is the unique edge of $G$ between $V(P_i)$ and $V(P_j)$

- $v$ has a neighbour in each of $P_1, P_2$ and $P_3$.

Our first lemma (theorem 2.4 of [5]) is well-known:

5.1 Let $G$ be Berge, and suppose $v$ can be linked onto a triangle $\{a_1, a_2, a_3\}$. Then $v$ is adjacent to at least two of $a_1, a_2, a_3$.

A prism is a graph consisting of two vertex-disjoint triangles $\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}$, and three paths $R_1, R_2, R_3$, where each $R_i$ has ends $a_i, b_i$, and for $1 \leq i < j \leq 3$ the only edges between $V(R_i)$ and $V(R_j)$ are $a_i a_j$ and $b_i b_j$. The three paths $R_1, R_2, R_3$ are said to form the prism. The prism is long if at least one of the three paths has length $> 1$. If $G$ is a graph, a prism in $G$ is an induced subgraph $K$ that is a prism. If $G$ is Berge, the three paths forming $K$ are either all even or all odd, and we call the prism even or odd respectively. A vertex $w \in V(G) \setminus V(K)$ is said to be major with respect to $K$ if it has at least two neighbours in each triangle of the prism.

If $F, K$ are induced subgraphs of $G$, a vertex in $V(K)$ is said to be an attachment of $F$ (or of $V(F)$) in $K$ if either it belongs to $V(F)$ or it has a neighbour in $V(F)$. If $K$ is a prism in $G$ with $R_1, R_2, R_3$ as before, a subset $X \subseteq V(K)$ is local with respect to $K$ if either $X \subseteq V(R_i)$ for some $i$, or $X$ is a subset of one of the triangles of $K$. If $f_1, \ldots, f_n$ is an induced path disjoint from $K$, we say that $f_1 \cdots f_n$ is a corner jump in position $a_1$ with respect to $K$ if $f_1$ is adjacent to $a_2, a_3$, and there is at least one edge between $f_n$ and $V(R_1) \setminus \{a_1\}$, and every edge between $\{f_1, \ldots, f_n\}$ and $V(K) \setminus \{a_1\}$ is between $f_1$ and $\{a_2, a_3\}$ or between $f_n$ and $V(R_1) \setminus \{a_1\}$. We define corner jumps in positions $a_2, a_3, b_1, b_2, b_3$ similarly. A corner jump means a path that is a corner jump in one of these six positions. Note that we are distinguishing between $f_1 \cdots f_n$ and $f_n \cdots f_1$ here.

We need theorem 10.1 of [5], specialized to $K_4$-free graphs, the following:

5.2 Let $R_1, R_2, R_3$ form a prism $K$ in a $K_4$-free Berge graph $G$, with triangles $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$, where each $R_i$ has ends $a_i$ and $b_i$. Let $F \subseteq V(G) \setminus V(K)$ be connected, such that its set of attachments in $K$ is not local. Then there exist $n \geq 1$ and an induced path $f_1 \cdots f_n$ with $f_1, \ldots, f_n \in F$, such that either:

- $n = 1$ and $f_1$ is major, or

- for some distinct $i, j \in \{1, 2, 3\}$, $f_1$ has two adjacent neighbours in $R_i$, and $f_n$ has two adjacent neighbours in $R_j$, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K)$, or

- $n \geq 2$, and for some distinct $i, j \in \{1, 2, 3\}$, $f_1$ is adjacent to $a_i, a_j$, and $f_n$ is adjacent to $b_i, b_j$, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K)$, or

- $f_1 \cdots f_n$ is a corner jump.

This has the following useful corollary.
5.3 Let $G$ be a $K_4$-free Berge graph containing no trestle, and let $C$ be a hole of $G$. Let $R_3$ be an induced path of $G$, with $V(R_3) \cap V(C) = \emptyset$, and with ends $a_3, b_3$. (Possibly $R_3$ has length zero.) Let $a_1a_2$ and $b_1b_2$ be disjoint edges of $C$, such that the only edges between $V(R_3)$ and $V(C)$ are $a_1a_3, a_2a_3, b_1b_3, b_2b_3$. Let $w \in V(G) \setminus (V(R) \cup V(C))$, and let $w$ be adjacent to $a_1, a_2$ and nonadjacent to at least two of $b_1, b_2, b_3$. Then $w$ has no neighbours in $C$ except $a_1, a_2$.

**Proof.** For $i = 1, 2$, let $R_i$ be the path of $C$ between $a_i, b_i$ not using the edge $a_1a_2$, and let $c_i$ be the neighbour of $a_i$ in $R_i$. Suppose $w$ has another neighbour in $V(C)$. Suppose first that $R_3$ has positive length, so $R_1, R_2, R_3$ form a prism $K$. The set of neighbours of $w$ in $K$ is not local, and so 5.2 implies that one of its outcomes holds if we set $n = 1$ and $f_1 = w$. Now the first outcome of 5.2 is false since $w$ has at most one neighbour in $\{b_1, b_2, b_3\}$, and the third is false since $n = 1$. Suppose the second holds. Then $w$ has exactly four neighbours in the hole $C$, namely $c_1, a_1, a_2, c_2$. Since $C$ is even, and

$$w-c_1-R_1-b_1-b_2-R_2-c_2-w$$

is not an odd hole, it follows that $C$ has length four; but then the prism is odd, so $R_3$ is odd, and

$$w-a_1-a_3-R_3-b_3-b_2-w$$

is an odd hole, a contradiction. Thus the fourth outcome holds. Since $w$ is adjacent to $a_1, a_2$ it follows that $w$ has neighbours in $V(R_3) \setminus \{a_3\}$, and has no other neighbours in $V(C)$, a contradiction.

We may therefore assume that $R_3$ has length zero, so $a_3 = b_3$. Suppose that $R_2$ has length one. Then since the subgraph induced on $(V(C) \setminus \{a_2, b_2\}) \cup \{w\}$ is not an odd hole, it follows that $C$ has length four; and since $w$ has more than two neighbours in $C$ and is nonadjacent to one of $b_1, b_2$, it follows that $G$ contains a trestle, a contradiction. Thus $R_2$ has length at least two, and similarly so does $R_1$. Since $a_3-a_2-R_2-b_2-a_3$ is a hole it follows that $R_2$ is even, and similarly $R_1$ is even. Consequently $R_1, R_2$ are both even. Suppose that for $i = 1, 2$, $w$ has a neighbour in $R_i$ different from $a_i$. Since $w$ cannot be linked onto $\{b_1, b_2, b_3\}$, we deduce that $c_1, a_1, a_2, c_2$ are the only neighbours of $w$ in $C$, and then either $C$ or the graph induced on $(V(C) \setminus \{a_1, a_2\}) \cup \{w\}$ is an odd hole. Thus we may assume that $w$ has no neighbour in $R_2$ different from $a_2$; and so it has a neighbour in $R_1$ different from $a_1$. Let $Q$ be an induced path between $b_1$ and $w$ with interior in $V(R_1)$. Since $w-a_2-R_2-b_2-b_1-Q-w$ is not an odd hole, it follows that $Q$ is even; but then $w-a_2-a_3-b_1-Q-w$ is an odd hole, a contradiction. This proves 5.3.

We use 5.2 to prove the following.

5.4 Let $G$ be a $K_4$-free Berge graph containing no trapeze, trestle, octahedron or extended 4-wheel. Let $K$ be a prism in $G$, and let $A, B$ and $R_i, a_i, b_i$ $(i = 1, 2, 3)$ be as before. Let $w \in V(G) \setminus V(K)$ be major with respect to $K$. Let $F \subseteq V(G) \setminus V(K)$ be connected, such that its set of attachments in $K$ is not local, and $w$ is anticomplete to $F$. Then there is an induced path $f_1 - \cdots - f_n$ with $n \geq 1$ and $f_1, \ldots, f_n \in F$, such that either:

- $n \geq 3$ is odd, and for some distinct $i, j \in \{1, 2, 3\}$, $f_1$ has two adjacent neighbours $c_i, d_i$ in $R_i$, and $f_n$ has two adjacent neighbours $c_j, d_j$ in $R_j$, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K)$, and $w$ is adjacent to all of $c_i, d_i, c_j, d_j$, or
• $K$ is even, and for some distinct $i, j \in \{1, 2, 3\}$, $f_1$ has two adjacent neighbours $c_i, d_i$ in $R_i$, and $f_n$ has two adjacent neighbours $c_j, d_j$ in $R_j$, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K)$, and $w$ is adjacent to $a_i, b_i, a_j, b_j$ and nonadjacent to every internal vertex of $R_i$ and of $R_j$, or

• $n \geq 2$, and for some distinct $i, j \in \{1, 2, 3\}$, $f_1$ is adjacent to $a_i, a_j$, and $f_n$ is adjacent to $b_i, b_j$, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K)$, and $w$ is adjacent to $a_i, a_j, b_i, b_j$, or

• $f_1 \cdots f_n$ is a corner jump in position $a_i$ say (or $b_i$, similarly). Moreover, if $w$ is adjacent to $a_i$, and therefore nonadjacent to $a_j$ for some $j \in \{1, 2, 3\} \setminus \{i\}$, then $R_i$ has length one, $w$ is adjacent to $b_i, b_j$, and $w$ has no neighbour in $R_j$ except $b_j$.

**Proof.** Let $f_1, \ldots, f_n$ be as in 5.2.

(1) The first outcome of 5.2 does not hold.

For suppose it does; thus $f_1$ is major. Let $\{u, v\} = \{w, f_1\}$; thus, $u, v$ are nonadjacent major vertices, and there is symmetry between $u, v$. Let $X$ be the set of vertices in $K$ adjacent to both $u, v$. Thus $A \cap X, B \cap X \neq \emptyset$. If $u, v$ have the same neighbours in $A \cup B$, then the subgraph induced on $X \cap (A \cup B)$ is either a 2-edge matching, or a 3-edge path, or a cycle of length four, and so $H$ contains a trapeze, trestle or octahedron, contrary to the hypothesis. So we may assume that $u, v$ have different neighbours in $A$; and since they both have exactly two neighbours in $A$ (because $G$ is $K_4$-free) we may assume that $u$ is adjacent to $a_1, a_3$, and $v$ is adjacent to $a_2, a_3$. Hence $a_3 \in X$. Since $G$ contains no hole of length five, every vertex in $X$ is adjacent to one of $a_1, a_2$. In particular $b_3 \notin X$, and for $i = 1, 2$, if $b_i \in X$ then $R_i$ has length one.

If $b_1, b_2 \in X$, then $R_1, R_2$ both have length one; but then the subgraph induced on

$$\{a_1, a_2, a_3, b_1, b_2, u, v\}$$

is an odd antihole, a contradiction. Thus we may assume (exchanging $u, v$ if necessary) that $b_2 \notin X$. Now also $b_3 \notin X$, so $b_1 \in X$ and therefore $R_1$ has length one. Moreover the subgraph induced on $\{u, f_1, b_2, b_3\}$ is a path of length three between $u$ and $f_1$. Thus every vertex in $X$ is adjacent to one of $b_2, b_3$; and since $a_3 \in X$, it follows that $R_3$ has length one. But then $a_1, b_3, u, v$ are all adjacent to both $a_3, b_1$, and so $G$ contains either a trapeze (if $v$ is adjacent to $b_3$) or an extended 4-wheel (if $u$ is adjacent to $b_3$). This proves (1).

(2) If the second outcome of 5.2 holds then the theorem holds.

For suppose, say, $f_1$ has two adjacent neighbours in $R_1$, and $f_n$ has two adjacent neighbours in $R_2$, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K)$. Let $c_1, d_1$ be the two neighbours of $f_1$ in $R_1$, where $a_1, c_1, d_1, b_1$ are in order in $R_1$, and choose $c_2, d_2 \in V(R_2)$ similarly. Suppose that $w$ is adjacent to all of $c_1, d_1, c_2, d_2$. Thus $n$ is odd, since $w-c_1-f_1-\cdots-f_n-d_2-w$ is not an odd hole. If $n = 1$, then the subgraph induced on the set of common neighbours of $f_1, w$ has two disjoint edges, and so $G$ contains a trapeze, trestle or octahedron, a contradiction. Thus $n \geq 3$ and the theorem holds. Consequently we may assume that $w$ is not adjacent to $c_1$ say, and so $w$ cannot be linked
onto the triangle \( \{c_1, d_1, f_1\} \). Suppose that \( w \) is adjacent to both \( c_2, d_2 \). From 5.3 applied to the hole induced on \( V(R_1) \cup V(R_2) \) and the path \( f_1 \cdots f_n \), it follows that \( w \) has no more neighbours in \( V(R_1 \cup R_2) \), and since \( w \) is adjacent to at least one of \( a_1, a_2 \) and at least one of \( b_1, b_2 \), we deduce that \( R_2 \) has length one, and \( w \) is adjacent to \( a_3, b_3 \). But then \( R_1 \) is odd (since \( R_2 \) is odd), and so \( w-a_3-a_1-R_1-b_1-b_2-w \) is an odd hole, a contradiction. Thus \( w \) is adjacent to at most one of \( c_2, d_2 \), and therefore cannot be linked onto \( \{c_2, d_2, f_n\} \).

For \( i = 1, 2 \), let \( C_i, D_i \) be the subpaths of \( R_i \) between \( a_i, c_i \) and between \( d_i, b_i \) respectively. Suppose that \( w \) has a neighbour in \( V(C_1) \setminus \{a_1\} \). Since \( w \) cannot be linked onto \( \{c_1, d_1, f_1\} \), it follows that \( w \) is nonadjacent to \( b_1, a_2 \). Since \( w \) is major, it is adjacent to \( b_2, b_3 \), and to \( a_1, a_3 \). Thus \( w \) can be linked onto \( \{c_2, d_2, f_n\} \), a contradiction. It follows that \( w \) has no neighbour in \( R_1 \), and similarly none in \( R_2 \).

Suppose that \( w \) is nonadjacent to both \( a_1, b_1 \). Then \( w \) is adjacent to \( a_2, a_3, b_2, b_3 \), and \( K \) is even. From the symmetry we may assume that \( a_2 \neq c_2 \); but \( a_2 \) can be linked onto \( \{c_2, d_2, f_n\} \), via paths with interiors in \( V(C_1) \cup \{f_1, \ldots, f_n\}, V(C_2) \) and \( \{w\} \cup V(D_2) \), a contradiction. Thus \( w \) is adjacent to at least one of \( a_1, b_1 \), and similarly to at least one of \( a_2, b_2 \). If \( w \) is adjacent to all of \( a_1, a_2, b_1, b_2 \) then the theorem holds, so we may assume that \( w \) is nonadjacent to \( b_1 \). Hence \( w \) is adjacent to \( a_1, b_2, b_3 \). From the hole \( w-a_1-R_1-b_1-b_3-w \) it follows that the prism is odd, and hence \( R_3 \) is odd. Let \( Q \) be the path

\[
a_1-C_1-c_1-f_1-\cdots-f_n-d_2-D_2-b_2.
\]

From the hole \( a_1-Q-b_2-b_3-R_3-a_3-a_1 \) it follows that \( Q \) is odd; but then \( w-a_1-Q-b_2-w \) is an odd hole, a contradiction. This proves (2).

(3) *If the third outcome of 5.2 holds then the theorem holds.*

Suppose the third outcome of 5.2 holds; so \( n \geq 2 \), and, say, \( f_1 \) is adjacent to \( a_1, a_2 \), and \( f_n \) is adjacent to \( b_1, b_2 \), and there are no other edges between \( \{f_1, \ldots, f_n\} \) and \( V(K) \). If \( w \) is adjacent to \( a_1, b_1, a_2, b_2 \), then the theorem holds, so we assume that \( w \) is nonadjacent to \( a_1 \) say. Hence \( w \) is adjacent to \( a_2, a_3 \). By 5.3 applied to the prism \( K' \) formed by \( R_1, R_2 \) and \( f_1, \ldots, f_n \), it follows that \( w \) is nonadjacent to one of \( b_1, b_2 \), say \( b_i \), and therefore adjacent to \( b_3 \). Since \( w-a_2-f_1-\cdots-f_n-b_i-b_3-w \) is not an odd hole, it follows that either \( n \) is even or \( a_2, b_2 \) are adjacent; and in either case \( K' \) is odd and therefore \( n \) is even.

For \( j = 1, 2 \), since \( w-a_2-f_1-\cdots-f_n-b_j-w \) is not an odd hole, it follows (from \( j = 1 \)) that \( b_1 \) is nonadjacent to \( w \), and (from \( j = 2 \)) that \( R_2 \) has length one. Since \( w-a_3-a_1-R_1-b_1-b_2-w \) is not an odd hole, \( w \) has a neighbour \( c_1 \in R_1^* \). Since \( w \) cannot be linked onto \( \{a_1, a_2, f_1\} \), it follows that \( c_1 \) is adjacent to \( b_1 \). But similarly \( c_1 \) is adjacent to \( a_1 \), contradicting that \( R_1 \) has odd length. This proves (3).

(4) *If the fourth outcome of 5.2 holds then the theorem holds.*

Suppose that \( f_1, \ldots, f_n \) is a corner jump in position \( a_3 \), say. If \( w \) is adjacent to both \( a_1, a_2 \) then the theorem holds, so we may assume that \( w, a_1 \) are nonadjacent. Thus \( w \) is adjacent to \( a_2, a_3 \). Let \( R_3' \) be an induced path between \( f_1 \) and \( b_3 \) with interior in \( \{f_2, \ldots, f_n\} \cup V(R_3) \). Then \( R_1, R_2, R_3' \) form a prism \( K' \), and by three applications of 5.3 applied to the three holes of this prism, we deduce that \( w \) is nonadjacent to one of \( b_1, b_2 \), and nonadjacent to one of \( b_2, b_3 \) (and hence adjacent to \( b_1, b_3 \)), and has no neighbours in \( R_1 \cup R_3' \) except \( b_1, b_3 \). Consequently \( K' \) (and therefore \( K \)) is odd. If \( a_3 \) has no
neighbour in $R'_3$, then $w-a_3-a_1-f_1-R'_3-b_3-w$ is an odd hole, a contradiction; so $a_3$ has a neighbour in $R'_3$, and hence there is an induced path $Q$ between $a_3, b_3$ with interior in $V(R'_3)$. In particular, $w$ has no neighbour in $Q'$, and $Q$ is odd; and since $w-a_3-Q-b_3-w$ is not an odd hole, we deduce that $a_3, b_3$ are adjacent. But then the theorem holds.

From 5.2 and (1)–(4), this proves 5.4. □

6 Prisms with balanced vertices

Let $K$ be a prism in a graph $G$, formed by paths $R_i$ with ends $a_i, b_i$ ($1 \leq i \leq 3$) as usual. We say a major vertex $w$ is balanced if there are two values of $i \in \{1, 2, 3\}$ such that $w$ is adjacent to both $a_i, b_i$; and $w$ is clear if it is anticomplete to $V(R_i)$ for some $i \in \{1, 2, 3\}$. (Thus a clear major vertex is balanced.) In this section we prove that if $G$ is a $K_4$-free Berge graph, containing no even pair and no trampoline, then no prism in $G$ has a balanced major vertex. A $4$-wheel is the graph obtained from a cycle of length four by adding one more vertex adjacent to every vertex of the cycle. We need:

6.1 Let $G$ be a $K_4$-free Berge graph containing no trapeze or trestle. Let $K$ be a prism in $G$, and let $A, B$ and $R_i, a_i, b_i$ ($i = 1, 2, 3$) be as before. Let $w \in V(G) \setminus V(K)$ be major with respect to $K$. Suppose that either $w$ is balanced, or $G$ does not contain a $4$-wheel. Let $w$ be nonadjacent to $a_3$, and let $a_3-p_1, \ldots, p_k-w$ be an induced path from $a_3$ to $w$. Suppose that the set of attachments in $K$ of \{p$1, \ldots, p$k$\} is local. Then $k$ is odd.

**Proof.** Suppose that $k$ is even. Let $X$ be the set of attachments in $K$ of \{p$1, \ldots, p$k$-1\}. For $i = 1, 2$, $a_i$ is adjacent to both $w, a_3$. In particular, $a_i \notin \{p_1, \ldots, p_k\}$ since $k$ is even. Moreover, since $w-a_i-a_3-p_1, \ldots, p_k-w$ is not an odd hole, it follows that $a_i$ has a neighbour in $\{p_1, \ldots, p_k\}$. Since $G$ is $K_4$-free, not both $a_1, a_2$ are adjacent to $p_k$; say $a_1$ is not adjacent to $p_k$ without loss of generality. Thus $a_1 \in X$; and since $a_3 \in X$ and $X$ is local, we deduce that $X \subseteq A$. In particular, $p_1, \ldots, p_k-1 \notin V(K)$. If $p_k \in V(K)$, then $p_k \in X$ since it is adjacent to $p_{k-1}$, and hence $p_k \in A$, which is impossible. Thus none of the vertices $p_1, \ldots, p_k, w$ belong to $V(K)$.

Now $w$ is adjacent to at least one of $b_2, b_3$; let $R$ be the induced path between $w$ and $a_3$ with interior in $V(R_3) \cup \{b_2\}$. Since $w-R-a_3-a_1-w$ is a hole, it follows that $R$ is even. Consequently

\[w-R-a_3-p_1, \ldots, p_k-w\]

is not a hole (since it would be odd), and since no vertex in $V(R) \setminus \{a_3\}$ belongs to $X$, it follows that $p_k$ has a neighbour in $V(R) \setminus \{a_3\}$. Let $R'$ be the induced path between $p_k$ and $a_3$ with interior in $V(R)$. Then $a_3-p_1, \ldots, p_k-R'-a_3$ is a hole, and so $R'$ is even. Consequently $w-p_k-R'-a_3-a_1-w$ is not a hole, and therefore $w$ has a neighbour in the interior of $R'$. We deduce that the neighbour of $w$ in $R$, and the neighbour of $p_k$ in $R'$, are the same vertex $q$ say. Suppose that $q = b_2$. Then $w, p_k$ are both anticomplete to $V(R_3)$, and therefore $R_3$ is even; and since

\[w-b_1-b_3-R_3-a_3-p_1, \ldots, p_k-w\]

is not a hole (because it would have odd length), and $b_1 \notin X$, we deduce that $w, p_k$ are both adjacent to $b_1$, and so $b_1, b_2, w, p_k$ are pairwise adjacent, a contradiction. Consequently $q \neq b_2$. Since we cannot
link \( a_1 \) onto \( \{w, p_k, q\} \), via \( a_1w \) and two paths with interiors in \( V(R_3), \{p_1, \ldots, p_{k-1}\} \) respectively, it follows that \( p_1 \) is the only neighbour of \( a_1 \) in \( \{p_1, \ldots, p_k\} \). Since \( G \) is \( K_4 \)-free, \( a_2 \) is nonadjacent to \( p_1 \), and so we can link \( a_2 \) onto \( \{w, p_k, q\} \); and so \( a_2, p_k \) are adjacent. Thus the set of attachments in \( K \) of \( \{p_k\} \) is not local.

Let us apply 5.2 setting \( F = \{p_k\} \). Now \( p_k \) is not major, since it has only one neighbour in \( A \), and the third outcome of 5.2 does not hold since \( |F| = 1 \). Suppose that the second outcome of 5.2 holds; so \( p_k \) has two adjacent neighbours in \( R_2 \) (namely, \( a_2 \) and its neighbour in \( R_2 \)) and two adjacent neighbours in \( R_3 \) (namely, \( q \) and its neighbour in \( R_3 \) between \( q \) and \( b_3 \); this is only possible if \( q \neq b_3 \)), and \( p_k \) has no other neighbours in \( V(K) \). But then we can link \( p_k \) onto \( \{a_1, a_3, p_1\} \), via paths with interiors in \( (V(R_2) \setminus \{a_2\}) \cup V(R_3), V(R_1) \setminus \{b_1\} \), and \( \{p_1, \ldots, p_{k-1}\} \), a contradiction. We deduce that the fourth outcome of 5.2 holds, and so the one-vertex path \( p_k \) is a corner jump. Since \( p_k \) has a neighbour in \( V(R_3) \setminus \{a_3\} \), and is adjacent to \( a_2 \) and not to \( a_1, a_3 \), it follows that \( p_k \) is a corner jump in position \( b_2 \), and \( q = b_3 \). Since \( p_k, w, b_1, b_3 \) are not all pairwise adjacent, it follows that \( w \) is nonadjacent to \( b_1 \), and therefore adjacent to \( b_2 \). But then \( w \) is not balanced with respect to \( K \), and yet the subgraph induced on \( B \cup \{w, p_k\} \) is a 4-wheel, a contradiction. This proves 6.1.

Next we show:

6.2 Let \( G \) be a \( K_4 \)-free Berge graph with no even pair and no trampoline. If \( K \) is a prism in \( G \), then no major vertex is balanced with respect to \( K \).

**Proof.** Suppose that there is a prism with a balanced major vertex; and if possible choose one with a clear major vertex. Thus we have chosen a vertex \( w \), and two paths \( R_1, R_2 \), with ends \( a_i, b_i \) for \( i = 1, 2 \), such that

- \( R_1, R_2 \) both have length at least one, and are disjoint, and \( w \notin V(R_1 \cup R_2) \)
- \( a_1a_2 \) and \( b_1b_2 \) are edges, and there are no other edges between \( V(R_1) \) and \( V(R_2) \)
- \( w \) is adjacent to \( a_1, a_2, b_1, b_2 \)
- there is a path \( R_3 \) with ends \( a_3, b_3 \), with \( V(R_3) \) disjoint from \( V(R_1 \cup R_2) \cup \{w\} \), such that \( a_3 \) is adjacent to \( a_1, a_2 \), and \( b_3 \) is adjacent to \( b_1, b_2 \), and there are no other edges between \( R_3 \) and \( R_1 \cup R_2 \)
- if there is a prism in \( G \) with a clear major vertex, then \( w \) has no neighbour in \( R_3 \).

Consequently, we may choose three sets \( A, B, C \), pairwise disjoint and each disjoint from \( V(R_1 \cup R_2) \cup \{w\} \), such that

- every vertex in \( A \) is complete to \( \{a_1, a_2\} \) and has no other neighbours in \( R_1 \cup R_2 \)
- every vertex in \( B \) is complete to \( \{b_1, b_2\} \) and has no other neighbours in \( R_1 \cup R_2 \)
- no vertex in \( C \) has a neighbour in \( R_1 \cup R_2 \)
- for every vertex \( v \in A \cup B \cup C \), there is an induced path containing \( v \) with one end in \( A \) and the other end in \( B \), and with interior in \( C \)
A, B \neq \emptyset

- if there is a prism in \( G \) with a clear major vertex, then \( w \) has no neighbour in \( C \).

Since \( G \) is \( K_4 \)-free, it follows that \( A, B \) are stable, and \( w \) is anticomplete to \( A \cup B \). Choose such a triple \((A, B, C)\) with \( A \cup B \cup C \) maximal. If \( R \) is an induced path with one end in \( A \) and the other end in \( B \), and with interior in \( C \), we call \( R \) a rung. Let \( W = A \cup B \cup C \cup V(R_1) \cup V(R_2) \).

(1) Let \( p_0-p_1-\cdots-p_k \) be an induced path such that \( p_0 \in A \) and \( p_1, \ldots, p_k \notin A \cup B \), and \( w \) is nonadjacent to \( p_0, \ldots, p_k \). Let \( X \) be the set of vertices in \( W \) that either belong to \( \{p_1, \ldots, p_k\} \) or are adjacent to some vertex in \( \{p_1, \ldots, p_k\} \). Then either \( X \subseteq A \cup B \cup C \), or \( X \subseteq A \cup \{a_1, a_2\} \).

For suppose not, and choose \( k \) minimum such that the claim is false. From the minimality of \( k \) it follows that \( p_1, \ldots, p_k \notin V(R_1 \cup R_2) \), and from the hypothesis we have \( p_1, \ldots, p_k \notin A \cup B \). (They might belong to \( C \), however.) For \( 1 \leq i \leq k \) let \( X_i \) denote the set of vertices in \( W \) that either belong to \( \{p_1, \ldots, p_k\} \) or are adjacent to some vertex in \( \{p_1, \ldots, p_k\} \). Thus \( X_1 = X \) and is not a subset of \( A \cup B \cup C \), and not a subset of \( A \cup \{a_1, a_2\} \). Since \( p_0 \in A \cap X_1 \), it follows that \( X_1 \) is not a subset of any of \( A \cup B \cup C \), \( A \cup \{a_1, a_2\} \), \( B \cup \{b_1, b_2\} \), \( V(R_1) \), \( V(R_2) \). Choose \( h \leq k \) maximum such that \( X_h \) is not a subset of any of these five sets.

Suppose that \( p_j \in C \) for some \( j \) with \( h \leq j \leq k \). Since \( X_h \not\subseteq A \cup B \cup C \), there exists \( i \) with \( h \leq i \leq k \) such that some vertex \( y \in V(R_1 \cup R_2) \) is adjacent to \( p_i \). Since one of \( p_1, \ldots, p_{k-1} \) either belongs to \( C \) or has a neighbour in \( C \), the minimality of \( k \) implies that \( i = k \). Since \( p_j \in C \) and therefore is nonadjacent to \( y \), we deduce that \( j < k \). But then \( p_j, y \in X_{j+1} \), contrary to the maximality of \( h \). This proves that \( p_h, \ldots, p_k \notin C \), and therefore \( p_h, \ldots, p_k \notin W \).

Choose a rung \( R_3 \) with ends \( a_3 \in A \) and \( b_3 \in B \), such that the set of attachments of \( \{p_h, \ldots, p_k\} \) in the prism \( K \) formed by \( R_1, R_2, R_3 \) is not local. By 4.1, 4.2, 4.4 and 4.3, \( G \) contains no trapeze, trestle, octahedron or extended 4-wheel. From 5.4, we deduce that one of the five outcomes of 5.4 holds; and from the minimality of \( k \) and the maximality of \( h \), the path \( f_1 \cdots f_k \) of 5.4 is either the path \( p_h \cdots p_k \) or its reverse.

Suppose the first outcome holds; then \( k - h + 1 \geq 3 \) is odd, and for some distinct \( i, j \in \{1, 2, 3\} \), \( p_h \) has two adjacent neighbours \( c_i, d_i \) in \( R_i \), and \( p_k \) has two adjacent neighbours \( c_j, d_j \) in \( R_j \), and there are no other edges between \( \{p_h, \ldots, p_k\} \) and \( V(K) \), and \( w \) is adjacent to all of \( c_i, d_i, c_j, d_j \). The minimality of \( k \) implies that not both \( i, j \in \{1, 2\} \); and so \( w \) has neighbours in \( R_3 \). Yet \( w \) is a clear major vertex with respect to the prism induced on \( V(R_i \cup R_j) \cup \{p_h, \ldots, p_k\} \), contrary to the choice of \( R_1, R_2, w \).

Suppose the second outcome of 5.4 holds; then \( K \) is an even prism, and for some distinct \( s, t \in \{1, 2, 3\} \), \( p_h \) has two adjacent neighbours \( c_s, d_s \) in \( R_s \), and \( p_k \) has two adjacent neighbours \( c_t, d_t \) in \( R_t \), and there are no other edges between \( \{p_h, \ldots, p_k\} \) and \( V(K) \), and \( w \) is adjacent to \( a_s, b_s, a_t, b_t \) and nonadjacent to every internal edge of \( R_s \) and of \( R_t \). Since \( w \) is balanced it follows that \( \{s, t\} = \{1, 2\} \), and since none of \( p_1, \ldots, p_{k-1} \) has a neighbour in \( V(R_1 \cup R_2) \setminus \{a_1, a_2\} \), it follows that \( h = k \). We may assume that for \( i = 1, 2 \), \( a_i, c_i, d_i, b_i \) are in order in \( R_i \). For \( i = 1, 2 \), let \( C_i, D_i \) be the subpaths of \( R_i \) between \( a_i, c_i \) and between \( d_i, b_i \) respectively. If \( a_1 \neq c_1 \), then we can link \( a_1 \) onto \( \{p_k, c_1, d_1\} \) via paths with interiors in \( \{w\} \cup V(D_1), V(C_1) \), and \( \{a_3\} \cup \{p_1, \ldots, p_k\} \), a contradiction. Thus \( a_1 = c_1 \) and similarly \( a_2 = c_2 \). Since \( K \) is even, it follows that

\[
p_k-d_1-D_1-b_1-b_2-D_2-d_2-p_k
\]
is an odd hole, a contradiction.

Suppose the third outcome of 5.4 holds; then \( k > h \), and since \( w \) is nonadjacent to \( a_3, b_3 \), one of \( p_h, p_k \) is adjacent to \( a_1, a_2 \), and the other to \( b_1, b_2 \), and there are no other edges between \( \{p_h, \ldots, p_k\} \) and \( V(K) \). From the minimality of \( k \), \( p_h \) is nonadjacent to both \( b_1, b_2 \); so \( p_k \) is adjacent to \( b_1, b_2 \), and \( p_h \) to \( a_1, a_2 \). But then we can add \( p_h \) to \( A \) and \( p_k \) to \( B \) and \( p_{h+1}, \ldots, p_{k-1} \) to \( C \), contrary to the maximality of \( A \cup B \cup C \).

Suppose the fourth outcome of 5.4 holds; then one of \( p_h, \ldots, p_k, p_{k+1}, \ldots, p_h \) is a corner jump in one of the six positions, say position \( x_i \in \{a_i, b_i\} \). There is no \( j \in \{1, 2, 3\} \setminus \{i\} \) such that \( w \) is adjacent to just one end of \( R_j \); and so from the fourth outcome of 5.4, it follows that \( w \) is nonadjacent to \( x_i \), and so \( i = 3 \). But then we can add \( p_h, \ldots, p_k, p_{k+1}, \ldots, p_h \) to \( A \) or \( B \) or \( C \) (in the appropriate way, depending whether \( x_3 = a_3 \) or \( b_3 \)), and depending whether the corner jump is \( p_k, \ldots, p_h \) or \( p_h, \ldots, p_k \) contrary to the maximality of \( A \cup B \cup C \).

We have shown then that none of the outcomes of 5.4 holds, which is impossible; and this proves (1).

(2) If \( P \) is an induced path with both ends in \( A \cup B \) such that \( w \) is anticomplete to \( V(P) \), then \( P \) has even length.

We proceed by induction on the length of \( P \). If some internal vertex of \( P \) belongs to \( A \cup B \), then the result follows from the inductive hypothesis, so we may assume that \( P \) is \( p_0p_1, \ldots, p_{k+1} \) say, where \( p_0 \in A \), and \( p_{k+1} \in A \cup B \), and \( p_1, \ldots, p_k \not\in A \cup B \). Let \( X \) be the set of vertices in \( W \) that belong to \( \{p_1, \ldots, p_k\} \) or have a neighbour in this set. By (1), either \( X \subseteq A \cup \{a_1, a_2\} \), or \( X \subseteq A \cup B \cup C \). Suppose first that \( p_{k+1} \in B \). Since \( p_{k+1} \in X \), it follows that \( X \subseteq A \cup B \cup C \), and so \( w-a_1-p_0-p_{k+1}-b_2-w \) is a hole, and therefore \( P \) has even length. Thus we may assume that \( p_{k+1} \in A \). If \( a_1-p_0-p_{k+1}-a_1 \) is a hole then again \( P \) has even length, so we may assume that \( a_1 \in X \); and so \( X \not\subseteq A \cup B \cup C \), and therefore \( X \subseteq A \cup \{a_1, a_2\} \). But there is an induced path \( Q \) joining \( p_0, p_{k+1} \) with interior in \( B \cup C \cup \{b_2\} \), and it has even length since it can be completed to a hole via \( p_{k+1}-a_1-p_0 \). Since \( P \cup Q \) is a hole, it follows that \( P \) has even length. This proves (2).

Since \( G \) has no even pair, there is an odd induced path between some vertex of \( A \cup B \) and \( w \). Choose such a path as short as possible. By (2), none of its internal vertices belong to \( A \cup B \). Let this path be \( a_3-p_1, \ldots, p_k-w \) say, where \( a_3 \in A_3 \). Choose a rung \( R_3 \) with \( a_3 \) as one end, and let \( K \) be the prism formed by \( R_1, R_2, R_3 \). By 6.1 applied to \( a_3-p_1, \ldots, p_{k-1} \), the set of attachments of \( \{p_1, \ldots, p_{k-1}\} \) in \( K \) is not local. But this contradicts (1). This proves 6.2.

A square in \( G \) is a hole of length four. We deduce:

6.3 Let \( G \) be a \( K_4 \)-free Berge graph with no even pair and no trampoline. Then \( G \) contains no 4-wheel.

**Proof.** Suppose that \( G \) contains a 4-wheel, and let \( a_1-b_1-a_2-b_2-a_1 \) be a square in \( G \), and let \( c \) be adjacent to \( a_1, a_2, b_1, b_2 \). Since \( a_1, a_2 \) is not an even pair, there is an odd induced path \( a_1-p_1, \ldots, p_{k-1}a_1 \); and therefore \( b_1, b_2, c \not\in \{p_1, \ldots, p_k\} \). Suppose that there is an edge \( uv \) of the path \( a_1-p_1, \ldots, p_{k-1}a_1 \) such that \( \{u, v\} \) is complete to \( \{b_1, b_2\} \). From the symmetry we may assume that \( u, v \neq a_2 \). Since \( \{a_2, c, u, v\} \) is complete to \( \{b_1, b_2\} \), and therefore includes no triangle, it follows that \( G \) contains
a trapeze, trestle, or octahedron, a contradiction. Thus there is no such edge $uv$. We claim that 
\{b_1, b_2\} is a leap for the path $a_1-p_1-\cdots-p_k-a_1$. This follows from 2.1 if $k \geq 3$, and so we may assume

that $k = 2$, and therefore neither of $p_1, p_2$ is complete to \{b_1, b_2\}. But each of $b_1, b_2$ is adjacent to at

least one of $p_1, p_2$ since $G$ has no hole of length five; and so again \{b_1, b_2\} is a leap. Thus we may

assume that $b_1$ is adjacent to $p_1$, and $b_2$ to $p_2$, and there are no other edges between \{b_1, b_2\} and

\{p_1, \ldots, p_k\}. But then the paths $p_1-\cdots-p_k, a_1b_2$ and $b_1a_2$ form a prism and $c$ is a balanced major

vertex with respect to it, contrary to 6.2. This proves 6.3. 

7 Prisms with major-general vertices

Let $K$ be a prism in a graph $G$, formed by paths $R_i$ with ends $a_i, b_i$ ($1 \leq i \leq 3$) as usual. A vertex $w \in V(G) \setminus V(K)$ is said to be major-general with respect to $K$ if it is major and there exists $i \in \{1, 2, 3\}$ such that $R_i$ has length at least two and $w$ is adjacent to both ends of $R_i$. Our next objective is to extend 6.2, proving the analogous theorem for major-general vertices, the following.

7.1 Let $G$ be a $K_4$-free Berge graph with no even pair and no trampoline. If $K$ is a prism in $G$, then no vertex is major-general with respect to $K$.

Proof. Suppose that there is a prism with a major-general vertex $w$. Then there is an induced path $R_3$ with length at least two, with ends $a_3, b_3$, and two other vertices $a_2, b_1$, and nine pairwise disjoint subsets $A_i, C_i, B_i$ ($1 \leq i \leq 3$) of $V(G) \setminus \{w\}$, satisfying

- $a_2-a_3-R_3-b_3-b_1$ is an induced path
- $A_i = \{a_i\}$ for $i = 2, 3$; $B_i = \{b_i\}$ for $i = 1, 3$; $C_3$ is the set of internal vertices of $R_3$
- for $1 \leq i < j \leq 3$, $A_i$ is complete to $A_j$, and $B_i$ is complete to $B_j$, and there are no other edges between $A_i \cup B_i \cup C_i$ and $A_j \cup B_j \cup C_j$
- for $1 \leq i \leq 3$ and every vertex $v \in A_i \cup B_i \cup C_i$, there is an $i$-rung containing $v$, where an $i$-rung

means an induced path with one end in $A_i$ and the other end in $B_i$, and with interior in $C_i$
- $w$ is adjacent to $a_2, a_3, b_1, b_3$, and
- $A_1, B_2$ are nonempty.

(To see this, note that since $w$ is major-general with respect to some prism, and not balanced, we may assume in the usual notation that $w$ is adjacent to $a_2, b_1, a_3, b_3$, and $R_3$ has length at least two, and then the claim follows.) Let $W$ be the union of the nine sets $A_i, C_i, B_i$ ($1 \leq i \leq 3$), and choose $A_1, C_1, C_2, B_2$ such that $W$ is maximal.

1. Let $p_0, p_1, \ldots, p_k$ be an induced path such that $p_0 \in A_1$ and $p_1, \ldots, p_k \notin A_1 \cup B_1$, and $w$ is nonadjacent to $p_0, \ldots, p_k$. Let $X$ be the set of vertices in $W$ that either belong to $\{p_1, \ldots, p_k\}$ or are adjacent to some vertex in $\{p_1, \ldots, p_k\}$. Then either $X \subseteq A_1 \cup B_1 \cup C_1$, or $X \subseteq A_1 \cup \{a_2, a_3\}$.

For suppose not, and choose $k$ minimum such that the claim is false, and choose $h \leq k$ as in
step (1) of the proof of 6.2. As in that proof, it follows that \(p_h, \ldots, p_k \notin W\), and there is a prism \(K\), formed by a 1-rung \(R_1\), a 2-rung \(R_2\), and the path \(R_3\), such that the set of attachments of \(\{p_h, \ldots, p_k\}\) in \(K\) is not local. Choose \(a_1, b_2\) such that for \(i = 1, 2, 3\), the ends of \(R_i\) are \(a_i, b_i\). Again, one of the outcomes of 5.4 holds.

The first outcome does not hold since \(G\) contains no prism with respect to which \(w\) is balanced, by 6.2. The second and third outcomes do not hold since \(w\) is not balanced with respect to \(K\). Thus the fourth outcome of 5.4 holds; so one of \(p_h \cdots - p_k, p_k \cdots - p_h\) is a corner jump in one of the six positions, say position \(x\).

Suppose first that \(x = a_1\), and so one of \(p_h \cdots - p_k, p_k \cdots - p_h\) is a corner jump in position \(a_1\) with respect to \(K\). If \(\{p_h, \ldots, p_k\}\) is anticomplete to \(B_2 \cup C_2\) then we can either add \(p_h\) to \(A_1\) and \(p_{h+1}, \ldots, p_k\) to \(C_1\), or add \(p_k\) to \(A_1\) and \(p_h, \ldots, p_{k-1}\) to \(C_1\) (depending whether \(p_h \cdots - p_k\) or \(p_k \cdots - p_h\) is the corner jump with respect to \(K\)), a contradiction to the maximality of \(W\). Thus there is a 2-rung \(R'_2\) with ends \(a_2, b'_2\) say, such that one of \(p_h, \ldots, p_k\) has a neighbour in \(V(R'_2) \setminus \{a_2\}\). From the minimality of \(k\), no vertex in \(\{p_1, \ldots, p_{k-1}\}\) has a neighbour in \(V(R'_2) \setminus \{a_2\}\); so \(p_k\) has such a neighbour. If \(p_k\) is adjacent to \(a_2, a_3\), then the prism formed by \(R_1, R'_2, R_3\) does not satisfy 5.3, since \(p_k\) has at most one neighbour in \(\{b_1, b'_2, b_3\}\). Thus \(h < k\), and \(p_h\) is adjacent to \(a_2, a_3\), and \(p_k\) has a neighbour in \(V(R_1) \setminus \{a_1\}\) and a neighbour in \(V(R'_2) \setminus \{a_2\}\). If \(p_k\) has a neighbour in \(R'_2\) different from \(b'_2\), we can link \(p_k\) onto \(\{p_h, a_2, a_3\}\) via paths with interiors in \(\{p_{h+1}, \ldots, p_{k-1}\}\), \(V(R'_2) \setminus \{b'_2\}\), and \(V(R_1) \setminus \{a_1\}\) \(\cup V(R_3)\), a contradiction. So \(b'_2\) is the only neighbour of \(p_k\) in \(R'_2\). But then we can link \(b'_2\) onto \(\{p_h, a_2, a_3\}\), via paths with interiors in \(V(R'_2)\), \(\{p_h, \ldots, p_k\}\) and \(V(R_3)\), a contradiction. Thus \(x \neq a_1\).

Suppose that \(x = b_2\). From the minimality of \(k\), no vertex in \(\{p_1, \ldots, p_{k-1}\}\) is adjacent to \(b_3\); so the corner jump is \(p_k \cdots - p_h\), and \(p_k\) is adjacent to \(b_1, b_3\), and \(p_h\) has a neighbour in \(V(R_2) \setminus \{b_2\}\). Then from the maximality of \(W\), there is a 1-rung \(R'_1\) with ends \(a'_1, b_1\), such that one of \(p_h, \ldots, p_k\), say \(p_i\), has a neighbour \(v \in V(R'_1) \setminus \{b_1\}\). If \(i = k\) then the prism formed by \(R'_1, R_2, R_3\) does not satisfy 5.3 (since \(p_k\) has at most one neighbour in \(\{a_1', a_2, a_3\}\)). Thus \(i < k\), and consequently \(h < k\). From the minimality of \(k\), \(p_h\) has no neighbour in \(R_2\) except possibly \(a_2\); and so \(p_h, a_2\) are adjacent. From the minimality of \(k\), since \(p_h\) is adjacent to \(a_2\), it follows that none of \(p_1, \ldots, p_{k-1}\) has a neighbour in \(B_1 \cup C_1\); and in particular \(v = a'_1\). But then the prism formed by \(R_1, R_3\) and the path \(a_2 \cdots p_h \cdots - p_k\) does not satisfy 5.3, since \(a'_1\) has at most one neighbour in \(\{b_1, b_3, p_k\}\) (since we have shown that \(p_i \neq p_k\) ). Thus \(x \neq b_2\).

If \(x = a_2\), there is a prism \(K'\) formed by \(R_1, R_3\) and a path starting with one of \(p_h \cdots - p_k, p_k \cdots - p_h\) and with final vertex \(b_2\), and with interior in \(V(R_2) \setminus \{a_2\}\); and this prism does not satisfy 5.3, a contradiction. Similarly \(x \neq b_1\); and so \(x \in \{a_3, b_3\}\). By the fourth outcome of 5.4, since \(w\) is adjacent to \(x\), it follows that \(R_3\) has length one, a contradiction. This proves (1).

(2) If \(P\) is an induced path with both ends in \(A_1 \cup B_2\) such that \(w\) is anticomplete to \(V(P)\), then \(P\) has even length.

We proceed by induction on the length of \(P\). If some internal vertex of \(P\) belongs to \(A_1 \cup B_2\), then the result follows from the inductive hypothesis, so we may assume that \(P\) is \(p_0 \cdots p_k\) say, where \(p_0 \in A_1\), and \(p_{k+1} \in A_1 \cup B_2\), and \(p_1, \ldots, p_k \notin A_1 \cup B_2\). Let \(X\) be the set of vertices in \(W\) that belong to \(\{p_1, \ldots, p_k\}\) or have a neighbour in this set. By (1), either \(X \subseteq A_1 \cup \{a_2, a_3\}\), or \(X \subseteq A_1 \cup B_1 \cup C_1\), and in particular, \(p_{k+1} \notin B_2\). Thus \(p_{k+1} \in A_1\). If \(a_2 \cdots p_0 \cdots - p_k \cdots - a_2\) is a hole then
again $P$ has even length, so we may assume that $a_2 \in X$; and so $X \not\subset A_1 \cup B_1 \cup C_1$, and therefore $X \subset A_1 \cup \{a_2, a_3\}$. But there is an induced path $Q$ joining $p_0, p_{k+1}$ with interior in $B_1 \cup C_1 \cup \{b_3\}$, and it has even length since it can be completed to a hole via $p_{k+1} - a_2 - p_0$. Since $P \cup Q$ is a hole, it follows that $P$ has even length. This proves (2).

Since $G$ has no even pair, there is an odd induced path between some vertex of $A_1 \cup B_2$ and $w$. Choose such a path as short as possible. By (2), none of its internal vertices belong to $A_1 \cup B_2$. Let this path be $a_1 - p_1 - \cdots - p_k - w$, where $a_1 \in A_1$ say. Choose a 1-rung $R_1$ with $a_1$ as one end, and choose a 2-rung $R_2$; and let $K$ be the prism formed by $R_1, R_2, R_3$. By 6.1 applied to $a_1 - p_1 - \cdots - p_k$, the set of attachments of $\{p_1, \ldots, p_{k-1}\}$ in $K$ is not local. But this contradicts (1). This proves 7.1.  

8 Line graphs

A cut of a graph $G$ is a partition $(A_1, X, A_2)$ of $V(G)$ such that $A_1, A_2$ are nonempty and $A_1$ is anticomplete to $A_2$; and it is a $k$-cut if $|X| \leq k$. We say $G$ is $k$-connected if $|V(G)| > k$ and there is no $(k - 1)$-cut.

A branch-vertex of a graph $H$ is a vertex with degree $\geq 3$; and a branch of $H$ means a maximal path $P$ in $H$ such that no internal vertex of $P$ is a branch-vertex. Let $J$ be a graph with minimum degree at least three. If $H$ is a subdivision of $J$ then $V(J)$ is the set of branch-vertices of $H$, and the branches of $H$ are in 1-1 correspondence with the edges of $J$ in the natural way.

If $H$ is a graph, then $L(H)$ denotes its line graph; thus $E(H) = V(L(H))$. If $J$ is 3-connected and $H$ is a bipartite subdivision of $J$, and $L(H)$ is an induced subgraph of $G$, we call $L(H)$ an appearance of $J$ in $G$. An appearance $L(H)$ of $J$ in $G$ is degenerate if $J = K_4$ and there is a cycle of $H$ of length four containing all the vertices of $J$, or $H = J = K_{3,3}$, and non-degenerate otherwise. In this section we prove the following.

8.1 Let $G$ be a 3-connected $K_4$-free Berge graph, containing no even pair and no trampoline, and no clique cutset. Suppose that there is an appearance of a 3-connected graph $J$ in $G$, nondegenerate if $J = K_4$. Then $G$ is the line graph of a bipartite graph.

If $L(H)$ is an appearance of $J$ in $G$, a vertex $w \in V(G) \setminus V(L(H))$ is major with respect to $L(H)$ if for each $v \in V(J) \subset V(H)$, there is at most one edge $x$ of $H$ incident with $v$ such that $w$ is nonadjacent to $x$ in $G$.

8.2 Let $G$ be a $K_4$-free Berge graph, containing no even pair and no trampoline. For every 3-connected graph $J$ and every appearance $L(H)$ of $J$ in $G$, no vertex is major with respect to $L(H)$.

Proof. Suppose that $w$ is major with respect to $L(H)$. There is a subgraph $H'$ of $H$ that is a bipartite subdivision of $K_4$, and $w$ is major with respect to $L(H')$. Thus if the theorem holds when $J = K_4$ then it holds in general. We therefore may assume that $J = K_4$. Let the four vertices of $J$ be $C_1, \ldots, C_4$. For all distinct $i, j \in \{1, \ldots, 4\}$, let $B_{ij} = B_{ji}$ be the branch of $H$ with ends $C_i, C_j$, let $e(i, j)$ be the edge of $B_{ij} = B_{ji}$ incident with $C_i$, and let $H_{ij} = H_{ji}$ be the subgraph of $H$ obtained by deleting the edges and interior vertices of $B_{ij}$. Let $N$ be the set of neighbours of $w$ in $V(L(H))$. Thus $N \subset E(H)$. For $1 \leq i \leq 4$, exactly two of the edges of $H$ incident with $C_i$ belong to $N$ (for at
least two are in \( N \) since \( w \) is major, and not all three since \( G \) is \( K_4 \)-free).

(1) For \( 1 \leq i < j \leq 4 \), \( N \) contains at least one of \( e(i, j), e(j, i) \).

For let \((i, j) = (1, 2)\) say. Suppose that no end-edge of \( B_{12} \) is in \( N \). Thus \( e(1, 3), e(1, 4), e(2, 3), e(2, 4) \in N \). Suppose first that \( B_{12} \) has length one, and let \( x \) be its unique edge. Then \( \{x, w\} \) is complete in \( G \) to \{\( e(1, 3), e(1, 4), e(2, 3), e(2, 4) \}\}, and so \( G \) contains a trapeze, trestle, or octahedron, a contradiction. Thus \( B_{12} \) has length at least two. Then \( L(H_{34}) \) is an induced subgraph of \( G \), and it is a prism (since \( B_{12} \) has length at least two). Moreover, since \( w \) is nonadjacent to both end-edges of \( B_{12} \), we deduce that \( w \) is a balanced major vertex with respect to this prism, contrary to 6.2. This proves (1).

We may assume that \( e(1, 2), e(1, 3) \in N \), and therefore \( e(1, 4) \notin N \). By (1), \( e(4, 1) \in N \). From the symmetry between \( e_2 \) and \( e_3 \), we may assume that \( e(4, 3) \notin N \), and hence \( e(3, 4), e(4, 2) \in N \).

Since \( L(H_{12}) \) is a prism (since \( B_{34} \) has at least two edges) and \( w \) is not major-general with respect to this prism, by 7.1, it follows that \( e(3, 1) \notin N \), and \( e(3, 2) \in N \), and \( B_{23} \), \( B_{24} \) both have length one (and so \( e(2, 1) \notin N \)). But then \( w \) is major-general with respect to the prism \( L(H_{24}) \), a contradiction. This proves 8.2.

An appearance \( L(H) \) of \( J \) in \( G \) is \textit{overshadowed} if there is a branch \( B \) of \( H \) with odd length \( \geq 3 \), and a vertex \( \text{win} V(G) \setminus V(L(H)) \), such that for each end \( b \) of \( B \) in \( H \), there is at most one edge of \( H \) that is incident with \( b \) in \( H \) and nonadjacent to \( w \) in \( G \).

\textbf{8.3} Let \( G \) be a \( K_4 \)-free Berge graph, containing no even pair and no trampoline. For every 3-connected graph \( J \), there is no overshadowed appearance of \( J \) in \( G \).

\textbf{Proof.} Suppose \( L(H) \) is an overshadowed appearance of \( J \) in \( G \), and let \( B, w \) be as above. Let the ends of \( B \) in \( H \) be \( b_1, b_2 \). Since \( J \) is 3-connected, there are three paths \( P_1, P_2, P_3 \) of \( H \) between \( b_1, b_2 \), vertex-disjoint except for \( b_1, b_2 \), where \( P_3 = B \). Let \( H' \) be the union of these paths; then \( L(H') \) is an even prism (since \( B \) has odd length) and \( w \) is a major (and therefore major-general) vertex with respect to it, contrary to 7.1. This proves 8.3.

Let \( J \) be a 3-connected graph. A \textit{J-strip system} \((S, N)\) in a graph \( G \) consists of a subset \( S_{uv} = S_{vu} \subseteq V(G) \) for each edge \( uv \) of \( J \), and a subset \( N_v \subseteq V(G) \) for each vertex \( v \) of \( J \), satisfying the following conditions:

- The sets \( S_{uv} \) (\( uv \in E(J) \)) are pairwise disjoint.
- For each \( u \in V(J) \), \( N_u \subseteq \bigcup(S_{uv} : v \in V(J) \) adjacent to \( u \)).
- For each \( uv \in E(J) \), every vertex of \( S_{uv} \) is in a \( uv \)-rung (a \textit{uv}-rung is an induced path \( R \) of \( G \) with ends \( s, t \) say, where \( V(R) \subseteq S_{uv} \), and \( s \) is the unique vertex of \( R \) in \( N_u \), and \( t \) is the unique vertex of \( R \) in \( N_v \)).
- If \( uv, wx \in E(J) \) with \( u, v, w, x \) all distinct, then there are no edges between \( S_{uv} \) and \( S_{wx} \).
- If \( uv, uw \in E(J) \) with \( v \neq w \), then \( N_u \cap S_{uv} \) is complete to \( N_u \cap S_{uw} \), and there are no other edges between \( S_{uv} \) and \( S_{uw} \).
• For each \( uv \in E(J) \) there is a special \( uv \)-rung such that for every cycle \( C \) of \( J \), the sum of the lengths of the special \( uv \)-rungs for \( uv \in E(C) \) has the same parity as \( |V(C)| \).

We define \( V(S, N) = \bigcup (S_{uv} : uv \in E(J)) \). If \( u, v \in V(J) \) are adjacent, we define \( N_{uv} = N_u \cap S_{uv} \). So every vertex of \( N_u \) belongs to \( N_{uv} \) for exactly one \( v \). Note that \( N_{uv} \) is in general different from \( N_{vu} \), but \( S_{uv} \) and \( S_{vu} \) mean the same thing.

If \( L(H) \) is an appearance of \( J \) in \( G \), then since \( L(H) \) is an induced subgraph of \( G \), there is a \( J \)-strip system \((S, N)\) in \( G \), defined by setting

- for each edge \( uv \) of \( J \), \( S_{uv} \) is the set of edges of the branch of \( H \) with ends \( u, v \)
- for each \( v \in V(J) \), \( N_v \) is the set of edges of \( H \) incident with \( v \) in \( H \).

We call this the **strip system of** \( H \).

A \( J \)-strip system \((S', N')\) in \( G \) *extends* a \( J \)-strip system \((S, N)\) in \( G \) if \( V(S, N) \subseteq V(S', N') \), and \( S_{uv} \cap V(S, N) = S_{uv} \) for every \( uv \in E(J) \), and \( N_v' \cap V(S, N) = N_v \) for every \( v \in V(J) \); and a \( J \)-strip system \((S, N)\) in \( G \) is **maximal** if there is no \( J \)-strip system in \( G \) that extends \((S, N)\).

**Proof of 8.1.**

Choose a 3-connected graph \( J \) maximal such that there is an appearance \( L(H) \) of \( J \) in \( G \), non-degenerate if \( J = K_4 \). (Thus \( E(H) \subseteq V(G) \).) We will prove that \( G = L(H) \). Since \( G \) is \( K_4 \)-free it follows that \( J \) has maximum degree three. Since \( L(H) \) is an appearance of \( J \) in \( G \), we may choose a maximal \( J \)-strip system \((S, N)\) that extends the strip system of \( H \).

1. For all \( uv \in E(J) \), all \( uv \)-rungs have lengths of the same parity.

This follows from theorem 8.1 of [5].

2. For every edge \( uv \) of \( J \), if some \( uv \)-rung has length zero then \( |S_{uv}| = 1 \).

For by 8.3 and theorem 8.2 of [5] it follows that every \( uv \)-rung has length zero. Suppose that \( x, y \in S_{uv} \) are distinct. Then \( x, y \) are both complete to \( N_u \setminus N_{uv} \) and both complete to \( N_v \setminus N_{vu} \); and so \( G \) contains a trapeze, trestle or octahedron, a contradiction. Thus \( |S_{uv}| = 1 \). This proves (2).

We say \( X \subseteq V(S, N) \) is **local** (with respect to the strip system) if either \( X \subseteq N_v \) for some \( v \in V(J) \), or \( X \subseteq S_{uv} \) for some edge \( uv \in E(J) \). Let \( \mathcal{F} \) be the set of all vertex sets of components of \( G \setminus V(S, N) \).

3. For each \( F \in \mathcal{F} \) the set of attachments of \( F \) in \( V(S, N) \) is local.

This follows from theorem 8.5 of [5], because of 8.2, 8.3, the choice of \( J \), and the maximality of the strip system, using that \( L(H) \) is non-degenerate if \( J = K_4 \), and that \((S, N)\) extends the strip system of \( H \).

4. For every edge \( uv \in E(J) \), \( |N_{uv}| = 1 \).
For we prove, by induction on the length of $P$, that if $P$ is an induced path with both ends in $N_{ax}$ for some edge $ax$ of $J$ then $P$ is even. Let $a \in V(J)$, with neighbours $x, y, z$ in $J$; and suppose that $P$ is an induced path of $G$ with both ends in $N_{ax}$. If some internal vertex of $P$ belongs to $N_{ax}$ the result follows from the inductive hypothesis, and if some vertex of $P$ is in $N_{ay} \cup N_{az}$ then $P$ has length two as required; so we may assume that $P^* \cap N_a = \emptyset$. Let the vertices of $P$ be $p_1, \ldots, p_k$ in order. Since $p_1, p_k \in N_{ax} \subseteq S_{ax}$, (2) implies that every $ax$-rung has positive length, and so $N_{ax} \cap N_x = \emptyset$. Let $F_1$ be the union of all $F \in \mathcal{F}$ such that every attachment of $F$ in $V(S, N)$ belongs to $N_a$, and let $F_2$ be the union of all $F \in \mathcal{F}$ such that every attachment of $F$ is in $S_{ax}$ and some attachment is not in $N_a$. From (3), every member of $\mathcal{F}$ with an attachment in $S_{ax} \setminus N_{xa}$ is a subset of one of $F_1, F_2$. Choose $c \in N_{ay}$.

Suppose first that some vertex of $P$ belongs to $F_1$. Choose $h, j$ with $1 \leq h < j \leq k$ and $j - h$ minimum such that $p_h, p_j \notin F_1$ and there exists $i$ with $h < i < j$ and $p_i \in F_1$. It follows that $p_h, p_j \in N_a$, and therefore $i = 1$ and $j = k$. Let $R, R'$ be $ax$-rungs containing $p_1, p_k$ respectively, and let $b \in N_x \setminus N_{xa}$. Then there is an induced path $Q$ between $p_1, p_k$ with interior in $V(R) \cup V(R') \cup \{b\}$, and we claim it is even. For if $b \in V(Q)$ then $Q$ is even since $R, R'$ have the same parity by (1); and if $b \notin V(Q)$ then $Q$ is even since $Q$ can be complete to a hole via $p_k$-$c$-$p_1$. Thus in either case $Q$ is even; but $P \cup Q$ is a hole, and so $P$ is even as required.

Thus we may assume that no vertex of $P$ belongs to $F_1$. If no vertex of $P$ is in $N_{xa}$, then $P^* \subseteq F_2 \cup (S_{ax} \setminus (N_{ax} \cup N_{xa}))$, and therefore $P$ can be completed to a hole via $p_k$-$c$-$p_1$, and so $P$ is even as required. Thus we may assume that there exist $h, j \in \{2, \ldots, k-1\}$, minimum and maximum respectively such that $p_h, p_j \in N_{ax}$. (Possibly $h = j$.) From the maximality of $V(S, N)$, the internal vertices of $p_1 \cdot \cdot \cdot p_h$ belong to $S_{ax}$ (for otherwise they could be added to $S_{ax}$), and so $p_1 \cdot \cdot \cdot p_h$ is an $ax$-rung, and so $p_j \cdot \cdot \cdot p_k$. Consequently their lengths have the same parity, by (1); and from the inductive hypothesis the subpath $p_h \cdot \cdot \cdot p_j$ has even length; and so $P$ has even length. This completes the proof that $P$ has even length.

We deduce that for each edge $uv$ of $J$, any two vertices in $N_{uv}$ would be an even pair, and so $|N_{uv}| = 1$. This proves (4).

Thus each $N_v$ is a clique. If there exists $F \in \mathcal{F}$ such that the set of attachments of $F$ in $V(S, N)$ is contained in some $N_v$, then $G$ admits a clique cutset, a contradiction. For each $uv \in E(J)$, let $A_{uv}$ be the union of $S_{uv}$ and all $F \in \mathcal{F}$ such that the set of attachments of $F$ in $V(S, N)$ is a subset of $S_{uv}$. It follows that the sets $A_{uv}$ ($uv \in E(J)$) are pairwise disjoint and have union $V(G)$.

(5) For each edge $uv$ of $J$, $|A_{uv}| \leq 2$.

For every path in $G$ between $A_{uv}$ and $V(G) \setminus A_{uv}$ contains a member of $N_{uv} \cup N_{vu}$. But by (4), $|N_{uv} \cup N_{vu}| = 2$, and since $G$ is 3-connected, it follows that $|A_{uv}| \leq 2$. This proves (5).

From (5) it follows that $G = L(H)$, and so $G$ is a line graph. This proves 8.1.

9 Degenerate $K_4$’s

In this section we extend 8.1 to include the case when $G$ contains an appearance of $K_4$, but all such appearances are degenerate. This case was excluded from 8.1 so that we could apply theorem 8.5 of
9.1 Let $G$ be a $K_4$-free Berge graph, containing no even pair or trampoline, and containing no appearance of $K_{3,3}$. Let $L(H)$ be a degenerate appearance of $J$ in $G$, where $J$ is isomorphic to $K_4$. Let $V(J) = \{c_1, \ldots, c_4\}$, and for $1 \leq i < j \leq 4$ let $B_{ij}$ be the branch of $H$ with ends $c_i, c_j$. Let $c_1-c_2-c_3-c_4-c_1$ be a cycle of $H$, and let $b_1, b_2, b_3, b_4$ be the unique edges of $B_{12}, B_{23}, B_{34}, B_{14}$ respectively. Then every path in $G$ between $E(B_{13})$ and $E(B_{24})$ contains one of $b_1, \ldots, b_4$.

Proof. First, we observe that $b_1-b_2-b_3-b_4-b_1$ is a square of $G$. Let the edges of $B_{13}$ be $p_1, \ldots, p_m$ in order; thus, $p_1-\cdots-p_m$ is an induced path $P$ of $G$, and $p_1$ is adjacent to $b_1, b_4$, and $p_m$ is adjacent to $b_2, b_3$. Similarly, let the edges of $B_{24}$ form an induced path $q_1-\cdots-q_n$ (which we call $Q$) in $G$, where $q_1$ is adjacent to $b_1, b_2$, and $q_n$ to $b_3, b_4$. Since $H$ is bipartite it follows that $m, n$ are even. Suppose there is a path of $G$ between $V(P)$ and $V(Q)$ containing none of $b_1, \ldots, b_4$, and choose a minimal such path. Thus we may assume that $r_1-\cdots-r_k$ is an induced path $R$, where $r_1, \ldots, r_k \notin V(P \cup Q) \cup \{b_1, b_2, b_3, b_4\}$, and $r_1$ has neighbours in $V(P)$ and $r_k$ has neighbours in $V(Q)$, and there are no other edges between $\{r_1, \ldots, r_k\}$ and $V(P \cup Q)$. Let us choose $H$ and $R$ such that $R$ has minimum length.

(1) If $b_1, b_2$ are anticomplete to $V(R)$, then $r_1$ has exactly two neighbours in $V(P)$ and they are adjacent.

For suppose that $b_1, b_2$ are nonadjacent to $r_1, \ldots, r_k$. If $r_1$ has a unique neighbour $r_0 \in V(P)$, we can link $r_0$ onto $\{b_1, b_2, q_1\}$, a contradiction; and if $r_1$ has two nonadjacent neighbour in $V(P)$, we can link $r_1$ onto the same triangle, again a contradiction. This proves (1).

(2) At least one of $b_1, \ldots, b_4$ has a neighbour in $V(R)$.

For suppose not. By (1), $r_1$ has exactly two neighbours in $V(P)$, and they are adjacent; and similarly $r_k$ has exactly two neighbours in $Q$, and they are adjacent. But then the restriction of $G$ to $V(P \cup Q \cup R) \cup \{b_1, \ldots, b_4\}$ is the line graph of a bipartite subdivision of $K_{3,3}$, contrary to the hypothesis. This proves (2).

(3) At least two of $b_1, \ldots, b_4$ have a neighbour in $V(R)$.

For suppose that $b_1$ has a neighbour in $V(R)$, and $b_2, b_3, b_4$ do not. By (1), $r_1$ is adjacent to one of $p_2, \ldots, p_m$; and so we can link $b_1$ onto $\{b_2, b_3, p_m\}$, a contradiction. This proves (3).

(4) Either $b_1, b_3$ both have neighbours in $V(R)$, or $b_2, b_4$ both have neighbours in $V(R)$.

For suppose not; then by (3) we may assume that $b_1, b_2$ have neighbours in $V(R)$ and $b_3, b_4$ do not. By (1) and the symmetry, it follows that $r_1$ has exactly two neighbours in $V(P)$ and they are adjacent. Choose $i \in \{1, \ldots, k\}$ minimum such that $r_i$ is adjacent to one of $b_1, b_2$. If $r_i$ is adjacent to $b_1$ and not to $b_2$, then we can link $b_1$ onto $\{b_2, b_3, p_m\}$, a contradiction; and similarly $r_i$ is adjacent to both $b_1, b_2$. Let $S$ be the induced path between $b_1, b_3$ with interior in $\{r_1, \ldots, r_i, p_2, \ldots, p_m\}$. Since $b_4$ is anticomplete to $S^*$, it follows that $S$ is even; and so $b_3-S-b_1-q_1-\cdots-q_n-b_3$ is not a hole. Hence
one of $r_1, \ldots, r_i$ has a neighbour in $Q$, and therefore $i = k$. Since $b_1, b_2, q_1, r_k$ are not all pairwise adjacent, it follows that $r_k$ is nonadjacent to $q_1$, and therefore $r_k$ is adjacent to one of $q_2, \ldots, q_n$. Moreover, from the minimality of $i$, it follows that $b_1, b_2$ are nonadjacent to $r_1, \ldots, r_{k-1}$. But then we can link $r_k$ onto $\{b_3, b_4, q_n\}$, via $r_k$-$b_1$-$b_3$ and two paths with interiors in $\{r_1, \ldots, r_{k-1}, p_2, \ldots, p_m\}$ and $\{q_2, \ldots, q_{n-1}\}$, a contradiction. This proves (4).

From (4) there is a subpath $S$ of $R$ containing neighbours either of both $b_1, b_3$ or of both $b_2, b_4$. Choose such a path as short as possible. From the symmetry we may assume it contains neighbours of both $b_1, b_3$, and so $V(S)$ is the interior of an induced path between $b_1, b_3$.

(5) $S = R$, and $S$ has even length.

Let $S$ be $s_1 \cdots s_t$ say, where $b_1$-$s_1$-$\cdots$-$s_t$-$b_3$ is an induced path. Suppose first that $S$ has odd length. It follows (since $b_2$-$b_1$-$s_1$-$\cdots$-$s_t$-$b_3$-$b_2$ is not an odd hole) that $b_2$, and similarly $b_4$, have neighbours in $V(S)$. From the minimality of $S$, and the symmetry between $c_2, c_4$, we may assume that $s_1$ is the unique vertex of $S$ adjacent to $b_2$, and $s_t$ is the unique vertex of $S$ adjacent to $b_4$. If $r_1 \notin V(S)$, then the subgraph induced on $V(P \cup S) \cup \{b_1, b_2, b_3, b_4\}$ is another degenerate appearance of $K_4$ in $G$, and there is a proper subpath of $R$ with attachments in $V(P)$ and $V(S)$, contrary to our choice of $H, R$. Thus $r_1 \in V(S)$, and so $r_1$ is one of $s_1, s_t$. Consequently $r_1$ is either complete to $\{b_1, b_2\}$ or to $\{b_3, b_4\}$, and we may assume the first from the symmetry. Since $S$ is odd, it follows that $r_1$ is nonadjacent to $b_3, b_4$, and (since $k > 1$, because $S$ is odd) $r_1$ is anticomplete to $V(Q)$. Since $b_2$-$b_3$-$b_4$-$p_1$-$r_1$-$b_2$ is not an odd hole, it follows that $r_1, p_1$ are nonadjacent, and so $r_1$ has a neighbour in $\{p_2, \ldots, p_m\}$; and hence we can link $b_1$ onto $\{b_3, b_4, q_m\}$ via $b_1$-$b_4$, $b_1$-$q_1$-$\cdots$-$q_n$ and a path between $b_1, b_3$ with interior in $\{r_1, p_2, \ldots, p_m\}$, a contradiction.

Thus $S$ is even. Since $b_1$-$s_1$-$s_t$-$b_3$-$p_m$-$P$-$p_1$-$b_1$ is not an odd hole, there are edges between $V(S)$ and $V(P)$, and so $r_1 \in V(S)$, and similarly $r_k \in V(S)$, and so $R = S$. This proves (5).

From (5) and the symmetry between $b_2, b_4$, we may assume that $b_1$-$r_1$-$\cdots$-$r_k$-$b_3$ is an induced path.

(6) $k > 1$.

For suppose that $k = 1$. Thus $r_1$ is adjacent to both $b_1, b_3$, and has neighbours in both $V(P), V(Q)$. By 8.2, $r_1$ is not major with respect to $L(H)$, and so from the symmetry we may assume that $r_1$ has at most one neighbour in $\{b_3, b_4, q_n\}$. Hence $r_1$ is nonadjacent to $b_4, q_n$. By 5.3 applied to the prism induced on $V(Q) \cup \{b_1, \ldots, b_4\}$, $r_1$ is nonadjacent to $b_2$. Since $r_1$ has a neighbour in $\{q_1, \ldots, q_{n-1}\}$ (because it is nonadjacent to $q_n$), we can link $r_1$ onto $\{b_1, b_2, q_1\}$, and so $r_1$ is adjacent to $q_1$. By 5.3 applied to the same prism as before, $r_1$ has no neighbours in $Q$ except $q_1$. Since $r_1$-$q_1$-$\cdots$-$q_{n-1}$-$b_4$-$P$-$r_1$ is not an odd hole, $r_1$ is nonadjacent to $p_1$. This restores the symmetry between $p_1, q_{n-1}$, and so from the symmetry $r_1$ is adjacent to $p_m$ and has no other neighbour in $P$. But then $b_1, q_1, p_m, b_3$ are all common neighbours of $r_1, b_2$, and so $G$ contains a trapeze, a contradiction. This proves (6).

(7) Not both $b_2, b_4$ have neighbours in $R$.

For suppose they do; then from the minimality of $S$ and (5), it follows that $S$ is the interior of an induced path between $b_2, b_4$. In particular, one of $b_2, b_4$ (say $b_i$) is adjacent to $r_k$ and not to
From (7) and the symmetry between \( b_2, b_4 \), we may assume that \( b_4 \) is anticomplete to \( V(R) \). Since \( b_1 - r_1 - \cdots - r_k - q_1 - b_1 \) is not an odd hole, and \( R \) is even of length at least two by (5) and (6), we deduce that \( r_k, q_1 \) are nonadjacent, and so \( r_k \) has a neighbour in \( \{ q_2, \ldots, q_n \} \). We can link \( r_k \) onto \( \{ q_n, b_3, b_4 \} \), via \( r_k - b_3, r_k - R - r_1 - b_1 - b_4 \), and a path between \( r_k, q_n \) with interior in \( \{ q_2, \ldots, q_{n-1} \} \); and so \( r_k, q_n \) are adjacent. Since \( r_k \) has at most one neighbour in \( \{ b_1, b_2, q_1 \} \), 5.3 applied to the prism induced on \( V(Q) \cup \{ b_1, \ldots, b_4 \} \) implies that \( r_k \) has no neighbours in \( Q \) except \( q_n \). If \( b_2 \) has a neighbour in \( V(R) \) we can link \( b_2 \) onto \( \{ b_3, q_n, r_k \} \), via \( b_2 - b_3, b_2 - q_1 - \cdots - q_n \) and a path with interior in \( V(R) \), and so \( b_2, r_k \) are adjacent; but then \( b_4 - q_n - r_k - b_2 - b_1 - b_4 \) is an odd hole, a contradiction. Thus \( b_2 \) is anticomplete to \( V(R) \). If \( r_1 \) has a neighbour in \( \{ p_2, \ldots, p_m \} \), then we can link \( r_1 \) onto \( \{ b_2, b_3, p_m \} \) via paths with interiors in \( \{ b_1 \}, V(R) \) and \( \{ p_2, \ldots, p_m \} \), a contradiction. Thus \( p_1 \) is the only neighbour of \( r_1 \) in \( P \). But then the subgraph induced on \( V(P \cup Q \cup R) \cup \{ b_1, b_3 \} \) is an even prism, and \( b_4 \) is major-general with respect to this prism, contrary to 7.1. This proves \( 9.1 \).

Now we prove the desired extension of 8.1, the following.

**9.2** Let \( G \) be a 3-connected \( K_4 \)-free Berge graph, containing no even pair and no trampoline, and no clique cutset. Suppose that there is an appearance of a 3-connected graph \( J \) in \( G \). Then \( G \) is the line graph of a bipartite graph.

**Proof.** Choose a 3-connected graph \( J \) maximal such that there is an appearance \( L(H) \) of \( J \) in \( G \). By 8.1, we may assume that \( J = K_4 \) and \( L(H) \) is degenerate. Let \( V(J) = \{ c_1, \ldots, c_4 \} \), and for \( 1 \leq i < j \leq 4 \) let \( B_{ij} \) be the branch of \( H \) with ends \( c_i, c_j \). Let \( B_{12}, B_{23}, B_{34}, B_{14} \) all have length one, and let \( C \) be the cycle of \( H \) with vertices \( c_1c_2c_3c_4c_1 \) in order.

Let us choose a maximal \( J \)-strip system \( (S, N) \) that extends the strip system of \( H \). For convenience we write \( N_i \) for \( N_{c_i} \) for \( 1 \leq i \leq 4 \), and \( S_{ij} \) for \( S_{c_ic_j} \) and \( N_{ij} \) for \( N_{c_ic_j} \) for all distinct \( i, j \in \{ 1, \ldots, 4 \} \). As in the proof of 8.1, for each \( uv \in E(J) \), every \( uv \)-rung has the same parity, and they either all have positive length zero or \( |S_{uv}| = 1 \). In particular, \( S_{12}, S_{23}, S_{34}, S_{14} \) each have a unique member. Let \( b_{i2} \) be the unique member of \( S_{12} \), and define \( b_{23}, b_{34}, b_{14} \) similarly.

We say \( X \subseteq V(S, N) \) is local (with respect to the strip system) if either \( X \subseteq N_v \) for some \( v \in V(J) \), or \( X \subseteq S_{uv} \) for some edge \( uv \in E(J) \).

(1) If \( F \subseteq V(G) \setminus V(S, N) \) is connected, then the set of attachments of \( F \) in \( V(S, N) \) is local.

For suppose not, and choose \( F \) minimal violating the claim. Let \( X \) be the set of attachments of \( F \) in \( V(S, N) \). By 9.1, we may assume that \( X \subseteq E(C) \cup S_{13} \). Since \( X \) is not local, \( X \not\subseteq S_{13} \), and so we may assume that \( b_{12} \in X \). Suppose that also \( b_{34} \in X \). From the minimality of \( F \), it follows that there is an induced path \( b_{12} - f_1 - \cdots - f_k - b_{34} \), where \( F = \{ f_1, \ldots, f_k \} \). Since the union of this path and a \( c_2c_4 \)-rung induces a hole, and all \( c_2c_4 \)-rungs are odd, it follows that \( k \) is even; and so \( b_{23}, b_{14} \) both have neighbours in \( F \). From the minimality of \( F \), \( f_1 \) is the unique neighbour in \( F \) of one of \( b_{23}, b_{14} \), and \( f_k \) is the unique neighbour of the other. If \( b_{14} \) is adjacent to \( f_1 \) then we can add \( f_1 \) to \( N_1 \) and add \( f_k \) to \( N_3 \), and add \( F \) to \( S_{13} \), contrary to the maximality of \( V(S, N) \). Thus \( b_{23} \) is adjacent to \( f_1 \), and \( b_{14} \) to \( f_k \), and \( k > 1 \). The minimality of \( F \) implies that no member of \( F \) has a neighbour in \( S_{13} \); but then we can add \( f_1 \) to \( N_2 \), add \( f_k \) to \( N_4 \), and add \( F \) to \( S_{24} \), again a contradiction.
This proves that \( b_{34} \notin X \). Suppose that \( b_{14} \in X \). Then similarly, \( b_{23} \notin X \). Since \( X \) is not local, it follows that \( X \cap S_{13} \notin N_{13} \). From the minimality of \( F \), it follows that there is an induced path \( f_1 \cdots f_k \), where \( F = \{ f_1, \ldots, f_k \} \), and \( f_1 \) is adjacent to \( b_{12}, b_{14} \), and \( f_k \) has neighbours in \( S_{13} \setminus N_1 \), and there are no other edges between \( V(S,N) \setminus N_{13} \) and \( F \). But then we can add \( f_1 \) to \( N_1 \) and \( F \) to \( S_{13} \), contrary to the maximality of \( V(S,N) \).

This proves that \( b_{14} \notin X \). Suppose that \( X \cap S_{13} \notin N_{13} \). Then there is an induced path between \( b_{12} \) and \( b_{34} \) with interior in \( F \cup (S_{13} \setminus N_{13}) \); this path is even since it can be completed to a hole via \( b_{34} - b_{14} - b_{12} \), and yet it can also be completed to a hole via a path between \( b_{34}, b_{12} \) with interior a \( c_2 c_4 \)-rung, giving an odd hole, a contradiction. Thus \( X \cap S_{13} \subseteq N_{13} \).

Since \( X \) is not local, and therefore \( X \notin N_1 \), it follows that \( b_{23} \in X \). But then similarly \( X \cap S_{13} \subseteq N_{31} \), and so \( X \cap S_{13} = \emptyset \), contradicting that \( X \) is not local. This proves (1).

Now the proof is completed just like the proof of 8.1, using (1) above as a substitute for statement (3) in that proof. This proves 9.2.

This has the following consequence.

**9.3** Let \( G \) be a 3-connected \( K_4 \)-free Berge graph, containing no even pair, no trampoline, and no clique cutset. If \( G \) contains an even prism, then \( G \) is the line graph of a bipartite graph.

**Proof.** By 9.2, we may assume that there is no appearance of \( K_4 \) in \( G \). Since \( G \) contains an even prism, we can choose in \( G \) a collection of nine sets

\[
\begin{align*}
A_1 & \quad C_1 & \quad B_1 \\
A_2 & \quad C_2 & \quad B_2 \\
A_3 & \quad C_3 & \quad B_3
\end{align*}
\]

with the following properties:

- all these sets are nonempty and pairwise disjoint
- for \( 1 \leq i < j \leq 3 \), \( A_i \) is complete to \( A_j \) and \( B_i \) is complete to \( B_j \), and there are no other edges between \( A_i \cup B_i \cup C_i \) and \( A_j \cup B_j \cup C_j \)
- for \( 1 \leq i \leq 3 \), every vertex of \( A_i \cup B_i \cup C_i \) belongs to an induced path between \( A_i \) and \( B_i \) with interior in \( C_i \)
- some induced path between \( A_1 \) and \( B_1 \) with interior in \( C_1 \) is even.

Choose these nine sets with maximal union, and let \( H \) be the subgraph of \( G \) induced on their union. Let us write \( S_i = A_i \cup B_i \cup C_i \) for \( 1 \leq i \leq 3 \). Let us say a subset \( X \subseteq V(H) \) is local if \( X \) is a subset of one of \( S_1, S_2, S_3 \). \( A_1 \cup A_2 \cup A_3 \) or \( B_1 \cup B_2 \cup B_3 \). By 7.1, there is no prism in \( G \) with a major-general vertex; so by the argument of step (2) of the proof of theorem 10.6 of [5], it follows that

(1) For every connected subset \( F \) of \( V(G) \setminus V(H) \), its set of attachments in \( H \) is local.

Now since \( G \) is 3-connected, it follows from (1) that for \( i = 1,2,3 \), at least one of \( A_i \), \( B_i \) has more than one member. Consequently we may assume that \( |A_1|, |A_2| > 1 \), from the symmetry. Since \( G \) is \( K_4 \)-free, \( A_1, A_2 \) are both stable; but then the subgraph induced on \( A_1 \cup A_2 \cup A_3 \) contains a 4-wheel, contrary to 6.3. This proves 9.3.

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10 Long prisms

Our next goal is to eliminate all prisms. A prism is long if it has more than six vertices, and short otherwise. In this section we eliminate long prisms, and in the next we eliminate short prisms.

Let $K$ be a short prism in $G$, and let $w$ be a major vertex with respect to $K$. Let $N$ be the set of vertices in $K$ adjacent to $w$, and let $x, y$ be the two vertices in $V(K) \setminus N$. We say $w$ separates $K$ if every path in $G$ between $x, y$ has a vertex in $N \cup \{w\}$. In this section we prove the following.

10.1 Let $G$ be a 3-connected $K_4$-free Berge graph, containing no even pair, no trampoline, and no clique cutset. Suppose that $G$ contains no even prism, and no appearance of $K_4$, and that $|V(G)| > 6$. Then

- $G$ contains no long prism,
- for every short prism $K$, every major vertex (with respect to $K$) separates $K$, and
- if there is a short prism then some short prism has a major vertex.

Proof. Let $K$ be a prism; we must show that $K$ is short, and every major vertex separates $K$, and some short prism has a major vertex. We can choose a collection of nine subsets of $V(G)$

$$
A_1 \quad C_1 \quad B_1 \\
A_2 \quad C_2 \quad B_2 \\
A_3 \quad C_3 \quad B_3
$$

with the following properties:

- all these sets are pairwise disjoint, and $A_1, A_2, A_3, B_1, B_2, B_3$ are nonempty,
- for $1 \leq i < j \leq 3$, $A_i$ is complete to $A_j$ and $B_i$ is complete to $B_j$, and there are no other edges between $A_i \cup B_i \cup C_i$ and $A_j \cup B_j \cup C_j$,
- for $1 \leq i \leq 3$, every vertex of $A_i \cup B_i \cup C_i$ belongs to an induced path between $A_i$ and $B_i$ with interior in $C_i$, and
- for $i = 1, 2, 3$ there is an induced path between $A_i$ and $B_i$ with interior in $C_i$, such that these three paths form the prism $K$.

Choose these nine sets with maximal union, and let $H$ be the subgraph of $G$ induced on their union. Let $A = A_1 \cup A_2 \cup A_3$, and define $B, C$ similarly. For $1 \leq i \leq 3$, let $S_i = A_i \cup B_i \cup C_i$, and let us say an induced path between $A_i$ and $B_i$ with interior in $C_i$ is an $i$-rung. Since $G$ contains no even prism, it follows that for $1 \leq i \leq 3$, every $i$-rung is odd. Let us say a subset $X \subseteq V(H)$ is local if $X$ is a subset of one of $S_1, S_2, S_3, A$ or $B$. We say $v \in V(G) \setminus V(H)$ is major with respect to $H$ if $v$ has neighbours in at least two of $A_1, A_2, A_3$ and at least two of $B_1, B_2, B_3$.

1. Let $F \subseteq V(G) \setminus V(H)$ be connected, and contain no major vertex. Let $X$ be the set of attachments of $F$ in $H$. Then $X$ is local.

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Suppose not, and choose $F$ minimal with this property. Thus we may choose an $i$-rung $R_i$ for $i = 1, 2, 3$, forming a prism $K'$ say, such that $X \cap V(K')$ is not local with respect to $K'$. For $i = 1, 2, 3$, let $R_i$ have ends $a_i \in A_i$ and $b_i \in B_i$. By 5.2, and the minimality of $F$, there is an induced path $f_1 \cdots f_n$ in $F$ with $n \geq 1$ and $F = \{f_1, \ldots, f_n\}$, such that (up to symmetry) either:

- $n = 1$ and $f_1$ is major with respect to $K'$, or
- for some distinct $i, j \in \{1, 2, 3\}$, $f_1$ has two adjacent neighbours in $R_i$, and $f_n$ has two adjacent neighbours in $R_j$, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K)$, or
- $n \geq 2$, and for some distinct $i, j \in \{1, 2, 3\}$, $f_1$ is adjacent to $a_i, a_j$, and $f_n$ is adjacent to $b_i, b_j$, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K)$, or
- $f_1 \cdots f_n$ is a corner jump.

The first is impossible since no vertex in $F$ is major with respect to $K'$ (since any such vertex would also be major with respect to $H$), and the second is impossible there is no appearance of $K_4$ in $G$. Suppose that the third holds, with $i = 1, j = 2$. It follows that $n$ is even. Suppose that there exists $a'_1 \in A_1 \setminus \{a_1\}$. If $f_1$ is adjacent to $a'_1$, then the subgraph induced on $\{a_1, a'_1, a_2, a_3, f_1\}$ is a 4-wheel, a contradiction. Thus $f_1, a'_1$ are nonadjacent. Let $R'_1$ be a 1-rung with ends $a'_1$ and $b'_1 \in B_1$. Since

$$f_1 \cdots f_n \cdot b_2 \cdot b'_1 \cdot R'_1 \cdot a'_1 \cdot a_3 \cdot a_1 \cdot f_1$$

is not an odd hole, it follows that $F$ is not anticomplete to $V(R'_1)$. Consequently the set of attachments of $F$ in the prism formed by $R'_1, R_2, R_3$ is not local with respect to this prism; and yet $f_1$ is nonadjacent to $a'_1$, and $F$ is anticomplete to $V(R_3)$, contrary to 5.2. Thus there is no such vertex $a'_1$, and hence $A_1 = \{a_1\}$, and similarly $A_2 = \{a_2\}$, and $B_i = \{b_i\}$ for $i = 1, 2$. But then we can add $f_1$ to $A_3$, and $f_n$ to $B_3$, and $f_2, \ldots, f_{n-1}$ to $C_3$, contrary to the maximality of $V(H)$. This proves that the third outcome above does not hold.

We deduce that the fourth holds, and, say, $f_1$ is adjacent to $a_1, a_2$, and there is at least one edge between $f_n$ and $V(R_3) \setminus \{a_3\}$, and there are no other edges between $\{f_1, \ldots, f_n\}$ and $V(K') \setminus \{a_3\}$. Let $R'_1$ be a 1-rung, with ends $a'_1 \in A_1$ and $b'_1 \in B_1$. Thus the set of attachments of $F$ in the prism formed by $R'_1, R_2, R_3$ is not local with respect to this prism, and so by 5.2 applied to this prism, there is a unique edge between $F$ and $V(R'_1)$, and either $f_1$ is adjacent to $a'_1$, or $f_n$ is adjacent to $b'_1$ and the only edges between $V(K') \cup V(R'_1)$ and $F$ are $f_1 a_1, f_1 a_2, f_n b'_1, f_n b_3$. Suppose the latter. Then $n$ is odd, since $f_1 \cdots f_n \cdot b_2 \cdot b_1 \cdot R_3 \cdot a_1 \cdot f_1$ is a hole. Since $b_1 \cdot R_1 \cdot a_1 \cdot f_1 \cdots f_n \cdot b'_1 \cdot b_2 \cdot b_1$ is not an odd hole, it follows that $b'_1$ is not anticomplete to $V(R'_1)$, and so there is a 1-rung with ends $a_1, b'_1$; but this is impossible from what we showed above, since there are two edges between this 1-rung and $F$. This proves that for every choice of $R'_1$ (with ends $a'_1, b'_1$ as above) $f_1$ is adjacent to $a'_1$ and there are no other edges between $F$ and $V(R'_1)$. Consequently, $f_1$ is complete to $A_1$, and there are no other edges between $F$ and $S_1$. Similarly, the analogous statement holds for $A_2, S_2$; but then we can add $f_1$ to $A_3$ and $f_2, \ldots, f_n$ to $C_3$, contrary to the maximality of $V(H)$. Thus there is no such $F$. This proves (1).

Let $W$ be the set of all major vertices with respect to $H$. From (1), we may partition $V(G) \setminus (V(H) \cup W)$ into five (possibly empty) sets $A_0, B_0, D_1, D_2, D_3$, pairwise anticomplete, such that
• every attachment of $A_0$ in $V(H)$ belongs to $A$, and every attachment of $B_0$ in $V(H)$ belongs to $B$.

• for $i = 1, 2, 3$, every attachment of $D_i$ in $V(H)$ belongs to $S_i$; and for every component $X$ of $D_i$, some attachment of $X$ in $V(H)$ does not belong to $A$, and some attachment does not belong to $B$.

(2) For $i = 1, 2, 3$, if $P$ is an induced path with both ends in $A_i$ or both ends in $B_i$, and with no vertex in $W$, then $P$ has even length.

Suppose not, and choose $i$ and $P$ such that $P$ is odd, with the length of $P$ as small as possible. We may assume that both ends of $P$ belong to $A_1$ say. If some internal vertex of $P$ belongs to $A_1$, then it divides $P$ into two subpaths, one of which is odd, contrary to the minimality of $P$. Thus no internal vertex of $P$ is in $A_1$. Since $A_2, A_3$ are complete to $A_1$, it follows that no vertex of $P$ is in $A_2 \cup A_3$. Let $P$ have vertices $p_1, \ldots, p_k$ say. Now there is an induced path $Q$ between $p_1, p_k$ with interior in $C_1 \cup B_1 \cup B_2$, since $p_1, p_k$ both belong to 1-rungs. Since $Q$ can be completed to a hole via $p_k-a_3-p_1$ (where $a_3 \in A_3$) it follows that $Q$ is even. Consequently the union of $P$ and $Q$ is not a hole, and so some internal vertex of $P$ is equal to or adjacent to some internal vertex of $Q$. Consequently $P^*$ is not a subset of $A_0$, and (since no attachment of $A_0$ belongs to $P^*$) it follows that $V(P) \cap A_0 = \emptyset$. Thus $p_2, p_{k-1} \in B_1 \cup C_1 \cup D_1$. If $p_2, \ldots, p_{k-1} \in C_1 \cup D_1$, then $P$ can be completed to a hole via $p_k-a_3-p_1$, where $a_3 \in A_3$, which is impossible since $P$ is odd. Thus there exist $i, j \in \{2, \ldots, k-1\}$ such that $p_i, p_j \in B_1$, minimum and maximum respectively. The path $p_1, \ldots, p_i$ is therefore a 1-rung, and so is $p_j, \ldots, p_k$; both these 1-rungs are odd, and so the path $p_1, \ldots, p_j$ is also odd (and in particular $p_i \neq p_j$) contrary to the minimality of $P$. This proves (2).

(3) $W \neq \emptyset$.

For suppose that $W = \emptyset$. By (2), since there is no even pair, it follows that $|A_i| = |B_i| = 1$ for $i = 1, 2, 3$. Since $G$ admits no clique cutset, and $A$ is a clique, it follows that $A_0 = \emptyset$, and similarly $B_0 = \emptyset$; and since $G$ is 3-connected, we deduce that $C_i \cup D_i = \emptyset$ for $1 \leq i \leq 3$. Hence $G$ has only six vertices, a contradiction. This proves (3).

(4) If $w \in V(G) \setminus V(H)$ is major with respect to $H$, then (up to symmetry) $w$ is complete to $A_1 \cup B_2$, and has a unique neighbour $a_3 \in A_3$ and $b_3 \in B_3$, and $a_3, b_3$ are adjacent, and every 3-rung contains one of $a_3, b_3$, and $|A_1| = |B_2| = 1$.

For let $X$ be the set of neighbours of $w$ in $V(H)$. We may assume that $X \cap A_1, X \cap A_3, X \cap B_3 \neq \emptyset$. Consequently $X \cap A_2 = \emptyset$, since $G$ is $K_4$-free. For $1 \leq i \leq 3$ let $R_i$ be an $i$-rung, with ends $a_i \in A_i$ and $b_i \in B_i$, such that $a_1, b_3 \in X$. Since $w-a_1-a_2-R_2-b_2-b_3-w$ is not an odd hole, it follows that $w$ has a neighbour in $V(R_2) \setminus \{a_2\}$. Thus $w$ can be linked onto $\{b_1, b_2, b_3\}$, and so one of $b_1, b_2 \in X$. If $b_2 \notin X$, then similarly $a_3 \in X$, and so $w$ is balanced with respect to the prism formed by $R_1, R_2, R_3$, a contradiction. Thus $b_2 \in X$, and so $X \cap B_1 = \emptyset$. Since this holds for all choices of $R_2$, we deduce that $B_2 \subseteq X$, and similarly $A_1 \subseteq X$. If there exist distinct $a_1, a_1' \in A_1$, then the subgraph induced on $\{a_1, a_1', w, a_2, a_3\}$ is a 4-wheel (where $a_2 \in A_2$ and $a_3 \in A_3 \cap X$), contrary to 6.3. Thus $|A_1| = 1$, and similarly $|B_2| = 1$. Let $A_1 = \{a_1\}$ and $B_2 = \{b_2\}$. If there exist distinct $a_3, a_3' \in A_3 \cap X$,
then the subgraph induced on \( \{a_3, a'_3, w, a_1, a_2\} \) is a 4-wheel (where \( a_2 \in A_2 \)), again a contradiction. Thus \( |A_3 \cap X| = 1 \), and similarly \( |B_3 \cap X| = 1 \). Let \( A_3 \cap X = \{a_3\} \) and \( B_3 \cap X = \{b_3\} \) say. Suppose there is a 3-rung \( R'_3 \) containing neither of \( a_3, b_3 \); let its ends be \( a'_3 \in A_3 \) and \( b'_3 \in B_3 \) say. Since \( w-a_1-a'_3-R'_3-b'_3-b_2-w \) is not an odd hole, \( w \) has a neighbour in the interior of \( R'_3 \); but then \( w \) can be linked onto \( \{a_1, a_2, a'_3\} \) (where \( a_2 \in A_2 \)), a contradiction. Thus every 3-rung contains one of \( a_3, b_3 \). Next, suppose that \( a_3, b_3 \) are nonadjacent, and let \( R_3 \) be a 3-rung containing \( a_3 \). Let \( b'_3 \) be its end in \( B_3 \). If \( b_3 = b'_3 \), then \( w \) is major-general with respect to the prism formed by \( R_3 \) and some 1-rung and 2-rung, contrary to 7.1. Thus \( b_3 \neq b'_3 \), and so \( b'_3 \notin X \). Moreover, we cannot choose a 3-rung with ends \( a_3, b_3 \), and so \( b_3 \) is anticomplete to \( V(R_3) \). Since we cannot link \( w \) onto \( \{b_1, b_2, b'_3\} \) (where \( b_1 \in B_1 \)), it follows that \( X \cap V(R_3) = \{a_3\} \). But then \( w-a_3-R_3-b'_3-b_1-b_3-w \) is an odd hole (where \( b_1 \in B_1 \)), a contradiction. This proves that \( a_3, b_3 \) are adjacent, and so this proves (4).

(5) \(|W| = 1\).

For suppose that \( u, v \) are distinct major vertices. By (4), we may assume that \( v \) is complete to \( A_1 \cup B_2 \), and has a unique neighbour \( a_3 \in A_3 \) and \( b_3 \in B_3 \), and \( a_3, b_3 \) are adjacent, and every 3-rung contains one of \( a_3, b_3 \), and \( |A_1| = |B_2| = 1 \). Let \( A_1 = \{a_1\} \) and \( B_2 = \{b_2\} \). Take a 3-colouring of \( G \). We may assume that every vertex in \( A_1 \) has colour \( i \), for \( i = 1, 2, 3 \). Since \( v \) has neighbours in \( A_1 \) and in \( A_2 \) it follows that \( v \) has colour \( 2 \); since \( b_3 \) is adjacent to \( v \) and \( a_3 \), we deduce that \( b_3 \) has colour \( 1 \); since \( b_2 \) is adjacent to \( b_3 \) and \( b_1 \), \( b_2 \) has colour \( 3 \); and therefore every vertex in \( B_1 \) has colour \( 2 \), and every vertex in \( B_3 \) has colour \( 1 \).

Suppose first that \( u \) has a neighbour in \( A_1 \) and one in \( B_1 \); thus \( u \) is adjacent to \( a_1 \) and to some \( b_1 \in B_1 \). Consequently \( a_1, b_1 \) are adjacent, by (4) applied to \( u \). Moreover, \( u \) has colour \( 3 \), and therefore \( u \) is anticomplete to \( A_3 \cup B_2 \). Hence by (4) applied to \( u \), \( u \) is adjacent to \( b_3 \), and so \( \{a_1, b_3\} \) is complete to \( \{u, v, a_3, b_1\} \), and \( G \) contains a trapeze or trestle, a contradiction. This proves that \( u \) is anticomplete to one of \( A_1, B_1 \), and similarly to one of \( A_2, B_2 \). By (4), \( u \) has neighbours in both \( A_3, B_3 \). It follows that \( u \) has colour \( 2 \), and therefore \( u, v \) are nonadjacent, and \( u \) is anticomplete to \( A_2 \cup B_1 \). By (4), \( u \) is adjacent to \( a_1, b_2 \). Now every 3-rung has a vertex adjacent to \( u \), by (4), and since \( a_3-b_3 \) is a 3-rung, we may assume from the symmetry that \( a_3 \) is adjacent to \( u \). Let \( b'_3 \) be the unique neighbour of \( u \) in \( B_3 \). If \( b_3 = b'_3 \) then the subgraph induced on \( \{u, v, a_1, a_3, b_2, b_3\} \) is a trestle, and if \( b_3 \neq b'_3 \) then the subgraph induced on \( \{a_3, b_2, b'_3, u, v\} \) is a trapeze, in either case a contradiction. Thus \(|W| \leq 1 \), and the result follows from (3). This proves (5).

Let \( W = \{w\} \). By (4) we may assume that \( w \) is complete to \( A_1 \cup B_2 \), and has a unique neighbour \( a_3 \in A_3 \) and \( b_3 \in B_3 \), and \( a_3, b_3 \) are adjacent, and every 3-rung contains one of \( a_3, b_3 \), and \( |A_1| = |B_2| = 1 \). Let \( A_1 = \{a_1\} \) and \( B_2 = \{b_2\} \).

(6) \( w \) is anticomplete to \( C_3 \cup D_3 \).

For let \( X \) be a component of \( C_3 \cup D_3 \), and suppose that \( w \) has a neighbour in \( X \). Let \( N \) be the set of all vertices not in \( X \) with a neighbour in \( X \); thus, \( w \in N \subseteq A_3 \cup B_3 \cup \{w\} \). Since \( \{w, a_3, b_3\} \) are pairwise adjacent and \( G \) does not admit a clique cutset, it follows that \( N \not\subseteq \{w, a_3, b_3\} \), and so we may assume that some \( b'_3 \in B_3 \setminus \{b_3\} \) belongs to \( N \). Choose \( b_1 \in B_1 \); then we can link \( w \) onto \( \{b_1, b_2, b'_3\} \), a contradiction. This proves (6).
(7) For each \(a_2 \in A_2\), every odd induced path between \(a_2\) and \(w\) contains a vertex in \(A_3 \setminus \{a_3\}\); and consequently \(|A_3|, |B_3| \geq 2\) and \(|A_2| = |B_1| = 1\).

For let \(a_2-p_1 \cdots p_k-w\) be an odd induced path, and suppose that \(p_1, \ldots, p_k \notin A_3 \setminus \{a_3\}\). Choose \(a_2\) and \(p_1, \ldots, p_k\) with \(k\) minimum. If some \(p_i \in A_2\), then none of \(p_1, \ldots, p_i\) is in \(W\), and so \(i\) is odd by (2); and so \(p_i \cdots p_k-w\) is an odd induced path, contrary to the minimality of \(k\). Thus \(p_1, \ldots, p_k \notin A_2\).

Since \(p_1\) is adjacent to \(a_2\) and \(p_1\) is nonadjacent to \(w\), it follows that either \(p_1 \in C_2 \cup D_2\) or \(p_1 \in A_0\) (since \(p_1 \notin A_2\) by hypothesis). If \(p_1 \in C_2 \cup D_2\), then since none of \(p_2, \ldots, p_{k-1}\) is adjacent to \(w\), and therefore none of \(p_2, \ldots, p_{k-1}\) belongs to \(A_2 \cup B_2 \cup \{w\}\), it follows that \(p_2, \ldots, p_{k-1} \in C_2 \cup D_2\).

Consequently \(p_k \in C_2 \cup D_2 \cup B_2\). But then \(a_2-p_1 \cdots p_k-w-a_3-a_2\) is an odd hole, a contradiction. Thus \(p_1 \in A_0\). Since \(p_2, \ldots, p_k\) are nonadjacent to \(a_2\) and therefore not in \(A\), it follows that \(p_2, \ldots, p_k \in A_0\). But there is an induced path \(Q\) between \(a_2\) and \(w\) with interior in \(C_2 \cup B_2\), since \(a_2\) belongs to a 2-rung with ends \(a_2, b_2\); and \(Q\) is even since \(a_2-Q-w-a_3-a_2\) is a hole; and so \(a_2-p_1 \cdots p_k-w-Q-a_2\) is an odd hole, a contradiction. This proves the first assertion of (7). Since \(w, a_3\) is not an even pair, we deduce that \(A_3 \setminus \{a_3\} \neq \emptyset\), and so \(|A_3| \geq 2\); and similarly \(|B_3| \geq 2\). Finally, note that if also \(|A_2| \geq 2\) then \(G|A\) contains a 4-wheel, a contradiction. This proves (7).

Let \(A_2 = \{a_2\}\) and \(B_1 = \{b_1\}\).

(8) \(C_1, D_1, C_2, D_2 = \emptyset\).

For let \(a'_3 \in A_3 \setminus \{a_3\}\). Since we cannot link \(w\) onto \(\{a_1, a_2, a'_3\}\), it follows that \(w\) is anticomplete to \(C_2 \cup D_2\); and since \(G\) is 3-connected, it follows that \(C_2 \cup D_2 = \emptyset\), and similarly \(C_1 \cup D_1 = \emptyset\). This proves (8).

(9) If \(|A_3| \geq 3\) then \(w\) is anticomplete to \(A_0\).

For by (8), \(a_2, b_2\) are adjacent. Suppose that \(w\) has a neighbour in some component \(X\) of \(A_0\). Since \(G\) admits no clique cutset, there is an attachment of \(X\) in one of \(\{a_2\}, A_3 \setminus \{a_3\}\), and so there is an induced path \(w-p_1 \cdots p_k-u\) between \(w\) and some \(u \in \{a_2\} \cup A_3 \setminus \{a_3\}\), with \(p_1, \ldots, p_k \in A_0\).

Choose \(u\) and \(p_1 \cdots p_k\) with \(k\) minimum. We claim that \(a_2, p_k\) are adjacent. For suppose not; then \(u \in A_3 \setminus \{a_3\}\). Since there is a 3-rung \(R_3\) with ends \(u\) and \(b_3\), and \(w-p_1 \cdots p_k-u-R_3-b_3-w\) is a hole by (6), it follows that \(k\) is odd; and since \(a_2\) is anticomplete to \(\{p_1, \ldots, p_k\}\) (by the minimality of \(k\), and since \(a_2, p_k\) are nonadjacent) and \(a_2, b_2\) are adjacent, it follows that \(w-p_1 \cdots p_k-u-a_2-b_2-w\) is an odd hole, a contradiction. Thus \(a_2, p_k\) are adjacent, and so \(w-p_1 \cdots p_k-a_2\) is an induced path. By (7), \(k\) is odd.

Let \(a'_3 \in A_3 \setminus \{a_3\}\); we claim that \(a'_3, p_k\) are adjacent. For suppose not; then from the minimality of \(k\), it follows that \(a'_3\) is anticomplete to \(\{p_1, \ldots, p_k\}\), and so \(w-p_1 \cdots p_k-a_2-a'_3-R_3-b_3-w\) is an odd hole (where \(R_3\) is a 3-rung with ends \(a'_3, b_3\)), a contradiction. This proves that \(p_k\) is complete to \(A_3 \setminus \{a_3\}\). Choose distinct \(x, y \in A_3 \setminus \{a_3\}\) (this is possible since \(|A_3| \geq 3\); then the subgraph induced on \(\{a_1, p_k, a_2, x, y\}\) is a 4-wheel, a contradiction. This proves (9).

(10) \(|A_3|, |B_3| = 2\), and \(C_3, D_3 = \emptyset\).

Suppose first that either
\[ |A_3| \geq 3, \text{ or} \]
\[ |A_3| = 2 \text{ and there is an induced path between the two members of } A_3 \text{ with interior in } C_3 \cup D_3. \]

Since \( G \) has no even pair, there is an odd induced path \( a_3-p_1-\cdots-p_k \) with \( p_k \in A_3 \setminus \{a_3\} \); choose such a path with \( k \) minimum. By (2) \( w \) belongs to this path, and since \( a_3 \) is adjacent to \( w \) it follows that \( w = p_1 \), and \( k \geq 3 \). If \( p_h \in A_3 \) for some \( h \) with \( 1 \leq h < k \), then \( h \geq 2 \), and \( w \) does not belong to the path \( p_h-\cdots-p_k \), and so this path is even by (2); and so \( a_3-p_1-\cdots-p_h \) is odd, contrary to the minimality of \( k \). Thus \( p_1, \ldots, p_{k-1} \notin A_3 \). Now since the path is induced, none of \( p_1, \ldots, p_{k-1} \in A \); and since \( w = p_1 \), none of \( p_3, \ldots, p_k \) is adjacent to \( w \), since \( w = p_1 \). Since \( p_{k-1} \in A_3 \), it follows that \( p_{k-1} \in A_0 \) or \( p_{k-1} \in B_3 \cup C_3 \cup D_3 \).

Suppose that \( p_{k-1} \in B_3 \cup C_3 \cup D_3 \). Now \( p_{k-1} \notin b_3 \) since \( p_{k-1} \) is not adjacent to \( a_3 \), and \( p_{k-1} \notin B_3 \setminus \{b_3\} \) since \( p_{k-1}p_k \) is not a 3-rung (because every 3-rung contains \( a_3 \) or \( b_3 \)). Thus \( p_{k-1} \in C_3 \cup D_3 \). Since \( w \) is anticomplete to \( C_3 \cup D_3 \), it follows that \( p_2 \notin C_3 \cup D_3 \), and so we may choose \( i \) with \( 2 \leq i \leq k - 1 \) maximum such that \( p_i \notin C_3 \cup D_3 \). It follows that \( i \leq k - 2 \), and so \( p_i+1 \in C_3 \cup D_3 \), and therefore \( p_i \in A_3 \cup B_3 \). Since \( 2 \leq i < k \) it follows that \( p_i \notin A_3 \), so \( p_i \in B_3 \). Since \( p_j \in C_3 \cup D_3 \) for \( i < j < k \), it follows that \( p_i-p_i+1-\cdots-p_k \) is an induced path with ends in \( B_3 \) and \( A_3 \), and with interior in \( C_3 \cup D_3 \). From the maximality of \( V(H) \), this path belongs to \( H \) and therefore is a 3-rung; so \( p_i = b_3 \), a contradiction since none of \( p_2, \ldots, p_k \) are adjacent to \( a_3 \).

Thus \( p_{k-1} \in A_0 \). Since none of \( p_2, \ldots, p_{k-1} \) belongs to \( A \cup \{w\} \), it follows that \( p_2, \ldots, p_{k-1} \in A_0 \). Since \( p_2 \) is adjacent to \( p_1 = w \), it follows that \( w \) has a neighbour in \( A_0 \), and so \( |A_3| = 2 \) by (9); and therefore \( A_3 = \{a_3, p_k\} \). By hypothesis there is an induced path \( Q \) between \( a_3, p_k \) with interior in \( C_3 \cup D_3 \). Since \( Q \) can be completed to a hole via \( p_k-a_1-a_3 \), it follows that \( Q \) is even; and so \( a_3-p_1-\cdots-p_k-Q-a_3 \) is an odd hole, a contradiction.

Thus the bulletted statements above are both false. In particular, \( |A_3| = 2 \), and similarly \( |B_3| = 2 \). If \( C_3 \cup D_3 = \emptyset \) then the claim holds; so we may assume (for a contradiction) that \( X \) is a component of \( C_3 \cup D_3 \). Let \( N \) be the set of vertices not in \( X \) with a neighbour in \( X \). Now \( N \subseteq A_3 \cup B_3 \) since \( w \) is anticomplete to \( C_3 \cup D_3 \) by (6), and \( N \not\subseteq \{a_3, b_3\} \) since \( G \) does not admit a clique cutset. Thus from the symmetry we may assume that \( a'_3 \in N \), for some \( a'_3 \in A_3 \setminus \{a_3\} \). Consequently \( A_3 = \{a_3, a'_3\} \). Since the second bulletted statement above is false, it follows that \( a_3 \notin N \). Since \( G \) is 3-connected, we deduce that \( N \not\subseteq \{a'_3, b_3\} \); choose \( b'_3 \in N \setminus \{a'_3, b_3\} \). Thus \( b'_3 \in B_3 \setminus \{b_3\} \). There is an induced path between \( a'_3, b'_3 \) with interior in \( X \), and hence this is a 3-rung (from the maximality of \( V(H) \)), contradicting that every 3-rung contains either \( a_3 \) or \( b_3 \). This proves (10).

Now (8) and (10) imply that \( C_1, C_2, C_3 \) are all empty; and so the prism \( K \) is short. This proves the first assertion of the theorem. Let \( K' \) be the subgraph induced on \( \{a_1, a_2, a_3, b_1, b_2, b_3\} \). Now suppose that \( x \) is a major vertex with respect to \( K \). Then \( x \) is major with respect to \( H \), and so \( x = w \), and therefore \( K = K' \) (since \( w \) is not major with respect to any other prism contained in \( H \)) and so \( w \) separates \( K \). Thus the second assertion of the theorem holds. Finally the third assertion holds since \( K' \) is a short prism and \( w \) is a major vertex with respect to it. This proves 10.1.

11 Short prisms

In this section we complete the elimination of prisms. We prove the following.
11.1 Let $G$ be a 3-connected $K_4$-free Berge graph, containing no even pair, no trampoline, and no clique cutset. Suppose that $G$ contains a prism. Then $G$ is the line graph of a bipartite graph.

**Proof.** If $|V(G)| = 6$ then since $G$ contains a prism, it follows that $G$ is a short prism and therefore the theorem holds; so we may assume that $|V(G)| > 6$. Suppose that $G$ contains no appearance of $K_4$ and no even prism. By 10.1, $G$ contains no long prism; and $G$ contains a short prism with a major vertex $v$ say. Let the short prism have vertex set $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ where $\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}$ are triangles, and for $1 \leq i, j \leq 3$ $a_i$ is adjacent to $b_j$ if and only if $i = j$. Since $w$ is not balanced by 6.2, we may assume that $w$ is adjacent to $a_1, b_2, a_3, b_3$.

Let us say a prism-sequence in $G$ is a sequence $v_1, \ldots, v_n$ of distinct vertices of $G$ such that $n \geq 7$ and for $1 \leq i < j \leq n$, $v_i, v_j$ are adjacent if and only if $j - i \in \{1, 2, 5\}$. We observe that the sequence $a_2, a_1, a_3, w, b_3, b_2, b_1$ is a prism-sequence. Let us choose a prism-sequence $v_1, \ldots, v_n$ in $G$ with $n$ maximum. Choose a 3-colouring of $G$. We may assume that $v_n$ has colour 1, and $v_{n-1}$ has colour 2; and so $v_{n-2}$ has colour 3, $v_{n-3}$ has colour 1, and so on.

Since $v_{n-3}, v_n$ is not an even pair, there is an odd induced path $P$ in $G$ between $v_{n-3}, v_n$. Now $v_{n-1}, v_{n-2}, v_{n-5}$ are all complete to $\{v_{n-3}, v_n\}$. By 2.4, $\{v_{n-1}, v_{n-2}, v_{n-5}\}$ contains a leap for $P$, and since $G$ contains no long prism it follows that $P$ has length three. Let $P$ have vertices $v_{n-3}, x, y, v_n$ in order. Thus one of $x, y$ has colour 2 and is adjacent to $v_{n-2}, v_{n-5}$ and not to $v_{n-1}$, and the other has colour 3 and is adjacent to $v_{n-1}$ and not to $v_{n-2}, v_{n-5}$. Suppose that $x$ has colour 3. Then $\{v_{n-2}, v_{n-5}\}$ is complete to $y, v_n, v_{n-3}, v_{n-4}$; and $y \neq v_{n-4}$ since $y$ is adjacent to $v_n$, so $y, v_n, v_{n-3}, v_{n-4}$ are all different, and the subgraph induced on $\{v_{n-2}, v_{n-5}, y, v_n, v_{n-3}, v_{n-4}\}$ is a trapeze, a contradiction. Thus $x$ has colour 2. Hence $x$ is adjacent to $v_{n-2}, v_{n-5}$ and not to $v_{n-1}$, and $y$ is adjacent to $v_{n-1}$ and not to $v_{n-2}, v_{n-5}$. The subgraph induced on $\{x, v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}\}$ is not a 4-wheel, and so $x \in \{v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}\}$; and since $x$ has colour 2 it follows that $x = v_{n-4}$. We deduce that $y$ is adjacent to $v_n, v_{n-1}, v_{n-4}$, and not to $v_{n-2}, v_{n-3}, v_{n-5}$. Consequently $y \neq v_{n-1}, v_{n-2}, v_{n-5}$, and since $y$ is adjacent to $v_n$, it follows that $y$ is different from $v_1, \ldots, v_n$. Now the subgraph induced on $\{v_{n-6}, v_{n-5}, v_{n-4}, v_{n-2}, v_{n-1}, v_n\}$ is a short prism $K$ say, and $v_{n-3}$ is a major vertex with respect to $K$. By 10.1 it follows that $v_{n-3}$ separates $K$, and so $y$ is anticomplete to $\{v_1, \ldots, v_{n-6}\}$. Hence the sequence $v_1, \ldots, v_n, y$ is a prism-sequence, contrary to the maximality of $n$.

This proves that $G$ contains either an appearance of $K_4$ and or an even prism. From 9.2 and 9.3 it follows that $G$ is the line graph of a bipartite graph. This proves 11.1.

We deduce:

11.2 Let $G$ be a 3-connected $K_4$-free Berge graph, containing no even pair, no trampoline, and no clique cutset. Suppose that $G$ contains a square. Then $G$ is the line graph of a bipartite graph.

**Proof.** Fix a three-colouring of $G$. Choose a square $a_1b_1a_2b_2a_1$, such that if possible $a_1, a_2$ have different colours. Since $b_1, b_2$ is not an even pair, there is an odd induced path $P$ between $b_1, b_2$. Let the vertices of $P$ be $p_1, \ldots, p_k$, where $p_1 = b_1$ and $p_k = b_2$. If $\{a_1, a_2\}$ contains a leap for this path, then $G$ contains a prism and the result follows from 11.1. Thus we suppose that $\{a_1, a_2\}$ contains no leap. By 2.4 it follows that $a_1, a_2$ have the same colour, say colour 1. From the choice of the
square \(a_1b_1b_2a_1\), it follows that there is no square in which some two nonadjacent vertices have different colours. In particular, \(b_1, b_2\) have the same colour, say colour 2.

We claim that some edge of \(P\) is complete to \(\{a_1, a_2\}\). For if \(k > 4\) this follows from 2.1, so we assume \(k = 4\). Since \(a_1p_1 \cdots p_4a_1\) is not an odd hole, \(a_1\) is adjacent to one of \(p_2, p_3\), and similarly so is \(a_2\). If neither of \(p_2, p_3\) is complete to \(\{a_1, a_2\}\), then \(\{a_1, a_2\}\) is a leap, a contradiction; so from the symmetry we may assume that \(p_2\) is complete to \(\{a_1, a_2\}\), and so the edge \(p_1p_2\) is complete to \(\{a_1, a_2\}\). This proves that some edge of \(P\) is complete to \(\{a_1, a_2\}\), and consequently there exists \(i\) with \(1 < i < k\) such that \(p_i\) is complete to \(\{a_1, a_2\}\) and \(p_i\) has colour different from 2. Now \(p_i\) is nonadjacent to one of \(b_1, b_2\), say \(b_j\); and so \(p_i-a_1-b_j-a_2-p_i\) is a square, and \(p_i, b_j\) have different colours, a contradiction. This proves 11.2.

\section{Odd wheels}

An \textit{odd wheel} in \(G\) is a pair \((C, W)\), where \(C\) is a hole in \(G\) of length at least six, and \(W \subseteq V(G) \setminus V(C)\) is an anticonnected set, such that there are at least three vertices in \(C\) complete to \(W\), and there is a path \(P\) of \(C\) with odd length at least three, such that the ends of \(P\) are not complete to \(W\) and the internal vertices are complete to \(W\). If we could eliminate odd wheels (as well as all the other things we have eliminated in this paper so far), it would follow that \(G\) is “impoverished”, and we could use a result of [3] to complete the proof of 1.3; but we were unable to do so. Instead we need a more complicated argument.

Let us say \(G\) is \textit{square-free} if it has no square. The graphs \(G\) that concern us at this stage are square-free, and therefore every odd wheel \((C, W)\) in \(G\) satisfies \(|W| = 1\) (let us call the member of \(W\) the \textit{hub} of the wheel); and we can prove that at least one vertex is not the hub of an odd wheel. Consequently we need to rework the result of [3], replacing the hypothesis that there is no odd wheel by the weaker hypothesis that some vertex is not the hub of an odd wheel. (In compensation, our graphs are \(K_4\)-free and square-free, so they are much more restricted than the graphs of [3],.) We begin by modifying some of the lemmas from [5].

First, we need a result proved in [5], the following (this result is not stated explicitly in that paper, but the proof of theorem 16.3 of [5] also proves the present statement.) A \textit{double diamond} is a graph defined in [5]; all that matters here is that no square-free graph has a double diamond.

\subsection*{12.1}

\textit{Let \(G\) be a Berge graph, containing no double diamond, and such that \(G\) and \(\overline{G}\) contain no long prism. Choose an anticonnected set \(W\) maximal such that there is an odd wheel \((C, W)\), and let \(N\) be the set of all vertices of \(G\) complete to \(W\). Then \(V(G) \setminus (N \cup W)\) is not connected.}

If \(G\) is square-free, then it contains no double diamond, and \(\overline{G}\) contains no long prism; and if \(w\) is a hub of an odd wheel in \(G\) then \(W = \{w\}\) is maximal such that there is an odd wheel \((C, W)\). Consequently 12.1 implies

\subsection*{12.2}

\textit{Let \(G\) be a square-free Berge graph containing no long prism, and let \((C, \{w\})\) be an odd wheel. Let \(N\) be the set of all neighbours of \(w\). Then \(V(G) \setminus (N \cup \{w\})\) is not connected.}

Furthermore, we have:
Proof. For \( X \subseteq V(G) \) we define \( N(X) \) to be the set of all vertices not in \( X \) but with a neighbour in \( X \), and \( M(X) = V(G) \setminus (X \cup N(X)) \). Since \( G \) is not complete, we may choose a nonempty connected set \( X \) such that \( M(X) \neq \emptyset \). Choose such a set \( X \) maximal. From the maximality of \( X \), if \( y \in N(X) \), then \( X \cup \{y\} \cup N(X) \setminus \{y\} = V(G) \), and so \( M(X) \) is complete to \( N(X) \). Since \( G \) has no clique cutset, there are two nonadjacent vertices in \( N(X) \); and since \( G \) is square-free it follows that \( M(X) \) is a clique. Suppose that \( |M(X)| \geq 2 \). Since \( G \) is \( K_4 \)-free, it follows that \( N(X) \) is stable, and \( |M(X)| = 2 \). But every induced path with both ends in \( N(X) \) is even (to see this, we may assume that no internal vertex is in \( N(X) \), and no vertex is in \( M(X) \), and then the path can be completed to a hole via a vertex of \( M(X) \)), and so any two vertices in \( N(X) \) are an even pair, a contradiction.

Thus \( |M(X)| = 1 \); let \( M(X) = \{w\} \) say. Then \( N(X) \) is the set of neighbours of \( w \), and \( V(G) \setminus (N(X) \cup \{w\}) \) is connected, since this is \( X \). This proves 12.3.

Next we need a modification of theorem 17.2 of [5], the following.

12.4 Let \( G \) be a square-free Berge graph containing no long prism. Let \( v_1v_2v_3v_4 \) be an induced path of \( G \), and let \( F, Y \) be disjoint subsets of \( V(G) \setminus \{v_1, \ldots, v_4\} \), such that \( F \) is connected and \( Y \) is anticonnected. Suppose that \( v_2, v_3 \) are complete to \( Y \) and anticonnected to \( F \), and \( v_1, v_4 \) are not complete to \( Y \) and not anticonnected to \( F \). Suppose also that every vertex in \( Y \) has a neighbour in \( F \). Then for each \( y \in Y \), there is an odd wheel in \( G \) with hub \( y \).

Proof.

1. If \( y \in Y \) is nonadjacent to both \( v_1, v_4 \), then there is an odd wheel \( (C, \{y\}) \), with \( V(C) \subseteq \{v_1, v_2, v_3, v_4\} \cup F \) and with \( v_1, \ldots, v_4 \in V(C) \).

For choose a minimal connected subset \( X \) of \( F \) containing neighbours of all three of \( v_1, v_4, y \). There is an induced path \( P \) between \( v_1, v_4 \) with interior in \( X \), and if \( y \) has a neighbour in \( P \) then the claim holds; so we assume (for a contradiction) that \( y \) is anticonnected to \( V(P) \). Let \( P \) have vertices \( v_1 = p_1 \cdots p_m = v_4 \). Choose a minimal path \( yq_1 \cdots q_n \) such that \( q_1, \ldots, q_n \in X \) and \( q_n \) has a neighbour in \( V(P) \). Choose \( i, j \in \{1, \ldots, m\} \), minimal and maximal respectively, such that \( q_1 \) is adjacent to \( p_i, p_j \). From the minimality of \( X \), at least one of \( v_1, v_4 \) is anticonnected to \( \{q_1, \ldots, q_n\} \), say \( v_4 \), and so \( j < m \). If \( j = 1 \) then we can link \( p_1 \) onto \( \{y, v_2, v_3\} \), a contradiction, so \( 2 \leq j \leq m - 1 \). From the minimality of \( X \), \( X = \{p_2, \ldots, p_{m-1}, q_1, \ldots, q_n\} \). If \( j > i + 1 \), we can delete \( p_{i+1} \) from \( X \); if \( j = i + 1 \), then \( G \) contains a long prism; and if \( j = i \), then we can link \( p_i \) onto \( \{y, v_2, v_3\} \), in each case a contradiction. This proves (1).

2. If there exists \( y \in Y \) nonadjacent to both \( v_1, v_4 \), then the theorem holds.

For let us prove that every member of \( Y \) is nonadjacent to both \( v_1, v_4 \), and then the theorem holds by (1). Suppose not; then since \( Y \) is anticonnected, there exist \( y, z \in Y \), nonadjacent, such that \( y \) is
nonadjacent to both \( v_1, v_4 \), and \( z \) is adjacent to at least one of \( v_1, v_4 \). By (1) there is an odd wheel \( (C, \{ y \}) \), with \( V(C) \subseteq \{ v_1, v_2, v_3, v_4 \} \cup F \) and with \( v_1, \ldots, v_4 \in V(C) \). Since \( G \) is square-free and \( y, z \) are nonadjacent, no vertex in \( V(C) \) is adjacent to both \( y, z \) except \( v_2, v_3 \). By theorem 2.10 of [5], either one of \( y, z \) has no neighbours in \( C \) except \( v_2, v_3 \), or each of \( y, z \) has no neighbours in \( C \) except for \( v_2, v_3 \) and one of \( v_1, v_4 \). The first is impossible, since \( (C, \{ y \}) \) is an odd wheel and \( z \) is adjacent to one of \( v_1, v_4 \); and the second is impossible since \( (C, \{ y \}) \) is an odd wheel. This proves (2).

We therefore assume that every vertex in \( Y \) is adjacent to one of \( v_1, v_4 \), for a contradiction. Let \( Y_1 \) be the set of vertices in \( Y \) that are adjacent to \( v_1 \), and \( Y_2 \) those adjacent to \( v_4 \). Since \( G \) is square-free it follows that \( Y_1, Y_2 \) are both cliques, and since \( |Y| > 1 \) (because \( v_1, v_4 \) are not complete to \( Y \)) and therefore every member of \( Y \) has a nonneighbour in \( Y \), it follows that \( Y_1 \cap Y_2 = \emptyset \). Since \( Y \) is anticonnected, there exist \( y_1 \in Y_1 \) and \( y_2 \in Y_2 \), nonadjacent. Thus \( y_1 \) is adjacent to \( v_1 \) and not \( v_4 \), and \( y_2 \) is adjacent to \( v_4 \) and not \( v_1 \).

(3) There is no path \( P \) between \( v_1 \) and \( v_4 \) with \( P^* \subseteq F \) such that \( y_1 \) or \( y_2 \) has a neighbour in \( P^* \).

For suppose there is such a path \( P \), and let \( C \) be the hole \( v_4-P-v_1-v_2-v_3-v_4 \). Then \( y_1, y_2 \) both have at least three neighbours in \( C \), and one of them has at least four; so by theorem 2.10 of [5] some vertex of \( C \) different from \( v_2, v_3 \) is complete to \( \{ y_1, y_2 \} \), which is impossible since \( G \) is square-free. This proves (3).

Now \( y_1, y_2 \) both have neighbours in \( F \). Choose a minimal connected subset \( X \) of \( F \) such that \( v_1, v_4 \) and at least one of \( y_1, y_2 \) has a neighbour in \( X \). There is a path \( P \) from \( v_1 \) to \( v_4 \) with interior in \( X \), and \( y_1, y_2 \) are anticomplete to \( P^* \), by (3). From the symmetry between \( y_1, y_2 \), we may assume that there is a path \( y_1-q_1-\cdots-q_n \) with \( q_1, \ldots, q_n \in X \), such that \( q_n \) has a neighbour in \( P^* \). Choose such a path with \( n \) minimum. Thus \( X = P^* \cup \{ q_1, \ldots, q_n \} \). Since \( X \) is connected, it follows from the minimality of \( X \) that \( y_1, y_2 \) are both anticomplete to \( X \setminus \{ q_1 \} \). There is an induced path between \( y_1, v_4 \) with interior in \( X \); it is even since it can be completed to a hole via \( v_4-v_3-y_1 \); and so it cannot be completed to a hole via \( v_1-v_2-y_2-y_1 \). Consequently \( y_2, q_1 \) are adjacent. But then \( q_1-y_1-v_2-y_2-q_1 \) is a square, a contradiction. This proves 12.4.

Now we apply these result to our even pairs problem. The main result of this section is the following. (The following proof is almost identical with the proof of the theorem 3.5 of [3], with a few small but critical differences.)

12.5 Let \( G \) be a \( K_4 \)-free square-free Berge graph, containing no long prism, with no even pair and no clique cutset. Then \( G \) is complete.

**Proof.** We suppose (for a contradiction) that \( G \) is not complete. Thus by 12.3, there is a vertex \( w \) such that if \( N \) denotes the set of neighbours of \( w \), and \( M = V(G) \setminus (N \cup \{ w \}) \) then \( M \) is connected and nonnull, and every vertex in \( N \) has a neighbour in \( M \). By 12.2, there is no odd wheel with hub \( w \). Let us say an outer path is an induced path of length at least two, with both ends in \( N \) and every internal vertex in \( M \).

Now \( N \) is not a clique, since \( G \) admits no clique cutset and \( M \neq \emptyset \); and since every vertex in \( N \) has a neighbour in \( M \), and \( M \) is connected, it follows that there is an outer path. Choose an outer
path $R$ of minimum length, with vertices $r_0-r_1-r_2-\cdots-r_k-r_{k+1}$ in order, say; thus $k$ is odd, since this path can be completed to a hole via $r_{k+1}-w-r_0$, and $k > 1$ since this hole is not a square. Let $A$ be the set of all vertices in $N$ that are adjacent to $r_1$ and to none of $r_2, \ldots, r_k$. (Thus $r_0 \in A$.) Similarly let $B$ be the set of all vertices in $N$ adjacent to $r_k$ and to none of $r_1, \ldots, r_{k-1}$. Since $G$ is square-free, $A, B$ are cliques; and if $a \in A$ and $b \in B$ then since $a-r_1-\cdots-r_k-b-a$ is not an odd hole, it follows that $a, b$ are nonadjacent. Thus $A$ is anticomplete to $B$.

(1) If $v \in N \setminus (A \cup B)$ then either $v$ has no neighbour in $R^*$ or $v$ is complete to $A \cup B$.

Suppose that $v$ has a neighbour in $R^*$, and has a nonneighbour $a \in A$ say. Choose $i \in \{1, \ldots, k\}$ minimum such that $v, r_i$ are adjacent. Then $i < k$ since $v \notin B$; and so $a-r_1-\cdots-r_i-v$ is an outer path of length less than that of $R$, a contradiction. This proves (1).

(2) Every induced path between $A$ and $B$ with no internal vertex in $A \cup B$ has even length.

For suppose that $P$ is a path between $A$ and $B$ of odd length, with no internal vertex in $A \cup B$. Let $P$ have vertices $p_0-\cdots-p_{n+1}$ in order, where $p_0 \in A$ and $p_{n+1} \in B$. Let us say a segment is a maximal subpath of $P$ such that all its vertices belong to $N$. Thus an edge of $P$ belongs to a segment if and only both its ends are in $N$, and then it belongs to a unique segment; and otherwise it belongs to a unique outer path included in $P$. Since all outer paths have even length and $P$ has odd length, it follows that some segment has odd length, say $p_i-\cdots-p_j$. By (1) $\{p_i, \ldots, p_j\} \cap P^*$ is anticomplete to $R^*$. Choose $i' \leq i$ maximum such that $p_{i'}$ has a neighbour in $R^*$ (this is possible since $p_0$ has a neighbour in $R^*$); and similarly choose $j' \geq j$ minimum such that $p_j'$ has a neighbour in $R^*$. Then $j' - i' \geq 2$. Choose a path $Q$ between $p_{i'}p_{j'}$ with interior in $R^*$, and let $P'$ be the subpath of $P$ between $p_{i'}p_{j'}$. Then $P' \cup Q$ is a hole $C$ say, and $p_i-\cdots-p_j$ is a maximal path of $C \setminus N$. If $j > i + 1$ then $C$ has length at least six and $(C, \{w\})$ is an odd wheel, a contradiction, so $j = i + 1$.

Suppose first that $i \geq 1$ and $j \leq n$. Then $i \geq 2$ and $j \leq n - 1$, and $i' < i$ and $j < j'$, and the path $p_{i-1}p_ip_{i+1}p_{i+2}$ (= $S$ say) is an induced subgraph of $C$. Both ends of $S$ have neighbours in the connected set $V(R) \cup \{p_0, \ldots, p_{i-2}, p_{i+3}, \ldots, p_{n+1}\}$ (= $F$ say) and the internal vertices of $S$ do not; and the ends of $S$ are nonadjacent to $w$, and the internal vertices are adjacent to $w$. But $w$ has a neighbour in $F$, and so by 12.4, there is an odd wheel with hub $w$, a contradiction. Thus we may assume that $i = 0$ and $j = 1$. Hence $p_1 \in N$, and therefore $p_1$ is anticomplete to $R^*$ by (1). Since $G$ is square-free, $r_1$ is nonadjacent to $p_2$. Thus $r_1p_0p_1p_2$ is a path $S$ say; its ends have neighbours in the connected set $\{r_2, \ldots, r_k+1\} \cup \{p_3, \ldots, p_{n+1}\}$, and its internal vertices do not; and its ends are nonadjacent to $w$, and its internal vertices are adjacent to $w$, and $w$ is adjacent to $p_{n+1}$. Again this is contrary to 12.4. This proves (2).

For $x, y \in A$, distinct, let us say an $(x,y)$-path is an induced path between $y$ and some vertex in $B$, such that $x$ is nonadjacent to every vertex of this path except $y$. If there is an $(x,y)$-path, we write $x \to y$. We need:

(3) If $x, y \in A$ are distinct, then not both $x \to y$ and $y \to x$.

For suppose that $P$ is an $(x,y)$-path and $Q$ is a $(y,x)$-path. We may assume that both $P, Q$ have only their final vertex in $B$. Moreover, since $x$ has no neighbours in $P$ except $y$, and $A$ is a clique, it
follows that \( y \) is the only vertex of \( P \) in \( A \), and similarly \( x \) is the only vertex of \( Q \) in \( A \). By (2), both \( P, Q \) have even length. Let \( P \) have vertices \( y-p_1-\cdots-p_m \) in order, and let \( Q \) have vertices \( x-q_1-\cdots-q_n \) in order. Since \( G \) is square-free, \( p_1, q_1 \) are nonadjacent, and so \( p_1-y-x-q_1 \) is a three-edge path. Let \( F \) be the union of the sets \( V(P) \setminus \{ y, p_1 \}, V(Q) \setminus \{ x, q_1 \} \) and \( V(R) \setminus \{ r_0, r_1 \} \). Then \( F \) is connected (since all three paths contain a vertex in \( B \)); and \( p_1, q_1 \) have neighbours in \( F \), and \( x, y \) do not. Moreover, \( \{ r_1, w \} \) is anticonnected (= \( Y \) say); and \( x, y \) are complete to \( Y \), and \( p_1, q_1 \) are not. (This last holds since if \( p_1 \in N \) then \( p_1 \) is nonadjacent to \( r_1 \) by (1).) Moreover, every vertex in \( Y \) has a neighbour in \( F \); for \( w \) is adjacent to \( r_{k+1} \in F \), and \( r_1 \) is adjacent to \( r_2 \in F \). This contradicts 12.4, since there is no odd wheel with hub \( w \); and therefore proves (3).

Since \( G \) is \( K_4 \)-free, it follows that \(|A|, |B| \leq 2\). From (3) there exists \( a \in A \) such that \( a \not\to a' \) for all \( a' \in A \setminus \{a\} \); that is, every induced path from \( a \) to \( B \) contains no vertex of \( A \) except \( a \). Similarly choose \( b \in B \) such that every induced path from \( A \) to \( b \) contains no vertex of \( B \) except \( b \). Suppose that \( P \) is an induced path between \( a, b \); then no internal vertex of \( P \) belongs to \( A \cup B \) from the choice of \( a, b \), and so \( P \) has even length by (2). Consequently \( (a, b) \) is an even pair, a contradiction.

We deduce that \( G \) is a clique. This completes the proof of 12.5.

Now we can complete the proof of the main theorem.

**Proof of 1.3.** Let \( G \) be a 3-connected \( K_4 \)-free Berge graph with no even pair and no clique cutset. If \( G \) contains a trampoline, then \( \overline{G} \) is a line graph by 3.1. Thus, we assume that \( G \) contains no trampoline. If \( G \) contains a prism or square, then \( G \) is the line graph of a bipartite graph, by 11.1 and 11.2. Thus we may assume that \( G \) contains no prism or square. By 12.5, \( G \) is complete, and hence is the line graph of a bipartite graph. This proves 1.3.

**References**


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