Simplicial vertices in graphs with no induced four-edge path or four-edge antipath, and the H_6-conjecture

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May 27, 2013

Abstract

Let $\mathcal{G}$ be the class of all graphs with no induced four-edge path or four-edge antipath. Hayward and Nastos [6] conjectured that every prime graph in $\mathcal{G}$ not isomorphic to the cycle of length five is either a split graph or contains a certain useful arrangement of simplicial and antisimplicial vertices. In this paper we give a counterexample to their conjecture, and prove a slightly weaker version. Additionally, applying a result of the first author and Seymour [1] we give a short proof of Fouquet's result [3] on the structure of the subclass of bull-free graphs contained in $\mathcal{G}$.

1 Introduction

All graphs in this paper are finite and simple. Let G be a graph. The *complement* $\overline{G}$ of G is the graph with vertex set $V(G)$, such that two vertices are adjacent in G if and only if they are non-adjacent in $\overline{G}$. For a subset X of $V(G)$, we denote by $G[X]$ *the subgraph of* G *induced by* X, that is, the subgraph of G with vertex set X such that two vertices are adjacent in $G[X]$ if and only if they are adjacent in G. Let H be a graph. If G has no induced subgraph isomorphic to H, then we say that G is H*-free*. If G is not H-free, G *contains* H, and *a copy of* H *in* G is an induced subgraph of G isomorphic to H. For a family $\mathcal{F}$ of graphs, we say that G is $\mathcal{F}$*-free* if G is F-free for every $F \in \mathcal{F}$.

We denote by P_{n+1} *the path with* $n+1$ *vertices and* n *edges*, that is, the graph with distinct vertices $\{p_0, ..., p_n\}$ such that p_i is adjacent to p_j if and only if $|i-j| = 1$. For a graph H, and a subset X of $V(G)$, if $G[X]$ is a copy of H in G, then we say that X *is an* H. By convention, when explicitly describing a path we will list the vertices in order. In this paper we are interested in understanding the class of $\{P_5, \overline{P_5}\}$-free graphs.

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Let A and B be disjoint subsets of $V(G)$. For a vertex $b \in V(G) \setminus A$, we say that b *is complete to* A if b is adjacent to every vertex of A, and that b *is anticomplete to* A if b is non-adjacent to every vertex of A. If every vertex of A is complete to B, we say A *is complete to* B, and that A *is anticomplete to* B if every vertex of A is anticomplete to B. If $b \in V(G) \setminus A$ is neither complete nor anticomplete to A, we say that b *is mixed on* A. A *homogeneous set* in a graph G is a subset X of $V(G)$ with $1 < |X| < |V(G)|$ such that no vertex of $V(G)\backslash X$ is mixed on X. We say that a graph is *prime* if it has at least four vertices, and no homogeneous set.

Let us now define the *substitution* operation. Given graphs H_1 and H_2, on disjoint vertex sets, each with at least two vertices, and $v \in V(H_1)$, we say that H *is obtained from* H_1 *by substituting* H_2 *for* v, or *obtained from* H_1 *and* H_2 *by substitution* (when the details are not important) if:

- $V(H) = (V(H_1) \cup V(H_2)) \setminus \{v\}$,
- $H[V(H_2)] = H_2$,
- $H[V(H_1) \setminus \{v\}] = H_1[V(H_1) \setminus \{v\}]$, and
- $u \in V(H_1)$ is adjacent in H to $w \in V(H_2)$ if and only if w is adjacent to v in H_1.

Thus, a graph G is obtained from smaller graphs by substitution if and only if G is not prime. Since P_5 and $\overline{P_5}$ are both prime, it follows that if H_1 and H_2 are $\{P_5, \overline{P_5}\}$-free graphs, then any graph obtained from H_1 and H_2 by substitution is $\{P_5, \overline{P_5}\}$-free. Hence, in this paper we are interested in understanding the class of prime $\{P_5, \overline{P_5}\}$-free graphs.

Let C_n denote the *cycle of length* n, that is, the graph with distinct vertices $\{c_1, ..., c_n\}$ such that c_i is adjacent to c_j if and only if $|i - j| = 1$ or $n - 1$. A theorem of Fouquet [3] tells us that:

1.1. *Any $\{P_5, \overline{P_5}\}$-free graph that contains C_5 is either isomorphic to C_5 or has a homogeneous set.*

That is, C_5 is the unique prime $\{P_5,\overline{P_5}\}$-free graph that contains C_5, and so we concern ourselves with prime $\{P_5,\overline{P_5}, C_5\}$-free graphs, the main subject of this paper.

Let G be a graph. A *clique* in G is a set of vertices all pairwise adjacent. A *stable set* in G is a set of vertices all pairwise non-adjacent. The *neighborhood* of a vertex $v \in V(G)$ is the set of all vertices adjacent to v, and is denoted $N(v)$. A vertex v is *simplicial* if $N(v)$ is a clique. A vertex v is *antisimplicial* if $V(G) \setminus N(v)$ is a stable set, that is, if and only if v is a simplicial vertex in the complement.

In [6] Hayward and Nastos proved:

1.2. *If G is a prime $\{P_5, \overline{P_5}, C_5\}$-free graph, then there exists a copy of P_4 in G whose vertices of degree one are simplicial, and whose vertices of degree two are antisimplicial.*

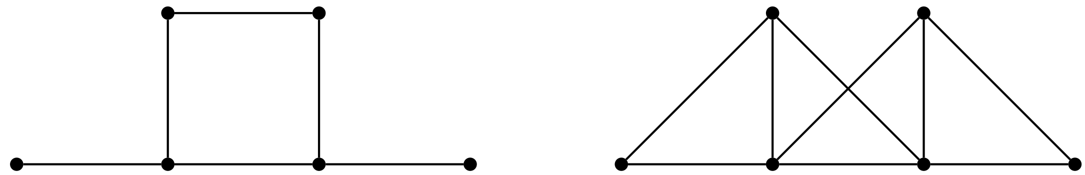

Figure 1: H_6 and $\overline{H_6}$.

A graph G is a *split graph* if there is a partition $V(G) = A \cup B$ such that A is a stable set and B is a clique. Földes and Hammer [2] showed:

1.3. *A graph G is a split graphs if and only if G is a $\{C_4, \overline{C_4}, C_5\}$-free graph.*

Drawn in Figure 1 with its complement, H_6 is the graph with vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $\{v_1v_2, v_2v_3, v_3v_4, v_2v_5, v_3v_6, v_5v_6\}$.

Hayward and Nastos conjectured the following:

1.4 (The H_6-Conjecture)**.** *If G is a prime $\{P_5, \overline{P_5}, C_5\}$-free graph which is not split, then there exists a copy of H_6 in G or $\overline{G}$ whose two vertices of degree one are simplicial, and whose two vertices of degree three are antisimplicial.*

First, in Figure 2 we provide a counterexample to 1.4. On the other hand, we prove the following slightly weaker version:

1.5. *If G is a prime $\{P_5, \overline{P_5}, C_5\}$-free graph which is not split, then there exists a copy of H_6 in G or $\overline{G}$ whose two vertices of degree one are simplicial, and at least one of whose vertices of degree three is antisimplicial.*

We say that a graph G *admits a* 1*-join*, if $V(G)$ can be partitioned into four non-empty pairwise disjoint sets (A, B, C, D), where A is anticomplete to $C \cup D$, and B is complete to C and anticomplete to D. In trying to use 1.5 to improve upon 1.1 we conjectured the following:

1.6. *If G is a $\{P_5, \overline{P_5}\}$-free graph, then either*

- *G is isomorphic to C_5, or*
- *G is a split graph, or*
- *G has a homogeneous set, or*
- *G or $\overline{G}$ admits a 1-join.*

However, 1.6 does not hold, and we give a counterexample in Figure 3.

The *bull* is a graph with vertex set $\{x_1, x_2, x_3, y, z\}$ and edge set $\{x_1x_2, x_2x_3, x_1x_3, x_1y, x_2z\}$. Lastly, applying a result of the first author and Seymour [1] we give a short proof of 1.1, and Fouquet's result [3] on the structure of $\{P_5, \overline{P_5}$,bull$\}$-free graphs.

This paper is organized as follows. Section 2 contains results about the existence of simplicial and antisimplicial vertices in $\{P_5, \overline{P_5}\}$-free graphs. In Section 3 we give a counterexample to the H_6-conjecture 1.4, and prove 1.5, a slightly weaker version of the conjecture. We also give a simpler proof of 1.2, and provide a counterexample to 1.6. Finally, in Section 4 we give a new proof of 1.1, and a structure theorem for $\{P_5, \overline{P_5}$,bull$\}$-free graphs.

2 Simplicial and Antisimplicial vertices

In this section we prove the following result:

2.1. *All prime $\{P_5, \overline{P_5}, C_5\}$-free graphs have both a simplicial vertex, and an antisimplicial vertex.*

Along the way we establish 2.9, a result which is helpful in finding simplicial and antisimplicial vertices in prime $\{P_5, \overline{P_5}\}$-free graphs.

Let G be a graph. We say G is *connected* if $V(G)$ cannot be partitioned into two disjoint sets anticomplete to each other. If $\overline{G}$ is connected we say that G is *anticonnected*. Let $X \subseteq Y \subseteq V(G)$. We say X is a *connected subset of* Y if $G[X]$ is connected, and that X is an *anticonnected subset of* Y if $G[X]$ is anticonnected. A *component* of X is a maximal connected subset of X, and an *anticomponent* of X is a maximal anticonnected subset of X.

First, we make the following three easy observations:

2.2. *If G is a prime graph, then G is connected and anticonnected.*

Proof. Passing to the complement if necessary, we may suppose G is not connected. Since G has at least four vertices, there exists a component C of $V(G)$ such that $|V(G) \setminus C| \geq 2$. However, then $V(G) \setminus C$ is a homogeneous set, a contradiction. This proves 2.2. □

We say a vertex $v \in V(G) \setminus X$ *is mixed on an edge of* X, if there exist adjacent $x, y \in X$ such that v is mixed on $\{x, y\}$. Similarly, a vertex $v \in V(G) \setminus X$ *is mixed on a non-edge of* X, if there exist non-adjacent $x, y \in X$ such that v is mixed on $\{x, y\}$.

2.3. *Let G be a graph, $X \subseteq V(G)$, and suppose $v \in V(G) \setminus X$ is mixed on X.*

1. *If X is a connected subset of $V(G)$, then v is mixed on an edge of X.*
2. *If X is an anticonnected subset of $V(G)$, then v is mixed on a non-edge of X.*

Proof. Suppose X is a connected subset of $V(G)$. Since v is mixed on X, both $X \cap N(v)$ and $X \setminus N(v)$ are non-empty. As $G[X]$ is connected, there exists an edge given by $x \in X \cap N(v)$ and $y \in X \setminus N(v)$, and v is mixed on $\{x, y\}$. This proves 2.3.1. Passing to the complement, we get 2.3.2.

□

2.4. *Let G be a graph, $X_1, X_2 \subseteq V(G)$ with $X_1 \cap X_2 = \emptyset$, and $v \in V(G) \setminus (X_1 \cup X_2)$.*

1. *If G is P_5-free, and X_1, X_2 are connected subsets of $V(G)$ anticomplete to each other, then v is not mixed on both X_1 and X_2.*
2. *If G is $\overline{P_5}$-free, and X_1, X_2 are anticonnected subsets of $V(G)$ complete to each other, then v is not mixed on both X_1 and X_2.*

Proof. Suppose G is P_5-free, X_1, X_2 are disjoint connected subsets of $V(G)$ anticomplete to each other, and v is mixed on both X_1 and X_2. By 2.3.1, v is mixed on an edge of X_1, given by say $x_1, y_1 \in X_1$ with v adjacent to x_1 and non-adjacent to y_1, and an edge of X_2, given by say $x_2, y_2 \in X_2$ with v adjacent to x_2 and non-adjacent to y_2. However, then $\{y_1, x_1, v, x_2, y_2\}$ is a P_5, a contradiction. This proves 2.4.1. Passing to the complement, we get 2.4.2.

□

As a consequence of 2.3 and 2.4 we obtain the following two useful results:

2.5. *Let u and v be non-adjacent vertices in a $\overline{P_5}$-free graph G, and let A be an anticonnected subset of $N(u) \cap N(v)$. Then no vertex $w \in V(G) \setminus (A \cup \{u, v\})$ can be mixed on both A and $\{u, v\}$.*

Proof. Since A and $\{u, v\}$ are disjoint anticonnected subsets of $V(G)$ complete to each other, 2.5 follows from 2.4.2.

□

2.6. *Let u, v and w be three pairwise non-adjacent vertices in a $\{P_5, \overline{P_5}\}$-free graph G such that w is mixed on an anticonnected subset A of $N(u) \cap N(v)$. Then no vertex $z \in N(w) \setminus (A \cup \{u, v\})$ can be mixed on $\{u, v\}$.*

Proof. Suppose there exists a vertex $z \in N(w) \setminus (A \cup \{u, v\})$ which is mixed on $\{u, v\}$, with say z adjacent to v and non-adjacent to u. Since w is mixed on A, by 2.3.2, it follows that w is mixed on a non-edge of A, given by say $x, y \in A$ with w adjacent to x and non-adjacent to y. By 2.5, z is not mixed on A. However, if z is anticomplete to A, then $\{y, u, x, w, z\}$ is a P_5, and if z is complete to A, then $\{x, y, w, u, z\}$ is a $\overline{P_5}$, in both cases a contradiction. This proves 2.6.

□

Now, we can start to prove 2.1.

2.7. *Let G be a prime $\{P_5, \overline{P_5}, C_5\}$-free graph. Then G has an antisimplicial vertex, or admits a 1-join.*

Proof. Suppose G does not admit a 1-join. Let W be a maximal subset of vertices that has a partition $A_1 \cup ... \cup A_k$ with $k \geq 2$ such that:

- $A_1, ..., A_k$ are all anticonnected subsets of $V(G)$, and
- $A_1, ..., A_k$ are pairwise complete to each other.

(1) $V(G) \setminus W$ is non-empty.

By 2.2, G is anticonnected, which implies that $V(G) \setminus W$ is non-empty. This proves (1).

(2) Every $v \in V(G) \setminus W$ is either anticomplete to or mixed on A_i for each $i \in \{1, ..., k\}$.

Suppose $v \in V(G) \setminus W$ is complete to some A_i. Take B to be the union of all the A_j to which v is complete. However, since $\{v\} \cup W \backslash B$ is anticonnected and complete to B, it follows that $W' = B \cup (\{v\} \cup W \backslash B)$ contradicts the maximality of W. This prove (2).

(3) If for some $i \in \{1, ..., k\}$, $v \in V(G) \setminus W$ is mixed on A_i, then v is anticomplete to $W \backslash A_i$.

By 2.4.2, any $v \in V(G) \setminus W$ is mixed on at most one A_i, and so together with (2) this proves (3).

(4) Every vertex in $V(G) \setminus W$ is mixed on exactly one A_i, for some $i \in \{1, ..., k\}$.

Suppose not. Let $X \subseteq V(G) \setminus W$ be the set of vertices anticomplete to W, which is non-empty by (2) and (3). By 2.2, G is connected, and so there exists an edge given by $v \in X$ and $u \in V(G) \setminus (X \cup W)$. By (2), u is mixed on some A_i, and so, by 2.3.2, u is mixed on a non-edge of A_i, given by say $x_i, y_i \in A_i$ with u adjacent to x_i and non-adjacent to y_i. However, by (3), u is anticomplete to $W \setminus A_i$, and so for $j \neq i$ and a vertex $z \in A_j$ we get that $\{v, u, x_i, z, y_i\}$ is a P_5, a contradiction. This proves (4).

And so, by (3) and (4), we can partition $V(G) = A_1 \cup ... \cup A_k \cup B_1 \cup ... \cup B_k$, where each B_i is the set of vertices mixed on A_i and anticomplete to $(A_1 \cup ... \cup A_k) \setminus A_i$.

(5) $B_1, ..., B_k$ are pairwise anticomplete.

Suppose for $i \neq j$, $b_i \in B_i$ is adjacent to $b_j \in B_j$. By 2.3.2, b_i is mixed on a non-edge of A_i, given by say $x_i, y_i \in A_i$ with b_i adjacent to x_i and non-adjacent to y_i. As b_j is mixed on A_j, there exists $x_j \in A_j$ non-adjacent to b_j, however then $\{b_j, b_i, x_i, x_j, y_i\}$ is a P_5, a contradiction. This proves (5).

(6) Exactly one B_i is non-empty.

By (1) and (4), at least one B_i is non-empty. Suppose for $i \neq j$, B_i and B_j are both non-empty. Then, by (5), $A = B_i$, $B = A_i$, $C = (A_1 \cup ... \cup A_k) \backslash A_i$ and $D = (B_1 \cup ... \cup B_k) \backslash B_i$ is a 1-join, a contradiction. This proves (6).

Hence, by (6), we may assume B_1 is non-empty while $B_2, \dots, B_k$ are all empty.

(7) $k = 2$ and $|A_2| = 1$.

Since $A_2 \cup ... \cup A_k$ is not a homogeneous set, (6) implies that $k = 2$ and $|A_2| = 1$. This proves (7).

Let a be the vertex in A_2.

(8) B_1 is a stable set.

Suppose not. Then there exists a component B of B_1 with $|B| > 1$. Since a is anticomplete to B_1, and B is a component of B_1, as G is prime, it follows that there exist $a_1 \in A_1$ which is mixed on B. Thus, by 2.3.1, a_1 is mixed on an edge of B, given by say $b, b' \in B$ with a_1 adjacent to b and non-adjacent to b'. Next, partition $A_1 = C \cup D$ with $C = A_1 \cap (N(b) \setminus N(b'))$ and $D = A_1 \setminus C$, where both C and D are non-empty, as $a_1 \in C$ and b' is mixed on A_1. Since A_1 is anticonnected there exists a non-edge given by $c \in C$ and $d \in D$. However, since $d \in D$, it follows that $\{d, a, c, b, b'\}$ is either a $P_5, \overline{P_5}$ or C_5, a contradiction. This proves (8).

Thus, by (8), a is an antisimplicial vertex. This proves 2.7.

□

Next, we observe:

2.8. *Let u and v be non-adjacent vertices in a prime $\overline{P_5}$-free graph G. Then either*

- *$N(u) \cap N(v)$ is a clique, or*
- *there exists a vertex $w \in V(G) \setminus (N(u) \cup N(v) \cup \{u, v\})$ which is mixed on an anticonnected subset of $N(u) \cap N(v)$.*

Proof. Suppose $N(u) \cap N(v)$ is a not clique. Then there exists an anticomponent A of $N(u) \cap N(v)$ with $|A| > 1$. Since $\{u, v\}$ is complete to $N(u) \cap N(v)$, and A is a anticomponent of $N(u) \cap N(v)$, as G is prime, it follows that there exists $w \in V(G) \setminus ((N(u) \cap N(v)) \cup \{u, v\})$ which is mixed on A. Thus, by 2.5, w is not mixed on $\{u, v\}$, and so w is anticomplete to $\{u, v\}$. This proves 2.8.

□

A useful consequence of 2.8 is the following:

2.9. *Let v be a vertex in a prime $\{P_5, \overline{P_5}\}$-free graph G.*

1. *If v is antisimplicial, and we choose u non-adjacent to v such that $|N(u) \cap N(v)|$ is minimum, then u is a simplicial vertex.*

2. *If v is simplicial, and we choose u adjacent to v such that $|N(u) \cup N(v)|$ is maximum, then u is an antisimplicial vertex.*

Proof. Suppose v is antisimplicial, we choose u non-adjacent to v such that $|N(u) \cap N(v)|$ is minimum, and u is not simplicial. Since v is antisimplicial, it follows that $N(u) \setminus N(v)$ is empty, and thus, as u is not simplicial, $N(u) \cap N(v)$ is not a clique. Hence, by 2.8, there exists some w, non-adjacent to both u and v, which is mixed on an anticonnected subset of $N(u) \cap N(v)$. However, then, by our choice of u, there exists a vertex $z \in N(v) \setminus N(u)$ adjacent to w, contradicting 2.6. This proves 2.9.1. Passing to the complement, we get 2.9.2.

□

2.10. *Let G be a prime $\{P_5, \overline{P_5}, C_5\}$-free graph. Then G has a simplicial vertex, or an antisimplicial vertex.*

Proof. Suppose G does not have an antisimpicial vertex. Then, by 2.7, it admits a 1-join (A, B, C, D).

(1) A and D are stable sets.

By symmetry, it suffices to argue that A is a stable set. Suppose not. Then there exists a component A' of A with $|A'| > 1$. Since $C \cup D$ is anticomplete to A, and A' is a component of A, as G is prime, it follows that there exists $b \in B$ which is mixed on A'. Thus, by 2.3.1, b is mixed on an edge of A', given by say $a, a' \in A'$ with b adjacent to a' and non-adjacent to a. By 2.2, G is connected, and so there exists an edge given by $c \in C$ and $d \in D$. However, then $\{a, a', b, c, d\}$ is a P_5, a contradiction. This proves (1).

Next, fix some $c \in C$, and choose a vertex $a \in A$ such that $|N(a) \cap N(c)|$ is minimum.

(2) a is a simplicial vertex.

Suppose not. Then, by (1), $N(a) \cap N(c) = N(a) \subseteq B$ is not a clique, and so, by 2.8, there exists w, non-adjacent to both a and c, which is mixed on an anticonnected subset of $N(a) \cap N(c)$. Since B is complete to C and anticomplete to D, it follows that w belongs to A. However, then, by our choice of a, there exists a vertex $z \in N(c) \setminus N(a)$ adjacent to w, contradicting 2.6. This proves (2).

This completes the proof of 2.10. □

Putting things together we can now prove 2.1.

Proof of 2.1. By 2.10, passing to the complement if necessary, there exists an antisimplicial vertex a. And so, by 2.9.1, if we choose s non-adjacent to a such that $|N(a) \cap N(s)|$ is minimum, then s is simplicial. This proves 2.1. □

3 The H_6-Conjecture

In this section we give a counterexample to the H_6-conjecture 1.4, and prove 1.5, a slightly weaker version of the conjecture. We also give a proof of 1.2, and provide a counterexample to 1.6.

We begin by establishing some properties of prime graphs. Recall the following theorem of Seinsche [7]:

3.1. *If G is a P_4-free graph with at least two vertices, then G is either not connected or not anticonnected.*

Together, 2.2 and 3.1 imply the following:

3.2. *Every prime graph contains P_4.*

Next, as first shown by Hoàng and Khouzam [4], we observe that:

3.3. *Let G be a prime graph.*

1. *A vertex $v \in V(G)$ is simplicial if and only if v is a degree one vertex in every copy of P_4 in G containing it.*

2. *A vertex $v \in V(G)$ is antisimplicial if and only if v is a degree two vertex in every copy of P_4 in G containing it.*

Proof. Both forward implications are clear. To prove the converse of 3.3.1, suppose there exists a vertex v which is not simplicial and yet is a degree one vertex in every copy of P_4 in G containing it. Then there exists an anticomponent A of $N(v)$ with $|A| > 1$. Since v is complete to A, and A is a anticomponent of $N(v)$, as G is prime, it follows that there exists $u \in V(G) \setminus (N(v) \cup \{v\})$ which is mixed on A. Thus, by 2.3.2, u is mixed on a non-edge of A, given by say $x, y \in A$ with u adjacent to x and non-adjacent to y. However, then $\{y, v, x, u\}$ is a P_4 with v having degree two, a contradiction. This proves 3.3.1. Passing to the complement, we get 3.3.2.

□

Finally, we observe that:

3.4. *Let G be a prime graph.*

1. *The set of antisimplicial vertices in G is a clique.*

2. *The set of simplicial vertices in G is a stable set.*

Proof. Suppose there exist non-adjacent antisimplicial vertices $a, a' \in V(G)$. Since a is antisimplicial, it follows that $N(a') \setminus N(a)$ is empty. Similarly, $N(a) \setminus N(a')$ is also empty. However, this implies that $\{a, a'\}$ is a homogeneous set in G, a contradiction. This proves 3.4.1. Passing to the complement, we get 3.4.2.

□

3.5. *Let G be a prime $\{P_5, \overline{P_5}, C_5\}$-free graph. Let A be the set of antisimplicial vertices in G, and let S be the set of simplicial vertices in G. Then $G[A \cup S]$ is a split graph which is both connected and anticonnected.*

Proof. 3.4 implies that $G[A \cup S]$ is a split graph, where A is a clique and S is a stable set. By 2.9.1, every vertex in A has a non-neighbor in S, and, by 2.9.2, every vertex in S has a neighbor in A. Thus, $G[A \cup S]$ is both connected and anticonnected. This proves 3.5.

□

We are finally ready to give a proof of 1.2, first shown in [6] by Hayward and Nastos.

3.6. *If G is a prime $\{P_5, \overline{P_5}, C_5\}$-free graph, then there exists a copy of P_4 in G whose vertices of degree one are simplicial, and whose vertices of degree two are antisimplicial.*

Proof. Let A be the set of antisimplicial vertices in G, and let S be the set of simplicial vertices in G. By 2.1, both A and S are non-empty. Hence, $G[A \cup S]$ is a graph with at least two vertices, which, by 3.5, is both connected and anticonnected, and so, by 3.1, it follows that $G[A \cup S]$ contains P_4. Since 3.4 implies that A is a clique and S is a stable set, it follows that every copy of P_4 in $G[A \cup S]$ is of the desired form. This proves 3.6.

□

Next, we turn our attention to the H_6-conjecture. A result of Hoàng and Reed [5] implies the following:

3.7. *If G is a prime $\{P_5, \overline{P_5}, C_5\}$-free graph which is not split, then G or $\overline{G}$ contains H_6.*

In hopes of saying more along these lines, motivated by 3.6 and 3.7, Hayward and Nastos posed 1.4, which we restate:

3.8 (The H_6-Conjecture)**.** *If G is a prime $\{P_5, \overline{P_5}, C_5\}$-free graph which is not split, then there exists a copy of H_6 in G or $\overline{G}$ whose two vertices of degree one are simplicial, and whose two vertices of degree three are antisimplicial.*

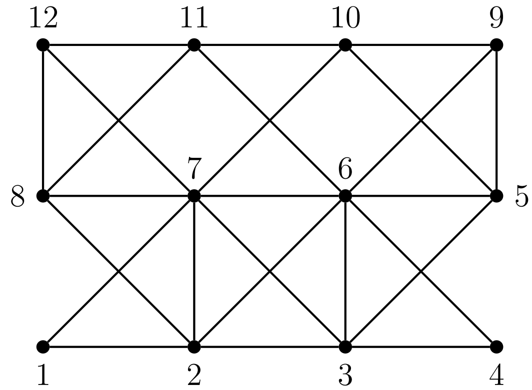

Figure 2: Counterexample to the H_6-conjecture, where additionally $\{2,3\}$ is complete to $\{9,10,11,12\}$.

In Figure 2 we give a counterexample to 3.8. The graph G in Figure 2 contains C_4, and so, by 1.3, is not split. The mapping $\phi : V(G) \to V(\overline{G})$

$$\phi := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 3 & 1 & 4 & 2 & 7 & 5 & 8 & 6 & 11 & 9 & 12 & 10 \end{pmatrix}$$

is an isomorphism between G and $\overline{G}$. Thus, as G is self-complementary, it suffices to check that G is P_5-free, which is straight forward, as is verifying that G is prime, and we leave the details to the reader. The set of simplicial vertices in G is $\{1,4\}$, and the set of antisimplicial vertices in G is $\{2,3\}$. However, no copy of C_4 in G contains $\{2,3\}$, and so there does not exist a copy of H_6 of the desired form.

However, all is not lost as we can prove 1.5, a slightly weaker version of the H_6-conjecture, which we restate:

3.9. *If G is a prime $\{P_5, \overline{P_5}, C_5\}$-free graph which is not split, then there exists a copy of H_6 in G or $\overline{G}$ whose two vertices of degree one are simplicial, and at least one of whose vertices of degree three is antisimplicial.*

Proof. By 3.6, there exist simplicial vertices s, s', and antisimplicial vertices a, a' such that $\{s, a, a', s'\}$ is a P_4 in G. Now, choose maximal subsets A of antisimplicial vertices in G, and S of simplicial vertices in G such that $a, a' \in A$, $s, s' \in S$, every vertex in A has a neighbor in S, and every vertex in S has a non-neighbor in A.

(1) Any graph containing a vertex which is both simplicial and antisimplicial is split.

By definition, if a vertex $v \in V(G)$ is both simplicial and antisimplicial, then $N(v)$ is a clique and $V(G) \setminus N(v)$ is a stable set. This proves (1).

(2) There exists no vertex $v \in V(G) \setminus (A \cup S)$ adjacent to a vertex $u \in S$ and non-adjacent to a vertex $w \in A$.

Suppose not. If u is adjacent to w, then $N(u)$ is not a clique, and if u is non-adjacent to w, then $V(G) \setminus N(w)$ is not a stable set, in both cases a contradiction. This proves (2).

By (1) and (2), we can partition $V(G) = A \cup S \cup B \cup C \cup D$, where B is the set of vertices complete to A and anticomplete to S, C is the set of vertices complete to A with a neighbor in S, and D is the set of vertices anticomplete to S with a non-neighbor in A. Recall 3.4 implies that A is a clique and S is a stable set.

(3) No vertex of $C \cup D$ is simplicial or antisimplicial.

Consider a vertex $c \in C$. Then there exists $s_c \in S$ adjacent to c. Hence, c is not antisimplicial, as otherwise we could add c to A contrary to maximality. By construction, s_c has a non-neighbor $a_c \in A$. Since c is complete to the A, it follows that $N(c)$ is not a clique, and thus c is not simplicial. Hence, C contains no simplicial or antisimplicial vertices. Passing to the complement, we get that no vertex in D is simplicial or antisimplicial. This proves (3).

(4) We may assume that C is a clique, and D is a stable set.

By symmetry, it is enough to argue that if D is not a stable set, then the theorem holds. Suppose we have an edge given by $x, y \in D$. By definition, any antisimplicial vertex is adjacent to at least one of x and y. And so, as x and y both have non-neighbors in A, there exists $a_x, a_y \in A$ such that a_x is adjacent to x and non-adjacent to y, and a_y is adjacent to y and non-adjacent to x. Since S is anticomplete to D, it follows that a_x and a_y do not have a common neighbor $s'' \in S$, as otherwise $\{a_x, y, s'', x, a_y\}$ is a $\overline{P_5}$. By construction, every vertex in A has a neighbor in S, and so there exists $s_x \in S$ adjacent to a_x and non-adjacent to a_y, and $s_y \in S$ adjacent to a_y and non-adjacent to a_x. However, then $\{s_x, a_x, a_y, s_y, x, y\}$ is a copy of H_6 in G of the desired form. Passing to the complement, we may also assume that C is a clique. This proves (4).

(5) For all $d \in D$ and $u \in A$, $N(d) \subseteq N(u) \cup \{u\}$.

By (4), $A \cup C$ is a clique and $D \cup S$ is a stable set. Thus, for any $d \in D$, it follows that $N(d) \subseteq A \cup B \cup C$. Since A is complete to B, it follows that any $a \in A$ is complete to $(A \setminus \{a\}) \cup B \cup C$. This proves (5).

(6) We may assume both C and D are empty.

By symmetry, it is enough to argue that if D is non-empty, then the theorem holds. Suppose D is non-empty, and choose $d \in D$ with $|N(d)|$ minimum. Then there exists $a_d \in A$ non-adjacent to d. By (3) and (5), $N(a_d) \cap N(d) = N(d)$ is not a clique, and

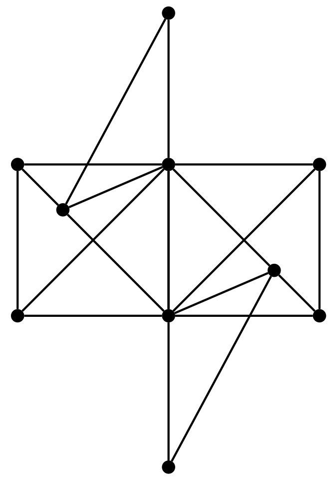

Figure 3: Counterexample to Conjecture 3.10.

so, by 2.8, there exists a vertex w, non-adjacent to both a_d and d, which is mixed on an anticonnected subset of $N(d)$. Since a_d is complete to $(A \setminus \{a_d\}) \cup B \cup C$, it follows that $w \in D \cup S$. If $w \in D$, then, by our choice of d, there exists $z \in N(w) \setminus N(d)$ which, by (5), is adjacent to a_d, contradicting 2.6. Hence, $w \in S$. Since w is mixed on an anticonnected subset of $N(d)$, by 2.3.2, w is mixed on a non-edge of $N(d)$, given by say $x, y \in N(d)$ with w adjacent to x and non-adjacent to y. Since $A \cup C$ is a clique, and B is complete to A and anticomplete to S, it follows that $x \in C$ and $y \in B$. By construction, every vertex in A has a neighbor in S, and so there exists $s_d \in S$ adjacent to a_d. Since s_d is mixed on $\{a_d, d\}$ and non-adjacent to y, 2.5 implies that s_d is anticomplete to $\{x, y\}$. However, then $\{s_d, a_d, x, w, y, d\}$ is a copy of H_6 in G of the desired form. Passing to the complement, we may also assume that C is empty. This proves (6).

By (6), since G is prime, it follows that $|B| \leq 1$, implying that G is a split graph, a contradiction. This proves 3.9.

$\square$

Another conjecture which seemed plausible for a while is 1.6, which we restate:

3.10. *If G is a $\{P_5, \overline{P_5}\}$-free graph, then either*

- *G is isomorphic to C_5, or*
- *G is a split graph, or*
- *G has a homogeneous set, or*
- *G or $\overline{G}$ admits a 1-join.*

However, with Paul Seymour we found the counterexample in Figure 3. The graph in Figure 3 contains C_4 and $\overline{C_4}$, and so, by 1.3, is not split; we leave the rest of the details to the reader.

4 $\{P_5, \overline{P_5}$,bull$\}$-free Graphs

In this section we give a short proof of 1.1, and of Fouquet's result 4.4 on the structure of $\{P_5, \overline{P_5}$,bull$\}$-free graphs. The following is joint work with Max Ehramn.

Let O_k be the bipartite graph on $2k$ vertices with bipartition $(\{a_1, \ldots a_k\}, \{b_1, \ldots b_k\})$ in which a_i is adjacent to b_j if and only if $i + j \geq k + 1$. If a graph G is isomorphic to O_k for some k, then we call G a *half graph*. Note that by construction half graphs are prime. In [1] the first author and Seymour proved:

4.1. *Let G be a graph, and let H be a proper induced subgraph of G. Assume that both G and H are prime, and that both G and $\overline{G}$ are not half graphs. Then there exists an induced subgraph H' of G, isomorphic to H, and a vertex $v \in V(G) \setminus V(H')$, such that $G[V(H') \cup \{v\}]$ is prime.*

Next, we give a proof of Fouquet's result 1.1, which we restate:

4.2. *If G is a prime $\{P_5, \overline{P_5}\}$-free graph which contains C_5, then G is isomorphic to C_5.*

Proof. Suppose not, and so C_5 is a proper induced subgraph of G. Since C_5 is self-complementary both G and $\overline{G}$ contain an odd cycle, hence are non-bipartite, and thus not half graphs. As C_5 is prime, by 4.1, there exists a subgraph H induced by $\{v_1, v_2, v_3, v_4, v_5\}$ isomorphic to C_5, and a vertex $v \in V(G) \backslash V(H)$ such that the subgraph of G induced by $V(H) \cup \{v\}$ is prime. Considering the complement, we may assume v is adjacent to at most two vertices in $V(H)$. To avoid a homogeneous set in $G[V(H) \cup \{v\}]$, by symmetry, the only possibilities are for $N(v) = \{v_1\}$, in which case $\{v, v_1, v_2, v_3, v_4\}$ is a P_5, or for $N(v) = \{v_1, v_2\}$, in which case $\{v, v_2, v_3, v_4, v_5\}$ is a P_5, in both cases a contradiction. This proves 4.2. □

Thus, to understand prime $\{P_5, \overline{P_5}$,bull$\}$-free graphs it is enough to study prime $\{P_5, \overline{P_5}, C_5$,bull$\}$-free graphs.

4.3. *If G is a prime $\{P_5, \overline{P_5}, C_5$,bull$\}$-free graph, then either G or $\overline{G}$ is a half graph.*

Proof. Suppose not. By 3.2, G contains P_4, which is isomorphic to O_2. Since G and $\overline{G}$ are not half graphs, it follows that P_4 is a proper induced subgraph of G. As P_4 is prime, by 4.1, there exists a subgraph H induced by $\{v_1, v_2, v_3, v_4\}$ isomorphic to P_4, and a vertex $v \in V(G) \backslash V(H)$ such that the subgraph of G induced by $V(H) \cup \{v\}$ is prime. Considering the complement, we may assume v is adjacent to at most two vertices in H. To avoid a homogeneous set in $G[V(H) \cup \{v\}]$, by symmetry, the only possibilities are for $N(v) = \{v_1\}$, in which case $\{v, v_1, v_2, v_3, v_4\}$ is a P_5, for $N(v) = \{v_1, v_4\}$, in which case

$\{v, v_1, v_2, v_3, v_4\}$ is a C_5, or for $N(v) = \{v_2, v_3\}$, in which case $\{v, v_1, v_2, v_3, v_4\}$ is a bull, in all cases a contradiction. This proves 4.3. □

Putting things together we obtain Fouquet's original structural result [3]:

4.4. *If G is a $\{P_5, \overline{P_5}, bull\}$-free graph, then either*

- $|V(G)| \leq 2$, *or*
- *G is isomorphic to C_5, or*
- *G has a homogeneous set, or*
- *G or $\overline{G}$ is a half graph.*

Proof. As all graphs on three vertices have a homogeneous set, 4.4 immediately follows from 4.2 and 4.3.

□

5 Acknowledgement

We would like to thank Ryan Hayward, James Nastos, Irena Penev, Matthieu Plumettaz, Paul Seymour, and Yori Zwols for useful discussions, and Max Ehramn for telling us about the H_6-conjecture.

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