Berge Trigraphs

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Abstract

A graph is *Berge* if no induced subgraph of it is an odd cycle of length at least five or the complement of one. In joint work with Robertson, Seymour, and Thomas we recently proved the Strong Perfect Graph Theorem, which was a conjecture about the chromatic number of Berge graphs. The proof consisted of showing that every Berge graph either belongs to one of a few basic classes, or admits one of a few kinds of decompositions. We used three kinds of decompositions: skew-partitions, 2-joins, and proper homogeneous pairs. At that time we were not sure whether all three decompositions were necessary. In this paper we show that the proper homogeneous pair decomposition is in fact unnecessary. This is a consequence of a general decomposition theorem for "Berge trigraphs".

A *trigraph* T is a generalization of a graph, where the adjacency of some vertex pairs is "undecided". A trigraph is *Berge* if however we decide the undecided pairs, the resulting graph is Berge.

We show that the decomposition result of [2] for Berge graphs extends (with slight modifications) to Berge trigraphs; that is for a Berge trigraph T, either T belongs to one of a few basic classes or T admits one of a few decompositions. Moreover, the decompositions are such that however we decide the undecided pairs of T, the resulting graph admits the same decomposition. This last property is crucial for the application.

The full proof of this result is over 200 pages long and was the author's PhD thesis. In this paper we present the parts that differ significantly from the proof of the decomposition theorem for Berge graphs, and only in the case needed for the application.

1 Introduction

We begin with some definitions. All graphs in this paper are simple and finite. The *complement* $\overline{G}$ of a graph G has the same vertex set as G, and two distinct vertices u, v are adjacent in $\overline{G}$ if and only if they are non-adjacent in G. A *hole* in G is an induced cycle of length at least 4. An *antihole* in G is an induced subgraph whose complement is a hole in $\overline{G}$.

A graph is called *Berge* if it contains no odd hole and no odd antihole. A *clique* in G is a subset of the vertex set every two members of which are adjacent. A graph G is *perfect* if its chromatic number equals the size of its maximum clique and the same holds for every induced subgraph of G. Since this equality does not hold for odd holes and antiholes, every perfect graph is Berge.

Recently in joint work with N.Robertson, P.Seymour, and R.Thomas [2] we were able to prove that the reverse statement holds as well—namely every Berge graph is perfect (this was conjectured by Berge in 1961 [1] and had become known as the Strong Perfect Graph Conjecture.) To show that, we proved a structure theorem for Berge graphs. This settled a conjecture by Conforti, Cornuéjols, and Vušković [6], asserting that every Berge graph either belongs to one of a few basic classes or admits one of a few kinds of decompositions (where the decompositions are such that they cannot occur in a minimal counterexample to the Strong Perfect Graph Conjecture).

In [3] the decomposition theorem of [2] is reproved in the more general setting of Berge *trigraphs*, namely graphs in which the adjacency of some vertex pairs is "undecided" (we give precise definitions later.) Parts of the proof are a rather straightforward generalization of [2], while in others new ideas were needed. The full proof is over 200 pages long. Our objective here is to present the novel parts of the proof (the rest is omitted for reasons of space).

This work is motivated by an application to Berge graphs proving that one of the decompositions used in [2] was unnecessary (explained later). For this application we can confine ourselves to the case when every vertex is incident with at most one "undecided" edge. In this paper we therefore only consider this case.

Most of the proof in [2] follows the paradigm bellow:

- find a subgraph of the Berge graph that has a certain "structure"
- using this structure, prove that the whole graph is either basic, or admits a decomposition.

At first it seems that instead of using trigraphs, one could redefine the "structure" to allow more flexibility, and say the whole proof in terms of graphs only. We would like to remark that despite a certain amount of effort invested in this approach, we were unable to come up with consistent ways to define the structures, and so the idea of using trigraphs seems crucial.

Let us start by stating the decomposition theorem of [2]. First we need some definitions. For a subset X of the vertex set of G we denote by $G|X$ the subgraph of G induced on X. The *line graph* $L(G)$ of a graph G is the graph whose vertex set is $E(G)$ in which two members of $E(G)$ are adjacent if and only if they share an end in G.

We need one other class of graphs, defined as follows. Let $m, n \geq 2$ be integers, and let $\{a_1, \ldots, a_m\}$, $\{b_1, \ldots, b_m\}$, $\{c_1, \ldots, c_n\}$, $\{d_1, \ldots, d_n\}$ be disjoint sets. Let G have vertex set their union, and edges as follows:

- a_i is adjacent to b_i for $1 \leq i \leq m$, and c_j is non-adjacent to d_j for $1 \leq j \leq n$

- there are no edges between $\{a_i, b_i\}$ and $\{a_{i'}, b_{i'}\}$ for $1 \le i < i' \le m$, and all four edges between $\{c_j, d_j\}$ and $\{c_{j'}, d_{j'}\}$ for $1 \le j < j' \le n$
- there are exactly two edges between $\{a_i, b_i\}$ and $\{c_j, d_j\}$ for $1 \le i \le m$ and $1 \le j \le n$, and these two edges are disjoint.

We call such a graph G a *double split graph*. Let us say a graph G is *basic* if either G or $\overline{G}$ is bipartite or is the line graph of a bipartite graph, or G is a double split graph. (Note that if G is a double split graph then so is $\overline{G}$.) It is easy to see that all basic graphs are perfect.

A *path* in G is an *induced* subgraph of G which is non-null, connected, acyclic, and in which every vertex has degree ≤ 2, and an *antipath* is an induced subgraph whose complement is a path. (Please note that this is different from the standard definition of a path in a graph, because of the requirement to be induced.) The *length* of a path is the number of edges in it (and the length of an antipath is the number of edges in its complement.) We therefore recognize paths and antipaths of length 0. A path is said to be *odd* if it has odd length, and *even* otherwise. If P is a path, P^* denotes the set of internal vertices of P, called the *interior* of P; and similarly for antipaths.

Now we turn to the various kinds of decomposition needed in [2]. First, a decomposition essentially due to Cornuéjols and Cunningham [7], a *proper 2-join* in G is a partition (X_1, X_2) of $V(G)$ so that there exist disjoint nonempty $A_i, B_i \subseteq X_i$ $(i = 1, 2)$ satisfying:

- every vertex of A_1 is adjacent to every vertex of A_2, and every vertex of B_1 is adjacent to every vertex of B_2,
- there are no other edges between X_1 and X_2,
- for $i = 1, 2$, every component of $G|X_i$ meets both A_i and B_i, and
- for $i = 1, 2$, if $|A_i| = |B_i| = 1$ and $G|X_i$ is an induced path joining the members of A_i and B_i, then it has odd length ≥ 3.

If $X, Y \subseteq V(G)$ are disjoint, we say X is *complete* to Y (or the pair (X, Y) is *complete*) if every vertex in X is adjacent to every vertex in Y; and we say X is *anticomplete* to Y if there are no edges between X and Y. The second decomposition used in [2] is a very slight variant of the "homogeneous sets" due to Chvátal and Sbihi [5] — a *proper homogeneous pair* is a pair of disjoint nonempty subsets (A, B) of $V(G)$, such that if A_1, A_2 denote respectively the sets of all A-complete and A-anticomplete vertices and B_1, B_2 are defined similarly, then:

- $A_1 \cup A_2 = B_1 \cup B_2 = V(G) \setminus (A \cup B)$ (and in particular every vertex in A has a neighbor and a non-neighbor in B and vice versa)
- the four sets $A_1 \cap B_1$, $A_1 \cap B_2$, $A_2 \cap B_1$, $A_2 \cap B_2$ are all nonempty.

Let A, B be disjoint subsets of $V(G)$. We say the pair (A, B) is *balanced* if there is no odd path between non-adjacent vertices in B with interior in A, and there is no odd antipath between adjacent vertices in A with interior in B. A set $X \subseteq V(G)$ is *connected* if $G|X$ is connected (so $\emptyset$ is connected); and *anticonnected* if $\overline{G}|X$ is connected.

The third kind of decomposition in [2] is due to Chvátal [4] — a *skew-partition* in G is a partition (A, B) of $V(G)$ so that A is not connected and B is not anticonnected. Skew-partitions pose a difficulty that the other two decompositions do not, for it had not been shown before [2] that a minimal counterexample to the strong perfect graph conjecture cannot admit a skew-partition. In [2] we solved this problem by confining ourselves to balanced skew-partitions, which do not present this difficulty. In fact, we proved the following:

1.1 *For every Berge graph G, either G is basic, or one of G, $\overline{G}$ admits a proper 2-join, or G admits a proper homogeneous pair, or G admits a balanced skew-partition.*

Our main result here, a structure theorem for Berge trigraphs, is similar to 1.1— we prove that every Berge trigraph either belongs to one of a few basic classes or has a decomposition. As a corollary we can prove a strengthening of the structure theorem for Berge graphs, the following:

1.2 *For every Berge graph G, either G is basic, or one of G, $\overline{G}$ admits a proper 2-join or G admits a balanced skew-partition.*

(Thus the proper homogeneous pair decomposition can be avoided.)

2 Trigraphs

A *trigraph* T is a 4-tuple $(V(T), E(T), S(T), N(T))$ where V is the *vertex set* of T and every unordered pair of vertices belongs to one of the three disjoint sets: the *strong edges* $E(T)$, the *strong non-edges* $N(T)$ and the *switchable pairs* $S(T)$. In this notation a graph can be viewed as a trigraph with $S(T) = \emptyset$.

A subtrigraph T' of T is a trigraph with $V(T') \subseteq V(T)$, and for two vertices $v_1, v_2 \in V(T')$, the pair v_1v_2 belongs to $E(T'), S(T')$ or $N(T')$ if it belongs to $E(T), S(T)$ or $N(T)$ respectively. For $X \subseteq V(T)$ we denote by $T|X$ the subtrigraph of T with vertex set X.

A *realization* of a trigraph T is a graph G on the same vertex set as T such that $E(G) = S' \cup E$ for some subset S' of $S(T)$. Let us denote by ${G_{S'}}^T$ the realization of T with the edge set $E \cup S'$. Sometimes we will describe a realization of T as an *assignment of values* to switchable pairs of T. In ${G_{S'}}^T$ the switchable pairs in S' are assigned the value "edge", and those in $S(T) \setminus S'$ — the value "non-edge". The realization with edge set $E \cup S(T)$ is called the *full realization* of T.

We say that two vertices u, v of a trigraph T are *weakly adjacent* if $uv \in E \cup S$, *weakly non-adjacent* if $uv \in N \cup S$, *strongly adjacent* if $uv \in E$, *strongly non-adjacent* if $uv \in N$. (So if u and v are both weakly adjacent and weakly non-adjacent then uv is a switchable pair.) We say u is a weak (strong) neighbor of v if u is weakly (strongly) adjacent to v. We say u is a weak (strong) non-neighbor of v if u is weakly (strongly) non-adjacent to v. A subset X of $V(T)$ is *weakly (strongly) stable* if every two members of X are weakly (strongly) non-adjacent, and it is a *weak (strong) clique* if every two members of it are weakly (strongly) adjacent. If $X, Y \subseteq V(T)$ are disjoint, we say X is *weakly (strongly) complete* to Y (or the pair (X, Y) is *weakly (strongly) complete*, X is *weakly (strongly) Y-complete*) if every vertex in X is weakly (strongly) adjacent to every vertex in Y; and we say X is *weakly (strongly) anticomplete* to Y (X is *weakly (strongly) Y-anticomplete*) if every vertex in X is weakly (strongly) non-adjacent to every vertex in Y. If G is a realization of T we say that X is *(G, Y)-complete* if X is Y-complete in G.

The *complement* $\overline{T}$ of a trigraph T is a trigraph on the same vertex set as T such that $E(\overline{T}) = N(T)$, $N(\overline{T}) = E(T)$, $S(\overline{T}) = S(T)$. (This definition generalizes the complement of a graph.)

We say that a trigraph T is *Berge* if every realization of T is a Berge graph. Since a graph is Berge if and only if its complement is, and the complement of every realization of T is a realization of $\overline{T}$, a trigraph is Berge if and only if its complement is. A trigraph is called *monogamous* if every vertex of it belongs to at most one switchable pair.

A trigraph is *weakly connected* if its full realization is a connected graph. A *component* of T is a connected component of the full realization of T. A subset X of the vertex set of T is said to be weakly connected if the trigraph $T|X$ is weakly connected. A trigraph is *weakly anticonnected* if its complement is weakly connected, and an *anticomponent* of T is a weakly connected component of $\overline{T}$. A subset X of the vertex set of T is said to be weakly anticonnected if the trigraph $T|X$ is weakly anticonnected. A *component (anticomponent)* of a set $X \subseteq V(T)$ is a maximal weakly connected (anticonnected) subset of X.

A *path* or *hole* in T is a realization of a subtrigraph of T which is a path or a hole. Two vertices of a path or a hole of T are called *consecutive*, if they are adjacent in the path or the hole, respectively. An *antipath* or an *antihole* in T is a path or hole in $\overline{T}$. Thus a trigraph is Berge if and only if it contains no odd hole or antihole.

2.1 *Let T be a Berge trigraph and let uv be a switchable pair in T. Then every even path between u and v has length* 2.

Proof. If P is an even path of length > 2 between u and v then u-P-v-u is an odd hole in T, a contradiction. This proves 2.1. ∎

A trigraph T is *bipartite* if its vertex set can be partitioned into two strongly stable sets. Every realization of a bipartite trigraph is a bipartite graph, and hence every bipartite trigraph is Berge, and so is the complement of a bipartite trigraph.

A trigraph T is a *line trigraph* if the full realization of T is the line graph of a bipartite graph with at least three vertices of degree at least three, and in addition, every weak clique of size at least 3 in T is a strong clique. The following is an easy fact about line trigraphs:

2.2 *Every line trigraph is Berge.*

Proof. Let T be a line trigraph, and let $S = S(T)$. We need to prove that for every subset S' of S the graph ${G_{S'}}^T$ is Berge. The proof is by induction on $|S \setminus S'|$. The base case holds since ${G_S}^T$ is the line graph of a bipartite graph. For the inductive step it is enough to prove that if ${G_{S'}}^T$ is Berge then ${G_{S' \setminus e}}^T$ is Berge for every $e \in S'$. Let e be the pair u, v. Suppose ${G_{S' \setminus e}}^T$ contains an odd hole H. Since ${G_{S'}}^T$ is Berge, both u and v belong to H. But that means that in ${G_{S' \setminus e}}^T$ there exist an even path between u and v. Since ${G_{S'}}^T$ is Berge, this path has length 2 and T contains a weak clique of size 3 which is not strong, a contradiction. Now assume that ${G_{S' \setminus e}}^T$ contains an odd antihole A. Since ${G_{S'}}^T$ is Berge, both u and v belong to A. But then there exists a vertex of T weakly adjacent to both u and v, contrary to the fact that every clique of size at least 3 in T is strong. This proves 2.2 ∎

Let us now define the trigraph analogue of the double split graph, namely the *double split trigraph*.

Let $m, n \geq 2$ be integers, and let $\{a_1, \ldots, a_m\}$, $\{b_1, \ldots, b_m\}$, $\{c_1, \ldots, c_n\}$, $\{d_1, \ldots, d_n\}$ be disjoint sets. Let T have vertex set their union, and

- a_i is weakly adjacent to b_i for $1 \leq i \leq m$, and c_j is weakly non-adjacent to d_j for $1 \leq j \leq n$

- $\{a_i, b_i\}$ is strongly anticomplete to $\{a_{i'}, b_{i'}\}$ for $1 \leq i < i' \leq m$, and the $\{c_j, d_j\}$ is strongly complete to $\{c_{j'}, d_{j'}\}$ for $1 \leq j < j' \leq n$
- there are exactly two strong edges and exactly two strong non-edges between $\{a_i, b_i\}$ and $\{c_j, d_j\}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, and the two strong edges are disjoint.

We now show that

2.3 *Every double split trigraph is Berge.*

Proof. It is again enough to prove that for every $S' \subseteq S(T)$, the graph $G_{S'}{}^T$ is Berge. Let S_1 be the set of all switchable pairs a_i, b_i with $1 \leq i \leq m$, and let S_2 be the set of all switchable pairs c_j, d_j with $1 \leq j \leq n$. Hence $S(T)T = S_1 \cup S_2$. The proof is by induction on $|S_1 \setminus S'| + |S_2 \cap S'|$ (the number of switchable pairs whose value in the realization $G_{S'}{}^T$ is different from their value in the "natural" realization of T which is a double split graph.) The base case holds since $G_{S_1}{}^T$ is a double split graph.

By passing to the complement, if necessary, it is enough to show that if $G_{S'}{}^T$ is Berge then $G_{S' \setminus e}{}^T$ is Berge for every e in $S' \cap S_1$. Let the e be a_1, b_1. Since $G_{S'}{}^T$ is Berge, if $G_{S' \setminus e}{}^T$ contains an odd hole or an odd antihole, then both a_1 and b_1 belong to it.

Assume first that $G_{S' \setminus e}{}^T$ contains an odd hole. Then $G_{S' \setminus e}{}^T$ contains an even path between a_1 and b_1. Since $G_{S'}{}^T$ is Berge, this path has length 2. But then T contains a vertex weakly adjacent to both a_1 and b_1, a contradiction.

Now assume that $G_{S' \setminus e}{}^T$ contains an odd antihole. But then again T contains a vertex weakly adjacent to both a_1 and b_1, a contradiction. This proves 2.3. ∎

In order to state the trigraph analogue of 1.1 we also need to define three sporadic Berge trigraphs $Spor_1, Spor_2, Spor_3$:

- $V(Spor_1) = \{x_1, x_2, x_3\}$
  $E(Spor_1) = N(Spor_1) = \emptyset$
  and $S(Spor_1) = \{x_i x_j \ : \ 1 \leq i < j \leq 3\}$
- $V(Spor_2) = \{x_1, x_2, x_3, x_4\}$
  $E(Spor_2) = \{x_2 x_4\}$
  $N(Spor_2) = \{x_3 x_4\}$
  and $S(Spor_2) = \{x_i x_j \ : \ 1 \leq i < j \leq 3\} \cup \{x_1 x_4\}$
- $V(Spor_3) = \{x_1, x_2, x_3, x_4, x_5\}$
  $E(Spor_3) = \{x_1 x_4, x_2 x_5\}$
  $N(Spor_3) = \{x_4 x_5, x_3 x_4, x_3 x_5\}$
  and $S(Spor_3) = \{x_i x_j \ : \ 1 \leq i < j \leq 3\} \cup \{x_2 x_4, x_1 x_5\}$.

Now we describe the decompositions that we need. First, a *proper 2-join* in T is a partition (X_1, X_2) of $V(T)$ so that there exist disjoint nonempty $A_i, B_i \subseteq X_i$ $(i = 1, 2)$ satisfying:

- no switchable pair meets both X_1 and X_2,

- every vertex of A_1 is strongly adjacent to every vertex of A_2, and every vertex of B_1 is strongly adjacent to every vertex of B_2,
- there are no other strong edges between X_1 and X_2, and
- for $i = 1, 2$, every component of $T|X_i$ meets both A_i and B_i,
- for $i = 1, 2$ $|X_i| \geq 3$, and
- for $i = 1, 2$, if $|A_i| = |B_i| = 1$, then the full realization of $T|X_i$ is not an even path joining the members of A_i and B_i.

Our second decomposition is a "proper homogeneous pair" in T. A *proper homogeneous pair* is a pair of disjoint nonempty subsets (A, B) of $V(T)$, such that if A_1, A_2 denote respectively the sets of all strongly A-complete and strongly A-anticomplete vertices and B_1, B_2 are defined similarly, then:

- $|A| > 1$ and $|B| > 1$,
- $A_1 \cup A_2 = B_1 \cup B_2 = V(G) \setminus (A \cup B)$ (and in particular every vertex in A has a weak neighbor and a weak non-neighbor in B and vice versa), and
- the four sets $A_1 \cap B_1$, $A_1 \cap B_2$, $A_2 \cap B_1$, $A_2 \cap B_2$ are all nonempty.

Let A, B be disjoint subsets of $V(T)$. We say the pair (A, B) is *balanced* if there is no odd path of length greater than 1 with ends in B and interior in A, and there is no odd antipath of length greater than 1 with ends in A and interior in B. A *skew-partition* is a partition (A, B) of $V(T)$ so that A is not weakly connected and B is not weakly anticonnected. The third kind of decomposition we use is a balanced skew-partition.

The three decompositions we just described generalize the decompositions that we used in [2], and in addition all the "important" edges and non-edges in those graph decompositions are required to be strong edges and strong non-edges of the trigraph, respectively.

We now describe two more kinds of decompositions, that have no analogue in the graph case. We remark that these decompositions are not needed when the trigraph in question is monogamous, which is the case we focus on in this paper, but we need them to state the full theorem.

The first one is a "1-separation". We say that a trigraph T admits a *1-separation* if there is a vertex v in T such that the trigraph $T|(V(T) \setminus v)$ is not weakly connected. It is easy to see that if T_1 and T_2 are two Berge trigraphs and T is obtained from T_1 and T_2 by identifying a vertex $v_1 \in V(T_1)$ with a vertex $v_2 \in V(T_2)$, then T is Berge.

The second one is the homogeneous set decomposition. We say that T admits a *homogeneous set decomposition* (U, V_E, V_S, V_N) if U, V_E, V_S, V_N partition the vertex set of T and

- $|U| > 1$
- for every $u \in U$ and $v \in V(T) \setminus U$ the pair uv is a strong edge if $v \in V_E$, a strong non-edge if $v \in V_N$ and a switchable pair if $v \in V_S$
- either $|V_S| \leq 1$ and no realization of $T|U$ contains a path of length 3 or
$V_S = \{a, b\}$, a is strongly complete to V_E and U is a strongly stable set.

We say that a trigraph T is obtained from the trigraph T_1 by *substituting* the trigraph T_2 for a vertex v of T_1, if T is obtained from T_1 by replacing v by a copy of T_2, and making all vertices of T_2 strongly and weakly adjacent (non-adjacent) to the strong and weak neighbors (non-neighbors) of v in T_1, respectively.

The homogeneous set decomposition preserves Bergeness in trigraphs in the following two senses:

2.4 *Let T_1 be a Berge trigraph and let v be a vertex of T that belongs to at most one switchable pair. Let T_2 be a Berge trigraph no realization of which contains a path of length 3. Then the trigraph obtained from T_1 by substituting T_2 for v is Berge.*

2.5 *Let T_1 be a Berge trigraph and let v be a vertex of T that belongs to exactly 2 switchable pairs, say va and vb, and assume that a is strongly adjacent to every strong neighbor of v. Let T_2 be a Berge trigraph with $E(T_2) = S(T_2) = \emptyset$. Then the trigraph obtained from T_1 by substituting T_2 for v is Berge.*

Proof of 2.4. Suppose T is not Berge. Then there exists a realization of T that contains an odd hole or an odd antihole. Since $\overline{T}$ can be obtained from $\overline{T_1}$ by substituting $\overline{T_2}$ for v, and no realization of $\overline{T_2}$ contains a path of length 3, passing to the complement if necessary we may assume that T contains an odd hole H.

Since T_1 is Berge, at least two vertices of H belong to $V(T_2)$. Let V_E be the set of strong neighbors of v in T_1, let V_N be the set of strong non-neighbors of v in T_1, and let $V_S = V(T_1) \setminus (\{v\} \cup V_E \cup V_N)$. By the hypothesis of the theorem, $|V_S| \leq 1$.

Since no realization of T_2 contains a path of length 3, $|V(H) \setminus V(T_2)| \geq 2$, moreover, at least two vertices of $V(H) \setminus V(T_2)$ have both a neighbor and a non-neighbor in $V(H) \cap T_2$, consequently they both belong to V_S, a contradiction. This proves 2.4. ∎

Proof of 2.5. Suppose T is not Berge. Then there exists a realization of T that contains an odd hole or an odd antihole. Since T_1 is Berge, at least two vertices of the odd hole or the odd antihole belong to $V(T_2)$. Let V_E be the set of strong neighbors of v in T_1, let V_N be the set of strong non-neighbors of v in T_1, and let $V_S = V(T_1) \setminus (\{v\} \cup V_E \cup V_N)$. (By the hypothesis of the theorem $V_S = \{a, b\}$ and a is strongly complete to V_E.)

Assume first that T contains an odd hole H. Since $|V(H) \cap V(T_2)| \geq 2$, there are three vertices in $V(H) \setminus V(T_2)$ with neighbors in $V(H) \cap V(T_2)$ in H. Since $|V(H)| > 4$, at most one vertex of H is in V_E and so H uses both a, b and $|V(H) \cap V_E| = 1$. But since a is strongly complete to V_E, H does not use a, a contradiction.

Now assume that T contains an odd antihole A. Since A contains no stable set of size 3, exactly two vertices of A belong to $V(T_2)$, and they are consecutive in the antihole. Let the vertices of A be $\{a_1, a_2, .., a_k\}$ in order such that a_1 and a_2 are in $V(T_2)$. Then $\{a_3, a_k\} = \{a, b\}$ (for each of the vertices a_3, a_k is adjacent to exactly one of a_1, a_2 in A.) Without loss of generality we may assume that $a_3 = a$. The vertex a_4 is adjacent to both a_1 and a_2 in A, and is different from b, since $k \geq 5$. Hence a_4 belongs to V_E. On the other hand a_4 is non-adjacent to a_3 in A, contrary to the fact that a is strongly complete to V_E. This proves 2.5. ∎

We are now ready to state the decomposition theorem for Berge trigraphs.

2.6 *Let T be a Berge trigraph. Then either*

- *T or $\overline{T}$ is either bipartite, or a line trigraph, or a double split trigraph, or*
- *T or $\overline{T}$ is one of the three sporadic trigraphs $Spor_1$, $Spor_2$, $Spor_3$, or*
- *T or $\overline{T}$ admits either a proper 2-join, or a balanced skew-partition, or a proper homogeneous pair, or*
- *T or $\overline{T}$ admits either a homogeneous set decomposition or a 1-separation.*

A full proof of 2.6 can be found in [3].

3 The application—graph decomposition

In this section we show how 2.6 can be used to obtain structural results for Berge graphs, namely we will use 2.6 to prove 1.2. In fact, to prove 1.2 it is enough to consider the class of monogamous Berge trigraphs. We say a trigraph T is *basic* if T is monogamous and one of T, $\overline{T}$ is a bipartite trigraph, a line trigraph or a double split trigraph. In this case we get a simpler decomposition theorem, the following:

3.1 *Let T be a monogamous Berge trigraph. Then either*

- *T is basic or*
- *T or $\overline{T}$ admits a proper 2-join, or*
- *T admits a balanced skew-partition.*

Clearly 3.1 implies 1.2 for, as we have already said before, a graph can be viewed as a special case of a trigraph with an empty set of switchable pairs, and in particular a monogamous trigraph.

As we shall see in this section, in order to prove 3.1 it is enough to prove the following

3.2 *Let T be a monogamous Berge trigraph. Then either*

- *T is basic or*
- *T or $\overline{T}$ admits a proper 2-join, or*
- *T admits a balanced skew-partition, or*
- *T admits a proper homogeneous pair.*

In the remainder of this section we prove 3.1 assuming 3.2. The full proof of 3.2 is in [3]. In fact, it follows from probing closer into the proof of 2.6. In sections 4—8 of this paper we will present the aspects of the proof of 3.2 that differ significantly from the proof of 1.1.

Proof of 3.1. Suppose the theorem is false and consider a counterexample T with $|V(T)|$ minimum. By 3.2 that means that T admits a proper homogeneous pair decomposition and satisfies none of the outcomes of 3.1. Let (A, B) be a proper homogeneous pair in T, let A_1, A_2 respectively be the sets of all strongly A-complete and strongly A-anticomplete vertices in T and let B_1, B_2 be defined similarly. Let $C = A_1 \cap B_2$, $D = A_2 \cap B_1$, $E = A_2 \cap B_2$ and $F = A_1 \cap B_1$. Let us define a new trigraph T' with $V(T') = C \cup D \cup E \cup F \cup \{a, b\}$ where $a, b \notin V(T)$ such that

- $T'|(C \cup D \cup E \cup F) = T|(C \cup D \cup E \cup F)$
- ab is a switchable pair
- for a vertex u in $C \cup D \cup E \cup F$, ua is a strong edge or a strong non-edge in T' if u is strongly complete or strongly anticomplete to A in T, respectively
- for a vertex u in $C \cup D \cup E \cup F$, ub is a strong edge or a strong non-edge in T' if u is strongly complete or strongly anticomplete to B in T, respectively.

Since $T'|(C \cup D \cup E \cup F) = T|(C \cup D \cup E \cup F)$ and the only switchable pair containing a or b is ab, T' is monogamous.

We claim that T' is Berge. Suppose T' contains an odd hole or an odd antihole H. Since T is Berge, H is not a realization of a subtrigraph of T, so $V(H) \cap \{a, b\} \neq \emptyset$. If $a \in V(H)$ and $b \notin V(H)$ then for any vertex $a' \in A$ the trigraph $T|((V(H) \setminus \{a\}) \cup \{a'\})$ has a realization as an odd hole or antihole, a contradiction. So both a and b are in H. Choose $a' \in A$ and $b' \in B$, weakly adjacent if a is adjacent to b in H, and weakly non-adjacent otherwise. Now the trigraph $T|((V(H) \setminus \{a, b\}) \cup \{a', b'\})$ has a realization as an odd hole or antihole in T, a contradiction.

From the definition of a proper homogeneous pair, A and B contain at least two vertices each, so $|V(T')| < |V(T)|$. By the minimality of T, the assertion of the theorem holds for the trigraph T', namely either T' is basic or T' admits a balanced skew-partition or one of T' or $\overline{T'}$ admits a proper 2-join. We show that in fact T' cannot be basic and every decomposition of T' extends to a decomposition of the same type in T, thus obtaining a contradiction to the assumption that T is a counterexample to the theorem.

The proof now breaks into cases according to the type of behavior of T'. We can cut down the number of cases by noticing that if T is a minimum size counterexample to the theorem, then so is $\overline{T}$ and the graph $(\overline{T})'$ obtained from $\overline{T}$ by the procedure described above is just $\overline{T'}$. So we may assume that T' is either bipartite, or a line trigraph, or a double split trigraph, or admits a balanced skew-partition, or a proper 2-join.

Case 1 T' is a bipartite trigraph.

This case is impossible, for $\{a, b, f\}$ is a weak clique of size 3 for any vertex $f \in F$.

Case 2 T' is a line trigraph.

This case is impossible since $\{a, b, f\}$ is a weak clique that is not a strong clique for every vertex $f \in F$.

Case 3 T' is a double split trigraph.

Then $V(T') = \{a_1, \dots, a_m\} \cup \{b_1, \dots, b_m\} \cup \{c_1, \dots, c_n\} \cup \{d_1, \dots, d_n\}$ for some integers $m, n \geq 2$, and the only possible switchable pairs in T are those of the form $a_i b_i$ for $1 \leq i \leq m$ and $c_j d_j$ for $1 \leq j \leq n$. So no switchable pair is contained in both a weak clique and a weak stable set of size 3. But in T', $\{a, b, f\}$ is a weak triangle for every vertex $f \in F$ and $\{a, b, e\}$ is a weak stable set of size 3 for every vertex $e \in E$, a contradiction. This finishes Case 3.

Case 4 T' admits a balanced skew-partition.

That means that $V(T')$ can be partitioned into two sets M and N, such that M is not weakly connected and N is not weakly anticonnected. Let

$$M' = \begin{cases} M & if \quad a \notin M \quad b \notin M \\ M \setminus \{a\} \cup A & if \quad a \in M, \quad b \notin M \\ M \setminus \{b\} \cup B & if \quad a \notin M, \quad b \in M \\ M \setminus \{a,b\} \cup A \cup B & if \quad a \in M, \quad b \in M \end{cases}$$

Let N' be defined similarly. Since a, b is a switchable pair, the vertices a and b either belong to the same component of M or to the same anticomponent of N or one of them is in M and the other one is in N. Consequently (M', N') is a skew-partition of T.

We now show that this skew-partition is balanced. Assume it is not. By passing to the complement if necessary we may assume that there exists a path $p_1\text{-}\cdots\text{-}p_k$ of odd length at least 3, with ends in N' and interior in M'. Let P be this path. Since P is not a realization of a subtrigraph of T', either $|V(P) \cap A| \geq 2$ or $|V(P) \cap B| \geq 2$. Let s, t be minimum and maximum such that $1 \leq s < t \leq k$ and $\{p_s, p_t\}$ is a subset of one of A, B, say B.

Since either $B \subseteq M'$ or $B \subseteq N'$, and the ends of P are in N' and the interior is in M', it follows that either p_s, p_t are both ends of P and $P^* \cap B = \emptyset$, or they both belong to P^*. In the first case define $a_1 = p_2$ and $a_2 = p_{k-1}$. In the second case define $a_1 = p_{s-1}$ and $a_2 = p_{t+1}$. Since P is a path of length at least three, in both cases a_1, a_2 are distinct and do not belong to B.

In both cases a_1 is adjacent in P to p_s and not p_t, and a_2 is adjacent in P to p_t and not p_s. Since every vertex in $V(T) \setminus (A \cup B)$ is either strongly complete to B or strongly anticomplete to B, both a_1 and a_2 belong to A. So from the choice of s, t we deduce that $s = 1, t = k$. Hence p_2, p_{k-1} belong to A and $P^* \cap B = \emptyset$. Since $p_1\text{-}P\text{-}p_k\text{-}d\text{-}p_1$ is not an odd hole for $d \in D$, it follows that P^* is not contained in A.

Let $2 \leq i \leq k$ be minimum such that p_i does not belong to A. Then $i > 2$. Since p_i is adjacent to p_{i-1} (which is in A) and not to p_1 (which is in B), p_i belongs to C. So p_i is complete in P to $\{p_2, p_{k-1}\}$ and since P is a path $i = 3$ and $k = 5$, contrary to the fact that P has odd length. This proves that the skew-partition (N', M') is balanced and finishes Case 4.

Case 5 T' admits a proper 2-join.

Let (X_1, X_2) be a proper 2-join in T. Since ab is a switchable pair, either both a and b belong to X_1, or they both belong to X_2. Without loss of generality we may assume that both a and b are in X_1. But then, since every vertex in A has a weak neighbor in B and vice versa, it follows that $((X_1 \setminus \{a, b\}) \cup A \cup B, X_2)$ is a proper 2-join in T, a contradiction. This finishes Case 5 and completes the proof of 3.1. ∎

4 Overview of the proof of 3.2

In this section we sketch the outline of the proof of 3.2. Similarly to [2] the idea of the proof is, given a trigraph T, to find small subtrigraphs F of it that would force T either to be basic or to admit a decomposition. The proof breaks into steps each of which is characterized by the subtrigraph F

that is considered at that step. Clearly, having proved that a certain subtrigraph F_1, if present in a Berge trigraph, forces it to either belong to a basic class or have a decomposition, we can from then on assume that all trigraphs in question do not contain F_1.

Many of the trigraphs F we use here correspond to the subgraphs considered in [2]— such as a trigraph that has a realization that is the line graph of a "substantial" bipartite graph, or as an "odd prism" (precise definitions that are important for us in this paper will be given later; for others we refer the reader to [2] or [3]). However, later in the proof, finding the right generalization of the subgraph used in [2] becomes more difficult, and sometimes we will need to deviate from the route of [2]. (Further complications arise if the trigraph in question is not monogamous (see [3]), but they are outside of the scope of this paper.)

Let $\mathcal{T}_1, \dots, \mathcal{T}_{13}$ be the following classes of monogamous Berge trigraphs:

- $\mathcal{T}_1$ is the class of all Berge trigraphs in which every appearance of K_4 is degenerate
- $\mathcal{T}_2$ is the class of all trigraphs T such that $T, \overline{T} \in \mathcal{T}_1$ and no subtrigraph of T has a realization isomorphic to $L(K_{3,3})$.
- $\mathcal{T}_3$ is the class of all Berge trigraphs T so that for every bipartite subdivision H of K_4, no subtrigraph of T or of $\overline{T}$ has a realization isomorphic to the line graph of H
- $\mathcal{T}_4$ is the class of all $T \in \mathcal{T}_3$ so that no subtrigraph of T is an even prism
- $\mathcal{T}_5$ is the class of all $T \in \mathcal{T}_3$ so that no subtrigraph of T or of $\overline{T}$ is a long prism
- $\mathcal{T}_6$ is the class of all $T \in \mathcal{T}_5$ such that no subtrigraph of T is isomorphic to a double diamond
- $\mathcal{T}_7$ is the class of all $T \in \mathcal{T}_6$ so that T and $\overline{T}$ do not contain odd wheels
- $\mathcal{T}_8$ is the class of all $T \in \mathcal{T}_7$ so that T and $\overline{T}$ do not contain pseudowheels
- $\mathcal{T}_9$ is the class of all $T \in \mathcal{T}_8$ such that T and $\overline{T}$ do not contain wheels
- $\mathcal{T}_{10}$ is the class of all $T \in \mathcal{T}_9$ such that, for every hole C in T of length ≥ 6 with an origin, no vertex of T is weakly adjacent to the origin and both of its weak neighbors in C, and the same holds in $\overline{T}$
- $\mathcal{T}_{11}$ is the class of all $T \in \mathcal{T}_{10}$ such that, for every hole C in T of length ≥ 6, no vertex of T has three consecutive weak neighbors in C, and the same holds in $\overline{T}$
- $\mathcal{T}_{12}$ is the class of all $T \in \mathcal{T}_{11}$ such that every antihole in T has length 4
- $\mathcal{T}_{13}$ is the class of all $T \in \mathcal{T}_{12}$ such that T contains no strong clique of size three.

The following are the main steps of the proof of 3.2

1. For every Berge trigraph T, either T is a line trigraph or T admits a proper 2-join or a balanced skew-partition, or $T \in \mathcal{T}_1$.
2. For every T with $T, \overline{T} \in \mathcal{T}_1$, either T or $\overline{T}$ is a line trigraph or one of $T, \overline{T}$ admits a proper 2-join, or T admits a balanced skew-partition, or $T \in \mathcal{T}_2$.

3. For every $T \in \mathcal{T}_2$, either T is a double split trigraph, or one of $T, \overline{T}$ admits a proper 2-join, or T admits a balanced skew-partition, or $T \in \mathcal{T}_3$.
4. For every $T \in \mathcal{T}_1$, either T is an even prism with exactly 9 vertices, or T admits a proper 2-join or a balanced skew-partition, or $T \in \mathcal{T}_4$.
5. For every T such that $T \in \mathcal{T}_4$ and $\overline{T} \in \mathcal{T}_4$, either one of $T, \overline{T}$ admits a proper 2-join, or T admits a proper homogeneous pair, or T admits a balanced skew-partition, or $T \in \mathcal{T}_5$.
6. For every $T \in \mathcal{T}_5$, either one of $T, \overline{T}$ admits a proper 2-join, or T admits a balanced skew-partition, or $T \in \mathcal{T}_6$.
7. For every $T \in \mathcal{T}_6$, either T admits a balanced skew-partition, or $T \in \mathcal{T}_7$.
8. For every $T \in \mathcal{T}_7$, either T admits a balanced skew-partition, or $T \in \mathcal{T}_8$.
9. For every $T \in \mathcal{T}_8$, either T admits a balanced skew-partition, or $T \in \mathcal{T}_9$.
10. For every $T \in \mathcal{T}_9$, either T admits a balanced skew-partition, or $T \in \mathcal{T}_{10}$.
11. For every $T \in \mathcal{T}_{10}$, either T admits a balanced skew-partition, or $T \in \mathcal{T}_{11}$.
12. For every $T \in \mathcal{T}_{11}$, either $T \in \mathcal{T}_{12}$ or $\overline{T} \in \mathcal{T}_{12}$.
13. For every $T \in \mathcal{T}_{12}$, either T admits a balanced skew-partition, or one of T, $\overline{T}$ is bipartite or $T \in \mathcal{T}_{13}$.
14. For every $T \in \mathcal{T}_{13}$, either T or $\overline{T}$ is bipartite, or T admits a balanced skew-partition.

Steps 1—8 of the proof are a rather straightforward generalization of the proof in [2], the details of which can be found in [3], and we omit them here. The rest of the proof (steps 9—14) is trickier, and does not follow the outline of [2] as closely. In the remainder of this paper we present that part of the proof, namely we prove

4.1 *For every $T \in \mathcal{T}_8$, either T admits a balanced skew-partition or one of $T, \overline{T}$ is bipartite.*

Statements 9—14 are proved in 6.21, 7.4, 7.6, 7.8, 8.5 and 8.6, and thus 4.1 follows.

Some of the theorems in this paper are proved by applying theorems from [2] to the right realization of a trigraph. For this reason we need the classification of Berge graphs used in [2]. Let $\mathcal{F}_1, \dots, \mathcal{F}_{11}$ be the classes of Berge graphs defined as follows

- $\mathcal{F}_1$ is the class of all Berge graphs G such that for every bipartite subdivision H of K_4, every induced subgraph of G isomorphic to $L(H)$ is degenerate
- $\mathcal{F}_2$ is the class of all graphs G such that $G, \overline{G} \in \mathcal{F}_1$ and no induced subgraph of G is isomorphic to $L(K_{3,3})$
- $\mathcal{F}_3$ is the class of all Berge graphs G so that for every bipartite subdivision H of K_4, no induced subgraph of G or of $\overline{G}$ is isomorphic to $L(H)$

- $\mathcal{F}_4$ is the class of all $G \in \mathcal{F}_3$ so that no induced subgraph of G is an even prism
- $\mathcal{F}_5$ is the class of all $G \in \mathcal{F}_3$ so that no induced subgraph of G or of $\overline{G}$ is a long prism
- $\mathcal{F}_6$ is the class of all $G \in \mathcal{F}_5$ such that no induced subgraph of G is isomorphic to a double diamond
- $\mathcal{F}_7$ is the class of all $G \in \mathcal{F}_6$ so that G and $\overline{G}$ do not contain odd wheels
- $\mathcal{F}_8$ is the class of all $G \in \mathcal{F}_7$ so that G and $\overline{G}$ do not contain pseudowheels
- $\mathcal{F}_9$ is the class of all $G \in \mathcal{F}_8$ such that G and $\overline{G}$ do not contain wheels
- $\mathcal{F}_{10}$ is the class of all $G \in \mathcal{F}_9$ such that, for every hole C in G of length ≥ 6, no vertex of G has three consecutive neighbors in C, and the same holds in $\overline{G}$
- $\mathcal{F}_{11}$ is the class of all $G \in \mathcal{F}_{10}$ such that every antihole in G has length 4.

5 Tools and some definitions

In this section we give some definitions and quote (without proofs) some lemmas from [3] that will be needed in the subsequent sections. Please note, that since a graph can be viewed as a trigraph with the set of switchable pairs empty, certain subtrigraphs defined here translate into subgraphs when used in the graph case. We start with three facts about common weak and strong neighbors of weakly anticonnected sets.

This is an easy variant of a theorem of Roussel and Rubio [8].

5.1 *Let T be a Berge trigraph, let X be a weakly anticonnected subset of $V(T)$, and P be a path in $T \setminus X$ with odd length, such that that both ends of P are weakly X-complete. Assume that for no edge e of P, both of its ends are weakly X-complete and the vertices of e in P^* are strongly X-complete. Then every weakly X-complete vertex has a strong neighbor in $V(P^*)$.*

A *prism* is a trigraph consisting of two vertex-disjoint weak triangles $\{a_1, a_2, a_3\}$, $\{b_1, b_2, b_3\}$ and three vertex disjoint subtrigraphs R_1, R_2, R_3, such that for $1 \leq i \leq 3$, $\{a_i, b_i\} \subseteq V(R_i)$ and R_i has a realization as a path with ends a_i, b_i; and for $1 \leq i < j \leq 3$ the only possible strong edges between $V(R_i)$ and $V(R_j)$ are a_ia_j and b_ib_j. The prism is *long* if $|V(R_i)| \geq 3$ for at least one value of i.

We remind the reader that $\mathcal{T}_5$ is the class of all monogamous Berge trigraphs T, such that no subtrigraph of T or $\overline{T}$ has a realization isomorphic to the line graph of a bipartite subdivision of K_4, and no subtrigraph of T or $\overline{T}$ is a long prism.

5.2 *Let $T \in \mathcal{T}_5$, let X be a weakly anticonnected subset of $V(T)$, and P be a path in $T \setminus X$ of odd length, such that both ends of P are weakly X-complete. Then either:*

1. *some edge e of P is weakly X-complete, moreover the vertices of e that belong to P^* are strongly X-complete or*
2. *P has length 3 and there is an odd antipath joining the internal vertices of P with interior in X.*

The *double diamond* is the trigraph with eight vertices $a_1, \dots, a_4, b_1, \dots, b_4$ and with the following adjacencies:

- $\{a_1, a_2\}$ is weakly complete to $\{a_3, a_4\}$ and weakly anticomplete to $\{b_3, b_4\}$.
- $\{b_1, b_2\}$ is weakly complete to $\{b_3, b_4\}$ and weakly anticomplete to $\{a_3, a_4\}$.
- a_1-b_1-b_2-a_2-a_1 is a hole.
- a_3-b_4-b_3-a_4-a_3 is an antihole.

$\mathcal{T}_6$ is the class of all $T \in \mathcal{T}_5$ such that no subtrigraph of T is isomorphic to the double diamond.

5.3 *Let T be a trigraph in $\mathcal{T}_6$ and let G be a realization of T. Let C be a hole in G, and let $X \subseteq V(G) \setminus V(C)$ be anticonnected in G. Let P be a path in C of length > 1 so that its ends are (G, X)-complete and its internal vertices are not. Then P has even length.*

The following is an easy corollary of 5.3.

5.4 *Let T be a trigraph in $\mathcal{T}_6$ and let G be a realization of T. Let C be a hole in G, and let $X \subseteq V(G) \setminus V(C)$ be anticonnected in G. Then either C contains an even number of (G, X)-complete edges, or C contains exactly one (G, X)-complete edge and exactly two (G, X)-complete vertices.*

Let us now mention two theorems from [2] that we will need. Both of them are results about graphs, and so for our purposes they will always be applied to a certain realization of a trigraph. Let C be a hole in a Berge graph G, and let $e = uv$ be an edge of it. Let u' be the neighbor of u in $C \setminus v$, and let v' be the neighbor of v in $C - u$. A *leap* for C (*in* G, *at* uv) is a pair of non-adjacent vertices a, b of G, so that there are exactly six edges between a, b and C, namely au, av, au', bu, bv, bv'. A *hat* for C (*in* G, *at* uv) is a vertex of G adjacent to u and v and to no other vertex of C.

5.5 *Let G be a Berge graph, let $X \subseteq V(G)$ be anticonnected, let C be a hole in $G \setminus X$ with length > 4, and let $e = uv$ be an edge of C. Assume that u, v are X-complete and no other vertex of C is X-complete. Then either X contains a hat for C at uv, or X contains a leap for C at uv.*

Let $\{x_1, x_2, x_3\}$ be a triangle in G. A *reflection* of this triangle is another triangle $\{y_1, y_2, y_3\}$ in G, disjoint from the first, so that for $1 \leq i \leq 3$ x_i is adjacent to y_i, and these are the only edges between $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$. A subset F of $V(G)$ is said to *catch* the triangle $\{x_1, x_2, x_3\}$ if it is connected and disjoint from the triangle, and x_1, x_2, x_3 all have neighbors in F.

5.6 *Let $X = \{x_1, x_2, x_3\}$ be a triangle in a graph $G \in \mathcal{F}_7$, and let $F \subseteq V(G) \setminus X$ catch X. Then either*

1. *some vertex of F has ≥ 2 neighbors in X or*
2. *F contains a reflection of X.*

Next we present a lemma about properties of skew-partitions in Berge trigraphs, essentially proved in [2] and [3].

5.7 *Let $T \in \mathcal{T}_6$, and assume that T admits no balanced skew-partition. Let $X, Y \subseteq V(T)$ be nonempty, disjoint, and strongly complete to each other.*

- *If $X \cup Y = V(T)$, then $\overline{T}$ is bipartite.*
- *If $X \cup Y \neq V(T)$, then $V(T) \setminus (X \cup Y)$ is weakly connected, and if in addition $|X| > 1$, then every vertex in X has a weak neighbor in $V(T) \setminus (X \cup Y)$.*

In particular, T admits no skew-partition.

A *wheel* in a trigraph T is a pair (C, Y), satisfying:

- C is a hole of length ≥ 6
- Y is a non-empty weakly anticonnected set disjoint from C
- there are two disjoint edges of C, both weakly Y-complete.

Note that if (C, Y) is a wheel in some realization of T, then it is a wheel in T. We say that (C, Y) is a *strong wheel* in T if it is a wheel and there are two disjoint edges of C, both strongly Y-complete. We call C the *rim* and Y the *hub* of the wheel. If H is a path or a hole, a maximal path in H whose vertices are all strongly Y-complete, is called a *segment* or *Y-segment* of H. A wheel (C, Y) is *odd* if some segment has odd length.

We conclude this section with another definition. A *pseudowheel* in a trigraph T is a triple (X, Y, P), satisfying:

- X, Y are disjoint nonempty weakly anticonnected subsets of $V(T)$, strongly complete to each other
- P is a path p_1-$\cdots$-p_n of $T \setminus (X \cup Y)$, where $n \geq 5$
- p_1, p_n are weakly X-complete and there are no strongly X-complete vertices in P^*
- p_1 is strongly Y-complete, and so is at least one other vertex of P; and p_2, p_n are not strongly Y-complete.

We remark that an odd wheel (C, Y) with a Y-segment S of length one can be viewed as a pseudowheel, by taking X to consist of one of the vertices of S. This is not true for wheels in general. So trigraphs containing no pseudowheels may still contain wheels.

6 General wheels

The goal of this section is to show that if a trigraph in $\mathcal{T}_8$ contains a wheel, then it admits a balanced skew-partition (that is to prove 6.21.) To do so we generalize the notion of a "wheel system" used in [2] to the trigraph case. The main part of this section will be devoted to proving by induction a theorem about wheel systems, which we later use in order to derive a contradiction by showing that for every wheel in T, there is another wheel whose hub is a proper superset of the hub of the first wheel. A similar approach was used in [2]. However the method does not carry over smoothly;

the main difficulty being the fact that two non-consecutive vertices in the rim of a wheel can be weakly-adjacent to each other. In order to handle this problem we need to introduce the notions of a "weak wheel system", and a "shadow" of a trigraph.

Let us start with some lemmas about wheels and pseudowheels.

6.1 *Let $T \in \mathcal{T}_6$. If (C, Y) is an odd wheel in some realization of T then (C, Y) is an odd wheel in T.*

Proof. Let G be a realization of T such that (C, Y) is an odd wheel in G. Then C is a hole in G and hence in T. The set Y is anticonnected in G, so it is weakly anticonnected in T. Since C contains two disjoint (G, Y)-complete edges, it contains two disjoint weakly Y-complete edges in T. So (C, Y) is a wheel in T and it remains to prove that it is an odd wheel.

Let c_1-…-c_k be an odd Y-segment of C in G. We claim that both c_1 and c_k are strongly Y-complete in T. For suppose c_1 is not strongly Y-complete. Let G' be the graph obtained from G by replacing the switchable pair of T between Y and c_1 by a non-edge. G' is a realization of T, and hence it is Berge. By 5.3 an even number of edges of C are (G, Y)-complete. However the number of (G', Y)-complete edges in C differs by 1 from the number of (G, Y)-complete edges in C, and C contains at least three (G', Y)-complete vertices, contrary to 5.3 applied in G'. This proves that c_1 is strongly Y-complete and similarly so is c_k.

Since c_1-…-c_k is a Y-segment of C in G, the vertex of C consecutive with c_1 and different from c_2 is not strongly Y-complete in T, and neither is the vertex of C consecutive with c_k and different from c_{k-1}. If $k = 2$ then c_1c_2 is an odd Y-segment of C and the theorem holds. So we may assume that $k \geq 4$. By 5.3 applied to a realization of T in which C is a hole and all switchable pairs containing vertices of Y are assigned the value "non-edge", c_1-…-c_k contains an odd Y-segment in this realization, and consequently it contains an odd Y-segment in T. This proves 6.1. ∎

By 6.1, if (C, Y) is a wheel in T and is an odd wheel in some realization of T then C contains an odd Y-segment in T. Therefore by 5.4 it contains at least two, so there exists two disjoint strongly Y-complete edges in C, and consequently (C, Y) is a strong wheel. However, (C, Y) being a general wheel in a realization of T (and therefore in T) does not imply that any edge of C is strongly Y-complete. For example, a hole c_1-…-c_{2n}-c_1 with $c_2, c_4, \ldots, c_{2n}$ strongly Y-complete and $c_1, c_3, \ldots, c_{2n-1}$ weakly and not strongly Y-complete is a wheel in T, but none of its edges are strongly Y-complete.

Let us say that distinct vertices u, v of the rim of a wheel (C, Y) have the *same wheel-parity* if there is a path of the rim joining them containing an even number of strongly Y-complete edges (and hence by 5.3 so does the second path, if u, v are not consecutive); and *opposite wheel-parity* otherwise.

If K is a subtrigraph of a trigraph T, and $F \subseteq V(T)$ is a weakly connected set disjoint from $V(K)$, a vertex in $V(K)$ is an *attachment* of F if it has a weak neighbor in F. The following was proved in [3]. The proof is a straightforward generalization of the proof of a corresponding statement in [2].

6.2 *Let $T \in \mathcal{T}_6$, and let (C, Y) be a strong wheel in T. Let $F \subseteq V(T) \setminus (V(C) \cup Y)$ be weakly connected, such that no vertex in F is strongly Y-complete. Let $X \subseteq V(C)$ be the set of attachments of F in C. Suppose that there exist vertices in X with opposite wheel-parity, and there are two vertices in X that are not consecutive in C. Then either:*

- *there is a vertex $v \in F$ so that C contains two disjoint strongly Y-complete edges that are also weakly $Y \cup \{v\}$-complete, or*
- *there is a vertex $v \in F$ with at least four weak neighbors in C, and a 3-vertex path c_1-c_2-c_3 in C, so that c_1, c_2, c_3 are all strongly Y-complete and weakly adjacent to v and every other weak neighbor of v in C has the same wheel-parity as c_1, or*
- *there is a 3-vertex path c_1-c_2-c_3 in C, all strongly Y-complete, and a path c_1-f_1-$\cdots$-f_k-c_3 with interior in F, such $\{f_1, \dots, f_k\}$ is strongly anticomplete to $V(C) \setminus \{c_1, c_2, c_3\}$.*

6.3 *Let $T \in \mathcal{T}_7$ and let G be a realization of T. If (X, Y, P) is a pseudowheel in some realization of T then (X, Y, P) is a pseudowheel in T. Moreover, P has even length at least 6, and contains an odd number, at least 3, of strongly Y-complete edges.*

Proof. Let G be a realization of T such that (X, Y, P) is a pseudowheel in G. Then the following conditions are satisfied:

- X, Y are disjoint nonempty weakly anticonnected subsets of $V(T)$, weakly complete to each other
- P is a path p_1-$\cdots$-p_n of $T \setminus (X \cup Y)$, where $n \geq 5$
- p_1, p_n are weakly X-complete and there are no strongly X-complete vertices in P^*
- p_1 is weakly Y-complete, and so is at least one other vertex of P; and p_2, p_n are not strongly Y-complete.

To prove that (X, Y, P) is a pseudowheel in T, we need to show that

- p_1 is strongly Y-complete, and so is at least one other vertex of P
- X and Y are strongly complete to each other.

(1) *p_1 is strongly Y-complete.*

Suppose it is not. Let G_1 be the graph obtained from G by deleting all edges of G between Y and p_1 that are switchable pairs of T. Then G_1 is another realization of T and hence by 6.1 $G_1 \in \mathcal{F}_7$. Moreover, none of p_1, p_n is (G_1, Y)-complete. By theorem 18.4 of [2] applied to G, P contains an odd number, at least 3, of (G, Y)-complete edges, and since p_2 is not (G, Y)-complete, P contains an odd number, at least 3, of (G_1, Y)-complete edges. But then, by theorem 18.3 of [2], an odd number of elements of $\{p_1, p_n\}$ is (G_1, Y)-complete, a contradiction. This proves (1).

(2) *At least one vertex of $P \setminus p_1$ is strongly Y-complete.*

Since (X, Y, P) is a pseudowheel in G, at least one vertex p of P^* is weakly Y-complete. Let G_2 be the realization of T defined as follows

- P is a path

- X is complete to $Y \cup \{p_1, p_n\}$
- $\{p\}$ is complete to Y
- assign the value "non-edge" to all remaining switchable pairs.

By 6.1 G_2 is a graph in $\mathcal{F}_7$, (X, Y, P) is a pseudowheel in G_2 and by theorem 18.4 of [2] P contains an odd number, at least three, of (G_2, Y)-complete edges, and in particular $P^* \setminus \{p\}$ contains a (G_2, Y)-complete vertex. In T it means that $P^* \setminus \{p\}$ contains a strongly Y-complete vertex and the result follows. This proves (2).

(3) *P has even length at least 6 and contains an odd number, at least 3, of strongly Y-complete edges.*

Let G_3 be a realization of G defined as follows

- P is a path
- X is complete to $Y \cup \{p_1, p_n\}$
- assign the value "non-edge" to all remaining switchable pairs.

By 6.1 G_3 is a graph in $\mathcal{F}_7$, (X, Y, P) is a pseudowheel in G_3 and by theorem 18.4 of [2] P has length at least 6 and contains an odd number, at least 3, of (G_3, Y)-complete edges, and consequently an odd number, at least 3, of strongly Y-complete edges in T. By theorem 18.3 of [2], P has even length. This proves (3).

(4) *X and Y are strongly complete to each other.*

Let G_3 be defined as before and let G_4 be a realization of T defined as follows

- P is a path
- X is complete to $\{p_1, p_n\}$
- assign the value "non-edge" to all remaining switchable pairs.

By 6.1 both G_3 and G_4 belong to $\mathcal{F}_7$. Let $1 \leq i \leq n$ be minimum such that $p_{i-1}p_i$ is a strongly Y-complete edge. Since p_2 and p_n are not strongly Y-complete, $4 \leq i \leq n-1$. Let P' be the path p_i-$\dots$-p_n. Since by (3) P contains at least 3 strongly Y-complete edges and p_n is not strongly Y-complete, P' has length at least 3. The only strongly Y-complete edge in P and not in P' is $p_{i-1}p_i$. Thus P' contains an even number of strongly Y-complete edges, and consequently P' contains an even number of (G_3, Y)-complete edges and an even number of (G_4, Y)-complete edges.

Suppose first that i is odd. Now X is (G_3, Y)-complete; the path $P' = p_i$-$\dots$-p_n has even length > 4; p_n is the unique (G_3, X)-complete vertex of P' and it is not (G_3, Y)-complete; p_i is (G_3, Y)-complete; and P' contains an even number, at least two, of (G_3, Y)-complete edges, contrary to theorem 18.2 of [2] applied to P', X and Y in G_3, since p_1 is a (G_3, X)-complete vertex, nonadjacent in G_3 to p_{n-1} and p_{n-2}. This proves that i is even.

Now we turn to arguing in the graph G_4. In G_4, X and Y are anticonnected sets; P' is an odd path of length at least 3; p_i is (G_4, Y)-complete; p_n is the unique (G_4, X)-complete vertex in P'; and

$p_1 \in V(T) \setminus (X \cup Y \cup V(P'))$ is $(G_4, X \cup Y)$-complete and $(G_4, V(P'))$-anticomplete; and P' contains an even number of (G_4, Y)-complete edges, and so by theorem 17.5 of [2] $X \cup Y$ is not anticonnected. It follows that X is (G_4, Y)-complete and so in T, X is strongly Y-complete. This proves (4).

From (1),(2) and (4) 6.3 follows. ∎

6.1 Wheel systems

We start by restating the definitions used in [2] for the graph case. Let G be a graph. A *frame* in G is a pair (z, A_0), where $z \in V(G)$, and A_0 is a non-empty connected subset of $V(G) \setminus z$, containing no neighbors of z. With respect to a given frame (z, A_0), a *wheel system* in G of *height* $t \geq 1$ is a sequence $x_0, \dots, x_t$ of distinct vertices of $G \setminus (A_0 \cup \{z\})$, satisfying the following conditions:

1. A_0 contains neighbors of x_0 and of x_1, and no vertex in A_0 is $\{x_0, x_1\}$-complete.
2. For $2 \leq i \leq t$, there is a connected subset of $V(G)$ including A_0, containing a neighbor of x_i, containing no neighbor of z, and containing no $\{x_0, \dots, x_{i-1}\}$-complete vertex.
3. For $1 \leq i \leq t$, x_i is not $\{x_0, \dots, x_{i-1}\}$-complete.
4. z is adjacent to all of $x_0, \dots, x_t$.

For $1 \leq i \leq t$ we define $X_i = \{x_0, \dots, x_i\}$, and we define A_i to be the maximal connected subset of $V(G)$ that includes A_0, is anticomplete to z, and contains no X_i-complete vertex. We call $A_1, \dots, A_t$ the *companion sets* of the wheel system. So for each i, $A_{i-1} \subseteq A_i$. Note that condition 2 above just says that x_i has a neighbor in A_{i-1}.

Wheel systems were an important tool for handling Berge graphs containing wheels in [2]. We will first define their analogue for the trigraph case, and then generalize it further, to obtain machinery powerful enough to handle wheels in trigraphs. Let T be a trigraph. A *frame* in T is a pair (z, A_0), where $z \in V(G)$, and A_0 is a nonempty weakly connected subset of $V(T) \setminus z$, containing no weak neighbors of z. With respect to a given frame (z, A_0), a *wheel system* in T of *height* $t \geq 1$ is a sequence $x_0, \dots, x_t$ of distinct vertices of $T \setminus (A_0 \cup \{z\})$, satisfying the following conditions:

1. A_0 contains two distinct vertices a_0, a_1 such that a_0 is weakly adjacent to x_0 and weakly non-adjacent to x_1; a_1 is weakly adjacent to x_1 and weakly non-adjacent to x_0; there is a path P_0 of the full realization of $T|A_0$ from a_0 to a_1 such that the set $\{x_0, x_1\}$ is weakly anticomplete to P_0^*; and no vertex in A_0 is strongly $\{x_0, x_1\}$-complete. We call such a triple (a_0, a_1, P_0) an *anchor* of the wheel system.
2. For $2 \leq i \leq t$, there is a weakly connected subset of $V(T)$ including A_0, containing a weak neighbor of x_i, strongly anticomplete to z, and containing no strongly $\{x_0, \dots, x_{i-1}\}$-complete vertex.
3. For $1 \leq i \leq t$, x_i is not strongly $\{x_0, \dots, x_{i-1}\}$-complete.
4. z is weakly adjacent to all of $x_0, \dots, x_t$.

Note that this definition is symmetric between x_0, x_1, so $x_1, x_0, x_2, \ldots, x_t$ is another wheel system. Let $x_0, \ldots, x_t$ be a wheel system of height t. For $1 \le i \le t$ we define $X_i = \{x_0, \ldots, x_i\}$, and we define A_i to be the maximal weakly connected subset of $V(T)$ that includes A_0, is strongly anticomplete to z, and contains no strongly X_i-complete vertex. We call $A_1, \ldots, A_t$ the *companion sets* of the wheel system. So for each i, $A_{i-1} \subseteq A_i$.

Let $x_0, \ldots x_t$ be a wheel system with respect to the frame (z, A_0) and let us define the *standard realization* of $T|(A_t \cup X_t \cup \{z\})$ to be the following:

- for $i = 0, 1$ if $x_i a_i$ is a switchable pair, assign the value "edge" to $x_i a_i$
- assign the value "non-edge" to all remaining switchable pairs between A_t and $\{x_0, x_1\}$
- for $2 \le i \le t$ assign the value "edge" to all switchable pairs between x_i and A_{i-1}, and the value "non-edge" to all remaining switchable pairs between x_i and A_t
- assign the value "edge" to all remaining switchable pairs containing vertices of A_t
- assign the value "edge" to all switchable pairs containing z
- assign the value "non-edge" to all remaining switchable pairs containing vertices of X_t.

6.4 *Let T be a Berge trigraph and let $x_0, \ldots, x_t$ be a wheel system in T with respect to the frame (z, A_0). Let G be a realization of T such that $G|(X_t \cup A_t \cup \{z\})$ is the standard realization of $T|(X_t \cup A_t \cup \{z\})$. Then $x_0, \ldots, x_t$ is a wheel system in G with the same companion sets.*

Proof. A_0 is connected in G and contains neighbors of x_0 and of x_1, and no vertex in A_0 is $(G, \{x_0, x_1\})$-complete. For $1 \le i \le t$, the vertex x_i is not $(G, \{x_0, \ldots, x_{i-1}\})$-complete and z is adjacent in G to all of $x_0, \ldots, x_t$.

(1) *No vertex of A_i is (G, X_i)-complete.*

Suppose there exists $1 \le i \le t$ such that some $a \in A_i$ is (G, X_i)-complete. Since $x_0, \ldots, x_t$ is a wheel system in T with respect to the frame (z, A_0) with companion sets $A_1, \ldots, A_t$, the vertex a is not a strong common neighbor of X_i in T. Let $0 \le j \le i$ be such that $x_j a$ is a switchable pair of T. Since T is monogamous, j is unique. Then $j \ge 2$ for $x_0 a_0$ and $x_1 a_1$ are the only switchable pairs of T between $\{x_0, x_1\}$ and A_t that are assigned the value "edge" in G. Since $a x_j$ is an edge of G, it follows that $a \in A_{j-1}$. But then A_{j-1} contains a strong common neighbor of X_{j-1} in T, a contradiction. This proves (1).

Now in G for $2 \le i \le t$, A_{i-1} is a connected subset of $V(G)$ including A_0, containing a neighbor of x_i, containing no neighbor of z, and by (1) containing no $(G, \{x_0, \ldots, x_{i-1}\})$-complete vertex. It remains to show that A_i is the maximal connected subset of $V(G)$ including A_0 and containing no neighbor of z and no (G, X_i)-complete vertex. Suppose there exists a proper superset A'_i of A_i with these properties. Then in T, A'_i is weakly connected, it includes A_0, contains no weak neighbor of z and no strong common neighbor of X_i, contrary to the maximality of A_i in T. This completes the proof of 6.4. ∎

Next we prove the base case of the inductive proof mentioned at the start of Section 6.

6.5 *Let $T \in \mathcal{T}_7$, and let x_0, x_1 be a wheel system of height one in T with respect to the frame (z, A_0). Let Y be a weakly anticonnected set disjoint from $A_1 \cup \{x_0, x_1\}$ and let $v \in V(T) \setminus (Y \cup A_1 \cup \{x_0, x_1, z\})$. Assume that*

- *x_0, x_1 are both strongly Y-complete,*
- *v is weakly adjacent to z, weakly non-adjacent to one of x_0, x_1, and is not strongly Y-complete,*
- *every vertex in Y that is weakly non-adjacent to v has a weak neighbor in A_1 and is weakly adjacent to z, and*
- *v has a weak neighbor in A_1.*

Then z is weakly Y-complete and there is a wheel (C, Y) in T with $x_0, x_1, z \in V(C) \subseteq \{x_0, x_1, z\} \cup A_1$.

Proof. Let G be a realization of T defined as follows:

- $G|(A_1 \cup \{x_0, x_1, z\})$ is the standard realization of $T|(A_1 \cup \{x_0, x_1, z\})$
- assign the value "non-edge" to all switchable pairs with both ends in Y
- assign the value "edge" to all remaining switchable pairs containing vertices of A_1
- assign the value "edge" to all remaining switchable pairs containing z
- assign the value "non-edge" to all remaining switchable pairs containing vertices of $Y \cup \{x_0, x_1\}$
- assign values to the remaining switchable pairs of T arbitrarily.

Since $T \in \mathcal{T}_7$ and G is a realization of T, $G \in \mathcal{F}_7$. Applying theorem 19.2 of [2] to G we deduce that z is (G, Y)-complete, and G contains a wheel (C, Y) with $x_0, x_1, z \in V(C) \subseteq \{x_0, x_1, z\} \cup A_1$. In T that means that z is weakly Y-complete, and (C, Y) is a wheel in T with $x_0, x_1, z \in V(C) \subseteq \{x_0, x_1, z\} \cup A_1$. This proves 6.5. ∎

We need two special kinds of wheel systems. Let $x_0, \dots, x_t$ be a wheel system in T, and define X_i, A_i as above. Let $Y \subseteq V(T)$ be nonempty and weakly anticonnected, such that Y is disjoint from $\{z, x_0, \dots, x_t\}$, $x_0, \dots, x_{t-1}$ are all strongly Y-complete and x_t is not. We say $x_0, \dots, x_t$ is a

- *Y-diamond* if $t \geq 3$, x_t is strongly X_{t-2}-complete, and x_t has a weak neighbor in A_{t-2}
- *Y-square* if $t \geq 3$, x_t is strongly adjacent to x_{t-1}, x_t has no weak neighbor in A_{t-2}, and there is a vertex in A_{t-1} weakly adjacent to x_t with a weak neighbor in A_{t-2}

The main result of this subsection is the following.

6.6 *Let $T \in \mathcal{T}_7$, let (z, A_0) be a frame, and let $Y \subseteq V(T) \setminus (A_0 \cup \{z\})$ be nonempty and weakly anticonnected. Suppose that there is either a Y-diamond or a Y-square in T. Then z is weakly Y-complete and T contains a wheel (C, Y).*

Proof. Let $x_0, \dots, x_t$ be a Y-diamond or a Y-square in T. Let G be the following realization of T.

- $G|(A_t \cup X_t \cup \{z\})$ is the standard realization of $T|(A_t \cup X_t \cup \{z\})$
- assign the value "edge" to all remaining switchable pairs containing vertices of $A \cup \{z\}$
- assign the value "non-edge" to all remaining switchable pairs containing vertices of $X_t \cup Y$
- assign values arbitrarily to all remaining switchable pairs of T.

By 6.1 G is a graph in $\mathcal{F}_7$ and $x_0, \dots, x_t$ is a Y-diamond or a Y-square in G. (It is straightforward to verify that $x_0, \dots, x_t$ is a wheel system of the same type in T and in G.) Applying theorem 20.1 of [2] we deduce that z is (G, Y)-complete and G contains a wheel (C, Y). Consequently z is weakly Y-complete in T and T contains a wheel (C, Y).This proves 6.6. ▮

6.2 Finding a wheel system

In this subsection we apply the results of the previous two subsections to prove a powerful statement about wheel systems that will be the engine behind almost all the remainder of the paper. First we need a few lemmas about subtrigraphs of T that are wheels in some realization of T.

6.7 *Let $T \in \mathcal{T}_7$ and let G be a realization of T containing a wheel (C, Y). Then all vertices of C that are not (G, Y)-complete have the same wheel-parity in G.*

Proof. Suppose there are two vertices of C that are not (G, Y)-complete and have opposite wheel-parity. Then each subpath of C between them contains an odd number of (G, Y)-complete edges, and consequently contains an odd Y-segment in G. But then (C, Y) is an odd wheel in G and by 6.1 (C, Y) is an odd wheel in T, contrary to the fact that $T \in \mathcal{T}_7$. This proves 6.7. ▮

6.8 *Let $T \in \mathcal{T}_7$ and let G be a realization of T containing a wheel (C, Y). Let e be a (G, Y)-complete edge of C and let c be an end of e. Assume that c is not strongly Y-complete in T. Then every vertex of C that is not (G, Y)-complete has wheel-parity opposite from c in G.*

Proof. Suppose the claim is false. Let the vertices of C be $c_1, \dots, c_k$ in order. We may assume the vertex c_1 is contained in a (G, Y)-complete edge of C, that c_1 is not strongly Y-complete in T, and that for some $2 \leq j \leq n$ the vertex c_j is not (G, Y)-complete and has the same wheel-parity as c_1 in G. Let j be minimum with this property. Since c_1 is not strongly Y-complete, there exists a vertex $y \in Y$ such that $c_1 y$ is a switchable pair of T. The graph $G' = G \setminus c_1 y$ is a realization of T, and hence it is Berge.

There are at least three (G', Y)-complete vertices in $V(C)$, so C contains an even number of (G', Y)-complete edges. Since C also contains an even number of (G, Y)-complete edges, and c_1 is in a (G, Y)-complete edge, it follows that it is in two (G, Y)-complete edges, and both c_2 and c_n are (G, Y)-complete. Hence $2 < j < n$.

By the minimality of j, c_{j-1} is (G, Y)-complete. Since c_1 and c_j have the same wheel-parity in G, the path c_1-$\dots$-c_{j-1} contains an even number of (G, Y)-complete edges. So the path c_2-$\dots$-c_{j-1} contains an odd number of (G', Y)-complete edges, and since c_1 and c_j are not (G', Y)-complete, in C contains an odd Y-segment in G' and at least three (G', Y)-complete vertices. So (C, Y) is an odd wheel in G'. But G' is a realization of T, and by 6.1 $G' \in \mathcal{F}_7$, a contradiction. This proves 6.8. ▮

6.9 *Let $T \in \mathcal{T}_7$ and let G be a realization of T containing a wheel (C, Y). Let $c_1, \ldots, c_n$ be the vertices of C in order. Assume there exist $1 \leq i < j \leq n$ such that*

- *c_i has wheel-parity opposite from some vertex of C that is not (G, Y)-complete*
- *$j - i > 1$ and $(i, j) \neq (1, n)$*
- *c_i is weakly adjacent to c_j.*

Then c_j is strongly Y-complete.

Proof. By 6.7 the vertex c_i has wheel-parity opposite from all vertices of C that are not (G, Y)-complete, and in particular c_i is (G, Y)-complete. Let P_1 and P_2 be the two subpaths of C joining c_i and c_j. Both P_1 and P_2 have odd length for otherwise one of c_i-P_1-c_j-c_i and c_i-P_2-c_j-c_i is an odd hole in T. Hence both c_i-P_1-c_j-c_i and c_i-P_2-c_j-c_i are holes in T. Let G' be the graph obtained from G by adding the edge $a_i a_j$.

Since C is a hole, $j - i > 1$ and c_i, c_j are weakly adjacent, $c_i c_j$ is a switchable pair of T, and so it is the unique switchable pair containing c_j. So in order to prove that c_j is strongly Y-complete, it is enough to show that it is weakly Y-complete, and in particular it is enough to prove that c_j is (G, Y)-complete.

Assume for a contradiction that c_j is not (G, Y)-complete. Then c_i and c_j have opposite wheel-parity in G and each of the paths P_1, P_2 contains an odd number of (G, Y)-complete edges. Since c_j is not (G, Y)-complete, the number of (G', Y)-complete edges in c_i-P_m-c_j-c_i equals the number of (G, Y)-complete edges in P_m (where $m = 1, 2$) and hence it is odd. By 5.4 applied in G', each c_i-P_m-c_j-c_i contains exactly one (G', Y)-complete edge and exactly two (G', Y)-complete vertices. Since c_i is (G', Y)-complete, both these edges are incident with c_i. But then C contains exactly two (G, Y)-complete edges and they are both incident with c_i, contrary to the fact that (C, Y) is a wheel in G. This proves that c_j is (G, Y)-complete and completes the proof of 6.9. ∎

Let Y be a nonempty weakly anticonnected subset of $V(T)$, let (z, A_0) be a frame with $A_0 \cup \{z\}$ disjoint from Y, and let $x_0, \ldots, x_{t+1}$ be a wheel system with respect to this frame. We say Y is a *hub* for the wheel system if $t \geq 1$, $z, x_0, \ldots, x_t$ are all strongly Y-complete and x_{t+1} is not.

Next we prove a technical result, various modifications of which will be used later. (Now we need to use that there are no pseudowheels, so we are back in $\mathcal{T}_8$.)

6.10 *Let $T \in \mathcal{T}_7$, let $Y \subseteq V(T)$ be nonempty and weakly anticonnected, and assume that there do not exist X, P such that (X, Y, P) is a pseudowheel in T. Let (z, A_0) be a frame with $Y \cap (A_0 \cup \{z\}) = \emptyset$, and let $x_0, \ldots, x_{t+1}$ be a wheel system with hub Y, and with $t \geq 2$. Define X_i, A_i as usual. Then either*

- *x_{t+1} has a weak neighbor in A_{t-1}, or*
- *some member of Y is weakly non-adjacent to x_{t+1} and has no weak neighbor in A_t, or*
- *there are ≥ 2 members of Y that are weakly non-adjacent to x_{t+1} and have no weak neighbor in A_{t-1}, or*
- *there is a wheel with hub Y in T.*

Proof. Let G be the following realization of T.

- $G|(A_t \cup X_t \cup \{z\})$ is the standard realization of $T|(A_t \cup X_t \cup \{z\})$
- assign the value "edge" to all remaining switchable pairs containing vertices of $A_t \cup \{z\}$
- assign the value "non-edge" to all remaining switchable pairs containing vertices of $X_{t+1} \cup Y$
- assign values arbitrarily to all remaining switchable pairs of T.

By 6.1 and 6.3 G is a graph in $\mathcal{F}_7$ and there do not exist X, P such that (X, Y, P) is a pseudowheel in G.. By 6.4 $x_0, \ldots, x_{t+1}$ is a wheel system with respect to the frame (z, A_0) in G with companion sets $A_1, \ldots, A_{t+1}$. Moreover Y is a hub for $x_0, \ldots, x_{t+1}$ in G. By theorem 21.2 of [2] applied in G one of the following outcomes holds:

- in G x_{t+1} has a neighbor in A_{t-1}, which in T means that x_{t+1} has a weak neighbor in A_{t-1}, so the theorem holds; or
- in G some member of Y is non-adjacent to x_{t+1} and has no neighbor in A_t, which in T means that some member of Y is weakly non-adjacent to x_{t+1} and has no weak neighbor in A_t and the theorem holds; or
- in G there are ≥ 2 members of Y that are non-adjacent to x_{t+1} and have no neighbor in A_{t-1}, which in T means that there are ≥ 2 members of Y that are weakly non-adjacent to x_{t+1} and have no weak neighbor in A_{t-1}, and the theorem holds; or
- in G there is a wheel with hub Y, and therefore there is a wheel with hub Y in T, and the theorem holds.

This proves 6.10. ∎

The first modification of 6.10 that we need is the following:

6.11 *Let $T \in \mathcal{T}_7$, let $Y \subseteq V(T)$ be nonempty and weakly anticonnected, and assume that there do not exist X, P such that (X, Y, P) is a pseudowheel in T. Let (z, A_0) be a frame with $Y \cap (A_0 \cup \{z\}) = \emptyset$, and let $x_0, \ldots, x_{t+1}$ be a wheel system with hub Y, where $t \geq 1$. Define A_i, X_i as usual, and assume that at most one member of Y has no weak neighbor in A_1. Suppose T contains no wheel with hub Y. Then there exists r with $1 \leq r \leq t$, and a member $y \in Y$, with the following properties:*

- *y is weakly non-adjacent to x_{t+1} and has no weak neighbor in A_r*
- *x_{t+1} has a weak neighbor in A_r, and a weak non-neighbor in X_r.*

Proof. We proceed by induction on t. If $t = 1$ then 6.5 implies that there exists $y \in Y$ weakly non-adjacent to x_{t+1} and with no weak neighbor in A_t, and the theorem holds. So we may assume $t \geq 2$. If x_{t+1} has no weak neighbor in A_{t-1}, then the result follows from 6.10, since at most one member of Y has no weak neighbor in A_{t-1}. So we assume that x_{t+1} has a weak neighbor in A_{t-1}. If x_{t+1} is strongly X_{t-1}-complete then

$$x_0, \ldots, x_{t+1}$$

is a Y-diamond, and we get a contradiction by 6.6. Thus x_{t+1} is not strongly X_{t-1}-complete, and so

$$x_0, \dots, x_{t-1}, x_{t+1}$$

is a wheel system with hub Y, and the result follows from the inductive hypothesis. This proves 6.11. ∎

Next we define a more general structure in a trigraph, called a *weak wheel system*, and an operation that transforms a trigraph T into its *shadow* T'. The shadow is another trigraph, in which a weak wheel system of T becomes a "regular" wheel system.

A *weak frame* in T is a pair (z, A_0), where $A_0 \subseteq V(T)$ is nonempty and weakly connected, and $z \in V(T) \setminus A_0$ is weakly anticomplete to A_0. (This differs from a frame in that z may have a weak neighbor in A_0.) We define a *weak wheel system* with respect to a weak frame (z, A_0) to be a sequence $x_0, x_1, \dots x_t$ satisfying conditions 1-4 in the definition of a wheel system, except condition 2 is replaced by the following condition 2':

2'. For $2 \leq i \leq t$, there is a weakly connected subset F of $V(T)$ including A_0, containing a weak neighbor of x_i and no strongly $\{x_0, \dots, x_{i-1}\}$-complete vertex, and such that $F \setminus A_0$ is strongly anticomplete to z.

Let $x_0, \dots, x_t$ be a weak wheel system of height t. For $1 \leq i \leq t$ we define $X_i = \{x_0, \dots, x_i\}$, and we define A_i to be the maximal weakly connected subset of $V(T)$ that includes A_0, such that $A_i \setminus A_0$ is strongly anticomplete to z, and contains no strongly X_i-complete vertex. We call $A_1, \dots, A_t$ the *companion sets* of the weak wheel system. So for each i, $A_{i-1} \subseteq A_i$. An anchor and a hub for such wheel system are defined as before.

Let $x_0, \dots x_t$ be a weak wheel system with respect to the weak frame (z, A_0, Y) and let us define the *standard realization* of $T|(A_t \cup X_t \cup \{z\})$ as before, except now we add the following:

- assign the value "non-edge" to all switchable pairs between z and A_0

We need to modify 6.10 further, for our final goal is to be able to apply an analogue of it to weak frames. To do that we define the "shadow" of T (with respect to a weak wheel system.) Let (z, A_0) be a weak frame in $T \in \mathcal{T}_8$ and let $x_0, \dots, x_s$ be a weak wheel system with $s \geq 1$, Let the *shadow* T' of T (with respect to the weak wheel system $x_0, \dots, x_s$) be the trigraph defined as follows.

First assume that z is not strongly anticomplete to A_0. Let u be the weak neighbor of z in A_0. Then uz is the only switchable pair containing u or z in T. Let

- $V(T') = V(T) \setminus \{u\} \cup \{u', u''\}$ where u', u'' are distinct vertices not in $V(T)$.
- $T'|(V(T) \setminus \{u\}) = T|(V(T) \setminus \{u\})$.
- for $v \in V(T) \setminus \{u\}$, if $uv \in E(T)$ then $u'v, u''v \in E(T')$ and if $uv \in N(T)$ then $u'v, u''v \in N(T')$.
- $u'z \in E(T')$, $u''z \in N(T')$, $u'u'' \in S(T')$.

For $0 \leq i \leq s$ let $A'_i = A_i \setminus \{u\} \cup \{u''\}$.

If z is strongly anticomplete to A_0, define the shadow T' of T to be T itself, and for $0 \leq i \leq s$ let $A'_i = A_i$.

Given a graph G and two vertices $x, y \in V(G)$ we say that x *dominates* y if x is adjacent to every neighbor of y different from x itself.

6.12 *Let T be a trigraph in $\mathcal{T}_8$ and let (z, A_0) be a weak frame such that z has a weak neighbor u in A_0. Let T' be the shadow of T relative to $x_0, \ldots, x_s$. Let $Y \subseteq V(T) \setminus (A_0 \cup \{z\})$ be nonempty and weakly anticonnected, such that $x_0, \ldots, x_s, u$ are strongly Y-complete and z is weakly Y-complete. Assume that T contains no wheel with hub Y. Then $T' \in \mathcal{T}_7$ and there do not exist X, P such that (X, Y, P) is a pseudowheel in T'.*

Proof. We may assume that z is not strongly anticomplete to A_0. Let u, u', u'' be defined as before. By 6.1 to prove that T' is in $\mathcal{T}_7$ it is enough to show that every realization of T' belongs to $\mathcal{F}_7$. We also need to show that there do not exist X, P such that (X, Y, P) is a pseudowheel in T'. Let G be a realization of T'. The graphs $G \setminus u'$ and $G \setminus u''$ are realizations of $T' \setminus u'$ and $T' \setminus u''$ respectively; and therefore isomorphic to realizations of T, and $T \in \mathcal{T}_8$. So by 6.1 both $G \setminus u'$ and $G \setminus u''$ are in $\mathcal{F}_7$. Since also in G the vertex u' dominates the vertex u'' and $deg_G(u'') = deg_G(u') - 1$, it follows that G is Berge. The only neighbor of u' in G that is different from and non-adjacent to u'' is z. Since none of the excluded subgraphs in the definition of $\mathcal{F}_1, \ldots, \mathcal{F}_6$ contains two vertices, one of which dominates the other and whose degrees differ by at most one, and $G \setminus u'$, $G \setminus u''$ belong to $\mathcal{F}_7 \subseteq \mathcal{F}_6$, it follows that $G \in \mathcal{F}_6$.

Next we show that G belongs to $\mathcal{F}_7$. Suppose not. Then G contains an odd wheel (C, Y'). Since $G \setminus u'$ and $G \setminus u''$ are in $\mathcal{F}_7$, it follows that both u', u'' are in $V(C) \cup Y'$. Since C is a hole in G, u' dominates u'' and $deg_G(u'') = deg_G(u') - 1$, not both u', u'' belong to $V(C)$. Assume that exactly one of u', u'' belongs to Y'. Since at least four vertices of C are (G, Y')-complete and $deg_G(u') = deg_G(u'') + 1$, it follows that $u' \in Y'$ and $u'', z \in V(C)$. Let c_1, c_2 be the two vertices of C consecutive with u''. Then the only neighbors of u' in C are u'', c_1, c_2, z, contrary to the fact that C contains an odd Y'-segment in G. This proves that both u', u'' belong to Y'. But now $(C, Y' \setminus \{u'\})$ is an odd wheel in G and u' is not in it, a contradiction. This proves that $G \in \mathcal{F}_7$.

Finally we show that there do not exist X, P such that (X, Y, P) is a pseudowheel in T'. Suppose T' contains a pseudowheel (X, Y, P). Since $Y \cap A_0 = \emptyset$ and $u \in A_0$, it follows that $u', u'' \notin Y$. Since $T \in \mathcal{T}_8$, 6.3 implies that both u', u'' are in $V(P) \cup X$. Now P is a path in T'; p_1, p_n are weakly X-complete and no vertex of P^* is strongly X-complete; u' is strongly adjacent to every weak neighbor of u'', and and u', u'' do not belong to any switchable pair of T' except u', u''. It follows that not both u', u'' belong to $V(P)$. Let G_1 be a realization of T' defined as follows:

- P is a path in G_1
- p_1, p_n are (G_1, X)-complete
- assign the value "non-edge" to all remaining switchable pairs.

Then G_1 belongs to $\mathcal{F}_7$ since it is a realization of T', and (X, Y, P) is a pseudowheel in G_1. Suppose exactly one of u', u'' is in $V(P)$. Then the other is in X. Since $Y \cup \{p_1, p_n\}$ is (G_1, X)-complete and no vertex of $V(P)$ is adjacent in G_1 to both p_1, p_n, we deduce that $z \in \{p_1, p_n\}$, $u' \in X$, $u'' \in \{p_2, p_{n-1}\}$ and u'' is (G_1, Y)-complete. Since p_2 is not (G_1, Y)-complete, it follows that $u'' = p_{n-1}$, $z = p_1$ and u' is adjacent in G_1 to p_{n-2} and to no vertex of $V(P) \setminus \{p_1, p_{n-2}, p_{n-1}, p_n\}$.

Let C' be the hole p_1-$\ldots$-p_{n-2}-u'-p_1. Since p_n is not strongly Y-complete, by 6.3 at least three edges of $P \setminus \{p_n\}$ are (G_1, Y)-complete, and so at least three edges of $C' \setminus \{p_1\}$ are weakly Y-complete, and (C', Y) is a wheel in T'. But then replacing u' by u gives a wheel in T with hub Y, a contradiction. This proves that none of u', u'' is in $V(P)$.

So $\{u', u''\} \subseteq X$. Let $\tilde{X} = X \setminus \{u', u''\} \cup \{u\}$. Now in T the vertices p_1, p_n are weakly $\tilde{X}$-complete and no vertex of P^* is strongly $\tilde{X}$-complete, $\tilde{X} \cup \{p_1\}$ and at least one other vertex of P are strongly Y-complete and p_2, p_n are not strongly Y-complete. Thus $(\tilde{X}, Y, P)$ is a pseudowheel in T, contrary to the fact that $T \in \mathcal{T}_8$. This proves 6.12. ∎

6.13 *Let T be a trigraph in $\mathcal{T}_8$, let (z, A_0) be a weak frame. Let $x_0, \dots, x_s$ be a weak wheel system relative to (z, A_0). Let T' be the shadow of T relative to $x_0, \dots, x_s$. Assume z has a weak neighbor u in A_0, and let u', u'' be as in the definition of the shadow. Then (z, A_0') is a frame in T'; $\{x_0, x_1, \dots, x_s, u'\}$ is a wheel system with respect to (z, A_0') in T' and with the usual notation $A_1', \dots, A_s'$ are companion sets for it.*

Proof. The set A_0 is weakly connected in T' and is strongly anticomplete to z. Thus (z, A_0') is a frame in T'. Let $x_{s+1} = u'$ and let (a_0, a_1, P_0) be an anchor of the weak wheel system $x_0, \dots, x_s$. We need to check that the four axioms of a wheel system are satisfied.

1. First we observe that $u \neq a_0$, for otherwise one of z-a_0-P_0-a_1-x_1-z and z-x_0-a_0-P_0-a_1-x_1-z would be an odd hole, a contradiction. Analogously, $u \neq a_1$. So the vertices a_0 and a_1 belong to $A_0' \cap A_0$. Since the strong and weak adjacencies between $A_0' \cap A_0$ and $\{x_0, x_1\}$ are the same in T' as they are in T, a_0 is weakly adjacent to x_0 and weakly non-adjacent to x_1 in T', a_1 is weakly adjacent to x_1 and weakly non-adjacent to x_0 in T'. Let P_0' be the path obtained from P_0 by replacing u by u'' if $u \in V(P_0)$, and otherwise let $P_0' = P_0$. Then P_0' is a path of the full realization of $T'|A_0'$ from a_0 to a_1, with $V(P_0') \subseteq A_0'$ and $\{x_0, x_1\}$ is weakly anticomplete to P_0^*. Since u, and therefore u'', is not strongly $\{x_0, x_1\}$-complete and $A_0' \setminus \{u''\}$ is contained in A_0, it follows that no vertex of A_0' is strongly $\{x_0, x_1\}$-complete. Thus the first axiom is satisfied.

2. For $1 \leq i \leq s$ we will show that the set A_i' is a weakly connected subset of $V(T')$ including A_0', strongly anticomplete to z in T' and containing a weak neighbor of x_{i+1} in T' and containing no strongly $\{x_0, \dots, x_i\}$-complete vertex. The first three assertions are obvious. The fourth assertion holds because for $1 \leq i \leq s-1$ the set A_i contains a weak neighbor of x_{i+1} in T, and x_{s+1} has a weak neighbor u'' in $A_0' \subseteq A_s'$. The final assertion holds because A_i is a companion set of the wheel system $x_0, \dots, x_s$ in T and the vertex u'' is not strongly $\{x_0, x_1\}$-complete by the first axiom. Thus the second axiom is satisfied.

3. For $1 \leq i \leq s$, x_i is not strongly $\{x_0, \dots, x_{i-1}\}$-complete because $x_0, \dots, x_s$ is a wheel system in T; and x_{s+1} is not strongly $\{x_0, x_1\}$-complete because u is not strongly $\{x_0, x_1\}$-complete in T. So the third axiom holds.

4. z is weakly adjacent to all of $x_0, \dots, x_{s+1}$ because $x_0, \dots, x_s$ is a weak wheel system in T with respect to the weak frame (z, A_0), and x_{s+1} is strongly adjacent to z in T'. So the fourth axiom holds.

It is clear that $A_1', \dots, A_s'$ are companion sets for this wheel system. This proves 6.13. ∎

6.14 *Let $T \in \mathcal{T}_8$, not admitting a balanced skew-partition. Let (z, A_0) be a weak frame, and let $x_0, \dots, x_s$ be a weak wheel system with $s \geq 1$ relative to this frame. Let $Y \subseteq V(T)$ be nonempty and weakly anticonnected, with $Y \cap (A_0 \cup \{z\}) = \emptyset$, such that $x_0, \dots, x_s$ are strongly Y-complete and z is* *weakly Y-complete. Let u be the weak neighbor of z in A_0 if one exists and assume that u is strongly Y-complete. Let T' be the shadow of T relative to the weak wheel system $x_0, \dots, x_s$. Then in T' there is a sequence $x_{s+1}, \dots, x_{t+1}$ with $t \geq s$ such that $x_0, \dots, x_{t+1}$ is a wheel system with respect to the frame (z, A'_0), with hub Y.*

Proof. If u exists, let u', u'' be defined as usual, and define $x_{s+1} = u'$ and $k = 1$. Otherwise let $k = 0$. By 6.13 $x_0, x_1, \dots, x_{s+k}$ is a wheel system with respect to (z, A'_0) in T'. Choose a sequence $x_{s+k+1}, \dots, x_t$, all strongly Y-complete and such that $x_0, \dots, x_t$ is a wheel system with respect to (z, A'_0) in T', with t maximum. So $t \geq 1$. Define X_i as usual. Let $A'_1, \dots A'_t$ be the companion sets for this wheel system.

Let V be the set of all strongly X_t-complete vertices in $V(T')$ different from z. Let $X = X' = X_t$ and $A = A'_t$ if $T = T'$; and let $X = (X_t \setminus \{u'\}) \cup \{u\}$, $X' = X_t \cup \{u''\}$ and $A = A'_t \setminus \{u''\}$ otherwise. We claim that every vertex in V is strongly X-complete in T. This is clear if $T = T'$, so we may assume $T \neq T'$. Since u'' is not strongly $\{x_0, x_1\}$-complete in T', $V \subseteq V(T)$, and hence in T every vertex of V is strongly X-complete, and in T' every vertex of V is strongly X'-complete.

Suppose every path in T' from z to A contains a vertex of $X' \cup V$ in its interior. Then every path in T from z to A contains a vertex of $X \cup V$ in its interior, contrary to 5.7, since T does not admit a balanced skew-partition. Hence in T' there is a path P from z to A, with interior disjoint from $X' \cup V$. From the maximality of A'_t, it follows that P has length 2. Let x_{t+1} be the vertex consecutive with z in P. So x_{t+1} has a weak neighbor in A'_t in T', and therefore $x_0, \dots, x_t, x_{t+1}$ is a wheel system in T'. From the maximality of t it follows that x_{t+1} is not strongly Y-complete, and therefore Y is a hub for this wheel system in T'. This proves 6.14. ∎

6.15 *Let $T \in \mathcal{T}_8$, not admitting a balanced skew-partition. Let (z, A_0) be a weak frame, and let $x_0, \dots, x_s$ be a weak wheel system with $s \geq 1$ relative to this frame. Let $Y \subseteq V(T)$ be nonempty, disjoint from $A_0 \cup \{z\}$ and weakly anticonnected, such that $x_0, \dots, x_s$ are strongly Y-complete and z is weakly Y-complete. Define A_i, X_i as usual, and assume that*

1. *every member of Y has a weak neighbor in A_s*
2. *at most one member of Y has no weak neighbor in A_1*
3. *if z has a weak neighbor $u \in A_0$, then u is strongly Y-complete*
4. *there is no wheel with hub Y in T.*

Then there exist r with $1 \leq r < s$, a member y of Y and a vertex v with the following properties:

- *y is weakly non-adjacent to v and has no weak neighbor in A_r*
- *v is weakly adjacent to z, and has a weak neighbor in A_r, and a weak non-neighbor in X_r.*

Proof. Let T' be the shadow of T relative to the weak wheel system $x_0, \dots, x_s$. By 6.14 in T' there is a sequence $x_{s+1}, \dots, x_{t+1}$ with $t \geq s$ such that $x_0, \dots, x_{t+1}$ is a wheel system with respect to the frame (z, A'_0), with hub Y. By 6.12 $T' \in \mathcal{T}_7$ and there do not exist X, P such that (X, Y, P) is a pseudowheel in T'. By 6.11 applied in T', there exists r with $1 \leq r \leq t$, and a member $y \in Y$, such that y is weakly non-adjacent to x_{t+1} and has no weak neighbor in A_r, and x_{t+1} has a weak neighbor in A_r, and a weak non-neighbor in X_r. Since every member of Y has a weak neighbor in A_s, it follows that $r < s$. In T that means that y is weakly non-adjacent to x_{t+1} and has no weak neighbor in A_r, and x_{t+1} has a weak neighbor in A_r, and a weak non-neighbor in X_r and the result follows. This proves 6.15. ∎

6.3 Wheels with tails

We start with some definitions. Let (C, Y) be a wheel in T. Following [2] we say that (C, Y) is *optimal* if there is no wheel (C', Y') with $Y \subset Y'$. However, in the trigraph setting we need a finer notion of optimality. We say that (C, Y) is *trioptimal* if there is no wheel (C', Y') such that

- $Y \subset Y'$ or
- $Y = Y'$ and C' contains fewer strongly Y-complete edges than C, or
- $Y = Y'$, the number of strongly Y-complete edges in C and C' is the same and C' contains fewer strongly Y-complete vertices than C.

We remark that if (C, T) is a trioptimal wheel in T, and G is a realization of T in which (C, Y) is a wheel, then (C, Y) is an optimal wheel in G.

A *kite* for (C, Y) is a vertex $y \in V(T) \setminus (Y \cup V(C))$, not strongly Y-complete, that has at least four weak neighbors in C, three of which are consecutive and Y-complete in some realization of T in which (C, Y) is a wheel.

Let G be a realization of T such that (C, Y) is a wheel in T, let $z \in V(C)$, and let x_0, x_1 be the vertices consecutive with z in the hole C. A path S of $G \setminus \{x_0, x_1\}$ with nonempty interior from z to $V(C) \setminus \{z, x_0, x_1\}$ is a *tail* for z (with respect to the wheel (C, Y) and the realization G) if

- only one vertex of $S \setminus z$ has a weak neighbor in $V(C) \setminus \{z, x_0, x_1\}$,
- x_0, z, x_1 are all (G, Y)-complete,
- there is a (G, Y)-complete edge in $C \setminus \{x_0, z, x_1\}$
- the vertex consecutive with z in S is strongly adjacent to x_0, x_1,
- no vertex of S is in Y,
- no vertex of $V(S) \setminus \{z\}$ is strongly Y-complete.

Please note that a tail in G with respect to (C, Y) is not necessarily a tail in T with respect to (C, Y) and G, for to be a tail in T the vertex consecutive with z is required to be strongly $\{x_0, x_1\}$-complete.

6.16 *Let $T \in \mathcal{T}_8$, and let (C, Y) be a wheel in a realization G of T, such that (C, Y) is a trioptimal wheel in T and not all vertices of C are (G, Y)-complete. Suppose $z \in V(C)$ has opposite wheel-parity in G from some vertex of C that is not (G, Y)-complete. Let x_0, x_1 be the vertices consecutive with z in C, and assume that z has a weak neighbor $c \in V(C) \setminus \{x_0, x_1, z\}$ Then if there is a tail for z with respect to (C, Y) and G, then some vertex of T is a kite for (C, Y); and if there is a kite y for (C, Y), such that y is weakly adjacent to x_0, x_1, z, then y is strongly adjacent to c.*

Proof. Let S be a path from z to $V(C) \setminus \{x_0, x_1, z\}$ which is either a tail for z or a two-edge path via a kite for (C, Y) that is weakly adjacent to x_0, x_1, z. Let y be the vertex of S consecutive with z, and let s be the unique vertex of $S \setminus z$ with a weak neighbor in $V(C) \setminus \{x_0, x_1, z\}$. By choosing G appropriately we may assume that no vertex of $S \setminus z$ is (G, Y)-complete.

Since z has opposite wheel-parity in G from some vertex of C that is not (G, Y)-complete, and x_0, x_1 have wheel-parity opposite from z in G, by 6.8 both x_0 and x_1 are strongly Y-complete, and by 6.9 c is strongly Y-complete.

For $i = 0, 1$ let P_i be the subpath of C between z and c containing x_i. By 2.1 both P_0, P_1 have length > 2. Let C_i be the hole z-P_i-c-z. Since each of the holes C_i contains at least three (G, Y)-complete vertices (namely z, x_i and c), each of these holes contains an even number of (G, Y)-complete edges. Since (C, Y) is a wheel and c, z are both (G, Y)-complete, at least one of C_0, C_1 contains two disjoint (G, Y)-complete edges, say C_0. But $x_1 \in V(C) \setminus V(C_0)$ is strongly Y-complete, so by the trioptimality of (C, Y) in T it follows that (C_0, Y) is not a wheel and P_0 has length three. If C_1 also contains two disjoint (G, Y)-complete edges, then from the symmetry P_1 has length three and all the vertices of C are (G, Y)-complete, a contradiction. So zx_1 is the only (G, Y)-complete edge in P_1.

Let c' be the vertex consecutive with both x_0, c in C. Then c' is weakly Y-complete in T and has the same wheel-parity as z in G. We claim that s has a weak neighbor in $V(P_1) \setminus \{z, x_1\}$. For suppose not. Then the only weak neighbor of s in $V(C) \setminus \{x_0, x_1, z\}$ is c'. Let S' be a path from c' to x_1 with interior in $V(S)$ and let $P_1' = P_1 \setminus \{z\}$. Then c'-c-P_1'-x_1-S'-c' is a hole in G, with at least three (G, Y)-complete vertices and exactly one (G, Y)-complete edge, contrary to 5.4. This proves that s has a weak neighbor in $V(P_1) \setminus \{z, x_1\}$.

We may assume that y is weakly non-adjacent to c, for otherwise y is a kite for (C, Y) and the theorem holds. Since T is monogamous, it follows that y is strongly non-adjacent to c. So there is a path M of length at least three from z to c with interior in $V(S) \cup {P_1}^* \setminus \{x_1\}$. Since z-M-c-z is not an odd hole in T, the path z-M-c has odd length, both its ends are weakly Y-complete in T and no edge of it is (G, Y)-complete. Applying 5.2 to this path and the weakly anticonnected set Y, we deduce that M has length three and every weakly Y-complete vertex in T is strongly adjacent to one of the interior vertices of M. Let m be the vertex of M^* different from y. Then c' is strongly adjacent to one of y, m. If c' is weakly adjacent to y, then $Y \cup \{y\}$ is weakly complete to $\{c', x_0, z, x_1\}$, and so $(C, Y \cup \{y\})$ is a wheel in T, contrary to the trioptimality of T. So c' is strongly non-adjacent to y, and therefore c' is strongly adjacent to m and hence $m \notin V(C)$; consequently S is a tail and so y has no weak neighbor in $V(C) \setminus \{x_0, x_1, z\}$, and we may assume that no vertex of T is a kite for (C, Y), for otherwise the theorem holds.

Since x_1-P_1'-c-m-y-x_1 is not an odd hole in T, m has a strong neighbor in P_1^*. Since m is not a kite, it is strongly non-adjacent to x_0. Let Q be an antipath joining m and y with nonempty interior in Y. Since y-Q-m-z-c-y is not an odd hole, Q has odd length. So x_0-m-Q-y-c' is an odd antipath in T of length at least five, all its interior vertices have weak neighbors in the weakly connected set P_1^* and the ends do not have any strong neighbors in it, contrary to 5.2 applied in $\overline{T}$. This proves 6.16. ∎

6.17 *Let $T \in \mathcal{T}_8$, not admitting a balanced skew-partition, and let (C, Y) be a trioptimal wheel in T, and assume that not all vertices of C are strongly Y-complete. Then there is no kite for (C, Y) and hence there is no kite in any realization of T in which (C, Y) is a wheel.*

Proof. Assume y is a kite for (C, Y). Let G be a realization of T such that

- (C, Y) is a wheel in G
- not all vertices of C are (G, Y)-complete
- all switchable pairs between Y and y are assigned the value "non-edge"
- all switchable pairs between $V(C)$ and y are assigned the value "edge"
- there is a subpath x_0-z-x_1 of C, all (G, Y)-complete and adjacent to y.

Let $w \in V(C)$ be a vertex that is not (G, Y)-complete. Define W to be the set of all vertices in $V(C)$ that have the same wheel-parity as w in G, and let $U = V(C) \setminus W$. By 6.7 all vertices of C that are not (G, Y)-complete belong to W.

(1) *y is strongly adjacent to x_0 and x_1.*

Suppose y is weakly non-adjacent to x_0. Let G' be the graph obtained from G by deleting the edge x_0y. Then G' is a realization of T and is therefore Berge. Since $(C, Y \cup \{y\})$ is not a wheel in G, x_0z and zx_1 are the only $(G, Y \cup \{y\})$-complete edges in C. By 5.4, z, x_1 are the only $(G', Y \cup \{y\})$-complete vertices in C. We can now apply 5.5 to C and the anticonnected set $Y \cup \{y\}$ in G'. Since in G' y has at least three neighbors in C, and the vertex x_0 is (G', Y)-complete, there is no hat for C in $Y \cup \{y\}$. So there is a leap, and since x_0 is (G', Y)-complete, it follows that in G' the only neighbors of y in C are z, x_1 and the neighbor of x_1 in $C \setminus z$. But then the hole C contains exactly three $(G, \{y\})$-complete edges, contrary to 5.4. This proves that y is strongly adjacent to x_0. By the symmetry, y is strongly adjacent to x_1 and (1) holds.

(2) *$z \in W$.*

Suppose not, then $z \in U$. Let $A_0 = V(C) \setminus \{z, x_0, x_1\}$. Then (z, A_0) is a weak frame in T and x_0, x_1 is a weak wheel system relative to it. Since $z \in U$ and x_0, x_1 have wheel-parity opposite from z in G, it follows that $x_0, x_1 \in W$ and by 6.8 both x_0 and x_1 are strongly Y-complete. It follows from (1) that x_0, x_1 are strongly $Y \cup \{y\}$-complete. Suppose z has a weak neighbor c in A_0. Since $z \in U$, 6.9 implies that c is strongly Y-complete. By 6.16 c is strongly adjacent to y; consequently c is strongly $Y \cup \{y\}$-complete. But now we get a contradiction applying 6.15, since every vertex of $Y \cup \{y\}$ has a weak neighbor in A_0. This proves (2).

From (2) $x_0 \in U$. Applying theorem 16.1 of [2] to the wheel (C, Y) and the vertex y in G, we deduce from the trioptimality of (C, Y) and the fact that y has at least four weak neighbors in $V(C)$, that all weak neighbors of y in $V(C) \setminus \{z\}$ have the same wheel-parity as x_0 in G, and so they all belong to U. So by 6.7 every weak neighbor of y in $V(C) \setminus \{x_0, z, x_1\}$ is (G, Y)-complete.

Let the vertices of the subpath P of C between x_0 and x_1 not containing z be $c_1, \dots, c_m$ in order, where $c_1 = x_0$ and $c_m = x_1$. By the hypothesis of the theorem there exists k such that $1 \le k \le m$ and c_k is not (G, Y)-complete, and therefore is non-adjacent to y in G. Let $i < k$ be maximum and $j > k$ be minimum such that y is weakly adjacent to c_i and c_j. Then c_i and c_j are both (G, Y)-complete and are both in U. By 6.7 $c_k \in W$, and so c_i, c_j have wheel-parity opposite from c_k in G. That means that each of the subpaths of c_i-$\dots$-c_k and c_k-$\dots$-c_j of P contains an odd number of (G, Y)-complete edges and therefore $j-i\geq 4$. Let C' be the hole y-c_i-$\dots$-c_k-$\dots$-c_j-y. Then (C',Y) is a wheel in G and c and y have opposite wheel-parity in with respect to it in G, contrary to 6.7, since neither of them is (G,Y)-complete. This completes the proof of 6.17. ∎

6.18 *Let $T\in\mathcal{T}_8$, let (C,Y) be a wheel in a realization G of T such that not all vertices of C are (G,Y)-complete, let (C,Y) be trioptimal in T, let $z\in V(C)$, and let x_0,x_1 be the vertices consecutive with z in C. Assume there exists a vertex $c\in V(C)$, not (G,Y)-complete, and such that z and c have opposite wheel-parity in G. Let S be a tail for z with respect to (C,Y) and G, and let y be the vertex adjacent to z in S. Let $A_0=V(C)\setminus\{z,x_0,x_1\}$. Assume z is strongly anticomplete to A_0, so (z,A_0) is a frame. Let $x_0,\dots,x_{t+1}$ be a wheel system with respect to (z,A_0), with hub $Y\cup\{y\}$. Define $A_1,\dots,A_{t+1}$ as usual. Then either y is strongly adjacent to x_{t+1}, or y has a weak neighbor in A_t.*

Proof. Since both x_0,x_1 have the same wheel-parity as c in G, it follows from 6.8 they they are both strongly Y-complete. Let G' be a realization of T such that:

- $G'|(A_{t+1}\cup X_{t+1}\cup\{z\})$ is the standard realization of the wheel system
- assign the value "non-edge" to those switchable pairs between Y and A_0 that are not edges in G
- assign the value "edge" to all remaining switchable pairs containing a vertex of $A_{t+1}\cup\{z\}$
- assign the value "non-edge" to all remaining switchable pairs containing a vertex of $Y\cup X_{t+1}$
- $G'|V(S)$ is the path S.

By 6.3 G' is a graph in $\mathcal{F}_8$ for it is a realization of T. Now (C,Y) is an optimal wheel in G'. By 6.17 G' contains no kite. Now S is a tail for z in G', and y is the neighbor of z in S, (z,A_0) is a frame in G' and by 6.4 $x_0,\dots,x_{t+1}$ is a wheel system with respect to it with companion sets $A_1,\dots,A_{t+1}$. By theorem 22.4 of [2] either y is adjacent to x_{t+1} in G', or y has a neighbor in A_t in G'. In T that means that either y is strongly adjacent to x_{t+1}, or y has a weak neighbor in A_t. This proves 6.18. ∎

We combine the previous result with 6.11 to prove the following.

6.19 *Let $T\in\mathcal{T}_8$, not admitting a balanced skew-partition, and let (C,Y) be a wheel in a realization G of T, such that (C,Y) is a trioptimal wheel in T, and assume that not all vertices of C are (G,Y)-complete. Suppose $z\in V(C)$ has opposite wheel-parity in G from some vertex of C that is not (G,Y)-complete. Then there is no tail for z with respect to (C,Y) and G.*

Proof. Suppose S is a tail for z; let y be the vertex adjacent to z in S, and let x_0,x_1 be the vertices consecutive with z in C. Let $A_0=V(C)\setminus\{z,x_0,x_1\}$.

By 6.16 and 6.17, z is strongly A_0-anticomplete, and so (z,A_0) is a frame. Now x_0,x_1 is a wheel system with respect to (z,A_0), and x_0,x_1 are strongly Y-complete by 6.8 and therefore strongly $Y\cup\{y\}$-complete by the definition of a tail.

By 6.14 there exists a wheel system $x_0,\dots,x_{t+1}$ relative to the frame (z,A_0) for which $Y\cup\{y\}$ is a hub. By the definition of a tail there is a (G,Y)-complete edge in A_0, and so all members of Y have weak neighbors in A_0. By 6.11, there exists r with $1 \leq r \leq t$, such that y is weakly non-adjacent to x_{t+1} and has no weak neighbor in A_r, and x_{t+1} has a weak neighbor in A_r, and a weak non-neighbor in X_r. Now $x_0, \ldots, x_r, x_{t+1}$ is a wheel system with hub $Y \cup \{y\}$, and S is a tail for z in T with respect to (C, Y) and G, contrary to 6.18. This proves 6.19. ∎

6.4 The end of a wheel

In this subsection we complete the proof of the fact that if a trigraph in $\mathcal{T}_8$ contains a wheel then it admits a balanced skew-partition.

6.20 *Let $T \in \mathcal{T}_8$, not admitting a balanced skew-partition, and let (C, Y) be a strong wheel in T that is trioptimal in T. Then there is a subpath c_1-c_2-c_3 of C such that c_1, c_2, c_3 are all strongly Y-complete, and a path c_1-p_1-$\cdots$-p_k-c_3 such that none of $p_1, \ldots, p_k$ are in $V(C) \cup Y$, none of them is strongly Y-complete, and none of them has a weak neighbor in $V(C) \setminus \{c_1, c_2, c_3\}$.*

Proof. Since (C, Y) is a strong wheel there are two non-consecutive strongly Y-complete vertices in C with opposite wheel-parity, say a, b. Since a and b are weakly non-adjacent, by 5.7 there is a path P in T with $P^* \neq \emptyset$ joining them, so that none of its interior vertices is in Y or is strongly Y-complete. There may be internal vertices of P that belong to C, but we may choose a subpath P' of P, with ends a', b' say, so that $a', b' \in V(C)$ have opposite wheel-parity in T and P' has minimum length. Suppose a', b' are consecutive in C. Then since they have opposite wheel-parity, they are both strongly Y-complete and therefore neither is in the interior of P, and so a, b are consecutive in C, a contradiction. So a', b' are not consecutive in C.

Next suppose a', b' are weakly adjacent in T. Let G be a realization of T in which C is a hole and all switchable pairs containing a vertex of Y are assigned the value "non-edge". Then (C, Y) is a wheel in G since it is a strong wheel in T. Now a', b' have opposite wheel-parity in G, and so by 6.7 one of them, say a', is (G, Y)-complete and therefore strongly Y-complete in T. Thus a' is not in the interior of P, and so b' is, and therefore b' is not strongly Y-complete in T. But a' and b' have opposite wheel-parity in G, contrary to 6.9. This proves that a', b' are strongly non-adjacent in T and so P'^* is nonempty.

Let F be the interior of P'; then no vertex of F is in $Y \cup V(C)$, no vertex of F is strongly Y-complete, and there are attachments of F in C which are not consecutive in C and have opposite wheel-parity. The result follows from 6.17 and 6.2 applied to F and the trioptimality of (C, Y). This proves 6.20. ∎

We can now prove the main result of this subsection.

6.21 *Let $T \in \mathcal{T}_8$, not admitting a balanced skew-partition; then there is no wheel in T.*

Proof. Let (C, Y) be a trioptimal wheel in T.

(1) *There exists a realization of T in which (C, Y) is a wheel and exactly four edges of C are Y-complete.*

If (C, Y) is not a strong wheel, then it contains at most two strongly Y-complete edges, and since (C, Y) is a wheel in T, there exists a realization of T in which (C, Y) is a wheel and C contains exactly four edges, and the statement holds. So we may assume (C, Y) is a strong wheel in T. By 6.20 there is a subpath c_1-c_2-c_3 of C such that c_1, c_2, c_3 are all strongly Y-complete, and a path c_1-p_1-$\cdots$-p_k-c_3 such that none of $p_1, \ldots, p_k$ are in $V(C) \cup Y$, none of them is strongly Y-complete, and none of them has a weak neighbor in $V(C) \setminus \{c_1, c_2, c_3\}$. Let C' be the hole formed by the union of the paths $C \setminus c_2$ and c_1-p_1-$\cdots$-p_k-c_3. Then it has length ≥ 6, and it contains fewer strongly Y-complete edges than C. From the choice of (C, Y) it follows that (C', Y) is not a wheel. Since C has at least four strongly Y-complete edges, and C' has only two fewer, we deduce that exactly four edges of C are strongly Y-complete. So a realization of T in which C is a hole and all switchable pairs meeting Y are assigned the value "non-edge" has the desired property. This proves (1).

Let G be a realization of T in which (C, Y) is a wheel and exactly four edges of C are (G, Y)-complete. Since (C, Y) is not an odd wheel, there are vertices $x_0, z, x_1, c_1, c_2, c_3$ of C, in order, and all distinct except possibly $x_1 = c_1$ or $c_3 = x_0$, so that the (G, Y)-complete edges in C are $x_0z, zx_1, c_1c_2, c_2c_3$.

(2) *There is no tail for z in T with respect to (C, Y) and G.*

Since z has wheel-parity in G opposite from some vertex of $V(C)$ that is not (G, Y)-complete, there is no tail for z in T with respect to (C, Y) and G by 6.19. This proves (2).

Let $A_0 = V(C) \setminus \{z, x_0, x_1\}$. By 6.8 x_0 and x_1 are strongly Y-complete. By 5.7 and since by 6.9 z is strongly anticomplete to all vertices of A_0 that are not strongly Y-complete, there is a path S of $T \setminus \{x_0, x_1\}$ with $S^* \neq \emptyset$ from z to A_0, such that no vertex in $V(S) \setminus \{z\}$ is in Y or strongly Y-complete. We may assume that S is a path of G and no vertex of S^* is (G, Y)-complete. Let y be the vertex adjacent to z in S.

(3) *y is weakly non-adjacent to at least one of x_0, x_1.*

For assume it is strongly adjacent to both. Then T is a tail for z with respect to (C, Y) and G (because at least one of the (G, Y)-complete edges c_1c_2, c_2c_3 belongs to $C \setminus \{x_0, z, x_1\}$). This contradicts (2), and therefore proves (3).

(4) *y has no weak neighbor in A_0.*

For suppose first that it has a weak neighbor in $A_0 \setminus c_2$, say c. Then c, z are not consecutive and have opposite wheel-parity in the wheel (C, Y) in G. Not both x_0, x_1 are strongly adjacent to y, by (3). Since $c \neq c_2$, it follows that c and the two vertices of C consecutive with it are not all strongly Y-complete. Let G' be a realization of T such that

- C is a hole
- assign the value "non-edge" to all switchable pairs meeting Y and $V(C) \setminus \{x_0, z, x_1, c_1, c_2, c_3\}$
- assign the value "edge" to all switchable pairs meeting Y and $\{x_0, z, x_1, c_1, c_2, c_3\}$
- assign the value "non-edge" to all switchable pairs containing y and meeting $Y \cup \{x_0, x_1\}$
- assign the value "edge" to all remaining switchable pairs containing y

- assign values to remaining switchable pairs arbitrarily.

Theorem 16.1 of [2] applied to G' implies that $(C, Y \cup \{y\})$ is a wheel in G', and therefore in T, contrary to the trioptimality of (C, Y). So y has no weak neighbor in $A_0 \setminus c_2$. Next suppose that y is weakly adjacent to c_2. From the symmetry we may assume that $x_0 \neq c_3$. Let Q be the path of $C \setminus z$ between x_0, c_3; so Q has length > 0, and even length by 5.3 applied in G. Since x_0-Q-c_3-c_2-y-x_0 is not an odd hole, it follows that y is strongly non-adjacent to x_0. But then the hole x_0-Q-c_3-c_2-y-z-x_0 is the rim of an odd wheel with hub Y, contrary to $T \in \mathcal{T}_8$. So y is strongly non-adjacent to c_2. This proves (4).

Let S have vertices z-y-v_1-$\cdots$-v_{n+1}, where $v_{n+1} \in A_0$. From (4), $n \geq 1$. By choosing S of minimum length we may assume that none of $y, v_1, \ldots, v_{n-1}$ have weak neighbors in A_0.

(5) *If $n = 1$ then no weak neighbor of v_1 in A_0 is strongly Y-complete.*

Suppose $v_2' \in A_0$ is strongly Y-complete and weakly adjacent to v_1. From the symmetry we may assume that $x_0 \neq c_3$. Let Q be the path of $C \setminus z$ between x_0, c_3; so Q has length > 0, and even length by 5.3 applied in G. Since y, v_1 are not strongly Y-complete, there is an antipath joining them with interior in Y, and it is odd since it can be completed to an antihole via v_1-z-v_2'-y. Hence every weakly Y-complete vertex is strongly adjacent to one of y, v_1, and since c_2, c_3 are weakly Y-complete and not adjacent to y by (4), it follows that v_1 is strongly adjacent to c_2, c_3, and so v_1 has two strong neighbors in C that are of opposite wheel-parity in G. For the same reason and by (3), v_1 is strongly adjacent to one of x_0, x_1, and therefore v_1 has two strong neighbors in C that are not consecutive in C.

- C is a hole
- assign the value "non-edge" to all switchable pairs meeting Y and $V(C) \setminus \{x_0, z, x_1, c_1, c_2, c_3\}$
- assign the value "edge" to all switchable pairs meeting Y and $\{x_0, z, x_1, c_1, c_2, c_3\}$
- assign the value "edge" to all switchable pairs containing v_1 and meeting $V(C)$
- assign the value "non-edge" to all switchable pairs containing v_1 and meeting Y
- assign values to remaining switchable pairs arbitrarily.

By theorem 16.1 of [2] applied in G' there are three consecutive vertices in C, all (G', Y)-complete (and therefore (G, Y)-complete) and weakly adjacent to v_1. Since there is no kite in T, v_1 has no other weak neighbor in C. Hence $x_1 = c_1$ and the neighbors of v_1 in C are c_1, c_2, c_3. Consequently x_0 is strongly adjacent to y; but then x_0-Q-c_3-v_1-y-x_0 is an odd hole, a contradiction. This proves (5).

(6) *One of x_0, x_1 is strongly anticomplete to $\{v_1, \ldots, v_n\}$ and is weakly non-adjacent to y.*

For let z-y-p_1-$\cdots$-p_k be a path P from z to some strongly Y-complete vertex $p_k \in A_0$, with interior in $A_0 \cup \{y, v_1, \ldots, v_n\}$ with no strongly Y-complete vertex in P^*. Since none of $y, v_1, \ldots, v_{n-1}$ have weak neighbors in A_0 it follows that $\{y, v_1, \ldots, v_n\} \subseteq \{y, p_1, \ldots, p_{k-1}\}$. From (5), $k \geq 3$. Since

$T \in \mathcal{T}_8$, $(Y, \{x_0, x_1\}, P)$ is not a pseudowheel. But the ends of the path P are weakly Y-complete and its internal vertices are not strongly Y-complete; the path has length ≥ 4; Y, z are weakly $\{x_0, x_1\}$-complete, and y, p_k are not strongly $\{x_0, x_1\}$-complete. So no other vertices of P are weakly $\{x_0, x_1\}$-complete. Let G' be a realization of G such that

- P is a path,
- assign the value "edge" to all switchable pairs meeting $Y \cup \{z\}$ and $\{x_0, x_1\}$
- assign the value "non-edge" to all switchable pairs meeting $\{x_0, x_1\}$ and $\{y, p_k\}$
- assign the value "non-edge" to all remaining switchable pairs containing a vertex of Y
- assign the value "edge" to all remaining switchable pairs containing a vertex of $\{x_0, x_1\}$
- assign values to remaining switchable pairs arbitrarily.

By theorem 2.11 of [2] applied in G' to the path P and the anticonnected sets Y and $\{x_0, x_1\}$, it follows that one of x_0, x_1 is strongly non-adjacent to all of $p_1, \ldots, p_{k-1}$ and weakly non-adjacent to y. Since $\{y, v_1, \ldots, v_n\} \subseteq \{y, p_1, \ldots, p_{k-1}\}$, this proves (6).

Let $F = \{y, v_1, \ldots, v_n\}$. From the symmetry we may assume that x_0 is (G, F)-anticomplete. Let Q be a path of G from x_0 to y with interior in $F \cup A_0$. It follows that Q has length at least three. Let C' be the hole z-y-Q-x_0-z; so C' has length ≥ 6. Suppose that x_0 is different from c_3 and so its neighbor in $C \setminus \{z\}$ is not (G, Y)-complete. Since (C', Y) is not an odd wheel, it follows that (C', Y) is not a wheel, and so no vertex of $C' \setminus \{z, x_0\}$ is strongly Y-complete. By 5.5 applied in G it follows that Y contains a leap or a hat in G. A leap would imply there are two vertices in Y, joined by an odd path of length ≥ 5 with interior in $F \cup A_0$. Hence its ends are strongly $\{x_0, x_1\}$-complete, and its internal vertices are not, contrary to 5.2. So Y contains a hat, that is there exists $y' \in Y$ that is $(G, C' \setminus \{z, x_0\})$-anticomplete. Let x' be the vertex of $C \setminus \{z\}$ consecutive with x_0. In G the set $F \cup A_0$ catches the triangle $\{x_0, y', z\}$ and we apply 5.6. In G, the only neighbor of z in $F \cup A_0$ is y, and by (4) y has at most one neighbor in $F \cup A_0$, and hence there is no reflection of $\{x_0, z, y'\}$ in $F \cup A_0$. So 5.6.1 must hold. But in G, y is non-adjacent to x_0, y', and so $F \cup A_0$ contains a common neighbor of x_0 and y'. However in G, the only neighbor of x_0 in $F \cup A_0$ is x' and x' is non-adjacent to y', a contradiction. This proves that $x_0 = c_3$, and therefore $x_1 \neq c_1$.

By exchanging x_0, x_1, we deduce that in G, x_1 has a neighbor in F. Therefore in G there are two attachments of F in C with opposite wheel-parity, and two that are non-adjacent. By (1), 6.17, the trioptimality of the wheel and theorem 16.2 of [2] applied to G, and since $x_0 = c_3$ is (G, F)-anticomplete, it follows that in G there is a path R between z, c_2 with interior in F, and no vertex of C has neighbors in the interior of R except z, c_2. But then the hole formed by the union of R and the path $C \setminus x_0$ is the rim of an odd wheel with hub Y in G, and therefore in T by 6.1, a contradiction. This proves 6.21. ∎

7 A hole with a triad

In this section we prove that if a trigraph in $\mathcal{T}_9$ contains a hole of length at least 6 and a vertex with 3 consecutive weak neighbors in the hole (we call this configuration "a hole with a triad"), then it admits a balanced skew-partition. Our proof here is different from the proof in [2]. Most of the "hard" theorems in this section are trivial in the graph case.

7.1 A hole with an original triad

Let C be a hole in T. We say that a vertex $z \in V(C)$ is an *origin* of C if it has exactly two weak neighbors in C— namely the two vertices of C consecutive with it. For a hole C with some origin z we say that a vertex $y \in V(T) \setminus V(C)$ is an *original triad* for the pair (C, z) if y is weakly adjacent to z and both of the vertices consecutive with it in C. The goal of this subsection is to prove that if a trigraph $T \in \mathcal{T}_9$ contains a hole with an original triad, then it admits a balanced skew-partition.

7.1 *Let X be a weakly anticonnected set and let P be a path of length 3 with vertices p_1, p_2, p_3, p_4 in order, such that p_1 is weakly non-adjacent to p_2 and both p_1 and p_4 are weakly X-complete. Then one of p_2, p_3 is strongly X-complete.*

Proof. Suppose none of p_2, p_3 is strongly X-complete and let Q be an antipath joining them with interior in X. Then p_2-Q-p_3-p_1-p_4-p_2 and p_2-Q-p_3-p_1-p_2 are both antiholes of different parity in T, a contradiction. This proves 7.1. ∎

7.2 *Let T be a trigraph in $\mathcal{T}_9$, let (z, A_0) be a frame in T and let $x_0, \dots, x_s$ be a wheel system with respect to it. Let the sets A_i, X_i be defined as usual. Assume that $y \in V(T) \setminus (A_0 \cup X_s \cup \{z\})$ is weakly $X_s \cup \{z\}$-complete, and with a weak neighbor in A_s. Then y is strongly X_s-complete.*

Proof. Suppose the result is false, namely y has a weak non-neighbor in X_s and assume s is minimum for which it is false. Let (a_0, a_1, P) be an anchor of the wheel system $x_0, \dots, x_s$.

(1) $s > 1$

For suppose $s = 1$. We observe that y is strongly non-adjacent to both a_0 and a_1 for otherwise z-x_0-a_0-P_0-a_1-x_1-z would be the rim of a wheel with hub $\{y\}$ contrary to the fact that T is in $\mathcal{T}_9$. From the symmetry we may assume y is weakly non-adjacent to x_0. Since T contains no odd wheel and y is strongly non-adjacent to a_1, y has no weak neighbor in P_0. Since y has a weak neighbor in A_1, there exists a path Q with interior in A_1 from y to P_0, such that y has exactly one weak neighbor q in Q and only the last vertex of Q belongs to $V(P_0)$. Let $F = V(P_0) \cup Q^* \cup \{x_0\}$. Let G be the following realization of T:

- z-x_0-a_0-P_0-a_1-x_1-z is a hole
- Q is a path
- assign the value "non-edge" to the switchable pair yx_0
- if qx_1 is a switchable pair of T, assign the value "non-edge" to it
- assign the value "edge" to all remaining switchable pairs of T.

Now G is in $\mathcal{F}_7$ and F catches the triangle $\{z, x_1, y\}$. Suppose q is non-adjacent to x_1. No vertex of F has two neighbors in $\{z, x_1, y\}$ in G, for x_0 is the unique neighbor of z and it is non-adjacent to both y and x_1, and q is only neighbor of y, and it is non-adjacent to x_1. So by 5.6 F contains a reflection of $\{z, x_1, y\}$ in G. That means that in G q is adjacent to x_0 and there exists a neighbor f of x_1, different from x_0, q, that is adjacent to both x_0 and q. Since x_0-q-y-x_1-a_1-P_0-a_0-x_0 is not an odd hole in G, we deduce that q has a neighbor in P_0. So $F = V(P_0) \cup \{q, x_0\}$, and so $f \in V(P_0)$. The only neighbor of x_0 in P_0 is a_0, so $f = a_0$, contrary to the fact that z-x_0-a_0-P_0-a_1-x_1-z is a hole in G. This proves that q is adjacent to x_1, and so q is strongly adjacent to x_1 in T.

Since q is not strongly $\{x_0, x_1\}$-complete, q is weakly non-adjacent to x_0 and there exists a path F' from x_0 to q with $\emptyset \neq F'^* \subseteq A_1$. But then y-q-F'-x_0-z-y and y-q-F'-x_0-y are holes of different parity, a contradiction. This proves (1).

(2) *y has no weak neighbor in A_{s-1}.*

Suppose it does. Since the theorem holds for the wheel system $x_0, \dots x_{s-1}$, it follows that y is strongly complete to X_{s-1} and weakly non-adjacent to x_s. Then

$$x_0, \dots, x_s$$

is a wheel system with hub y and by 6.11 there exists r with $1 \leq r \leq s-1$ such that y has no weak neighbor in A_r and x_s has a weak neighbor in A_r, and a weak non-neighbor in X_r.

Since y has a weak neighbor in A_{s-1}, $r \leq s-2$ and $s > 2$. If y has a weak neighbor in A_{s-2} then the wheel system $x_0, \dots, x_{s-2}, x_s$ satisfies the hypotheses of the theorem, and it has height $< s$, so y is strongly adjacent to x_s, a contradiction. So y has no weak neighbor in A_{s-2} and we may assume that $r = s-2$.

Let Q be a path from y to a vertex with a weak neighbor in A_{s-2}, such that y has a unique weak neighbor q in Q and only the last vertex of Q has a neighbor in A_{s-2}, and $V(Q) \setminus \{y\} \subseteq A_{s-1}$. Then Q has length at least 1. Let $F = A_{s-2} \cup V(Q) \setminus \{y\}$. Then F is a weakly connected set and both y and x_s have weak neighbors in it. Hence $F \cup \{x_s\}$ contains a path P from y to x_s and since P has two completions of different parity: y-P-x_s-z-y and y-P-x_s-y, P has length 2. Since q is the unique weak neighbor of y in F, x_s is weakly adjacent to q.

But $q \in A_{s-1}$, and hence q is not strongly X_{s-1}-complete. Let W be an antipath from q to x_s with nonempty interior in X_{s-1}. Suppose W is odd. Then y-x_s-W-q-z is an odd antipath of length at least 5, all its interior vertices have weak neighbors in the weakly connected set $(A_{s-2} \cup V(Q)) \setminus \{q, y\}$ and z and y do not, contrary to 5.2 applied in $\overline{T}$. So W is even. Then y-x_s-W-q is an odd antipath. Suppose q has no weak neighbor in A_{s-2}. Then all interior vertices of y-x_s-W-q have weak neighbors in A_{s-2} and its ends do not and y is weakly adjacent to x_s, contrary to 5.2 and 7.1 applied in $\overline{T}$. So q has a weak neighbor in A_{s-2}. Since $q \notin A_{s-2}$, it follows that q is strongly X_{s-2}-complete, and since $q \in A_{s-1}$, we deduce that q is weakly non-adjacent to x_{s-1}. Let S be a path from q to x_{s-1} with nonempty interior in A_{s-2}. Then y-q-S-x_{s-1}-y is a hole, and hence S is even. So q-S-x_{s-1}-z is an odd path, its ends are weakly X_{s-2}-complete and none of its interior vertices are strongly X_{s-2}-complete (for they all belong to $A_{s-2} \cup \{x_{s-1}\}$), so by 5.2, S has length 2. Let t be the middle vertex of S.

Let U be an antipath joining t and x_{s-1} with interior in X_{s-2}. Then U is odd for it can be completed to an antihole via t-z-q-x_{s-1}.

We claim that $s > 3$. For suppose $s = 3$. Since U is odd, $x_0, x_1 \in V(U)$. Let G be a realization of T in which x_0-a_0-P_0-a_1-x_1-z-x_0 is a hole and x_{s-1}-U-t-z-q-x_{s-1} is an antihole; then G violates theorem 15.7 of [2]. This proves that $s > 3$.

Next we claim that x_{s-1} has a weak neighbor in A_{s-3}. For suppose it does not. Then by 5.2 applied in $\overline{T}$ to the odd antipath U and the weakly connected set A_{s-3}, and since z is weakly complete to U^* and strongly anticomplete to A_{s-3}, t has a weak neighbor in A_{s-3}. Since t does not belong to A_{s-3} (for t is weakly adjacent to x_{s-1}), t is strongly X_{s-3}-complete, and since $t \in A_{s-2}$, t is weakly non-adjacent to x_{s-2}. Since x_{s-2}-q-t-x_{s-1}-z-x_{s-2} is not an odd hole, x_{s-1} is strongly adjacent to x_{s-2}. But then

$$x_0, \dots, x_{s-1}$$

is a $\{q\}$-square, contrary to 6.6. This proves that x_{s-1} has a weak neighbor in A_{s-3}. If x_{s-1} is strongly X_{s-3}-complete then

$$x_0, \dots, x_{s-1}$$

is a $\{q\}$-diamond, contrary to 6.6, so x_{s-1} has a weak non-neighbor in X_{s-3}.

Next we show that q has a weak neighbor in A_{s-3}. For suppose it does not. Let M be a path with $V(M) \subseteq A_{s-2} \cup \{q\}$ from q to a vertex with a weak neighbor in A_{s-3} such that q has a unique weak neighbor in M and only the last vertex of M has a weak neighbor in A_{s-3}. Then both q and x_{s-1} have weak neighbors in $F' = (A_{s-3} \cup V(M)) \setminus \{q\}$, so there exists a path joining q and x_{s-1} with interior in F', and we may assume that S is such a path, and the vertex t belongs to $V(M)$. Now y-t-U-x_{s-1}-q is an odd antipath of length at least 5, all its interior vertices have weak neighbors in $F' \setminus \{t\}$ and the ends do not, contrary to 5.2 applied in $\overline{T}$. This proves that q has a weak neighbor in A_{s-3}.

Now we claim that $x_0, \dots, x_{s-3}, x_{s-1}, x_{s-2}, x_s$ is a wheel system. Certainly $x_0, \dots, x_{s-3}$ is a wheel system. The vertex x_{s-1} has a weak neighbor in A_{s-3} and a weak non-neighbor in X_{s-3}, so $x_0, \dots, x_{s-3}, x_{s-1}$ is a wheel system, and hence so is $x_0, \dots, x_{s-3}, x_{s-1}, x_{s-2}$. The vertex q is not strongly $X_{s-3} \cup \{x_{s-1}\}$-complete, for it belongs to A_{s-1} and is strongly X_{s-2}-complete. But then since q has a weak neighbor in A_{s-3}, it follows that q belongs to one of the companion sets of the wheel system $x_0, \dots, x_{s-3}, x_{s-1}$; x_s is weakly adjacent to q and is not strongly X_{s-2}-complete. This proves that $x_0, \dots, x_{s-3}, x_{s-1}, x_{s-2}, x_s$ is a wheel system. If x_s is strongly $X_{s-3} \cup \{x_{s-1}\}$-complete, then

$$x_0, \dots, x_{s-3}, x_{s-1}, x_{s-2}, x_s$$

is a $\{y\}$-diamond, contrary to 6.6. If x_s is not strongly $X_{s-3} \cup \{x_{s-1}\}$-complete, then $x_0, \dots, x_{s-3}, x_{s-1}, x_s$ is a wheel system, it satisfies the hypotheses of the theorem and has height $< s$, so y is strongly adjacent to x_s, a contradiction. This completes the proof of (2).

From (2) y has no weak neighbor in A_{s-1}. Let P be a path from y to a vertex with a weak neighbor in A_{s-1}, such that y has a unique weak neighbor in P, only the last vertex p of P has a neighbor in A_{s-1} and $V(P) \setminus \{y\} \subseteq A_s$. Then p is strongly X_{s-1}-complete.

(3) *P has length* 1.

Suppose P has length at least 2. Let p' be the weak neighbor of y in P. Since $p \neq p'$, p' has no weak neighbor in A_{s-1}. Let Q be an antipath from p' to y with interior in X_s. Assume first Q is odd. All internal vertices of Q have weak neighbors in the weakly connected set A_{s-1}, the ends y and p' do not, and y is weakly adjacent to the vertex of Q consecutive with it, contrary to 5.2 and 7.1. So Q is even. But then z-p'-Q-y is an odd antipath, all its internal vertices have weak neighbors in the weakly connected set $A_{s-1} \cup V(P) \setminus \{p', y\}$, the ends y and z do not, and y is weakly adjacent to the vertex of Q consecutive with it, contrary to 5.2 and 7.1 applied in $\overline{T}$. This proves (3).

The vertex p is weakly non-adjacent to x_s for it belongs to A_s and is strongly X_{s-1}-complete. So there exists a path R with nonempty interior in A_{s-1} joining p and x_s.

(4) *x_s is strongly adjacent to y and R has length* 2.

If $x_s y$ is a switchable pair then the path y-p-R-x_s has length > 2 and has two completions of different parity: y-z-x_s and y-x_s, a contradiction. So x_s is strongly adjacent to y. Since p-R-x_s-y-p is a hole, R is even, and so the path p-R-x_s-z is an odd path, both ends of which are weakly X_{s-1}-complete and none of its internal vertices is strongly X_{s-1}-complete (for they all belong to $A_{s-1} \cup \{x_s\}$), so by 5.2 p-R-x_s-z has length 3 and R has length 2. This proves (4).

Let t be the middle vertex of the path R. Let Q be an antipath between t and and x_s with interior in X_{s-1}. Then Q is odd for it can be completed to an antihole via t-z-p-x_s. Let G be a realization of T in which x_0-a_0-P_0-a_1-x_1-z-x_0 is a hole and x_s-Q-t-z-p-x_s is an antihole. By theorem 15.7 of [2], this hole and antihole meet in at most two vertices, and so not both x_0 and x_1 belong to Q. Hence $s \geq 3$ since Q is odd.

(5) *x_s has a weak neighbor in A_{s-2}.*

Suppose x_s is strongly anticomplete to A_{s-2}. We claim that t has a weak neighbor in A_{s-2}, for otherwise the odd antipath Q, the weakly connected set A_{s-2} and the vertex z contradict 5.1 applied in $\overline{T}$ (for z is a vertex weakly complete to $V(Q^*)$ and strongly anticomplete to A_{s-2}). So t has a weak neighbor in A_{s-2}, and hence t is strongly X_{s-2}-complete and weakly non-adjacent to x_{s-1}. Since x_s-t-p-x_{s-1}-z-x_s is not an odd hole, x_s is strongly adjacent to x_{s-1}. But then the wheel system

$$x_0, \ldots x_s$$

is a $\{p\}$-square, contrary to 6.6. This proves (5).

(6) *p has a weak neighbor in A_{s-2}.*

For suppose it does not. Let S be a path from p to a vertex with a weak neighbor in A_{s-2} such that p has a unique weak neighbor in S, $V(S) \setminus \{p\}$ is a subset of A_{s-1}, and only the last vertex of S has a weak neighbor in A_{s-2}. Then both p and x_s have weak neighbors in $F = (A_{s-2} \cup V(S)) \setminus \{p\}$, so there exists a path joining p and x_s with interior in F, and we may assume that R is such a path, and the vertex t belongs to $V(S)$. Now y-t-Q-x_s-p is an odd antipath of length at least 5, all its interior vertices have weak neighbors in $F \setminus \{t\}$ and the ends do not, contrary to 5.2 applied in $\overline{T}$. This proves (6).

If x_s is strongly X_{s-2}-complete, then by (5) the wheel system

$$x_0, \ldots, x_s$$

is a $\{p\}$-diamond, contrary to 6.6. So x_s is not strongly X_{s-2}-complete and $x_0, \ldots, x_{s-2}, x_s$ is a wheel system. From (6) p has a weak neighbor in A_{s-2} and is weakly non-adjacent to x_s, so p belongs to one of the companion sets of this wheel system. So $x_0, \ldots, x_{s-2}, x_s$ satisfies the hypotheses of the theorem and has height $< s$, and hence y is strongly $X_{s-2} \cup \{x_s\}$-complete, and so it is weakly non-adjacent to x_{s-1}.

Since both p and x_s have weak neighbors in A_{s-2}, we may assume that the interior of the path R is contained in A_{s-2}, that is $t \in A_{s-2}$. Both t and x_s have weak non-neighbors in X_{s-2} and so we may assume that the interior of the antipath Q joining them is a subset of X_{s-2}.

We claim that $x_0, \ldots, x_{s-2}, x_s, x_{s-1}$ is a wheel system. We have already shown that $x_0, \ldots, x_{s-2}, x_s$ is a wheel system. The vertex x_{s-1} is not strongly X_{s-2}-complete and not strongly A_{s-2}-anticomplete, so $x_0, \ldots, x_{s-2}, x_s, x_{s-1}$ is a wheel system. This wheel system has height s and it satisfies the hypothesis of the theorem and not the conclusion, since y is not strongly adjacent to x_{s-1}. By (4) applied to the wheel system $x_0, \ldots, x_{s-2}, x_s, x_{s-1}$ is strongly adjacent to x_{s-1}, a contradiction. This completes the proof of 7.2. ▮

Next we need another transformation of 6.10.

7.3 *Let $T \in \mathcal{T}_9$, admitting no balanced skew-partition, let (z, A_0) be a frame and $x_0, \ldots, x_s$ a wheel system with respect to it, and define X_i, A_i as usual. Then there is no vertex $y \in V(T) \setminus \{z, x_0, \ldots, x_s\}$ that is weakly $\{z, x_0, \ldots, x_s\}$-complete and has a weak neighbor in A_s.*

Proof. For suppose there is such a frame, wheel system, and vertex y, and choose them with s minimum (it is important here that we minimize over all choices of the frame, not just of the wheel system); say (z, A_0), $x_0, \ldots, x_s$ and y respectively. By 7.2 y is strongly $x_0, \ldots, x_s$-complete. By 6.15, there exists r with $1 \leq r < s$, and a vertex v such that y is weakly non-adjacent to v and has no weak neighbor in A_r, and v is weakly adjacent to z, and has a weak neighbor in A_r, and a weak non-neighbor in X_r. Since $x_0, \ldots, x_s, v$ is a wheel system, it follows from 7.2 that y is strongly non-adjacent to v. Then (y, A_0) is a frame, and $x_0, \ldots, x_r$ is a wheel system with respect to it. The vertex z is weakly $\{y, x_0, \ldots, x_r\}$-complete and has a neighbor in A'_r (namely v), where A'_r is the maximal weakly connected subset of $V(T)$ including A_0 and containing no weak neighbor of y and no strongly X_r-complete vertex. But this contradicts the minimality of s. This proves 7.3. ▮

Now we can prove the main result of this subsection.

7.4 *Let $T \in \mathcal{T}_9$, admitting no balanced skew-partition, and let C be a hole in T of length ≥ 6 with origin z. Then there is no vertex of $T \setminus V(C)$ that is an original triad for (C, z).*

Proof. Suppose that there is such a vertex, say y, and let it be weakly adjacent to $x_0, z, x_1 \in V(C)$, where x_0-z-x_1 is a subpath of C. Let $A_0 = V(C) \setminus \{z, x_0, x_1\}$. Since z is an origin for C, (z, A_0) is a frame. By 7.3 applied to (z, A_0) and x_0, x_1, it follows that y has no other weak neighbor in C. Choose t maximum so that there is a sequence $x_2, \ldots, x_t$ with the following properties:

- for $2 \leq i \leq t$, there is a weakly connected subset A_{i-1} of $V(T)$ including A_{i-2}, containing a weak neighbor of x_i, no weak neighbor of z or y, and no strongly $\{x_0, \ldots, x_{i-1}\}$-complete vertex,
- for $1 \leq i \leq t$, x_i is not strongly $\{x_0, \ldots, x_{i-1}\}$-complete, and
- $x_0, \ldots, x_t$ are weakly $\{y, z\}$-complete.

Since T admits no balanced skew-partition, by 5.7 there is a path P from $\{z, y\}$ to A_0, disjoint from $\{x_0, \ldots, x_t\}$ and containing no strongly $\{x_0, \ldots, x_t\}$-complete vertex in its interior. Choose such a path of minimum length. From the symmetry between z, y we may assume its first vertex is y; say the path is y-p_1-$\cdots$-p_{k+1}, where $p_{k+1} \in A_0$. From the minimality of the length of P it follows that z is strongly non-adjacent to all of $p_2, \ldots, p_{k+1}$. If z is weakly adjacent to p_1 then we may set $x_{t+1} = p_1$, contrary to the maximality of t. So $p_1, \ldots, p_{k+1}$ are all strongly non-adjacent to z. Hence (z, A_0) is a frame, and $x_0, \ldots, x_t$ is a wheel system with respect to it. The vertex y is weakly adjacent to all of $z, x_0, \ldots, x_t$, and there is a weakly connected subset of $V(T)$ including A_0, containing a weak neighbor of y, no weak neighbor of z, and no strongly $\{x_0, \ldots, x_t\}$-complete vertex. But this contradicts 7.3. This proves 7.4. ∎

7.2 The end of a hole with a triad

In this subsection we prove that if a trigraph in $\mathcal{T}_9$ contains a hole of length at least 6 and a vertex with three consecutive weak neighbors in the hole, then it admits a balanced skew-partition. By 7.4 we may assume that $T \in \mathcal{T}_{10}$.

7.5 *Let $T \in \mathcal{T}_{10}$. Let C be a hole of length at least 6 with vertices $c_1, \ldots, c_{2k}$ in order and let $y \in V(T) \setminus V(C)$ be weakly adjacent to c_1, c_2, c_3. Then $k = 3$, the pair c_2c_5 is switchable, y is strongly complete to $\{c_1, c_3, c_5\}$ and both y and c_2 are strongly anticomplete to $\{c_4, c_6\}$.*

Proof. Let $C_{odd} = \{c_{2i+1} \ : \ 0 \leq i < k\}$ and let $C_{even} = \{c_{2i} \ : \ 1 < i \leq k\}$. First we claim that c_2 is strongly C_{even}-anticomplete. Clearly c_2 is weakly C_{even}-anticomplete for C is a hole. Since there exists a subpath of C between c_2 and c_{2i} of length > 2, 2.1 implies that c_2, c_{2i} is not a switchable pair. This proves that c_2 is strongly C_{even}-anticomplete.

Now we show that y is strongly C_{even}-anticomplete. Suppose it is not. Let $1 < i \leq k$ be minimum such that y is weakly adjacent to c_{2i}. Let $1 \leq j < i$ be maximum such that y is weakly adjacent to c_{2j+1}. Since y-c_{2j+1}-$\ldots$-c_{2i-1}-c_{2i}-y is not an odd hole, we deduce that $j = i - 1$. But then $\{y\}$ is a hub for a wheel with rim C, contrary to the fact that $T \in \mathcal{T}_{10}$. This proves that y is strongly C_{even}-anticomplete.

Next we show that both y and c_2 are weakly C_{odd}-complete. Suppose not. Let $0 \leq i < k$ be minimum such that c_{2i+1} is not weakly $\{c_2, y\}$-complete. Then $i > 1$ for c_3 is weakly $\{c_2, y\}$-complete, and c_{2i-1} is weakly $\{c_2, y\}$-complete. Let c_{2i}-$\ldots$-c_m be a minimal subpath of $C \setminus c_2$ such that both c_2 and y have a weak neighbor in $\{c_{2i}, c_{2i+1}, \ldots, c_m\}$. Then c_m is weakly adjacent to one of c_2, y and some $x \in \{c_2, y\}$ is strongly anticomplete to $\{c_{2i}, c_{2i+1}, \ldots, c_{m-1}\}$. But then $C' = x$-c_{2i-1}-c_{2i}-$\ldots$ c_m-x is a hole of length at least 6 with origin x. Let $\{x'\} = \{c_2, y\} \setminus \{x\}$. Then x' is weakly adjacent to x and c_{2i-1} and has a weak neighbor in $V(C') \setminus \{x, c_{2i-1}\}$. We have already shown that x' is strongly non-adjacent to c_{2i}, so since $(C', \{x'\})$ is not an odd wheel, x' is weakly adjacent to c_m. But then x' is an original triad for the hole C', contrary to the fact that $T \in \mathcal{T}_{10}$. This proves that both y and c_2 are weakly C_{odd}-complete. Since c_2 belongs to at most one switchable pair and C is a hole, $k = 3$ and $C_{odd} = \{c_1, c_3, c_5\}$.

Now it remains to show that y is strongly C_{odd}-complete. y is strongly adjacent to c_5 because c_2c_5 is a switchable pair, y is weakly adjacent to c_5 and c_5 belongs to most one switchable pair in T. If y is weakly non-adjacent to c_3 then y-c_2-c_3-c_4-c_5-y is an odd hole. So y is strongly adjacent to c_3, and by symmetry to c_1, and hence y is strongly C_{odd}-complete. This completes the proof of 7.5. ∎

7.6 *Let $T \in \mathcal{T}_{10}$. Let C be a hole of length at least 6 with vertices $c_1, \dots, c_{2k}$ in order and let $y \in V(T) \setminus V(C)$ be weakly adjacent to c_1, c_2, c_3. Then T admits a balanced skew-partition.*

Proof. By 7.5 $k = 3$, the pair c_2c_5 is switchable, y is strongly complete to $\{c_1, c_3, c_5\}$ and both y and c_2 are strongly anticomplete to $\{c_4, c_6\}$. So c_2c_5 is the unique switchable pair containing c_2. In particular, y is strongly adjacent to c_2.

Let Y be a maximal weakly anticonnected set including y such that $\{c_1, c_2, c_3, c_5\}$ is strongly Y-complete. By 7.5 applied to every member of Y, the set Y is strongly $\{c_4, c_6\}$-anticomplete. Let X be the set of strong common neighbors of Y. Suppose there exists a path P in T from c_2 to c_5 with $P^* \neq \emptyset$ and with no interior vertex in $Y \cup X$. Since c_2 is weakly adjacent to c_5, it follows that either P has length 2 or it is odd. Since c_2 and c_5 are strongly Y-complete and no vertex in the interior of P is, it follows from 5.2 that P has length 2 or 3. Let p, p' be the neighbors in P of c_5 and c_2 respectively.

(1) *P does not have length* 2*.*

If P has length 2, then $p = p'$ is weakly (and therefore strongly) adjacent to both c_2 and c_5, and hence $p \notin V(C)$. If p is weakly adjacent to both c_1, c_3 then by 7.5 applied with $y = p$, p is strongly $\{c_1, c_2, c_3, c_5\}$-complete, and p is not strongly Y-complete, so $Y \cup \{p\}$ contradicts the maximality of Y.

Both subpaths of C between c_2 and c_5 have odd length, so they each contains an edge, e_1 and e_2 respectively, with both ends weakly adjacent to p. Since $(C, \{p\})$ is not a wheel in T, C does not contain two disjoint edges with both ends weakly adjacent to p, so e_1 and e_2 share an end, and hence they are both contain c_5, and in particular p is weakly adjacent to c_4 and c_6. By 7.5 applied to C and p with c_5 in place of c_2 and p in place of y, we deduce that p is strongly $\{c_2, c_4, c_6\}$-complete and strongly $\{c_1, c_3\}$-anticomplete. Let $y' \in Y$ be a weak non-neighbor of p. Then $A = c_3\text{-}p\text{-}y'\text{-}c_4\text{-}c_2\text{-}c_5\text{-}c_3$ is an antihole and $|V(A) \cap V(C)| > 2$, contrary to theorem 15.7 of [2] applied to a realization of T in which C is a hole and A is an antihole. This proves (1).

(2) *P does not have length* 3*.*

If $c_2\text{-}P\text{-}c_5$ has length 3 then $P = c_2\text{-}p'\text{-}p\text{-}c_5$. Let W be an antipath joining p and p' with interior in Y. Then $A = p'\text{-}W\text{-}p\text{-}c_2\text{-}c_5\text{-}p'$ is an antihole. The vertex c_4 is strongly $V(W^*)$-anticomplete and is weakly non-adjacent to c_2. If c_4 is weakly non-adjacent to p then $(A, \{c_4\})$ is a wheel in $\overline{T}$, contrary to the fact that $T \in \mathcal{T}_{10}$. So p is strongly adjacent to c_4, and by symmetry p is strongly adjacent to c_6. By 7.5 applied to C and p with c_5 in place of c_2 and p in place of y, we deduce that p is strongly adjacent to c_2, contrary to the fact that P is a path in T. This proves (2).

It follows from (1) and (2) that no such path P exists, and so by 5.7 T admits a balanced skew-partition. ∎

7.3 Hole and antihole

In this subsection we prove a useful corollary of 7.6. We start with a lemma.

7.7 *Let $T \in \mathcal{T}_{11}$, and suppose C and D are a hole and an antihole of length ≥ 6 in T respectively. Then $|V(C) \cap V(D)| \leq 2$.*

Proof. Let $c_1, \ldots, c_{2k}$ be the vertices of C in order and $d_1, \ldots, d_{2l}$ be the vertices of D in order. If there exists a realization of T in which C is a hole and D is an antihole, then the result follows from theorem 15.7 of [2]. So we may assume that either there exist two consecutive vertices of C that are also consecutive in D, or there exist two non-consecutive vertices of C that are non-consecutive in D.

(1) *For all i, j with $1 \leq i < j \leq k$, $c_{2i}c_{2j} \in N(T)$ and $c_{2i-1}c_{2j-1} \in N(T)$.*

This follows immediately from 2.1.

Suppose first that there exist two consecutive vertices of C that are also consecutive in D. In this case we may assume that $c_1 = d_1$ and $c_2 = d_2$. From (1) for all $c \in V(C) \setminus \{c_1, c_2\}$, c is strongly anticomplete to at least one of c_1, c_2. On the other hand for all $d \in V(D) \setminus \{d_1, d_2, d_3, d_{2l}\}$, d is weakly $\{d_1, d_2\}$-complete. So

$$V(C) \cap V(D) \subseteq \{d_1, d_2, d_3, d_{2l}\}.$$

Similarly

$$V(C) \cap V(D) \subseteq \{c_1, c_2, c_3, c_{2k}\}.$$

Note that by (1) $c_1c_3, c_2c_{2k} \in N(T)$. If $|V(C) \cap V(D)| \geq 3$ then we may assume $\{c_1, c_2, c_3\} \subseteq V(C) \cap V(D)$ and so $c_3 = d_{2l}$. But then the vertex d_4 is is weakly adjacent to c_1, c_2, c_3 contrary to the fact that $T \in \mathcal{T}_{11}$.

So we may assume that there exist two non-consecutive vertices of C that are non-consecutive in D. Let c_1, c_j be such vertices and we may assume $c_1 = d_1$. Since $c_1c_j \in S(T)$ we deduce from (1) that $j = 2m$ where $2 \leq m \leq k-1$ and $c_{2m} = d_{2n}$ for some $2 \leq n \leq l-1$. Every vertex in $V(D) \setminus \{d_1, d_2, d_{2l}, d_{2n}, d_{2n-1}, d_{2n+1}\}$ is weakly adjacent to both c_1 and c_{2m} and by 5.3 it is weakly adjacent to both ends of at least two edges of C, contrary to the fact that $T \in \mathcal{T}_{11}$. Since D has length at least 6, we deduce that $l = 3$, $n = 2$. Similarly $k = 3$ and $m = 2$. Since $|V(C) \cap V(D)| \geq 3$, we may assume from the symmetry that $d_2 \in V(C)$ and $d_2 = c_3$. Since by (1) d_6 is strongly adjacent to both $d_2 = c_3$ and $d_4 = c_4$, it follows that $d_6 \notin V(C)$. Similarly $c_5 \notin V(D)$. If d_6 is weakly adjacent to c_5, then d_6 has three consecutive weak neighbors in C, contrary to the fact that $T \in \mathcal{T}_{11}$. So we may assume that d_6 is strongly non-adjacent to c_5, and then c_5 has three consecutive weak non-neighbors in D, contrary to the fact that $T \in \mathcal{T}_{11}$. This proves 7.7. ▮

7.8 *Let $T \in \mathcal{T}_{11}$; then T does not contain both a hole of length ≥ 6 and an antihole of length ≥ 6.*

Proof. Let C be a hole and D an antihole, both of length ≥ 6. Let $W = V(C) \cap V(D)$, $A = V(C) \setminus W$, and $B = V(D) \setminus W$. Let W, A, B have cardinality w, a, b respectively. Let there be

p_e strong edges between A and W,
q_e strong edges between B and W,
r_e strong edges between A and B,
t_e strong edges with both ends in W,

p_s switchable pairs between A and W,
q_s switchable pairs between B and W,
r_s switchable pairs between A and B,
t_s switchable pairs with both ends in W,

p_n strong non-edges between A and W,
q_n strong non-edges between B and W,
r_n strong non-edges between A and B, and
t_n strong non-edges with both ends in W.

By 5.3, and since $T \in \mathcal{T}_{11}$, every vertex in B has at most $\frac{1}{2}(a+w)$ weak neighbors in C, and every vertex in A has at most $\frac{1}{2}(b+w)$ weak non-neighbors in D, so

$$q_e + q_s + r_e + r_s + p_s + p_n + r_n + r_s \leq \frac{1}{2}(a+w)b + \frac{1}{2}(b+w)a.$$

Also, every vertex in W has at most two strong neighbors in $A \cup W$ and at most two strong non-neighbors in $B \cup W$, so

$$p_e + 2t_e + q_n + 2t_n \leq 2w.$$

Also, since by 7.7 $w \leq 2$,

$$2t_s \leq w \leq 2w.$$

Summing, we obtain

$$p_e + p_s + p_n + q_e + q_s + q_n + r_e + 2r_s + r_n + 2t_e + 2t_n + 2t_s \leq ab + \frac{1}{2}bw + \frac{1}{2}aw + 4w.$$

But

$$p_e + p_s + p_n + q_e + q_s + q_n + r_e + r_s + r_n + 2t_e + 2t_s + 2t_n = ab + aw + bw + w(w-1),$$

so

$$\frac{1}{2}aw + \frac{1}{2}bw + w(w-1) \leq 4w,$$

that is,

$$w(a + b + 2w - 10) \leq 0.$$

Since $a + w, b + w \geq 6$, it follows that $w = 0$, and so C, D are disjoint. Moreover, equality holds throughout this calculation, so every vertex in D is weakly adjacent to exactly half the vertices of C and weakly non-adjacent to exactly half of the vertices of C and vice versa. Consequently every vertex in D is strongly adjacent to exactly half the vertices of C and strongly non-adjacent to exactly half of the vertices of C and vice versa.

By 5.3, and since $T \in \mathcal{T}_{11}$, it follows that for each $v \in D$, its strong neighbors in C are pairwise non-adjacent in C. Let C have vertices $c_1, \ldots, c_m$ in order, and let D have vertices $d_1, \ldots, d_n$. So for every vertex of D, its set of strong neighbors in $V(C)$ is either the set of all c_i with i even, or the set of all c_i with i odd, and the same with C, D exchanged. We may assume that c_1 is strongly adjacent to d_1. Hence the strong edges between $\{c_1, c_2, c_4, c_5\}$ and $\{d_1, d_2, d_4, d_5\}$ are $c_1d_1, c_1d_5, c_2d_2, c_2d_4, c_4d_2, c_4d_4, c_5d_1, c_5d_5$; and so the subtrigraph $T|\{c_1, c_2, c_4, c_5, d_1, d_2, d_4, d_5\}$ is the double diamond, contrary to $T \in \mathcal{T}_{11}$. This proves 7.8. ∎

8 The end

The objective of the remainder of the paper is to prove the following:

8.1 *Let $T \in \mathcal{T}_{12}$; then either T or $\overline{T}$ is bipartite, or T admits a balanced skew-partition.*

8.2 *Let $T \in \mathcal{T}_{12}$, admitting no balanced skew-partition. Let X, Y be disjoint weakly anticonnected subsets of $V(T)$, weakly complete to each other, and let $p_1\text{-}\cdots\text{-}p_n$ be a path P of $T \setminus (X \cup Y)$, with $n \geq 2$, such that p_1 is weakly X-complete and none of $p_2, \ldots, p_n$ is strongly X-complete; and p_n is weakly Y-complete and none of $p_1, \ldots, p_{n-1}$ is strongly Y-complete. Then there is no $z \in V(T) \setminus (X \cup Y \cup \{p_1, \ldots, p_n\})$, weakly complete to $X \cup Y$, and weakly anticomplete to p_1, p_n.*

Proof. Suppose such z exists. Since T is monogamous, we may assume that z is strongly non-adjacent to p_n. Choose X maximal subject to being weakly anticonnected, weakly complete to $Y \cup \{p_1, z\}$ and such that none of $p_2, \ldots, p_n$ is strongly X-complete.

(1) *Y is strongly X-complete.*

Suppose Y is not strongly X-complete. We claim that in this case $n > 2$. The set $X \cup Y$ is now weakly anticonnected, and so if $n = 2$ there exists an antipath of length > 2 from p_1 to p_2 with interior in $X \cup Y$. This antipath can be completed through $p_1\text{-}z\text{-}p_2$ to an antihole of length > 4, contrary to the fact that $T \in \mathcal{T}_{12}$. This proves that $n > 2$.

Let G_1 be a realization of T defined as follows:

- assign the value "edge" to all switchable pairs xy such that $x \in X$ and $y \in Y$
- assign the value "edge" to all switchable pairs xp_1 such that $x \in X$
- assign the value "edge" to all switchable pairs yp_n such that $y \in Y$
- assign the value "edge" to all switchable pairs vz such that $v \in X \cup Y$
- assign the value "non-edge" to all remaining switchable pairs containing a vertex of $X \cup Y$
- assign the value "non-edge" to all switchable pairs vz with $v \in V(P)$
- P is a path in G_1.

Then $G_1 \in \mathcal{F}_{11}$, and p_1 and p_n are respectively the unique (G_1, X)-complete and (G_1, Y)-complete vertices of the path P. It follows from theorems 2.6 and 2.9 of [2] applied to the sets X, Y, the path P and the vertex z in G_1 that n is even.

Now let G_2 be a realization of T obtained from G_1 by changing to "non-edge" the value of all switchable pairs xy with $x \in X$ and $y \in Y$. So $G_2 \in \mathcal{F}_{11}$, and again p_1 and p_n are respectively the unique (G_2, X)-complete and (G_2, Y)-complete vertices of the path P. This contradicts theorem 17.5 of [2] applied to the sets X, Y, the path P and the vertex z in G_2. This proves (1).

(2) *There exists a path Q in T from z to p_1, with nonempty interior, so that none of its internal vertices is in X or is strongly X-complete.*

Let U be the set of strong common neighbors of X and let $W = V(T) \setminus \{X \cup U\}$. Then by 5.7 W is weakly connected and if $|U| > 1$ then every vertex of U has a weak neighbor in W. Since by (1) $Y \subseteq U$ and p_1, z both belong to $U \cup W$, if they are strongly non-adjacent the claim follows. If p_1, z is a switchable pair of T, then since T is monogamous both p_1 and z belong to U, and again the claim follows. This proves (2).

Since no vertex of $P \setminus p_1$ is strongly X-complete, we may choose Q as in (2) so that if z has a weak neighbor in $\{p_2, \dots, p_{n-1}\}$ then $V(Q) \subseteq \{z, p_1, \dots, p_{n-1}\}$ and so that there exists a realization of T in which both P and Q are paths. Let G be a realization of T in which P and Q are paths and otherwise defined as follows:

- assign the value "edge" to all switchable pairs xp_1 such that $x \in X$
- assign the value "edge" to all switchable pairs yp_n such that $y \in Y$
- assign the value "edge" to all switchable pairs vz such that $v \in X \cup Y$
- assign the value "non-edge" to all remaining switchable pairs containing a vertex of $X \cup Y$
- assign values to all remaining switchable pairs arbitrarily.

Then $G \in \mathcal{F}_{11}$. In G the connected subset $V(Q \setminus z) \cup \{p_1, \dots, p_n\}$ ($= F$ say) contains a (G, X)-complete vertex, a (G, Y)-complete vertex, and a $(G, \{z\})$-complete vertex. The only (G, X)-complete vertex in F is p_1, and that is not (G, Y)-complete or $(G, \{z\})$-complete; so by theorem 24.4 of [2] some vertex in F is (G, Y)-complete and adjacent to z in G. If z has a neighbor in $\{p_1, \dots, p_n\}$, then $V(Q) \subseteq \{z, p_1, \dots, p_n\}$, and so p_n is the only vertex of F that is (G, Y)-complete; and it is not adjacent to z, a contradiction. So z has no neighbor in $\{p_1, \dots, p_n\}$, and therefore only one vertex in F is adjacent to z, the neighbor of z in Q, say q, and q is (G, Y)-complete. In T it means that q is strongly Y-complete. Hence q is strongly non-adjacent to p_1, for otherwise we could add q to X, contrary to the maximality of X. Consequently Q has length > 2. This contradicts theorem 24.3 of [2] applied to Q,X and any vertex $y \in Y$ in G. This proves 8.2. ∎

We can now prove the following:

8.3 *Let $T \in \mathcal{T}_{12}$, admitting no balanced skew-partition, and let C be a hole. If $z \in V(T) \setminus V(C)$ has two weak neighbors in C that are consecutive in C, then C has length 4 and z two strong neighbors in C that are not consecutive. In particular, T contains no antipath of length 4.*

Proof. Let C be the hole with vertices $p_1, \dots, p_{n+2}$ in order, and assume some $z \in V(T) \setminus V(C)$ is weakly adjacent to p_{n+1}, p_{n+2}. By 8.2, taking $X = \{p_{n+1}\}$ and $Y = \{p_{n+2}\}$, we deduce that z is strongly adjacent to at least one of p_1, p_n, say p_1. Since $T \in \mathcal{T}_{12}$ it follows that C has length 4. Since $zp_1 \in E(T)$, we may assume that zp_3 is a switchable pair, for otherwise the theorem holds. Since T is monogamous, zp_4 is a strong edge of T. Now applying 8.2 with $X = \{p_1\}$ and $Y = \{p_4\}$ we deduce that z is strongly adjacent to p_2 and the claim follows. This proves 8.3. ∎

8.4 *Let $T \in \mathcal{T}_{12}$, admitting no balanced skew-partition. Let X_1, X_2, X_3 be pairwise disjoint, nonempty, weakly anticonnected subsets of $V(T)$, strongly complete to each other. Let $F \subseteq V(T) \setminus (X_1 \cup X_2 \cup X_3)$*

be weakly connected, so that for at least two values of $i \in \{1,2,3\}$, every member of X_i has a weak neighbor in F. Let G be a realization of T in which X_1, X_2, X_3 are anticonnected and F is connected, and for at least two values of $i \in \{1,2,3\}$, every member of X_i has a neighbor in F. Then in G the set F contains a vertex complete to at least two of X_1, X_2, X_3.

Proof. This proof is identical to the proof of theorem 24.7 in [2], except 8.3 is used instead of theorem 24.6 of [2]. This proves 8.4. ∎

8.5 *Let $T \in \mathcal{T}_{12}$ and assume T contains a strong triangle. Then either $\overline{T}$ is bipartite or T admits a balanced skew-partition.*

Proof. Suppose not. T contains a strong triangle, and so we may choose disjoint nonempty weakly anticonnected sets $X_1, \ldots, X_k$, strongly complete to each other, with $k \geq 3$. Choose these with maximal union. Let $F = V(T) \setminus \bigcup_{i=1}^{k} X_i$.

(1) *No vertex of F is strongly complete to two of $X_1, X_2, \ldots, X_k$.*

Suppose $w \in F$ is strongly complete to two of $X_1, X_2, \ldots, X_k$. We may assume that w is strongly complete to $X_1, \ldots, X_i$ say where $2 \leq i \leq k$, and not strongly complete to $X_{i+1}, \ldots, X_k$. Define

$$X'_{i+1} = X_{i+1} \cup \ldots \cup X_k \cup \{w\};$$

then the sets $X_1, \ldots, X_i, X'_{i+1}$ violate the optimality of the choice of $X_1, \ldots, X_k$. This proves (1).

Let G be a realization of T defined as follows:

- assign the value "non-edge" to all switchable pairs of T with both ends in X_i for $1 \leq i \leq k$
- assign the value "edge" to all switchable pairs of T with both ends in F
- for all $x \in X_1 \cup X_2 \cup \ldots \cup X_k$ such that x has a strong neighbor in F, assign the value "non-edge" to all switchable pairs xf with $f \in F$
- if $x \in X_1 \cup X_2 \cup \ldots \cup X_k$ is weakly anticomplete to F, assign the value "edge" to the unique switchable pair between x and F
- assign values to all remaining switchable pairs arbitrarily.

(2) *Either $\overline{T}$ is bipartite or some vertex of F is complete in G to two of $X_1, \ldots, X_k$.*

Let N be the set of all strongly X_k-complete vertices in T. If $X_k \cup N = V(T)$, then by 5.7 $\overline{T}$ is bipartite and the statement holds. So we may assume that $X_k \cup N \neq V(T)$. By 5.7, the set $V(T) \setminus (X_k \cup N)$ is weakly connected and every vertex of N has a weak neighbor in it. It follows that F is weakly connected and all vertices of $X_1 \cup X_2$ have a weak neighbor in it. In G it means that F is connected and every vertex of $X_1 \cup X_2$ has a neighbor in F. By 8.4 some vertex $v \in F$ is complete in G to two of X_1, X_2, X_k. This proves (2).

By (2) there exists a vertex $v \in F$ that is complete in G to $X_1 \cup X_2$, say. It follows from (1) that in T, v is weakly and not strongly complete to $X_1 \cup X_2$. Since T is monogamous, v has a weak non-neighbor in at most one of X_1, X_2, and we may assume that v is strongly X_2-complete and has a weak non-neighbor $x \in X_1$. By the definition of G that means that x has no strong neighbor in F. Let N' be the set of all strongly X_2-complete vertices. As before we may assume that $X_2 \cup N' \neq V(T)$. On the other hand $X_2 \cup N' \setminus \{x\}$ is not weakly anticonnected (since X_2 is an anticomponent of it) and $V(T) \setminus (X_2 \cup N' \setminus \{x\})$ not weakly connected (for it is a subset of $\{x\} \cup F \setminus \{v\}$ containing x and a vertex of $F \setminus \{v\}$), so by 5.7 T admits a balanced skew-partition. This proves 8.5. ∎

8.5 completes the proof of the analogue of 8.1 in [2], for a Berge graph containing no strong triangle is bipartite. In the trigraph case, however, another step is required:

8.6 *Let $T \in \mathcal{T}_{13}$; then either T or $\overline{T}$ is bipartite, or T admits a balanced skew-partition.*

Proof. We may assume that T admits no balanced skew-partition and is not bipartite, so T contains a weak triangle $\{x_1, x_2, x_3\}$, and by 8.5 and since T is monogamous, we may assume that $x_1x_2 \in S(T)$ and $x_1x_3, x_2x_3 \in E(T)$. Let X_3 the set of all vertices of that are weakly complete to $\{x_1, x_2\}$.

(1) *X_3 is weakly anticonnected and strongly $\{x_1, x_2\}$-complete.*

Since T is monogamous, every vertex in X_3 is strongly adjacent to both x_1 and x_2, and since $T \in \mathcal{T}_{13}$, no two vertices of T are strongly adjacent. This proves (1).

(2) *Every path from x_1 to x_2 with nonempty interior in $V(T) \setminus X_3$ contains a strong common neighbor of X_3 in its interior.*

Let P be such a path and assume P^* contains no strongly X_3-complete vertex. From the definition of X_3, P does not have length 2. Hence by 2.1 P is odd. By 5.2 P has length 3. Let the vertices of P be x_1-p_1-p_2-x_2 in order. Let Q be an antipath joining p_1 and p_2 with interior in X_3. Then p_1-Q-p_2-x_1-x_2-p_1 is an antihole in T contrary to the fact that $T \in \mathcal{T}_{13}$. This proves (2).

Let N be the set of all strongly X_3-complete vertices. If $X_3 \cup N = V(T)$, then by 5.7 $\overline{T}$ is bipartite and the theorem holds. Now 5.7 implies that there is a path in T from x_1 to x_2 with nonempty interior in $V(T) \setminus (X_3 \cup N)$, contrary to (2). This proves 8.6. ∎

Now 8.1 follows from 8.5 and 8.6.

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