The structure of bull-free graphs I — three-edge-paths with centers and anticenters

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May 6, 2006; revised October 6, 2009

Abstract

The *bull* is a graph consisting of a triangle and two disjoint pendant edges. A graphs is called *bull-free* if no induced subgraph of it is a bull. This is the first paper in a series of three. The goal of the series is to explicitly describe the structure of all bull-free graphs. In this paper we study the structure of bull-free graphs that contain as induced subgraphs three-edge-paths P and Q, and vertices $c \notin V(P)$ and $q \notin V(Q)$, such that c is adjacent to every vertex of $V(P)$ and a has no neighbor in $V(Q)$. One of the theorems proved in an earlier version of this paper is used in [5] in order to prove that every bull-free graph on n vertices contains either a clique or a stable set of size $n^{\frac{1}{4}}$, thus settling the Erdös-Hajnal conjecture [8] for the bull.

1 Introduction

All graphs in this paper are finite and simple. The *bull* is a graph with vertex set $\{x_1, x_2, x_3, y, z\}$ and edge set

$$\{x_1x_2, x_2x_3, x_1x_3, x_1y, x_2z\}.$$

Let G be a graph. We say that G is *bull-free* if no induced subgraph of G is isomorphic to the bull. The complement of G is the graph $\overline{G}$, on the same vertex set as G, and such that two vertices are adjacent in G if and only if they are non-adjacent in $\overline{G}$. A *clique* in G is a set of vertices, all pairwise adjacent. A *stable set* in G is a clique in $\overline{G}$. A clique of size three is called a *triangle* and a stable set of size three is a *triad*.

*Most of this research was conducted during the period the author served as a Clay Mathematics Institute Research Fellow. Partially supported by NSF grant DMS-0758364.

Bull-free graphs received quite a bit of attention in the past, mostly in connection with perfect graphs (a graph is called *perfect* if for all its induced subgraphs the chromatic number equals the size of the maximum clique). In [7], Chvátal and Sbihi studied the structure of bull-free graphs with no induced odd cycles of length at least five or their complements, and proved that Berge's Strong Perfect Graph Conjecture [2] holds for bull-free graphs. Later Reed and Sbihi [12] gave a polynomial time algorithm for recognizing perfect bull-free graphs. Both these results were obtained many years before the proof of the Strong Perfect Graph Conjecture [4] and the recognition algorithm for perfect graphs [3]. Today, one of the main open question in the theory of perfect graphs is that of finding a polynomial time combinatorial coloring algorithm. Bull-free perfect graphs is one a few classes of perfect graphs for which there has been progress on this question [9, 10, 11]

This a the first paper in a series of three. In this series we depart from *perfect* bull-free graphs and study the structure of general bull-free graphs. The main result of the series is an explicit description the structure of all bull-free graphs. In general, describing the structure of a graph with a certain induced subgraph excluded is an interesting, but apparently very difficult question. In [6] Seymour and the author were able to describe the structure of all graphs that do not contain a claw ($K_{1,3}$) as an induced subgraph, but for most graphs the question is wide open. However, the result of this series, as well as the theorem in [6] and some others suggest that excluding a certain graph as induced subgraph may have a global structural impact. Another reason for our interest in studying the structure of graphs with certain induced subgraphs excluded is the following conjecture of Erdös and Hajnal [8]:

1.1 *For every graph H, there exists $\delta(H) > 0$, such that if G is a graph and no induced subgraph of G isomorphic to H, then G contains either a clique or a stable set of size $|V(G)|^{\delta(H)}$.*

This conjecture is also concerned with the global effect that excluding an induced subgraph has on a graph: a graph with an excluded induced subgraph is conjectured to be very different from a random graph, where the expected size of a largest clique and a largest stable set is logarithmic in the number of vertices [1].

Let G be a graph. For a subset A of $V(G)$ and a vertex $b \in V(G) \setminus A$, we say that b is *complete* to A if b is adjacent to every vertex of A, and that b is *anticomplete* to A if b is not adjacent to any vertex of A. For two disjoint subsets A and B of $V(G)$, A is *complete* to B if every vertex of A is complete to B, and A is *anticomplete* to B every vertex of A is anticomplete to B. For a subset X of $V(G)$, we denote by $G|X$ the subgraph induced by G on X, and by $G \setminus X$ the subgraph induced by G on $V(G) \setminus X$.

In this paper we study bull-free graphs that contain as induced subgraphs three-edge-paths P and Q, and vertices $c \notin V(P)$ and $q \notin V(Q)$, such that c is complete to $V(P)$, and a is anticomplete to $V(Q)$. We prove that every such graph either belongs a certain basic class, or admits a decomposition. This information is used in later papers of the series.

A *hole* in a graph is an induced cycle of length at least four. A *homogeneous set* in a graph G is a proper subset X of $V(G)$ such that every vertex of $V(G) \setminus X$ is either complete or anticomplete to X. We say that a graph G admits a *homogeneous set decomposition* if there is a homogeneous set X in G with $1 < |X| < V(G)$. In an earlier version of this paper we proved the following result that allowed the author, jointly with Safra, to prove 1.1 for the case when H is a bull:

1.2 *Let G be a bull-free graph and assume that G contains a hole H of length at least five, and vertices $c, a \in V(G) \setminus V(H)$ such that c is complete to $V(H)$ and a is anticomplete to $V(H)$. Then G admits a homogeneous set decomposition.*

[5] contains the proof of 1.2, and so we omit it from this paper.

This paper is organized as follows. In Section 2 we define an object called "trigraph", which is a generalization of a graph, and is more convenient for stating the main result of this series of papers. We define "bull-free trigraphs", and prove two easy lemmas about their properties. Section 3 contains further definitions needed to state the main theorem of this paper, as well as the main theorem itself (3.2.) The proof of 3.2 occupies Sections 4–5. The proof consists of a few steps. At each step we assume that a bull-free graph (in fact, trigraph) G contains a certain graph F as an induced subgraph, and then, analyzing how the rest of G attaches to F, we prove that one of the outcomes of 3.2 holds. More precisely, the steps are: 4.1, 5.2, 5.7, 5.8, and, finally, 3.2.

2 Trigraphs

In order to prove our main result, we consider objects, slightly more general than bull-free graphs, that we call "bull-free trigraphs". A *trigraph* G consists of a finite set $V(G)$, called the *vertex set* of G, and a map $\theta : V(G)^2 \to \{-1, 0, 1\}$, called the *adjacency function*, satisfying:

- for all $v \in V(G)$, $\theta_G(v, v) = 0$
- for all distinct $u, v \in V(G)$, $\theta_G(u, v) = \theta_G(v, u)$
- for all distinct $u, v, w \in V(G)$, at most one of $\theta_G(u, v), \theta_G(u, w) = 0$.

Two distinct vertices of G are said to be *strongly adjacent* if $\theta(u, v) = 1$, *strongly antiadjacent* if $\theta(u, v) = -1$, and *semi-adjacent* if $\theta(u, v) = 0$. We say that u and v are *adjacent* if they are either strongly adjacent, or semi-adjacent; and *antiadjacent* of they are either strongly antiadjacent, or semi-adjacent. If u and v are adjacent (antiadjacent), we also say that u is *adjacent (antiadjacent) to* v, or that u is a *neighbor (antineighbor)* of v. Similarly, if u and v are strongly adjacent (strongly antiadjacent), then u is a *strong neighbor (strong antineighbor)* of v. Let $E(G)$ be the set of all strongly adjacent pairs of G, $N(G)$ the set of all strongly antiadjacent pairs of G, and $S(G)$ the set of all pairs $\{u, v\}$ of vertices of G, such that u and v are distinct and semi-adjacent. Thus, a trigraph G is a graph if $S(G)$ empty.

Let G be a trigraph. The complement $\overline{G}$ of G is a trigraph with the same vertex set as G, and adjacency function $\overline{\theta} = -\theta$. For $v \in V(G)$ let $N(v)$ denote the set of all vertices in $V(G) \setminus \{v\}$ that are adjacent to v, and let $S(v)$ denote the set of all vertices in $V(G) \setminus \{v\}$ that are strongly adjacent to v. Let $A \subset V(G)$ and $b \in V(G) \setminus A$. We say that b is *strongly complete* to A if b is strongly adjacent to every vertex of A, b is *strongly anticomplete* to A if b is strongly antiadjacent to every vertex of A, b is *complete* to A if b is adjacent to every vertex of A, and b is *anticomplete* to A if b is antiadjacent to every vertex of A. For two disjoint subsets A, B of $V(G)$, B is *strongly complete (strongly anticomplete, complete, anticomplete)* to A if every vertex of B is strongly complete (strongly anticomplete, complete, anticomplete, respectively) to every vertex of A. We say that b is *mixed* on A if b is not strongly complete and not strongly anticomplete to A. A *clique* in G is a set of vertices all pairwise adjacent, and a *strong clique* is a set of vertices all pairwise strongly adjacent. A *stable set* is a set of vertices all pairwise antiadjacent, and a *strongly stable set* is a set of vertices all pairwise strongly antiadjacent. A (strong) clique of size three is a *(strong) triangle* and a (strong) stable set of size three is a *(strong) triad*. For $X \subset V(G)$ the trigraph *induced by* G *on* X (denoted by $G|X$) has vertex set X, and adjacency function that is the restriction of θ to X^2. Isomorphism between trigraphs is defined in the natural way, and for two trigraphs G and H we say that H is an *induced subtrigraph* of G (or G *contains* H *as an induced subtrigraph*) if H is isomorphic to $G|X$ for some $X \subseteq V(G)$. We denote by $G \setminus X$ the trigraph $G|(V(G) \setminus X)$.

A *bull* is a trigraph with vertex set $\{x_1, x_2, x_3, v_1, v_2\}$ such that $\{x_1, x_2, x_3\}$ is a triangle, v_1 is adjacent to x_1 and antiadjacent to x_2, x_3, v_2, and v_2 is adjacent to x_2 and antiadjacent to x_1, x_3. For a trigraph G, a subset X of $V(G)$ is said to be a *bull* if $G|X$ is a bull. We say that a trigraph is *bull-free* if no induced subtrigraph of it is a bull, or, equivalently, no subset of its vertex set is a bull.

Let G be a trigraph. An induced subtrigraph P of G with vertices $\{p_1, \ldots, p_k\}$ is a *path* in G if either $k = 1$, or for $i, j \in \{1, \ldots, k\}$, p_i is adjacent to p_j if $|i - j| = 1$ and p_i is antiadjacent to p_j if $|i - j| > 1$. Under these circumstances we say that P is a path *from* p_1 *to* p_k, its *interior* is the set $P^* = V(P) \setminus \{p_1, p_k\}$, and the *length* of P is $k - 1$. We also say that P is a $(k-1)$-edge-path. Sometimes, we denote P by p_1-...-p_k. An induced subtrigraph H of G with vertices $h_1, \dots, h_k$ is a *hole* if $k \geq 4$, and for $i, j \in \{1, \dots, k\}$, p_i is adjacent to p_j if $|i-j| = 1$ or $|i-j| = k-1$; and p_i is antiadjacent to p_j if $1 < |i-j| < k-1$. The *length* of a hole is the number of vertices in it. Sometimes we denote S by h_1-...-h_k-h_1. An *antipath* (*antihole*) is a path (hole) in $\overline{G}$.

Let G be a trigraph, and let $X \subseteq V(G)$. Let G_c be the graph with vertex set X, and such that two vertices of X are adjacent in G_c if and only if they are adjacent in G, and let G_a be be the graph with vertex set X, and such that two vertices of X are adjacent in G_a if and only if they are strongly adjacent in G. We say that X (and $G|X$) is *connected* if the graph G_c is connected, and that X (and $G|X$) is *anticonnected* if $\overline{G_a}$ is connected. A *connected component* of X is a maximal connected subset of X, and an *anticonnected component* of X is a maximal anticonnected subset of X. For a trigraph G, if X is a component of $V(G)$, then $G|X$ is a component of G.

We finish this section by two easy observations.

2.1 *If G be a bull-free trigraph, then so is $\overline{G}$.*

Proof. 2.1 follows from the fact that the complement of a bull is also a bull. ∎

2.2 *Let G be a trigraph, let $X \subseteq V(G)$ and $v \in V(G) \setminus X$. Assume that $|X| > 1$ and v is mixed on X. Then there exist vertices $x_1, x_2 \in X$ such that v is adjacent to x_1 and antiadjacent to x_2. Moreover, if X is connected, then x_1 and x_2 can be chosen adjacent.*

Proof. If v has a strong neighbor in X, let X_1 be the set of strong neighbors of v is X; and if v is anticomplete to X, let X_1 be the set of vertices of X that are semi-adjacent to v. Since v is mixed on X, it follows that X_1 is non-empty. Let $X_2 = X \setminus X_1$. Since v is mixed on X, $|X| > 1$, and v is semi-adjacent to at most one vertex of $V(G) \setminus \{v\}$, it follows that, in both cases, $X_2 \neq \emptyset$. Now every choice of $x_1 \in X_1$ and $x_2 \in X_2$ satisfies the first assertion of the theorem. If X is connected, it follows that there exist $x_1 \in X_1$ and $x_2 \in X_2$ that are adjacent, and therefore the second assertion of the theorem holds. This proves 2.2. ∎

3 The main theorem

Let G be a trigraph and let $S \subseteq V(G)$. A *center* for S is a vertex of $V(G) \setminus S$ that is complete to S, and an *anticenter* for S is a vertex of $V(G) \setminus S$ that is anticomplete to S. A vertex of G is a *center (anticenter)* for an induced subgraph H of G if it is a center (anticenter) for $V(H)$. In this section we state our main result, which is that every bull-free trigraph, that contains both a three-edge-path with a center and a three-edge-path with an anticenter, either belongs to a certain basic class, or admits a decomposition. We start by describing our basic trigraphs.

The class $\mathcal{T}_0$. Let G be the trigraph with vertex set

$$\{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2\}$$

and adjacency as follows: $\{b_1, b_2, c_1, c_2\}$ is a strong clique; a_1 is strongly adjacent to b_1, b_2 and semi-adjacent to c_1; a_2 is strongly adjacent to c_1, c_2 and semi-adjacent to b_1; d_1 is strongly adjacent to a_1, a_2; d_2 is either strongly adjacent or semi-adjacent to d_1; and all the remaining pairs are strongly antiadjacent. Let X be a subset of $\{b_1, b_2, c_1, c_2\}$ such that $|X| \leq 1$. Then $G \setminus X \in \mathcal{T}_0$.

We observe the following:

3.1 *Every trigraph in* $\mathcal{T}_0$ *is bull-free.*

Proof. We use the notation from the definition of $\mathcal{T}_0$. Let $G \in \mathcal{T}_0$. We may assume that $X = \emptyset$. Suppose there is a bull B in G. Let $B = \{v_1, v_2, v_3, v_4, v_5\}$, where the pairs $v_1v_2, v_2v_3, v_2v_4, v_3v_4, v_4v_5$ are adjacent, and all the remaining pairs are antiadjacent. Since $G \setminus \{d_2\}$ has no triad, it follows that $d_2 \in B$. Since every vertex in B has a neighbor in B, it follows that $d_1 \in B$. Since d_2 is in no triangle in G, we deduce that $d_2 \in \{v_1, v_5\}$, and from the symmetry we may assume that $d_2 = v_1$. Then $d_1 = v_2$, contrary to the fact that d_1 is in no triangle in G. This proves 3.1. ∎

Next let us define some decompositions. A proper subset X of $V(G)$ is a *homogeneous set* in G if every vertex of $V(G) \setminus X$ is either strongly complete or strongly anticomplete to X. We say that G admits *a homogeneous set decomposition*, if there is a homogeneous set X in G with $1 < |X| < |V(G)|$.

For two disjoint subsets A and B of $V(G)$, the pair (A, B) is a *homogeneous pair* in G, if A is a homogeneous set in $G \setminus B$ and B is a homogeneous set in $G \setminus A$. We say that the pair (A, B) is *tame* if

- $|V(G)| - 2 > |A| + |B| > 2$, and
- A is not strongly complete and not strongly anticomplete to B.

A trigraph G admits a *homogeneous pair decomposition* if there is a tame homogeneous pair in G.

In this paper we need a special kind of a homogeneous pair. Let (A, B) be a homogeneous pair in G. Let C be the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to A and strongly anticomplete to B, D the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to B and strongly anticomplete to A, E the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to $A \cup B$, and F the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly anticomplete to $A \cup B$. We say that (A, B) is *a homogeneous pair of type zero in* G if

- $D = \emptyset$, and
- some member of C is antiadjacent to some member of E, and
- A is a strongly stable set, and
- $|C \cup E \cup F| > 2$, and
- $|B| = 2$, say $B = \{b_1, b_2\}$, and b_1 is strongly adjacent to b_2, and
- let $\{i, j\} = \{1, 2\}$. Let A_i be the set of vertices of A that are adjacent to b_i. Then $A_1 \cap A_2 = \emptyset$, $A_1 \cup A_2 = A$, $1 \leq |A_i| \leq 2$, and if $|A_i| = 2$, then one of the vertices of A_i is semi-adjacent to b_i, and
- if $|A_1| = |A_2| = 1$, then F is non-empty.

Please note that every homogeneous pair of type zero is tame in both G and $\overline{G}$, and therefore if there is a homogeneous pair of type zero in either G or $\overline{G}$, then G admits a homogeneous pair decomposition. The main result of this paper is the following:

3.2 *Let G be a bull-free trigraph. Let P and Q be paths of length three, and assume that there is a center for P and an anticenter for Q in G. Then either*

- *G admits a homogeneous set decomposition, or*
- *G admits a homogeneous pair decomposition, or*
- *G or $\overline{G}$ belongs to $\mathcal{T}_0$.*

For future use, we also need to consider a more restricted class of bull-free trigraphs. We say that a trigraph G is *elementary* if there does not exist a path P of length three in G, such that some vertex c of $V(G) \setminus V(P)$ is a center for P, and some vertex a of $V(G) \setminus V(P)$ is an anticenter for P. We prove the following decomposition theorem for bull-free trigraphs that are *not* elementary:

3.3 *Let G be a bull-free trigraph that is not elementary. Then either*

- *one of $G, \overline{G}$ belongs to $\mathcal{T}_0$, or*
- *one of $G, \overline{G}$ contains a homogeneous pair of type zero,or*
- *G admits a homogeneous set decomposition.*

4 Stars and leaves

Let G be a trigraph, and let h_1-h_2-h_3-h_4-h_5-h_1 be a hole in G, say H. For $i \in \{1, \dots, 5\}$ let L_i be the set of all vertices in $V(G) \setminus V(H)$ that are adjacent to h_i and anticomplete to $V(H) \setminus \{h_i\}$, let S_i be the set of all vertices in $V(G) \setminus V(H)$ that are complete to $V(G) \setminus \{h_i\}$, and antiadjacent to h_i, and let C_i be the set of vertices that are complete to $\{h_{i-1}, h_{i+1}\}$, and anticomplete to $\{h_{i-2}, h_{i+2}\}$ (here addition and subtraction are *mod* 5). We call a vertex of L_i a *leaf at* h_i, a vertex of S_i a *star at* h_i, and a vertex of C_i a *clone at* h_i. For $i, j \in \{1, \dots, 5\}$ we say that $x \in L_i \cup S_i$ and $y \in L_j \cup S_j$ are in *the same position (with respect to H)* if $i = j$ and in *different positions (with respect to H)* if $i \neq j$. We observe that since every vertex of G is semi-adjacent to at most one other vertex of G, $(\bigcup_{i=1}^5 L_i) \cap (\bigcup_{i=1}^5 S_j) = \emptyset$.

The goal of this section is to prove the following:

4.1 *Let G be a bull-free trigraph, and let H be a hole of length five in G. If there exist both a leaf and a star with respect to H in G, then either*

- *G admits a homogeneous set decomposition, or*
- *there is a homogeneous pair of type zero either in G or in $\overline{G}$, or*
- *G or $\overline{G}$ belongs to $\mathcal{T}_0$.*

We break the proof into two parts, 4.2 and 4.3 below.

4.2 *Let G be a bull-free trigraph and let H be a hole of length five in G. Then there do not exist a leaf and a star in different positions with respect to H.*

Proof. Suppose for a contradiction that there exist a star s and a leaf l, in different positions with respect to H. Since in $\overline{G}$, H is a hole of length five and s and l are a leaf and a star, respectively, in different positions with respect to H, by 2.1, passing to the complement if necessary, we may assume that s is non-adjacent to l. Let h_1-$\dots$-h_5-h_1 be the vertices of H. We may assume that $s \in S_1$, and, from the symmetry, $l \in L_2 \cup L_3$. But now, if $l \in L_2$, then $\{l, h_2, h_3, s, h_5\}$ is a bull; and if $l \in L_3$, then $\{h_1, h_2, s, h_3, l\}$ is a bull, in both cases a contradiction. This proves 4.2. ∎

4.3 *Let G be a bull-free trigraph, let H be a hole of length five in G, and let l be a leaf and s be a star, in the same position with respect to H. Then either*

- *G admits a homogeneous set decomposition, or*
- *there is a homogeneous pair of type zero either in G or in $\overline{G}$, or*
- *G or $\overline{G}$ belongs to $\mathcal{T}_0$.*

Proof. Let the vertices of H be h_1-$\dots$-h_5-h_1. We may assume that $l \in L_1$ and $s \in S_1$. We may assume that G does not admit a homogeneous set decomposition.

(1) l is strongly anticomplete to $\{h_2, h_5\}$ and s is strongly complete to $\{h_3, h_4\}$.

By 2.1, it is enough to prove that l is strongly anticomplete to $\{h_2, h_5\}$. Suppose not. From the symmetry we may assume that l is adjacent to h_2. But then $\{h_5, h_1, l, h_2, h_3\}$ is a bull, a contradiction. This proves (1).

For $i \in \{1, \dots, 5\}$, let C_i be the set of vertices of $V(G) \setminus V(H)$ that are clones at h_i.

(2) Let $x \in C_2$. Then $x \notin S_1$, x is strongly complete to S_1, and x is strongly anticomplete to $L_1 \setminus \{x\}$.

Since x is antiadjacent to h_4, (1) implies that $x \notin S_1$. Since $\{h_5, h_1, u, x, h_3\}$ is not a bull for any $u \in L_1 \setminus \{x\}$, it follows that x is strongly anticomplete to $L_1 \setminus \{x\}$.

Assume first that x is adjacent to h_2. In this case, again by (1), $x \notin L_1$. Suppose that x is antiadjacent to some $s_1 \in S_1$. Now, if s_1 is antiadjacent to l, then $\{l, h_1, x, h_2, s_1\}$ is a bull, and if s_1 is adjacent to l, then $\{l, s_1, h_4, h_3, x\}$ is a bull, in both cases a contradiction.

So we may assume that x is strongly antiadjacent to h_2. Now, since $\{x, h_3, h_2, v, h_5\}$ is not a bull for any $v \in S_1$, it follows that x is strongly complete to S_1, and This proves (2).

(3) If $C_2 \neq \emptyset$, then every vertex of $V(G) \setminus (C_2 \cup \{h_2, h_3\})$ is either strongly complete or strongly anticomplete to $C_2 \cup \{h_2\}$.

Let $c_2, c_2' \in C_2 \cup \{h_2\}$, and suppose that some $x \in V(G) \setminus (C_2 \cup \{h_2, h_3\})$ is adjacent to c_2' and antiadjacent to c_2. Let C be the hole $G|((V(H) \setminus \{h_2\}) \cup \{c_2\})$, and C' the hole $G|((V(H) \setminus \{h_2\}) \cup \{c_2'\})$. By (2), $c_2, c_2' \notin S_1$, S_1 is strongly complete to C_2, and therefore every vertex of S_1 is a star at h_1 with respect to both C and C'. Assume that $x \in S_1$. Then $c_2 = h_2$, and x is semi-adjacent to c_2. By (2) applied to the hole C', we deduce that l is not a leaf for C', and therefore, from (1), $l = c_2'$ and c_2' is semi-adjacent to h_3. But now $\{h_2, h_3, l, x, h_5\}$ is a bull, a contradiction. Therefore $x \notin S_1$. By 4.2, there is no leaf for C or C' in any position different from h_1.

Assume first that c_2 is adjacent to c_2'. Since $\{h_5, h_1, c_2, c_2', h_3\}$ and $\{h_1, c_2', c_2, h_3, h_4\}$ are not bulls, it follows that c_2 is strongly complete to $\{h_1, h_3\}$. Similarly, c_2' is strongly complete to $\{h_1, h_3\}$, and hence $x \notin \{h_1, h_3\}$. In particular, this implies that $c_2, c_2' \notin L_1$, and so, from (1), every vertex of L_1 is a leaf at h_1 for both C and C'. By (2) applied to C and C', it follows that L_1 is strongly anticomplete to $\{c_2, c_2'\}$, and so $x \notin L_1$.

Since every vertex of L_1 is a leaf at h_1 for both C and C', by 4.2, there is no star for C or C' in any other position. In particular, c_2' is not a star at h_4 or h_5 with respect to C, and so c_2' is strongly anticomplete to $\{h_4, h_5\}$. Similarly, c_2' is strongly anticomplete to $\{h_4, h_5\}$. This proves that $x \notin V(H)$.

Suppose x is antiadjacent to h_1. Since $\{x, c_2', c_2, h_1, h_5\}$ is not a bull, it follows that x is strongly adjacent to h_5. We claim that x is strongly adjacent to h_4. If x is antiadjacent to h_3, the claim follows since $\{x, c_2', c_2, h_3, h_4\}$ is not a bull; and if x is strongly adjacent to h_3, the claim follows since $\{h_1, c_2', x, h_3, h_4\}$ is not a bull. But now, since $\{h_3, h_4, x, h_5, h_1\}$ is not a bull, x is strongly adjacent to h_3, and therefore, with respect to C', x is a star at h_1, and c_2 is a clone at c_2', but x is antiadjacent to c_2, contrary to (2). This proves that x is strongly adjacent to h_1.

Next assume that x is antiadjacent to h_3. Since $\{h_5, h_1, x, c_2', h_3\}$ is not a bull, it follows that x is strongly adjacent to h_5. Since $\{c_2, h_1, x, h_5, h_4\}$ is not a bull, we deduce that x is strongly adjacent to h_4. But now x is a star at h_3 with respect to C', a contradiction. This proves that x is strongly adjacent to h_3. Now, if x is adjacent to both h_4 and h_5, then x is a star at c_2 with respect to C, if x is adjacent to h_4 and strongly antiadjacent to h_5, then x is a star at h_5 with respect to C', if x is adjacent to h_5 and strongly antiadjacent to h_4, then x is a star at h_4 with respect to C', and if x is strongly antiadjacent to both h_4 and h_5, then $x \in C_2 \cup \{h_2\}$, in all cases a contradiction. This proves that c_2 is strongly antiadjacent to c_2'.

Since $\{c_2', h_1, c_2, h_5, h_4\}$ and $\{c_2', h_3, c_2, h_4, h_5\}$ are not bulls, it follows that c_2 is strongly anticomplete to $\{h_4, h_5\}$, and similarly, c_2' is strongly anticomplete to $\{h_4, h_5\}$. Since there is no leaf at h_3 with respect to C or C', it follows that h_1 is strongly complete to $\{c_2, c_2'\}$. This proves that $x \notin V(H)$.

Suppose x is adjacent to h_1. Since $\{c_2, h_1, c_2', x, h_4\}$ is not a bull, x is strongly antiadjacent to h_4. Assume that x is antiadjacent to h_3. Since $\{h_3, c_2', x, h_1, h_5\}$ is not a bull, it follows that x is strongly adjacent to h_5. But now $\{c_2, h_1, x, h_5, h_4\}$ is a bull, a contradiction. So x is strongly adjacent to h_3. Since $\{c_2, h_3, c_2', x, h_5\}$ is not a bull, we deduce that x is strongly antiadjacent to h_5. But now $x \in C_2$, a contradiction. This proves that x is strongly antiadjacent to h_1. A similar argument shows that x is strongly antiadjacent to h_3. Now, if x is complete to $\{h_4, h_5\}$, then $\{h_3, h_4, x, h_5, h_1\}$ is a bull, if x is anticomplete to $\{h_4, h_5\}$ then x is a leaf at c_2' with respect to C', and if x is adjacent to one of h_4, h_5 and antiadjacent to the other, then x is a leaf at one of h_4, h_5 with respect to C, in all cases a contradiction. This proves (3).

(4) $C_2 \cup \{h_2\}$ is a strongly stable set, $|C_2| \leq 1$, and if $|C_2| = 1$, then h_3

is semi-adjacent to a member of $C_2 \cup \{h_2\}$.

Suppose $C_2 \cup \{h_2\}$ is not a strongly stable set, and let X be a component of $C_2 \cup \{h_2\}$ with $|X| > 1$. Since X is not a homogeneous set in G, it follows that some vertex $v \in V(G) \setminus X$ is mixed on X. By (3), $v = h_3$. By 2.2, there exist $x, x' \in X$ such that h_3 is adjacent to x and antiadjacent to x', and x is adjacent to x'. But now $\{h_3, x, x', h_1, h_5\}$ is a bull, a contradiction. This proves that $C_2 \cup \{h_2\}$ is a strongly stable set.

Let C' be the set of vertices of $C_2 \cup \{h_2\}$ that are strongly adjacent to h_3. By (3), and since $C_2 \cup \{h_2\}$ is strongly stable, it follows that C' is a homogeneous set in G, and so $|C'| \leq 1$. Since every vertex of $C_2 \cup \{h_2\}$ is adjacent to h_3, and since h_3 is semi-adjacent to at most one vertex of G, it follows that $|(C_2 \cup \{h_2\}) \setminus C'| \leq 1$, and therefore $|C_2| \leq 1$. Moreover, if $|C_2| = 1$, then, since $|C'| \leq 1$, it follows that $(C_2 \cup \{h_2\}) \setminus C' \neq \emptyset$, and therefore h_3 is semi-adjacent to a member of $C_2 \cup \{h_2\}$. This proves (4).

(5) Either $C_2 = \emptyset$*, or* $C_3 = \emptyset$*.*

Suppose $C_2 \neq \emptyset$. By (4), it follows that h_3 is semi-adjacent to a member of $C_2 \cup \{h_2\}$, say c_2. If $C_3 \neq \emptyset$, then c_2 is mixed on $C_3 \cup \{h_3\}$, contrary to (2) applied in $\overline{G}$. Therefore $C_3 = \emptyset$. This proves (5).

(6) Either $C_2 \cup C_5 = \emptyset$*, or* $C_3 \cup C_4 = \emptyset$*.*

Suppose $C_2 \cup C_5 \neq \emptyset$. We may assume that $C_2 \neq \emptyset$. Then, by (5), $C_3 = \emptyset$. If $C_4 \neq \emptyset$, then, applying (5) in $\overline{G}$, we deduce that $C_2 = \emptyset$, a contradiction. So $C_4 = \emptyset$, and (6) follows.

In view of (6), passing to $\overline{G}$ if necessary, we may assume that $C_3 = C_4 = \emptyset$.

(7) $(C_2 \cup C_5 \cup \{h_2, h_5\}, \{h_3, h_4\})$ *is a homogeneous pair in* G*.*

Since $\{l, h_1, h_5, h_4, h_3\}$ is not a bull, it follows that h_1 is strongly antiadjacent to h_4, and from the symmetry h_1 is strongly antiadjacent to h_3. By 2.1 this implies that h_1 is strongly complete to $\{h_2, h_5\}$. Now, by (3), it follows that h_1 is strongly complete to $C_2 \cup C_5 \cup \{h_2, h_5\}$.

Suppose (7) is false. Then there exists $x \in V(G) \setminus (C_2 \cup C_5 \cup \{h_2, h_3, h_4, h_5\}$ that is mixed on either $\{h_3, h_4\}$, or $C_2 \cup C_5 \cup \{h_2, h_5\}$. Suppose first that x is mixed on $\{h_3, h_4\}$. From the symmetry we may assume that x is adjacent to h_4 and antiadjacent to h_3. Since if $x \in L_1$, then $x \in C_5$, we deduce, using (1), that $x \notin L_1 \cup S_1 \cup V(H)$.

Suppose x is adjacent to h_5. Since $\{h_3, h_4, x, h_5, h_1\}$ is not a bull, we deduce that x is strongly adjacent to h_1. Since by (4) and symmetry $C_5 \cup \{h_5\}$ is a strongly stable set, it follows that $x \notin C_5$, and therefore x is strongly adjacent to h_2. But then $x \in S_3$, contrary to 4.2. This proves that x is strongly antiadjacent to h_5. By 4.2, x is strongly adjacent to at least one of h_1, h_2, and since $C_3 = \emptyset$ and $x \notin C_5$, it follows that x is strongly complete to $\{h_1, h_2\}$. But now $\{h_5, h_1, x, h_2, h_3\}$ is a bull, a contradiction. This proves that x is not mixed on $\{h_3, h_4\}$, and therefore x is mixed on $C_2 \cup C_5 \cup \{h_2, h_5\}$.

By (3) and since $h \notin \{h_3, h_4\}$, we may assume that x is strongly complete to $C_2 \cup \{h_2\}$, and strongly anticomplete to $C_5 \cup \{h_5\}$. Suppose first that x is strongly anticomplete to $\{h_3, h_4\}$. Then, by 4.2, x is strongly adjacent to h_1. But now $\{h_5, h_1, x, h_2, h_3\}$ is a bull, a contradiction. Since x is not mixed on $\{h_3, h_4\}$, it follows that x is strongly complete to $\{h_3, h_4\}$. Since $C_3 = \emptyset$, it follows that x is strongly adjacent to h_1. But now $x \in S_5$, contrary to 4.2. This proves (7).

Now let $A_1 = C_2 \cup \{h_2\}$, $A_2 = C_5 \cup \{h_5\}$, $b_1 = h_3$, $b_2 = h_4$, $A = A_1 \cup A_2$, and $B = \{b_1, b_2\}$. Let C be the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to A and strongly anticomplete to B, D the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to B and strongly anticomplete to A, E the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly complete to $A \cup B$, and F the set of vertices of $V(G) \setminus (A \cup B)$ that are strongly anticomplete to $A \cup B$. Since $\{h_2, h_3, d, h_4, h_5\}$ is a bull for every $d \in D$, it follows that $D = \emptyset$. Since A is a strongly stable set, it follows that $S_1 \cap (A \cup B) = \emptyset$, and so $s \in E$. Since $h_1 \in C$, it follows that some vertex of C is antiadjacent to some vertex of E. If $|C \cup E \cup F| > 2$, then G admits a homogeneous pair decomposition of type zero. So we may assume that $|C| = |E| = 1$. Thus $C = \{h_1\}$, and $E = \{s\}$, but now $\overline{G} \in \mathcal{T}_0$. This proves 4.3. ▮

Now 4.1 follows from 4.2 and 4.3.

5 Paths of length three

In this section we prove 3.2 which we restate:

5.1 *Let G be a bull-free trigraph. Let P and Q be paths of length three, and assume that there is a center for P and an anticenter for Q in G. Then either*

- *G admits a homogeneous set decomposition, or*
- *G admits a homogeneous pair decomposition, or*
- *G or $\overline{G}$ belongs to $\mathcal{T}_0$.*

We remind the reader that a trigraph G is *elementary* if there does not exist a path P of length three in G, such that some vertex c of G is a center for P, and some vertex a is an anticenter for P. First we prove 3.3, which we restate:

5.2 *Let G be a bull-free trigraph that is not elementary. Then either*

- *one of $G, \overline{G}$ belongs to $\mathcal{T}_0$, or*
- *one of $G, \overline{G}$ contains a homogeneous pair of type zero,or*
- *G admits a homogeneous set decomposition.*

Proof. By 4.1 we may assume that there is no hole of length five in G or $\overline{G}$ with both a leaf and a star. Let p_1-p_2-p_3-p_4 be a path in G, say P, and let c be a center and a an anticenter for P (such P, c, and a exist since G is not elementary).

(1) If a is adjacent to c, then c is a strong center for P.

Since $\{a, c, p_3, p_2, p_1\}$ is not a bull, it follows that c is strongly adjacent to p_1, and form the symmetry c is strongly adjacent to p_4; and since $\{a, c, p_1, p_2, p_3\}$ is not a bull, it follows that c is strongly adjacent to p_3, and from the symmetry to p_2. This proves (1).

(2) Let $x \in V(G) \setminus V(P)$. Then either

1. *there exist $u, v, w \in V(P)$ such that u-v-w is a path, x is adjacent to u and v and antiadjacent to w, or*
2. *there exist $u, v, w \in V(P)$ such that u-v-w is an antipath, x is adjacent to u and antiadjacent to v and w, or*
3. *x is strongly adjacent to p_1 and p_4, and strongly antiadjacent to p_2 and p_3, or*
4. *x is a strong center or a strong anticenter for $V(P)$.*

Suppose x fails to satisfy (2.1)—(2.4). Then x is not a strong center and not a strong anticenter for $V(P)$. Suppose x is antiadjacent to p_1. Then x is strongly antiadjacent to at least one of p_2, p_3 for otherwise (2.1) holds with $u = p_3$, $v = p_2$ and $w = p_1$. Suppose x is adjacent to p_3, and therefore strongly antiadjacent to p_2. Then x is strongly adjacent to p_4, for otherwise (2.2) holds with $u = p_3$, $v = p_1$ and $w = p_4$. But now (2.1) holds with $u = p_4$, $v = p_3$ and $w = p_2$. This proves that x is strongly antiadjacent to p_3. If x is adjacent to p_4 then (2.2) holds with $u = p_4$, $v = p_1$ and $w = p_3$, therefore x is strongly antiadjacent to p_4. Now, switching the roles of p_1 and p_4, we deduce that x is strongly antiadjacent to p_1 and p_2. But then x is a strong anticenter for $V(P)$, a contradiction. This proves that x is strongly adjacent to p_1, and, by the symmetry, to p_4. Now p is adjacent to at least one of p_2, p_3 for otherwise (2.3) holds, and antiadjacent to at least one of p_2, p_3, for otherwise x is a strong center for P. From the symmetry we may assume that x is adjacent to p_2 and antiadjacent to p_3. But now (2.1) holds with $u = p_1$, $v = p_2$ and $w = p_3$. This proves (2).

(3) Let $x \in V(G) \setminus (V(P) \cup \{a\})$, and assume that x is not a strong center and not a strong anticenter for $V(P)$. Assume also that c is adjacent to a. Then $x \neq c$ and x is strongly adjacent to c.

By (1), c is a strong center for P, and therefore $x \neq c$. Assume for a contradiction that x is antiadjacent to c. By (2), one of (2.1)—(2.3) holds for x. Suppose first that (2.1) holds, and let u, v, w be as in (2.1). Since $\{x, v, w, c, a\}$ is not a bull, it follows that x is strongly adjacent to a. But now $\{a, x, u, v, w\}$ is a bull, a contradiction. Next assume that (2.2) holds and let u, v, w be as in (2.2). Now $\{x, u, w, c, v\}$ is a bull, a contradiction. This proves that (2.4) holds, and so p_1-p_2-p_3-p_4-x-p_1 is a hole of length five in G, say H, and c is a star for H. Consequently, a is not a leaf for H, and so a is strongly antiadjacent to x. But now $\{x, p_1, p_2, c, a\}$ is a bull, a contradiction. This proves (3).

Let C be the set of all strong centers for $V(P)$, A the set of all strong anticenters for $V(P)$, and let $M = V(G) \setminus (A \cup C \cup V(P))$. Since if G admits a homogeneous set decomposition then so does $\overline{G}$, p_2-p_4-p_1-p_3 is a path in $\overline{G}$ with center a and anticenter c, and by 2.1, we may assume by (1), passing to the complement if necessary, that $C \neq \emptyset$.

(4) If A is empty then the theorem holds.

If a has an antineighbor in C, then, by (1) applied in $\overline{G}$ it follows that $a \in A$, a contradiction. So a is strongly complete to C, and therefore, by (3) applied to every vertex of C, we deduce that C is strongly complete to $M \setminus \{a\}$. But now $V(P) \cup M$ is a homogeneous set of size at least five in G, and $C \subseteq V(G) \setminus (V(P) \cup M)$. So G admits a homogeneous set decomposition. This proves (4).

In view of (4) we may assume that A is non-empty. This restores the symmetry between G and $\overline{G}$. We observe that either every vertex in C has a neighbor in A, or every vertex in A has an antineighbor in C. From this, passing to the complement if necessary, we may assume that every vertex of C has a neighbor in A.

(5) C is strongly complete to M.

Let $c' \in C$ and $m \in M$, and let a' be a neighbor of c' in A. Now (3) applied with $c = c'$, $a = a'$ and $x = m$ implies that c is strongly adjacent to m. Since c and m were chosen arbitrarily, (5) follows.

Let A' be the set of vertices a' in A such that for some $m \in M$, there exists a path from a' to m with interior in A.

(6) A' is strongly complete to C.

Let k be an integer, let $a_1, \ldots, a_k \in A'$ and $m \in M$ and let m-a_1-$\ldots$-a_k be a path. We prove by induction on k that a_k is strongly complete to C. By (5) C is strongly complete to M. Suppose first that $k = 1$. By (1) and since $m \in M$, one of the following three cases holds:

Case 1. There exist $u, v, w \in V(P)$ such that u-v-w is a path, m is adjacent to u and v and antiadjacent to w. In this case, since $\{a_1, m, u, c, w\}$ is not a bull for any $c \in C$, it follows that a_1 is strongly complete to C.

Case 2. There exist $u, v, w \in V(P)$ such that u-v-w is an antipath, m is adjacent to u and antiadjacent to v and w. In this case, since $\{a_1, m, u, c, v\}$ is not a a bull for any $c \in C$, it follows that a_1 is strongly complete to C.

Case 3. m is strongly adjacent to p_1, p_4 and strongly antiadjacent to p_2, p_3. In this case, since $\{a_1, m, p_1, c, p_3\}$ is not a a bull for any $c \in C$, it follows that a_1 is strongly complete to C.

So we may assume that $k > 1$, and $\{a_1, \ldots, a_{k-1}\}$ is strongly complete to C. Let $a_0 = m$. Then a_{k-2} is defined, there exists $p \in V(P)$, antiadjacent to a_{k-2}, and $V(P)$ is anticomplete to $\{a_{k-1}, a_k\}$. But now, since $\{p, c, a_{k-2}, a_{k-1}, a_k\}$ is not a bull for any $c \in C$, it follows that C is strongly complete to a_k. This proves (6).

By the definition of A', every vertex of $A \setminus A'$ is strongly anticomplete to $V(P) \cup M \cup A'$, and by (5) and (6), C is strongly complete to $V(P) \cup M \cup A'$. Since $C \neq \emptyset$, we deduce that $V(P) \cup M \cup A' \neq V(G)$. But now $V(P) \cup M \cup A'$ is a homogeneous set of size at least four in G, and therefore G admits a homogeneous set decomposition. This proves 5.2. ▮

We can now strengthen 4.1:

5.3 *Let G be an elementary bull-free trigraph and let H be a hole of length five in G. If there is a leaf l for H, and some vertex c of $V(G) \setminus V(H)$ has at least three neighbors in $V(H)$, then G admits a homogeneous set decomposition.*

Proof. Suppose G does not admit a homogeneous set decomposition.

(1) Let H' be a hole of length five in G with a leaf l'. Then no vertex of $V(G) \setminus V(H')$ has four neighbors in $V(H')$.

Suppose some vertex c' of $V(G) \setminus V(H')$ has at least four neighbors in $V(H')$. Let the vertices of H' be h_1-h_2-h_3-h_4-h_5-h_1. Since every vertex of G is semi-adjacent to at most one other vertex of G, $l' \neq c'$. Let $i \in \{1, \ldots, 5\}$ be such that l' is a leaf at h_i. Since G is elementary, it follows that c' has a strong antineighbor in $V(H') \setminus \{h_i\}$. But now we get a contradiction to 4.2. This proves (1).

Let the vertices of H be h_1-h_2-h_3-h_4-h_5-h_1. Since every vertex of G is semi-adjacent to at most one other vertex of G, $l \neq c$. Let $i \in \{1, \ldots, 5\}$ be such that l is a leaf at h_i. By (1), c has exactly three neighbors in $V(H)$, and therefore for some $j \in \{1, \ldots, 5\}$, c is adjacent to h_j and to h_{j+1} (where $h_6 = h_1$). Since $\{h_{j-1}, h_j, c, h_{j+1}, h_{j+2}\}$ is not a bull, it follows that c is strongly adjacent to at least one of h_{j-1}, h_{j+2} (here we add subscripts *mod 5*). So we may assume that c is adjacent to h_5, h_1, h_2 and strongly antiadjacent to h_3 and h_4. Let X be the set of all vertices of G that are complete to $\{h_2, h_5\}$ and strongly anticomplete to $\{h_3, h_4\}$. Then $h_1, c \in X$. Let C be the component of X such that $h_1, c \in C$ (such a component exists since c is adjacent to h_1.)

(2) l is strongly complete or strongly anticomplete to C.

Suppose not. Since $|C| > 1$ and C is connected, by 2.2, we can choose distinct vertices $c_1, c_2 \in C$, such that l is adjacent to c_1 and antiadjacent to c_2, and c_1 is adjacent to c_2. Since l is a leaf for H, we may assume from the symmetry that l is antiadjacent to h_2, h_3. But now $\{l, c_1, c_2, h_2, h_3\}$ is a bull, a contradiction. This proves (2).

Since $1 < |C| < |V(G)|$, it follows that C is not a homogeneous set in G, and so there exists a vertex $x \in V(G) \setminus C$ that is mixed on C. Then $x \neq h_3, h_4$. Since $|C| > 1$ and C is connected, by 2.2, we can choose distinct vertices $c_1, c_2 \in C$, such that x is adjacent to c_1 and antiadjacent to c_2, and c_1 is adjacent to c_2.

Since $\{h_2, c_1, c_2, h_5, h_4\}$ is not a bull, it follows that $x \neq h_2$, and, from the symmetry, $x \neq h_5$. Let H' be the hole c_1-h_2-h_3-h_4-h_5-c_1. It follows from (2) that l is a leaf for H', and therefore, by (1), x does not have four neighbors in $V(H')$.

Suppose that x is antiadjacent to h_5. Since $\{x, c_1, c_2, h_5, h_4\}$ is not a bull, it follows that x is strongly adjacent to h_4. If x is antiadjacent to h_2, then, since $\{x, c_1, c_2, h_2, h_3\}$ is not a bull, it follows that x is strongly adjacent to h_3, and so $\{h_2, h_3, x, h_4, h_5\}$ is a bull, a contradiction. This proves that x is strongly adjacent to h_2. Since $\{h_5, c_1, x, h_2, h_3\}$ is not a bull, we deduce that x is strongly adjacent to h_3. But now x has four neighbors in $V(H')$, a contradiction. This proves that x is strongly adjacent to h_5, and from the symmetry to h_2. Since x does not have four neighbors in $V(H')$, it follows that x is strongly anticomplete to $\{h_3, h_4\}$. Consequently $x \in C$, which is a contradiction. This proves 5.3. ∎

Let us now prove two easy but useful lemmas:

5.4 *Let G be an elementary bull-free trigraph, let P be a path of length three with vertices p_1-p_2-p_3-p_4, let c be a strong center for $V(P)$, and let $q \in V(G) \setminus (V(P) \cup \{c\})$ be antiadjacent to c. Then one of the following holds:*

1. *q is strongly adjacent to p_1, p_2 and antiadjacent to p_3, p_4, or q is strongly adjacent to p_3, p_4 and antiadjacent to p_1, p_2, or*
2. *q is strongly adjacent to p_1, p_2, p_3 and antiadjacent to p_4, or q is strongly adjacent to p_2, p_3, p_4 and antiadjacent to p_1, or*
3. *q is strongly adjacent to p_1, p_4 and antiadjacent to p_2, p_3, or*
4. *q is strongly adjacent to p_1, p_2, p_4 and antiadjacent to p_3, or q is strongly adjacent to p_1, p_3, p_4 and antiadjacent to p_2, or*
5. *q is a strong center for P.*

In particular, q has at least two strong neighbors in $V(P)$.

Proof. Since G is elementary, it follows that q is not an anticenter for P, and therefore q has at least one strong neighbor in $V(P)$. From the symmetry, we may assume that q is strongly adjacent to one of p_1, p_2. Now, since $G|(\{p_1, p_2, p_4, q, c\})$ is not a bull, it follows that q is strongly adjacent to at least two of the vertices p_1, p_2, p_4. Assume first that q has exactly two strong neighbors in $V(P)$. If q is strongly adjacent to p_1, p_2, then (5.4.1) holds, if q is strongly adjacent to p_1, p_4, then (5.4.3) holds, and if q is strongly adjacent to p_2, p_4, then $\{q, p_4, p_3, c, p_1\}$ is a bull. This proves that q does not have exactly two strong neighbors in $V(P)$.

If q has four strong neighbors in $V(P)$, then (5.4.5) holds, and so we may assume that q has exactly three strong neighbors in $V(P)$. It follows that q is strongly adjacent to at least one of p_3, p_4, and so the symmetry of the path has been restored. From the symmetry we may assume that p_1 is a strong neighbor of q. Now the set of strong neighbors of q in $V(P)$ is either $\{p_1, p_2, p_3\}$ and (5.4.2) holds, or $\{p_1, p_2, p_4\}$ and (5.4.4) holds, or $\{p_1, p_3, p_4\}$ and again (5.4.4) holds, This proves 5.4. ∎

5.5 *Let G be an elementary bull-free trigraph, let P be a path of length three with vertices p_1-p_2-p_3-p_4, let c be a strong center for $V(P)$. Let $q \in V(G) \setminus (V(P) \cup \{c\})$, and suppose that there does not exist $i \in \{1, \ldots, 4\}$, such that $G|((V(P) \setminus \{p_i\}) \cup \{q\})$ is a path of length three. Then q is a strong center for P.*

Suppose that q is not a strong center for P. Since G is elementary, it follows that q is not an anticenter for P, and therefore q has a strong neighbor in $V(P)$. Assume that q is strongly adjacent to p_2. If q is antiadjacent to p_4, then, since $\{p_1, p_2, q, p_3, p_4\}$ is not a bull in G, it follows that one of $G|((V(P) \setminus \{p_1\}) \cup \{q\})$ and $G|((V(P) \setminus \{p_2\}) \cup \{q\})$ is a path of length three, a contradiction. So q is strongly adjacent to p_4. Since $G|((V(P) \setminus \{p_3\}) \cup \{q\})$ is not a a path of length three, it follows that q is strongly adjacent to p_1. Since q is not a strong center for P, we deduce that q is antiadjacent to p_3. But now $G|((V(P) \setminus \{p_2\}) \cup \{q\})$ is a path of length three, a contradiction. This proves that q is antiadjacent to p_2, and from the symmetry to p_3. Since q has a strong neighbor in $V(P)$, we may assume form the symmetry that q is strongly adjacent to p_1. But now $G|((V(P) \setminus \{p_4\}) \cup \{q\})$ is a path of length three, a contradiction. This proves 5.5. ∎

The following is another result about paths of length three with centers and anticenters, that is a step to proving 5.1.

5.6 *Let G be a bull-free trigraph, and let P and Q be paths of length three in G. If some vertex of G is a center for P and an anticenter for Q then G is not elementary.*

Proof. Let P be the path p_1-p_2-p_3-p_4 and Q the path q_1-q_2-q_3-q_4. Suppose some vertex $c \in V(G) \setminus (V(P) \cup V(Q))$ is a center for P and an anticenter for Q, and G is elementary.

(1) Every vertex of $V(Q) \setminus V(P)$ has at least two strong neighbors in $V(P)$.

Let $q \in V(Q) \setminus V(P)$. Then q is antiadjacent to c, and (1) follows from 5.4.

By 2.1, applying (1) in $\overline{G}$, we deduce that every vertex of $V(P) \setminus V(Q)$ has at least two strong antineighbors in $V(Q)$.

(2) $|V(P) \cap V(Q)| = 1$.

Since every vertex is semi-adjacent to at most one other vertex of G, and c is complete to $V(P)$ and anticomplete to $V(Q)$, it follows that $|V(P) \cap V(Q)| \leq 1$. Suppose $V(P) \cap V(Q) = \emptyset$. Let $A = \{\{p, q\} : p \in V(P)$ and $q \in V(Q)\}$. By (1) and the remark following (1), since $|A| = 16$, it follows that $|A \cap E(G)| = 8$, $|A \cap N(G)| = 8$ and $A \cap S(G) = \emptyset$. Thus every vertex of P has two strong neighbors and two strong antineighbors in $V(Q)$, and every vertex of Q has two strong neighbors and two strong antineighbors in $V(P)$.

For $q \in V(Q)$ let $P(q)$ denote the set of strong neighbors of q in $V(P)$. Since $\{p_1, p_2, q, p_3, p_4\}$ is not a bull for any $q \in V(Q)$, it follows that $P(q) \neq \{p_2, p_3\}$ for all $q \in V(Q)$. Since $\{q, p_1, p_2, c, p_4\}$ is not a bull for any $q \in V(Q)$, it follows that $P(q) \neq \{p_1, p_3\}$, and from the symmetry, $P(q) \neq \{p_2, p_4\}$, for all $q \in V(Q)$.

So for every $q \in V(Q)$, either $P(q) = \{p_1, p_2\}$, or $P(q) = \{p_1, p_4\}$, or $P(q) = \{p_3, p_4\}$. If for some $q \in V(Q)$, $P(q) = \{p_1, p_4\}$, then, since every vertex of $V(P)$ has two strong antineighbors in $V(Q)$, it follows that at least one vertex of $V(Q)$ is strongly anticomplete to $\{p_1, p_4\}$, a contradiction. Consequently, for every $q \in V(Q)$, either $P(q) = \{p_1, p_2\}$, or $P(q) = \{p_3, p_4\}$.

Let $Q_1 = \{q \in V(Q) : P(q) = \{p_1, p_2\}\}$ and $Q_2 = \{q \in V(Q) : P(q) = \{p_3, p_4\}\}$. Since every vertex of $V(P)$ has two strong antineighbors in $V(Q)$, it follows that $|Q_1| = |Q_2| = 2$. But now, since for $q_1 \in Q_1$ and and $q_2 \in Q_2$, $\{q_1, p_2, c, p_3, q_2\}$ is not a bull, we deduce that Q_1 is strongly complete to Q_2, contrary to the fact that Q is a path. This proves (1).

Let $V(P) \cap V(Q) = \{p\}$. Then c is semi-adjacent to p. From the symmetry we may assume that $p \in \{p_1, p_2\} \cap \{q_1, q_2\}$.

(2) $p \notin \{p_1\} \cap \{q_1\}$ and $p \notin \{p_2\} \cap \{q_2\}$.

Passing to the complement by 2.1, it is enough to prove that $p \notin \{p_1\} \cap \{q_1\}$. Suppose $p = p_1 = q_1$. If q_2 is adjacent to p_2, then, since $\{q_3, q_2, p, p_2, c\}$ is not a bull, we deduce that q_3 is strongly adjacent to p_2, and, consequently, p_2 has at most one strong antineighbor in $V(Q)$, a contradiction. So q_2 is strongly antiadjacent to p_2. Since $\{q_2, p, p_2, c, p_4\}$ is not a bull, it follows that q_2 is strongly adjacent to p_4. By (1), q_3 has two strong neighbors in $V(P)$. If q_3 is antiadjacent to p_4, then q_3 is strongly complete to $\{p_2, p_3\}$, and $\{p, p_2, q_3, p_3, p_4\}$ is a bull, a contradiction. So q_3 is strongly adjacent to p_4. Since $\{p, q_2, p_4, q_3, q_4\}$ is not a bull, it follows that p_4 is strongly adjacent to q_4. But now p_4 has at most one strong antineighbor if $V(Q)$, a contradiction. This proves (2).

(3) $p \notin \{p_1\} \cap \{q_2\}$.

Let $\{x, y\} = \{q_1, q_3\}$. Since $\{x, p, p_2, c, p_4\}$ and $\{y, p, p_2, c, p_4\}$ are not bulls, it follows that each of x, y is strongly adjacent to at least one of p_2, p_4. Since p_2 has at least two strong antineighbors in $V(Q)$, we may assume from the symmetry that x is strongly antiadjacent to p_2 and, therefore, x is strongly adjacent to p_4. Since $\{x, p, y, p_2, c\}$ is not a bull, it follows that y is strongly antiadjacent to p_2, and, therefore, y is strongly adjacent to p_4. By (1), q_4 has two strong neighbors in $V(P)$, and since $\{p, p_2, q_4, p_3, p_4\}$ is not a bull, it follows that q_4 is strongly adjacent to p_4. But now p_4 has at most one strong antineighbor in $V(Q)$, a contradiction. This proves (3).

(4) $p \notin \{p_2\} \cap \{q_1\}$.

If q_2 is adjacent to p_1, then, since $\{q_3, q_2, p, p_1, c\}$ is not a bull, it follows that p_1 is strongly adjacent to q_3, and so p_1 has at most one strong antineighbor in $V(Q)$, a contradiction. So q_2 is strongly antiadjacent to p_1. Since $\{q_2, p, p_1, c, p_4\}$ is not a bull, it follows that q_2 is strongly adjacent to p_4. Since p_4 has two strong antineighbors in $V(Q)$, it follows that p_4 has a strong antineighbor x in $\{q_3, q_4\}$. By (1), x is strongly adjacent to p_1. But now, $\{x, p_1, p, c, p_4\}$ is a bull, a contradiction. This proves (4).

Now 5.6 follows from (2), (3) and (4). ∎

A *tray* is a trigraph with vertex set

$$\{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2\},$$

and such that the following pairs of vertices are adjacent:

$$a_1a_2, a_1b_1, a_1b_2, a_1c_1, a_1c_2, a_2b_2, a_2c_1, a_2c_2, b_1b_2, b_1d_1, b_1d_2, b_2d_1, b_2d_2, c_2d_2$$

and all the remaining pairs are antiadjacent. Let A and B be disjoint subsets of $V(G)$. We say that the pair (A, B) is *triangle connected* if for every partition (A_1, A_2) of A with both A_1 and A_2 non-empty, there exist vertices $a_1 \in A_1$, $a_2 \in A_2$ and $b \in B$, such that a_1, a_2, b is a triangle, and not both a_1 and a_2 are strongly complete to B; and the same with A and B exchanged. In particular, if T is a tray, then the pair $(\{a_1, a_2\}, \{b_1, b_2\})$ is triangle connected. We use the notion of being triangle connected to prove the following:

5.7 *Let G be an elementary bull-free trigraph, and assume that some induced subtrigraph of G is a tray. Then G admits either a homogeneous set decomposition, or a homogeneous pair decomposition.*

Proof. Let $X = \{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2\}$ be a subset of $V(G)$ such that $G|X$ is a tray, such that the following pairs of vertices are adjacent:

$$a_1a_2, a_1b_1, a_1b_2, a_1c_1, a_1c_2, a_2b_2, a_2c_1, a_2c_2, b_1b_2, b_1d_1, b_1d_2, b_2d_1, b_2d_2, c_2d_2$$

and all the remaining pairs are antiadjacent. Suppose G does not admit a homogeneous set decomposition or a homogeneous pair decomposition. Let A, B be two subsets of $V(G)$ such that

1. $a_1, a_2 \in A$ and $b_1, b_2 \in B$,
2. the pair (A, B) is triangle connected
3. A is complete to $\{c_1, c_2\}$ and anticomplete to $\{d_1, d_2\}$,
4. B is complete to $\{d_1, d_2\}$ and anticomplete to $\{c_1, c_2\}$,
5. $A \cup B$ is maximal subject to (1)-(4).

(1) A is strongly complete to $\{c_1, c_2\}$ and strongly anticomplete to $\{d_1, d_2\}$; and B is strongly complete to $\{d_1, d_2\}$ and strongly anticomplete to $\{c_1, c_2\}$.

Suppose $c \in \{c_1, c_2\}$ has an antineighbor $a' \in A$. Let $d \in \{d_1, d_2\}$ be an antineighbor of c. Let $A' = \{a'\}$ and $A'' = A \setminus A'$. Since (A, B) is triangle connected, there exist $a'' \in A''$ and $b \in B$, such that $\{a', a'', b\}$ is a triangle. But then $\{c, a'', a', b, d\}$ is a bull, a contradiction. This proves that A is strongly complete to $\{c_1, c_2\}$, and form the symmetry, B is strongly complete to $\{d_1, d_2\}$.

Next suppose that $d \in \{d_1, d_2\}$ has a neighbor $a \in A$. Let $d' = \{d_1, d_2\} \setminus \{d\}$. Since (A, B) is triangle connected, a has a neighbor $b \in B$. But now $\{c_1, a, d, b, d'\}$ is a bull, a contradiction. This proves that A is strongly anticomplete to $\{d_1, d_2\}$, and form the symmetry, B is strongly anticomplete to $\{c_1, c_2\}$. This proves (1).

(2) Let $x \in V(G) \setminus (A \cup B \cup \{c_1, c_2, d_1, d_2\})$. If x has a neighbor in A, then x is not complete to $\{c_1, c_2, d_2\}$; and if x has a neighbor in B, then x is not complete to $\{c_2, d_1, d_2\}$.

Form the symmetry it is enough to prove the first assertion of (2). Suppose x has a neighbor $a \in A$. We observe that c_1-a-c_2-d_2 is a path of length three and d_1 is an anticenter for it. Now, since G is elementary, it follows that that x is not a center for $\{c_1, a, c_2, d_2\}$, and therefore x is not complete to $\{c_1, c_2, d_2\}$. This proves (2).

(3) Let $a \in A$, $b \in B$ and $x \in V(G) \setminus (A \cup B)$ be a triangle. Then either

- *x is complete to $\{c_1, c_2\}$ and anticomplete to $\{d_1, d_2\}$, or*
- *x is complete to $\{d_1, d_2\}$ and anticomplete to $\{c_1, c_2\}$.*

Suppose x is antiadjacent to c_1. Since $\{c_1, a, x, b, d_1\}$ is not a bull and $\{c_1, a, x, b, d_2\}$ is not a bull, it follows that x is strongly adjacent to d_1 and d_2, and therefore, by (2) x is strongly antiadjacent to c_2, and the second outcome of (3) holds.

So we may assume that x is strongly adjacent to c_1, and from the symmetry to d_1. From (2) and the symmetry we may assume that x is antiadjacent to d_2. But now $\{d_2, b, d_1, x, c_1\}$ is a bull, a contradiction. This proves (3).

Since if (A, B) is a homogeneous pair in G, then it is a tame homogeneous pair, and since G does not admits a homogeneous pair decomposition, we may assume from the symmetry that there exists a vertex $x \in V(G) \setminus (A \cup B)$, such that x has a neighbor and an antineighbor in A. Since (A, B) is triangle connected, there exist vertices $a, a' \in A$ and $b \in B$, such that $\{a, a', b\}$ is a triangle, x is adjacent to a and antiadjacent to a', and $\{a, a'\}$ is not strongly complete to B.

(4) x is strongly complete to $\{d_1, d_2\}$ and strongly anticomplete to $\{c_1, c_2, b\}$.

Suppose x is adjacent to b. Then one of the two outcomes of (3) holds. Assume first that x is complete to $\{d_1, d_2\}$, and anticomplete to $\{c_1, c_2\}$. We claim that the pair $(A, B \cup \{x\})$ is triangle connected. Since (A, B) is triangle connected, all we need to check is that the condition is satisfied for the partition $(B, \{x\})$ of $B \cup \{x\}$. But $\{b, x, a\}$ is a triangle, and x has an antineighbor a' in A, and the claim follows. Consequently, the pair $(A, B \cup \{x\})$ contradicts the choice of (A, B). This proves that the other outcome of (3) holds for x, and x is complete to $\{c_1, c_2\}$, and anticomplete to $\{d_1, d_2\}$. Suppose that $\{a, x\}$ is not strongly complete to B. In this case, since $\{a, x, b\}$ is a triangle, it follows that the pair $(A \cup \{x\}, B)$ is triangle connected, contrary to the choice of (A, B). Consequently, $\{a, x\}$ is strongly complete to B, and therefore a' has an antineighbor $b' \in B$. But now $\{a', a, x, b', d_1\}$ is a bull, a contradiction. This proves that x is antiadjacent to b.

Now, since $\{x, a, a', b, d\}$ where $d \in \{d_1, d_2\}$ is not a bull, it follows that x is strongly complete to $\{d_1, d_2\}$. By (2), x has a strong antineighbor $c \in \{c_1, c_2\}$. If x also has a neighbor $c' \in C$, then $\{c, a, c', x, d_1\}$ is a bull, a contradiction. So x is strongly anticomplete to $\{c_1, c_2\}$. This proves (4).

Let A_1 be the set of neighbors of x in A, and $A_2 = A \setminus A_1$, and let B_1, B_2 be defined similarly.

(5) $B_1 = \emptyset$.

Suppose $B_1 \neq \emptyset$. By (4) $B_2 \neq \emptyset$, and so the symmetry between A and B has been restored. But now (4) applied with the roles of A and B reversed implies that x is strongly complete to $\{c_1, c_2\}$ and strongly anticomplete to $\{d_1, d_2\}$, a contradiction. This proves (5).

To complete the proof let B' be the set of strong neighbors of a' in B. Since $\{x, a, c_1, a', b'\}$ is not a bull for any $b' \in B'$, it follows from (5) that a is strongly complete to B'. Consequently, since $\{a, a'\}$ is not strongly complete to B, it follows that $B' \neq B$. Since (A, B) is triangle connected, $|B| > 1$, and a' has a neighbor and an antineighbor in B, it follows that there exist vertices $u, v \in B$, such that u is adjacent to v, and a' is adjacent to u and antiadjacent to v. But now (5) implies that $\{a', u, v, d_1, x\}$ is a bull, a contradiction. This proves 5.7. ▮

5.7 allows us to prove the following, which is the last step before proving 5.1:

5.8 *Let G be an elementary bull-free trigraph and let P and Q be paths of length three in G, such that there exist a center c for P, and an anticenter a*

for Q. If $c \in V(Q)$, or $a \in V(P)$, then either G admits a homogeneous set decomposition or a homogeneous pair decomposition, or G contains a tray.

Proof. Suppose no induced subtrigraph of G is a tray, and G does not admit a homogeneous set decomposition or a homogeneous pair decomposition. Passing to the complement by 2.1, it is enough to show that $c \notin V(Q)$. Assume for a contradiction that $c \in V(Q)$. We may assume that P and Q are chosen with $V(P) \cup V(Q)$ minimal, subject to the condition that $V(Q)$ contains a center for P. Let the vertices of P be p_1-p_2-p_3-p_4, and the vertices of Q be q_1-q_2-q_3-q_4. From the symmetry we may assume that $c \in \{q_1, q_2\}$.

(1) $\{q_4, a\} \cap V(P) = \emptyset$.

Suppose not. Let $\{q_4, a\} = \{x, y\}$. Since $c \in \{q_1, q_2\}$, it follows that every vertex of $\{x, y\} \cap V(P)$ is semi-adjacent to c. Since c is semi-adjacent to at most one vertex of $V(G)$, we may assume that $x \in V(P)$, $y \notin V(P)$ and x is semi-adjacent to c. From the symmetry we may assume that $x = p_i$ and $i \in \{1, 2\}$. By 5.4 and since x is antiadjacent to y, and y is antiadjacent to c, it follows that y is strongly adjacent to p_{i+2}. Since $\{x, p_{i+1}, c, p_{i+2}, y\}$ is not a bull, it follows that y is strongly adjacent to p_{i+1}. Let $d = \{q_1, q_2\} \setminus \{c\}$. Since d is antiadjacent to both x and y, and p_{i+1} is adjacent to both x and y, it follows that $d \neq p_{i+1}$. Since $\{d, c, x, p_{i+1}, y\}$ is not a bull, it follows that d is strongly adjacent to p_{i+1}. Now p_{i+1} is strongly complete to $\{q_1, q_2\}$, and therefore $p_{i+1} \notin V(Q)$. But p_{i+1} is adjacent to a and has three neighbors in $V(Q)$, contrary to 5.4 applied in $\overline{G}$. This proves (1).

(2) If $c = q_1$, then at least one of q_2, q_3 is in $V(P)$.

Suppose $c = q_1$ and $\{q_2, q_3\} \cap V(P) = \emptyset$. By the minimality of $V(P) \cup V(Q)$ and since c is adjacent to q_2, it follows from 5.5 that q_2 is a strong center for P, and in particular, q_2 is strongly complete to $\{p_1, p_4\}$. By (1), $a \notin V(P)$. By 5.4, since a is antiadjacent to $c = q_1$, we may assume that a is adjacent to p_1. Now, since $\{a, p_1, q_1, q_2, q_3\}$ is not a bull, p_1 is strongly adjacent to q_3. But p_1 is adjacent to a and has three neighbors in $V(Q)$, contrary to 5.4 applied in $\overline{G}$. This proves (2).

(3) If $c = q_1$, then $q_3 \notin V(P)$.

Suppose $c = q_1$ and $q_3 \in V(P)$. From the symmetry, we may assume that $q_3 = p_i$ with $i \in \{1, 2\}$. By (1), $\{a, q_4\} \cap V(P) = \emptyset$. By 5.4 and since a is antiadjacent to q_3 and c, it follows that a is strongly adjacent to p_{i+2}. Since $\{q_3, p_{i+1}, q_1, p_{i+2}, a\}$ is not a bull, it follows that a is strongly adjacent to p_{i+1}. So $p_{i+1} \notin V(Q)$. Now p_{i+1} is complete to $\{a, q_1, q_3\}$, and therefore by 5.4 applied in $\overline{G}$, it follows that p_{i+1} is strongly antiadjacent to q_4. But now $\{q_4, q_3, q_1, p_{i+1}, a\}$ is a bull, a contradiction. This proves (3).

(4) $c \neq q_1$.

Suppose $c = q_1$. Then by (1),(2) and (3) $q_2 \in V(P)$, and $q_3, q_4, a \notin V(P)$. From the symmetry, we may assume that $q_2 \in \{p_1, p_2\}$.

Assume first that $q_2 = p_1$. By 5.4, $\{q_4, a\}$ is strongly complete to $\{p_3, p_4\}$. By 5.4 applied in $\overline{G}$, it follows that p_4 is strongly anticomplete to $\{q_2, q_3\}$. But now q_1-q_2-q_3-q_4-p_4-q_1 is a hole of length 5, say H; a is a leaf for H and p_3 is adjacent to q_1, q_4, p_4, contrary to 5.3. This proves that $q_2 \neq p_1$, and therefore $q_2 = p_2$.

Let $\{q_4, a\} = \{x, y\}$. By 5.4, since $\{x, y\}$ is anticomplete to q_2, it follows that $\{x, y\}$ is strongly complete to p_4. Suppose x is antiadjacent to p_3. Then by 5.4, x is strongly adjacent to p_1. Since $\{x, p_4, y, p_3, q_2\}$ is not a bull, it follows that y is strongly antiadjacent to p_3, and 5.4 implies that y is strongly adjacent to p_1. But now, in $\overline{G}|(V(P) \cup \{x\})$ is a hole, say H, q_1 is a leaf for H in $\overline{G}$, and y has at least three neighbors in $V(H)$ in $\overline{G}$, contrary to 5.3 applied in $\overline{G}$. This proves that x, and from the symmetry y, is strongly adjacent to p_3. But now p_3 is strongly adjacent to a and has three neighbors in $V(Q)$ (namely q_1, q_2, q_4), contrary to 5.4 applied in $\overline{G}$. This proves (4).

Now it follows from (4) that $c = q_2$.

(5) $\{q_1, q_3\} \cap V(P) \neq \emptyset$.

Suppose $\{q_1, q_3\} \cap V(P) = \emptyset$. By 5.5, it follows from the minimality of $V(P) \cap V(Q)$, the fact that q_2 is complete to $\{q_1, q_3\}$, and (4), that both q_1 and q_3 are centers for P. Therefore, $\{q_1, q_3\}$ is strongly complete to $\{p_1, p_4\}$. By 5.4 and since q_4 is antiadjacent to q_2, it follows that q_4 is adjacent to at least one of p_1, p_4, and from the symmetry we may assume to p_1. But now p_1 is a center for Q, contrary to the fact that G is elementary, since a is an anticenter for Q. This proves (5).

Let $\{q_1, q_3\} = \{x, y\}$.

(6) $\{q_1, q_3\} \subseteq V(P)$.

Suppose not. By (5), we may assume that $x \in V(P)$ and $y \notin V(P)$. From the symmetry we may assume that $x \in \{p_1, p_2\}$. Suppose first that $x = p_1$. If y is adjacent to both p_2 and p_4, then x-p_2-y-p_4 is a path of length three, and q_2 is a center for it, contrary to the minimality of $V(P) \cup V(Q)$. So y has at least one strong antineighbor in $\{p_2, p_4\}$.

Assume that y is adjacent to p_3. Since, by the minimality of $V(P) \cup V(Q)$ it follows that x-p_2-p_3-y is not a path, we deduce that y is strongly adjacent to p_2, and therefore y is strongly antiadjacent to p_4. But now $\{p_1, p_2, y, p_3, p_4\}$ is a bull, a contradiction. This proves that y is strongly antiadjacent to p_3.

Since a is antiadjacent to x, it follows from 5.4 that a is strongly adjacent to p_3 and p_4. Suppose a is antiadjacent to p_2. Since $\{a, p_3, p_2, q_2, y\}$ is not a bull, it follows that y is strongly adjacent to p_2, and therefore y is strongly antiadjacent to p_4. Since both x-p_2-p_3-p_4 and y-p_2-p_3-p_4 are paths, q_2 is a center for each of them, and q_4 is antiadjacent to q_2 and to one of x, y, it follows from 5.4 that q_4 is strongly adjacent to p_3 and p_4. Since p_3 and p_4 are adjacent to a, it follow from 5.4 applied in $\overline{G}$ that x is strongly antiadjacent to p_3 and p_4. Now there is symmetry between x and y, and so we may assume that q_4 is adjacent to y and antiadjacent to x. Since $\{x, p_2, q_4, p_3, a\}$ is not a bull, it follows that q_4 is strongly antiadjacent to p_2. But now $G|(V(P) \cup V(Q) \cup \{a\})$ is a tray, a contradiction. This proves that a is strongly adjacent to p_2.

Since p_2 is complete to $\{a, x, q_2\}$, it follows from 5.4 applied in $\overline{G}$ that p_2 is strongly antiadjacent to both y and q_4. Since G is elementary, it follows that y is not an anticenter for $V(P)$, and therefore y is strongly adjacent to p_4. Since q_4 is antiadjacent to q_2 and p_2, 5.4 implies that q_4 is strongly adjacent to p_4. Since a is antiadjacent to q_2 and x, 5.4 implies that a is strongly adjacent to p_4. But now p_4 is adjacent to q_2, y, q_4 and a, contrary to 5.4 applied in $\overline{G}$. This proves that $x \neq p_1$.

Consequently, $x = p_2$. Suppose y is antiadjacent to p_1. It follows from the minimality of $V(P) \cup V(Q)$ that y is strongly antiadjacent to p_3. Since G is elementary, we deduce that y is not an anticenter for P, and so y is strongly adjacent to p_4. But now x-p_3-p_4-y is a path, and q is a center for it, contrary to the minimality of $V(P) \cup V(Q)$. This proves that y is strongly adjacent to p_1. Again by the minimality of $V(P) \cup V(Q)$, we deduce that y is adjacent to p_3. Since p_1 has at least three neighbors $V(Q)$ and $p_1 \notin V(Q)$, 5.4 applied in $\overline{G}$, implies that p_1 is strongly antiadjacent to a. Since a is antiadjacent to both p_1 and q_2, 5.4 implies that a is strongly adjacent to p_3. However, p_3 has three neighbors in $V(Q)$, contrary to 5.4 applied in $\overline{G}$. This proves (6).

(7) $x \neq p_1$.

Suppose $x = p_1$. By 5.4 and since a is antiadjacent to both x and q_2, it follows that a is strongly adjacent to both p_3 and p_4. This, together with (6), implies that $y = p_2$, and so p_1 is antiadjacent to p_2. But now $\{p_1, q_2, p_2, p_3, a\}$ is a bull, a contradiction. This proves (7).

Now it follows from (6), (7) and the symmetry that $\{x, y\} \cap \{p_1, p_4\} = \emptyset$, and therefore $\{x, y\} = \{p_2, p_3\}$. This implies that p_2 is antiadjacent to p_3. By 5.4 and since a is antiadjacent to q_2, p_2, p_3, it follows that a is strongly adjacent to p_1 and p_4. But now $\{a, p_1, p_2, q_2, p_3\}$ is a bull, a contradiction. This completes the proof of 5.8. ∎

We can now prove 5.1.

Proof of 5.1. Let the vertices of P be p_1-p_2-p_3-p_4, let c be a center for P and a an anticenter for Q. By 5.2, we may assume that G is elementary, and by 5.7 we may assume that there is no tray in G. By 5.6, it follows that $a \neq c$. By 2.1, passing to $\overline{G}$ if necessary, we may assume that c is adjacent to a. Therefore, 5.8 implies that there does not exist $i \in \{1, 2, 3, 4\}$ such that $G|((V(P) \setminus \{p_i\}) \cup \{a\})$ is a path of length three. Consequently, 5.5 implies that a is a strong center for P, contrary to 5.6, since a is also an anticenter for Q. This proves 5.1. ∎

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