Claw-free Graphs. VII. Quasi-line graphs

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Abstract

A graph is a *quasi-line graph* if for every vertex v, the set of neighbours of v is expressible as the union of two cliques. Such graphs are more general than line graphs, but less general than claw-free graphs. Here we give a construction for all quasi-line graphs; it turns out that there are basically two kinds of connected quasi-line graphs, one a generalization of line graphs, and the other a subclass of circular arc graphs.

1 Introduction

Let G be a graph. (All graphs in this paper are finite and simple.) If $X \subseteq V(G)$, the subgraph $G|X$ *induced on* X is the subgraph with vertex set X and edge-set all edges of G with both ends in X. ($V(G)$ and $E(G)$ denote the vertex- and edge-sets of G respectively.) We say that $X \subseteq V(G)$ is a *claw in* G if $|X| = 4$ and $G|X$ is isomorphic to the complete bipartite graph $K_{1,3}$. We say G is *claw-free* if no $X \subseteq V(G)$ is a claw in G.

In the earlier papers of this sequence, we gave a construction for all claw-free graphs; we proved that every claw-free graph can be built by piecing together building blocks from some explicitly-described classes. See [5] for a survey of this material.

A graph G is a *quasi-line graph* if for every vertex v, the set of neighbours of v can be partitioned into two sets A, B in such a way that A and B are both cliques. (Note that there may be edges between A and B.) Thus all line graphs are quasi-line graphs, and all quasi-line graphs are claw-free, but both converse statements are false. Quasi-line graphs make an interesting half-way stage between line graphs and claw-free graphs; for instance, a number of theorems about line graphs extend to quasi-line graphs and yet not to claw-free graphs in general.

The purpose of this paper is to give a construction for all quasi-line graphs in the same way as the previous papers of this sequence gave a construction for all claw-free graphs. For the most part, we just specialize the earlier theorem; we have to understand which graphs built from our earlier construction are quasi-line graphs. Mostly this is straightforward, but there is some difficulty when the stability number is small. For instance, all graphs with stability number two are claw-free, and such graphs were one of our "building block" types; but they are not all quasi-line, and it is non-trivial to figure out which such graphs are indeed quasi-line. A similar (but easier) situation arises with stability number three, as we shall see. Most of the work of this paper arises from trying to analyse the cases when stability number is at most three.

To state the main theorem we need a number of definitions. First, as in the earlier papers, we work with slightly more general objects than graphs, that we call "trigraphs". A *trigraph* G consists of a finite set $V(G)$ of vertices, and a map $\theta_G : V(G)^2 \to \{1, 0, -1\}$, satisfying:

- for all $v \in V(G)$, $\theta_G(v, v) = 0$
- for all distinct $u, v \in V(G)$, $\theta_G(u, v) = \theta_G(v, u)$
- for all distinct $u, v, w \in V(G)$, at most one of $\theta_G(u, v), \theta_G(u, w) = 0$.

For distinct u, v in $V(G)$, we say that u, v are *strongly adjacent* if $\theta_G(u, v) = 1$, *strongly antiadjacent* if $\theta_G(u, v) = -1$, and *semiadjacent* if $\theta_G(u, v) = 0$. We say that u, v are *adjacent* if they are either strongly adjacent or semiadjacent, and *antiadjacent* if they are either strongly antiadjacent or semiadjacent. Also, we say u is *adjacent* to v and u is a *neighbour* of v if u, v are adjacent (and a *strong neighbour* if u, v are strongly adjacent); u is *antiadjacent* to v and u is an *antineighbour* of v if u, v are antiadjacent (and a *strong antineighbour* if u, v are strongly antiadjacent).

For a vertex a and a set $B \subseteq V(G) \setminus \{a\}$, we say that a is *complete* to B or *B-complete* if a is adjacent to every vertex in B; and that a is *anticomplete* to B or *B-anticomplete* if a is antiadjacent to every vertex in B. For two disjoint subsets A and B of $V(G)$ we say that A is *complete*, respectively *anticomplete*, to B, if every vertex in A is complete, respectively anticomplete, to B. (We sometimes say A is B-complete, or the pair (A, B) is complete, meaning that A is complete to B.) Similarly, we say that a is *strongly complete* to B if a is strongly adjacent to every member of B, and so on. Let us say a trigraph G is *connected* if there is no partition (V_1, V_2) of $V(G)$ such that $V_1, V_2 \neq \emptyset$ and V_1 is strongly anticomplete to V_2. A *clique* in G is a subset $X \subseteq V(G)$ such that every two members of X are adjacent, and a *strong clique* is a subset such that every two of its members are strongly adjacent. A subset of $V(G)$ is *stable* if every two of its members are antiadjacent, and *strongly stable* if every two of its members are strongly antiadjacent. A trigraph G is *quasi-line* if for every vertex v, the set of neighbours of v is the union of two strong cliques. Our objective is to describe all quasi-line trigraphs.

We say a trigraph H is a *thickening* of a trigraph G if for every $v \in V(G)$ there is a nonempty subset $X_v \subseteq V(H)$, all pairwise disjoint and with union $V(H)$, satisfying the following:

- for each $v \in V(G)$, X_v is a strong clique of H
- if $u, v \in V(G)$ are strongly adjacent in G then X_u is strongly complete to X_v in H
- if $u, v \in V(G)$ are strongly antiadjacent in G then X_u is strongly anticomplete to X_v in H
- if $u, v \in V(G)$ are semiadjacent in G then X_u is neither strongly complete nor strongly anticomplete to X_v in H.

This thickening is *non-trivial* if $|V(H)| > |V(G)|$.

Let Σ be a circle, and let $F_1, \ldots, F_k \subseteq \Sigma$ be homeomorphic to the interval $[0, 1]$, such that no two of $F_1, \ldots, F_k$ share an end-point. Now let $V \subseteq \Sigma$ be finite, and let G be a trigraph with vertex set V in which, for distinct $u, v \in V$,

- if $u, v \in F_i$ for some i then u, v are adjacent, and if also at least one of u, v belongs to the interior of F_i then u, v are strongly adjacent
- if there is no i such that $u, v \in F_i$ then u, v are strongly antiadjacent.

Such a trigraph G is called a *circular interval trigraph*, and if in addition no three of $F_1, \ldots, F_k$ have union Σ, we say G is a *long circular interval trigraph*. It is easy to see that circular interval trigraphs are quasi-line.

The same construction, using a line rather than a circle, yields the "linear interval trigraphs". More precisely, we say G is a *linear interval trigraph* if its vertex set can be numbered $\{v_1, \ldots, v_n\}$ in such a way that for $1 \leq i < j < k \leq n$, if v_i, v_k are adjacent then v_j is strongly adjacent to both v_i, v_k. Given such a trigraph G and numbering $v_1, \ldots, v_n$ with $n \geq 2$, we call $(G, \{v_1, v_n\})$ a *linear interval stripe* if no vertex is semiadjacent to v_1 or to v_n, and v_1, v_n are strongly antiadjacent, and there is no vertex adjacent to both v_1, v_n.

A *spot* is a pair (G, Z) such that G has three vertices say v, z_1, z_2, and v is strongly adjacent to z_1, z_2, and z_1, z_2 are strongly antiadjacent, and $Z = \{z_1, z_2\}$.

Let G be a circular interval trigraph, and let $\Sigma, F_1, \ldots, F_k$ be as in the corresponding definition. Let $z \in V(G)$ belong to at most one of $F_1, \ldots, F_k$; and if $z \in F_i$ say, let no vertex be an endpoint of F_i. We call the pair $(G, \{z\})$ a *bubble*.

If H is a thickening of G, where X_v ($v \in V(G)$) are the corresponding subsets, and $Z \subseteq V(G)$ and $|X_v| = 1$ for each $v \in Z$, let Z' be the union of all X_v ($v \in Z$); we say that (H, Z') is a *thickening* of (G, Z).

Here is a construction; a trigraph G that can be constructed in this manner is called a *linear interval join.*

- Start with a trigraph H_0 that is a disjoint union of strong cliques. Let $X_1, \ldots, X_k \subseteq V(H_0)$ be pairwise disjoint strongly stable sets, each of cardinality one or two, and with union $V(H_0)$.
- For $1 \le i \le k$, let (G_i, Y_i) be either a spot, or a thickening of a bubble, or a thickening of a linear interval stripe, where $H_0, G_1, \ldots, G_k$ are pairwise vertex-disjoint, and such that $|X_i| = |Y_i|$ for $1 \le i \le k$; and for each i, take a bijection between X_i and Y_i.
- We define $H_1, \ldots, H_k$ recursively as follows. For $1 \le i \le k$, having defined H_{i-1}, let H_i be the trigraph obtained from the disjoint union of H_{i-1} and G_i by making the neighbour set of x in H_{i-1} strongly complete to the neighbour set of y in G_i, and then deleting x, y, for each $x \in X_i$ and its mate $y \in Y_i$. (The order of these operations does not affect the final outcome.)
- Let $G = H_k$.

Note that if each (G_i, Y_i) is a spot, then the trigraph we construct is a line graph of a multigraph. Now we can state our main theorem:

1.1 *Every connected quasi-line trigraph is either a linear interval join or a thickening of a circular interval trigraph.*

2 Quasi-line trigraphs with no triad

If G is a trigraph and $X \subseteq V(G)$, we define the trigraph $G|X$ *induced on* X as follows. Its vertex set is X, and its adjacency function is the restriction of θ_G to X^2. Isomorphism for trigraphs is defined in the natural way, and if G, H are trigraphs, we say that G *contains* H and H is an *subtrigraph* of G if there exists $X \subseteq V(G)$ such that H is isomorphic to $G|X$. Let us say an *anticycle* in a trigraph G is a subtrigraph C with vertex set $\{v_1, \ldots, v_k\}$, where $k \ge 3$, v_i, v_{i+1} are antiadjacent for $1 \le i < k$, and v_1, v_k are antiadjacent; we call k the *length* of the anticycle, and say the anticycle is *odd* if k is odd. A vertex v is a *centre* for an anticycle C if $v \notin V(C)$ and v is adjacent to every vertex of C. Thus, G is quasi-line if and only if no odd anticycle has a centre.

A *triad* in a trigraph G means a stable set with cardinality three. A *claw* in a trigraph G is a subset $\{a_0, a_1, a_2, a_3\} \subseteq V(G)$, such that $\{a_1, a_2, a_3\}$ is a triad and a_0 is complete to $\{a_1, a_2, a_3\}$. If no subset of $V(G)$ is a claw, we say that G is *claw-free*.

A 5-*wheel* is a trigraph with six vertices $v_1, \ldots, v_6$, where for $1 \le i < j \le 5$, if $j - i \in \{1, 4\}$ then v_i, v_j are adjacent, and if $j - i \in \{2, 3\}$ then v_i, v_j are antiadjacent, and v_6 is adjacent to all of $v_1, \ldots, v_5$. (For the reader's convenience, we follow the convention that when we list the vertices of a 5-wheel, we list them in the order just given.)

In [4] we showed that every claw-free trigraph can be built by piecing together trigraphs from some explicitly-described basic classes, and much of the proof of 1.1 consists of figuring out which trigraphs in these basic classes are quasi-line. One basic class was the class of all trigraphs with no triad; all such trigraphs are claw-free, but mostly they are not quasi-line, so we begin in this section by studying these.

Two strongly adjacent vertices of a trigraph G are called *twins* if (apart from each other) they have the same neighbours and the same antineighbours in G, and if there are two such vertices, we say “G admits twins”.

Let A, B be disjoint subsets of $V(G)$. The pair (A, B) is called a *homogeneous pair* in G if A, B are strong cliques, and for every vertex $v \in V(G) \setminus (A \cup B)$, v is either strongly A-complete or strongly A-anticomplete and either strongly B-complete or strongly B-anticomplete. Let (A, B) be a homogeneous pair, such that A is neither strongly complete nor strongly anticomplete to B, and at least one of A, B has at least two members. In these circumstances we call (A, B) a *W-join*.

We say a trigraph is *slim* if it does not admit twins or a W-join. Every trigraph G is a thickening of a slim trigraph H, and if G is quasi-line then so is H, so we may normally confine ourselves to slim trigraphs.

Let H be a graph, and let G be a trigraph with $V(G) = E(H)$. We say that G is a *line trigraph* of H if for all distinct $e, f \in E(H)$:

- if e, f have a common end in H then they are adjacent in G, and if they have a common end of degree at least three in H, then they are strongly adjacent in G
- if e, f have no common end in H then they are strongly antiadjacent in G.

We will show:

2.1 *Let G be a slim quasi-line trigraph with no triad. Then either G is a line trigraph of a subgraph of K_5, or G is a circular interval trigraph.*

We begin with:

2.2 *Let G be a slim quasi-line trigraph with no triad, and let $v_1, \ldots, v_8 \in V(G)$ be distinct, such that $\{v_1, v_2, v_5\}, \{v_2, v_3, v_6\}, \{v_3, v_4, v_7\}, \{v_4, v_1, v_8\}$ and $\{v_5, v_6, v_7, v_8\}$ are cliques, and every pair of vertices in $\{v_1, \ldots, v_8\}$ not contained in one of these five cliques is antiadjacent. Then G is a line trigraph of a subgraph of K_5.*

Proof. Since $\{v_1, v_2, v_7\}$ is not a triad, v_1, v_2 are strongly adjacent; since $\{v_1, v_3, v_5\}$ is not a triad, v_1, v_5 are strongly adjacent; since $\{v_4, v_5, v_6\}$ is not a triad, v_5, v_6 are strongly adjacent; and since $\{v_1, v_5, v_6, v_7, v_4, v_8\}$ does not induce a 5-wheel, v_5, v_7 are strongly adjacent. Since $\{v_2, v_3, v_4, v_8, v_5, v_1\}$ does not induce a 5-wheel, v_1, v_3 are strongly antiadjacent; and since $\{v_1, v_2, v_3, v_7, v_8, v_6\}$ does not induce a 5-wheel, v_1, v_6 are strongly antiadjacent. From the symmetry it follows that every pair of distinct members of $\{v_1, \ldots, v_8\}$ are either strongly adjacent or strongly antiadjacent. Consequently the subtrigraph induced on $\{v_1, \ldots, v_8\}$ is a line trigraph of a graph H with five vertices $h_1, \ldots, h_5$ and eight edges

$$h_1h_2, h_2h_3, h_3h_4, h_1h_4, h_2h_5, h_3h_5, h_4h_5, h_1h_5$$

(in order corresponding to $v_1, \ldots, v_8$). For $1 \leq i < j \leq 5$, let f_{ij} be the edge of H with ends h_i, h_j if it exists. (Thus we have renamed the vertices $v_1, \ldots, v_8$ in the f_{ij} notation, since this is more convenient.) For each $v \in V(G) \setminus E(H)$, we say that v is of *ij-type (with respect to H)*, where $1 \leq i < j \leq 5$, if for every edge $f_{i'j'}$ of H, v is strongly adjacent to $f_{i'j'}$ if and only if $\{i, j\} \cap \{i', j'\} \neq \emptyset$, and otherwise v is strongly antiadjacent to $f_{i'j'}$.

(1) *For every vertex $v \in V(G) \setminus E(H)$ there exist $i < j$ such that v is of ij-type.*

For let N be the set of neighbours of v in $E(H)$, and let M be the set of antineighbours of v

in $E(H)$. Since there is no triad, not both $f_{12}, f_{34} \in M$, and not both $f_{23}, f_{14} \in M$, so we may assume that $f_{12}, f_{23} \notin M$. Suppose that $f_{34} \in M$. Since $\{v, f_{34}, f_{25}\}$ is not a triad, $f_{25} \notin M$, and similarly $f_{15} \notin M$. Suppose in addition that $f_{35} \in N$. Since $\{v, f_{23}, f_{34}, f_{45}, f_{15}, f_{35}\}$ does not induce a 5-wheel, it follows that $f_{45} \notin M$. Since $\{f_{12}, f_{23}, f_{35}, f_{45}, f_{14}, v\}$ does not induce a 5-wheel, $f_{14} \notin N$; since $\{v, f_{14}, f_{35}\}$ is not a triad, $f_{35} \notin M$; and since $\{f_{12}, f_{23}, f_{34}, f_{45}, f_{15}, v\}$ does not induce a 5-wheel, $f_{34} \notin N$. But then v is of 25-type. We may therefore assume that $f_{35} \notin N$. Since $\{v, f_{14}, f_{35}\}$ is not a triad, $f_{14} \notin M$; since $\{f_{15}, f_{25}, f_{23}, f_{34}, f_{14}, v\}$ does not induce a 5-wheel, $f_{34} \notin N$; and since $\{v, f_{25}, f_{35}, f_{34}, f_{14}, f_{45}\}$ does not induce a 5-wheel, $f_{45} \notin N$. But then v is of 12-type.

We may therefore assume that $f_{34} \notin M$, and similarly that $f_{14} \notin M$. Now not both $f_{15}, f_{25} \in N$, since $\{f_{15}, f_{25}, f_{23}, f_{34}, f_{14}, v\}$ does not induce a 5-wheel; so from the symmetry, we may assume that $f_{15}, f_{35} \notin N$. Since $\{v, f_{12}, f_{25}, f_{35}, f_{34}, f_{23}\}$ does not induce a 5-wheel, $f_{25} \notin M$, and similarly $f_{45} \notin M$; but then v is of 24-type. This proves (1).

(2) *For all distinct $v, v' \in V(G) \setminus E(H)$, if v, v' are of ij-type and $i'j'$-type respectively, then v, v' are strongly adjacent if $\{i,j\} \cap \{i',j'\} \neq \emptyset$, and otherwise v, v' are strongly antiadjacent.*

For suppose first that $h_{i'}, h_{j'}$ are adjacent in H, and let H' be the graph obtained from H by deleting the edge $f_{i'j'}$ and adding a new edge v' with ends $h_{i'}, h_{j'}$. Then $E(H') \subseteq V(G)$, and the subtrigraph induced on $E(H')$ is a line trigraph of H'. By (1) applied to v and H', there exist a, b with $1 \leq a < b \leq 5$ such that v is of ab-type with respect to H'; that is, for $1 \leq c < d \leq 5$ with $(c,d) \neq (i',j')$, v is strongly adjacent to f_{cd} if and only if $\{a,b\} \cap \{c,d\} \neq \emptyset$, and otherwise v is strongly antiadjacent to f_{cd}; and v is strongly adjacent to v' if and only if $\{a,b\} \cap \{i',j'\} \neq \emptyset$, and otherwise v, v' are strongly antiadjacent. We claim that $\{a,b\} = \{i,j\}$. There is a cycle C of H with length five, not using the edge $f_{i'j'}$. Consequently there are two vertices $x_1, x_2 \in \{h_a, h_b, h_i, h_j\}$ such that each of x_1, x_2 is adjacent in C to a vertex not in $\{h_a, h_b, h_i, h_j\}$. Let f be an edge of C with ends x_1 and some vertex not in $\{h_a, h_b, h_i, h_j\}$. Since v has ij-type with respect to H, it follows that v, f are strongly adjacent in G if and only if $x_1 \in \{h_i, h_j\}$. But also, since v has ab-type with respect to H', and the graphs H, H' differ only by exchange of the edges $f_{i',j'}, v'$, and these edges are different from f, it follows that v, f are strongly adjacent in G if and only if $x_1 \in \{h_a, h_b\}$. Consequently $x_1 \in \{h_i, h_j\}$ if and only if $x_1 \in \{h_a, h_b\}$; but $x_1 \in \{h_a, h_b, h_i, h_j\}$, and so $x_1 \in \{h_a, h_b\} \cap \{h_i, h_j\}$. The same holds for x_2, and so $\{i,j\} = \{a,b\}$ as claimed. But we saw that v is strongly adjacent to v' if and only if $\{a,b\} \cap \{i',j'\} \neq \emptyset$, and otherwise v, v' are strongly antiadjacent; and so in this case (2) holds.

We may therefore assume that $h_{i'}, h_{j'}$ are nonadjacent in H, and similarly h_i, h_j are nonadjacent in H. Thus $(i,j), (i',j') \in \{(1,3), (2,4)\}$, and we may assume from the symmetry that $(i',j') = (1,3)$. If also $(i,j) = (1,3)$, then v, v' are strongly adjacent since $\{v, v', f_{25}\}$ is not a triad. If $(i,j) = (2,4)$ then v, v' are strongly antiadjacent since otherwise $\{v', f_{12}, f_{25}, f_{45}, f_{34}, v\}$ induces a 5-wheel. This proves (2).

From (2) it follows that if $v \in V(G) \setminus E(H)$ has ij-type, then h_i, h_j are nonadjacent in H, since otherwise v, f_{ij} would be twins; and so every vertex in $V(G) \setminus E(H)$ has 13-type or 24-type. Moreover, any two vertices of the same type are twins, so there is at most one of each type, and it follows that G is a line trigraph of a subgraph of K_5. This proves 2.2. ∎

Proof of 2.1. If $V(G)$ is expressible as the union of two strong cliques, then since G is slim it follows that $|V(G)| \leq 2$ and the theorem holds. Thus we may assume that G is not the union of two strong cliques, and so G contains an anticycle of odd length. Choose n minimum such that n is odd and there is an anticycle of length n. Since there is no triad it follows that $n \geq 5$. From the minimality of n we have:

(1) *Let v_1-v_2-$\cdots$-v_n-v_1 be an anticycle of length n. Then for $1 \leq i < j \leq n$, v_i and v_j are strongly adjacent unless $j - i = 1$ or $(i,j) = (1,n)$.*

(2) *Let v_1-v_2-$\cdots$-v_n-v_1 be an anticycle C of length n. For every vertex $v \in V(G)$, either v is antiadjacent to a unique vertex of C, or there are exactly two vertices in C antiadjacent to v (and different from v), say v_i, v_j; and in this case either $j = i+2 \mod n$ or $j = i-2 \mod n$.*

The claim is clear if $v \in V(C)$, so we assume that $v \notin V(C)$. Let I be the set of $i \in \{1,\dots,n\}$ such that v, v_i are antiadjacent. Since v is not a centre for the odd anticycle, it follows that $I \neq \emptyset$, and we may assume that $1 \in I$. If $I = \{1\}$ then the claim holds, so we assume that there exists $i \in I \setminus \{1\}$. Now one of v-v_1-$\cdots$-v_i-v, v-v_i-v_{i+1}-$\cdots$-v_n-v_1-v is an odd anticycle, and from the choice of n it has length at least n; and so either i is even and $i+1 \geq n$, or i is odd and $n-i+3 \geq n$. Consequently $i \in \{3, n-1\}$, and so $I \subseteq \{1,3,n-1\}$. If $3, n-1 \in I$, then v-v_3-v_4-$\cdots$-v_{n-1}-v is an odd anticycle of length $n-2$, which is impossible; so $I = \{1,3\}$ or $I = \{1,n-1\}$. This proves (2).

(3) *Let v_1-v_2-$\cdots$-v_n-v_1 be an anticycle of length n. There do not exist $u_2, u_3 \in V(G) \setminus \{v_1,\dots,v_n\}$ such that v_1-u_2-u_3-v_4-$\cdots$-v_n-v_1 is an odd anticycle and the pairs u_2v_3 and v_2u_3 are adjacent.*

For suppose that such u_2, u_3 exist. Let us say a *square* is a set $\{a,b,c,d\}$ of four distinct vertices, such that

- a, b are antiadjacent to v_1 and strongly complete to $\{v_4, v_5, \dots, v_n\}$,
- c, d are antiadjacent to v_4 and strongly complete to $\{v_5, \dots, v_n, v_1\}$
- the pairs bc, ad are adjacent, and ac, bd are antiadjacent.

Since $\{a, b, v_1\}$ is not a triad, it follows that a, b are strongly adjacent, and similarly so are c, d. (We follow the convention that when we list the elements of a square, the element written first corresponds to a in the conditions above, and so on.)

Thus $\{u_2, v_2, u_3, v_3\}$ is a square. Consequently we may choose disjoint sets A, B with $|A|, |B| \geq 2$, such that

- A is anticomplete to v_1 and strongly complete to $\{v_4, v_5, \dots, v_n\}$
- B is anticomplete to v_4 and strongly complete to $\{v_5, \dots, v_n, v_1\}$
- for every partition of A or B into two nonempty subsets, there is a square included in $A \cup B$ that has nonempty intersection with both subsets, and
- subject to these conditions $A \cup B$ is maximal.

Since there is no triad, it follows that A, B are strong cliques. Since (A, B) is not a W-join, we may assume from the symmetry that there exists $v \in V(G) \setminus (A \cup B)$ with a neighbour and an antineighbour in A; and since $|A| \geq 2$ we may partition A into two nonempty subsets, the first only containing neighbours of v, and the second only containing antineighbours. Consequently we may choose a square $\{a, b, c, d\}$ such that v, a are antiadjacent and v, b are adjacent. Since $\{v, a, c\}$ is not a triad it follows that v, c are strongly adjacent. Let C be the anticycle v_1-a-c-v_4-$\cdots$-v_n-v_1, and let C' be the anticycle v_1-b-d-v_4-$\cdots$-v_n-v_1. Since b is not a centre for C, it follows that v_1, b are strongly antiadjacent, and so $v \neq v_1$. By (1), the only vertices in $V(C) \setminus \{a\}$ antiadjacent to a are v_1, c, and $v \neq c$ by hypothesis, so $v \notin V(C)$, and therefore $v \notin V(C')$.

Suppose that d, v are antiadjacent. Since v-a-v_1-b-d-v is an anticycle of odd length, it follows that $n = 5$. By (2) applied to C and to C', it follows that v is strongly adjacent to v_1, v_4. If v is antiadjacent to v_5 then $\{b, v, v_1, d, v_5, c\}$ induces a 5-wheel; and if v is adjacent to v_5, then the subtrigraph induced on $\{v_1, d, a, v_4, c, v_5, b, v\}$ satisfies the hypotheses of 2.2, and so G is a line trigraph of a subgraph of K_5 and the theorem holds. Thus we may assume that d, v are strongly adjacent.

Let M be the set of antineighbours of v in $V(C)$. Since $a \in M$, (2) implies that M is one of $\{a\}, \{a, v_n\}, \{a, v_4\}$. If $M = \{a\}$ then v is a centre for C', which is impossible. If $M = \{a, v_n\}$, then v-a-c-v_4-$\cdots$-v_n-v is an odd anticycle with centre b, which is impossible. Thus $M = \{a, v_4\}$, and so $\{a, b, v, d\}$ is a square. But then we can add v to B, contrary to the maximality of $A \cup B$. This proves (3).

For the remainder of the proof, let us fix an anticycle C of length n (we recall that n was chosen earlier), and it is convenient to number its vertices using even subscripts $c_2, c_4, \ldots, c_{2n}$, and *not* in the usual order; we number the vertices (so that consecutive vertices are antiadjacent) as

$$c_{n+1}\text{-}c_2\text{-}c_{n+3}\text{-}c_4\text{-}\cdots\text{-}c_{2n-2}\text{-}c_{n-1}\text{-}c_{2n}\text{-}c_{n+1}.$$

Thus for $1 \leq i < j \leq 2n$ with i, j even, c_i and c_j are antiadjacent if and only if $j - i = n - 1$ or $j - i = n + 1 \mod 2n$. (We read all subscripts modulo $2n$ through the remainder of this proof.) For $1 \leq i \leq 2n$ with i even, let A_i be the set of all vertices antiadjacent to both c_{i+n-1}, c_{i+n+1} (and therefore strongly adjacent to every other vertex of C, by (2)); and for $1 \leq i \leq 2n$ with i odd, let A_i be the set of all vertices antiadjacent to c_{i+n} and strongly adjacent to every other vertex of C. Thus $c_i \in A_i$ for $1 \leq i \leq 2n$ with i even; and the sets $A_1, \ldots, A_{2n}$ are pairwise disjoint, and have union $V(G)$ by (2). Moreover, each A_i is a strong clique, since there is no triad in G. (The reader may find it helpful to visualize the sets $A_1, \ldots, A_{2n}$ arranged in a circle in the order $A_1, \ldots, A_{2n}$; our goal is to refine this circular order by ordering the members of each set A_i to obtain a representation of G as a circular interval trigraph.)

(4) *For $1 \leq i, j \leq 2n$ with $i \neq j$, if $u \in A_i$ and $v \in A_j$ are antiadjacent then $j - i$ is one of $n-2, n-1, n, n+1, n+2$.*

To see this, suppose first that one of i, j is even; say $i = 2$. Now C has vertices

$$c_{n+1}\text{-}c_2\text{-}c_{n+3}\text{-}c_4\text{-}\cdots\text{-}c_{2n-2}\text{-}c_{n-1}\text{-}c_{2n}\text{-}c_{n+1}$$

in order, and so

$$c_{n+1}\text{-}u\text{-}c_{n+3}\text{-}c_4\text{-}\cdots\text{-}c_{2n-2}\text{-}c_{n-1}\text{-}c_{2n}\text{-}c_{n+1}$$

is also an anticycle of length n, say C'. Since u, v are antiadjacent, (2) tells us that the set of antineighbours of v in C' is one of $\{u, c_{2n}\}, \{u\}, \{u, c_4\}$. Consequently the set of antineighbours of v in C is one of

$$\emptyset, \{c_{2n}\}, \{c_2, c_{2n}\}, \{c_2\}, \{c_2, c_4\}, \{c_4\}.$$

The first is impossible by (2), and the others imply that v belongs to $A_n, A_{n+1}, A_{n+2}, A_{n+3}, A_{n+4}$ respectively. Thus the claim holds if i is even.

We may therefore assume that i is odd, and similarly j is odd. We may assume that $i = 1$, and we therefore need to show that j is one of $n, n+2$. Suppose not; then from the symmetry we may assume that $j \geq n+3$. But then $j \geq n+4$ since j is odd, and

$$v\text{-}c_{j-n}\text{-}c_{j+1}\text{-}c_{j+1-n}\text{-}\cdots\text{-}c_{n-1}\text{-}c_{2n}\text{-}c_{n+1}\text{-}u\text{-}v$$

is an odd anticycle of length $2n+4-j \leq n$. Thus equality holds, since C is an odd anticycle of minimum length; and so $j = n+4$. But then

$$c_{n+1}\text{-}u\text{-}v\text{-}c_4\text{-}c_{n+5}\text{-}\cdots\text{-}c_{2n-2}\text{-}c_{n-1}\text{-}c_{2n}\text{-}c_{n+1}$$

is an anticycle, and $\{u, v\}$ is complete to $\{c_2, c_{n+3}\}$, contrary to (3). This proves (4).

(5) *For $1 \leq i \leq 2n$, if $u \in A_i$, then*

- *u is strongly anticomplete to A_{n+i}*
- *u is either strongly complete to A_{n+i+2} or strongly anticomplete to A_{n+i+1}, and*
- *u is either strongly complete to A_{n+i-2} or strongly anticomplete to A_{n+i-1}.*

For suppose first that i is even, say $i = n+1$. Suppose that $v \in A_1$, and so v is strongly adjacent to every vertex of C except c_1. Now

$$u\text{-}c_2\text{-}c_{n+3}\text{-}c_4\text{-}\cdots\text{-}c_{2n-2}\text{-}c_{n-1}\text{-}c_{2n}\text{-}u$$

is an anticycle of length n, say C', and v is strongly adjacent to all its vertices except possibly u. Since G is quasi-line, v has a strong antineighbour in C', and hence u, v are strongly antiadjacent. This proves the first statement when i is even.

Next suppose that u has an antineighbour $v \in A_3$ and a neighbour $w \in A_2$. Since $v \in A_3$ and therefore is strongly complete to every vertex of C except c_{n+3}, it follows that $u \neq c_{n+1}$. But

$$c_{n+1}\text{-}w\text{-}c_{n+3}\text{-}c_4\text{-}\cdots\text{-}c_{2n-2}\text{-}c_{n-1}\text{-}c_{2n}\text{-}c_{n+1}$$

is an anticycle of length n, and so is

$$u\text{-}v\text{-}c_{n+3}\text{-}c_4\text{-}\cdots\text{-}c_{2n-2}\text{-}c_{n-1}\text{-}c_{2n}\text{-}u,$$

and $\{u, v\}$ is complete to $\{w, c_{n+1}\}$, contrary to (3). This proves the second assertion when i is even. The third assertion follows from the symmetry.

Now suppose that i is odd, say $i = 1$. We have already seen that A_1 is strongly anticomplete to A_{n+1}, so the first assertion holds. For the second, assume that u has an antineighbour $v \in A_{n+3}$ and a neighbour $w \in A_{n+2}$. Since $\{v, w, c_2\}$ is not a triad, v, w are strongly adjacent. But

$$c_{n+1}\text{-}u\text{-}v\text{-}c_4\text{-}\cdots\text{-}c_{2n-2}\text{-}c_{n-1}\text{-}c_{2n}\text{-}c_{n+1}$$

is an anticycle of length n, and w is a centre for it, a contradiction. This proves the second statement, and again the third follows by symmetry. This proves (5).

(6) *For $1 \leq i \leq 2n$, and $j \in \{i+n-2, i+n-1, i+n, i+n+1, i+n+2\}$, there do not exist distinct $a, b \in A_i$ and $c, d \in A_j$ such that the pairs ac, bd are antiadjacent and ad, bc are adjacent.*

For this is clear if $j = i + n$, since A_i is strongly anticomplete to A_{i+n} by (5). From the symmetry we may assume that $j = i+n+1$ or $i+n+2$. Suppose first that i is even, say $i = 2$, and so $j \in \{n+3, n+4\}$. In both cases c, d are antiadjacent to c_4, and so

$$c_{n+1}\text{-}a\text{-}c\text{-}c_4\text{-}\cdots\text{-}c_{2n-2}\text{-}c_{n-1}\text{-}c_{2n}\text{-}c_{n+1}$$

and

$$c_{n+1}\text{-}b\text{-}d\text{-}c_4\text{-}\cdots\text{-}c_{2n-2}\text{-}c_{n-1}\text{-}c_{2n}\text{-}c_{n+1}$$

are anticycles of length n, and the pairs ad and bc are adjacent, contrary to (3).

Now suppose that i is odd, say $i = 1$, and therefore $j \in \{n+2, n+3\}$. If $j = n+3$ then the same two anticycles given above still violate (3), so we may assume that $j = n+2$. Let us say a *rectangle* is a set $\{p, q, r, s\}$ of four distinct vertices, such that

- $p, q \in A_1$,
- $r, s \in A_{n+2}$, and
- the pairs qr, ps are adjacent, and pr, qs are antiadjacent.

By hypothesis there is a rectangle, and so we may choose disjoint sets A, B with $|A|, |B| \geq 2$, such that

- $A \subseteq A_1$, and $B \subseteq A_{n+2}$, and $|A|, |B| \geq 2$
- for every partition of A or B into two nonempty subsets, there is a rectangle included in $A \cup B$ that has nonempty intersection with both subsets, and
- subject to these conditions $A \cup B$ is maximal.

Since (A, B) is not a W-join, we may assume from the symmetry that there exists $v \in V(G) \setminus (A \cup B)$ with a neighbour and an antineighbour in A; and since $|A| \geq 2$ we may choose a rectangle $\{p, q, r, s\}$ such that v, p are antiadjacent and v, q are adjacent. It follows that $v \notin V(C)$ (since p, q are strongly antiadjacent to c_{n+1} and strongly adjacent to all other vertices of C). Since v, p are antiadjacent and $p \in A_1$, (4) implies that v belongs to one of $A_{n-1}, A_n, A_{n+1}, A_{n+2}, A_{n+3}$. If $v \in A_{n-1} \cup A_n \cup A_{n+1}$, then v, c_{2n} are antiadjacent, and so

$$v\text{-}p\text{-}r\text{-}c_2\text{-}c_{n+3}\text{-}c_4\text{-}\cdots\text{-}c_{2n-2}\text{-}c_{n-1}\text{-}c_{2n}\text{-}v$$

is an odd anticycle of length $n+2$, and q is a centre for it, a contradiction. Thus v belongs to one of A_{n+2}, A_{n+3}, and in particular v, c_2 are antiadjacent; and therefore v is strongly adjacent to both r, s since there is no triad. If $v \in A_{n+2}$ then $\{p, q, v, s\}$ is a rectangle, and so we may add v to B, contrary to the maximality of $A \cup B$. If $v \in A_{n+3}$ then

$$c_{n+1}\text{-}p\text{-}v\text{-}c_4\text{-}\cdots\text{-}c_{2n-2}\text{-}c_{n-1}\text{-}c_{2n}\text{-}c_{n+1}$$

is an anticycle of length n with a centre s, a contradiction. This proves (6).

(7) *For $1 \le i \le 2n$, there do not exist distinct $v, w \in A_i$ such that some vertex $u \in A_{n+i-2} \cup A_{n+i-1}$ is adjacent to w and antiadjacent to v, and some vertex $x \in A_{n+i+1} \cup A_{n+i+2}$ is adjacent to w and antiadjacent to v.*

For suppose that such u, x exist. First suppose that i is even, say $i = 2$. Thus $u \in A_n \cup A_{n+1}$ and $x \in A_{n+3} \cup A_{n+4}$, and so

$$u\text{-}v\text{-}x\text{-}c_4\text{-}c_{n+5}\text{-}c_6\text{-}\cdots\text{-}c_{2n-2}\text{-}c_{n-1}\text{-}c_{2n}\text{-}u$$

is an anticycle of length n with a centre w, a contradiction. Next suppose that i is odd, say $i = 1$. Thus $u \in A_{n-1} \cup A_n$ and $x \in A_{n+2} \cup A_{n+3}$. Since $v \in A_1$ and so is strongly adjacent to every vertex of C except c_{n+1}, it follows that $u, x \notin V(C)$. Hence and

$$u\text{-}v\text{-}x\text{-}c_2\text{-}c_{n+3}\text{-}c_4\text{-}\cdots\text{-}c_{2n-2}\text{-}c_{n-1}\text{-}c_{2n}\text{-}u$$

is an anticycle of length $n+2$ with a centre w, a contradiction. This proves (7).

From (5), (6), (7), for $1 \le i \le 2n$ we can order A_i as $\{v_1, \ldots, v_k\}$ say, such that for $1 \le h < j \le k$, every vertex in $A_{n+i-2} \cup A_{n+i-1}$ that is adjacent to v_j is strongly adjacent to v_h, and every vertex in $A_{n+i+1} \cup A_{n+i+2}$ that is adjacent to v_h is strongly adjacent to v_j. We call this the *natural order* of A_i. Take a circle Σ, and $2n$ disjoint line segments $L_1, \ldots, L_{2n}$ from Σ in order. For each i, let us map the members of A_i injectively into L_i in their natural order. This gives a representation of G as a circular interval trigraph. This proves 2.1. ∎

3 Isolated triads

A triad T in a quasi-line trigraph G is *isolated* if T is disjoint from every other triad. It follows that every vertex in $V(G) \setminus T$ has two strong neighbours and one strong antineighbour in T. In this section we show:

3.1 *Let G be a quasi-line trigraph with an isolated triad T, such that there is no W-join (P, Q) with $P, Q \subseteq V(G) \setminus T$. Then G is a circular interval trigraph.*

Proof. Let $T = \{t_1, t_2, t_3\}$ be a isolated triad. For $i = 1, 2, 3$, let C_i be the set of all vertices in $V(G) \setminus T$ that are strongly antiadjacent to t_i and (therefore) strongly adjacent to the other two members of T. Thus C_1, C_2, C_3, T are pairwise disjoint and have union $V(G)$. We observe first that

C_1, C_2, C_3 are strong cliques; for if say $x, y \in C_1$ are antiadjacent, then $\{x, y, t_1\}$ is a triad with nonempty intersection with T, contrary to the hypothesis.

Reading subscripts modulo 3, for $x \in V(G) \setminus C_i$ we define $N_i(x)$ to be the set of neighbours of x in C_i, and $M_i(x)$ to be the set of antineighbours of x in C_i.

(1) *For $i = 1, 2, 3$, if $u, v \in C_i$ then one of $N_{i+1}(u) \cap M_{i+1}(v), N_{i-1}(u) \cap M_{i-1}(v) = \emptyset$.*

For suppose that $x \in N_{i+1}(u) \cap M_{i+1}(v)$ and $y \in N_{i-1}(u) \cap M_{i-1}(v)$. Since $\{u, v, x, y\}$ is not a claw it follows that x, y are adjacent. But then $\{v, t_{i-1}, x, y, t_{i+1}, u\}$ induces a 5-wheel, a contradiction. This proves (1).

(2) *For all distinct $i, j \in \{1, 2, 3\}$, if $u, v \in C_i$ are distinct then one of $N_j(u) \cap M_j(v), N_j(v) \cap M_j(u) = \emptyset$.*

For we may assume that $i = 1$ and $j = 2$. Let us say a *square* is a set $\{a, b, c, d\}$ of four distinct vertices, with $a, b \in C_1$ and $c, d \in C_2$, such that the pairs ac, bd are adjacent, and the pairs ad, bc are antiadjacent. Suppose that there is a square. Consequently we may choose disjoint sets A_1, A_2 with $|A_1|, |A_2| \geq 2$, such that

- $A_1 \subseteq C_1$, and $A_2 \subseteq C_2$
- for every partition of A_1 or A_2 into two nonempty subsets, there is a square included in $A_1 \cup A_2$ that has nonempty intersection with both subsets, and
- subject to these conditions $A_1 \cup A_2$ is maximal.

Since (A_1, A_2) is not a W-join (by hypothesis), we may assume (by the symmetry between C_1, C_2) that there exists $z \in V(G) \setminus (A_1 \cup A_2)$ with a neighbour and an antineighbour in C_1. Hence $z \neq t_1, t_2, t_3$. Since $|A_1| > 1$, we may choose a square $\{a, b, c, d\}$ such that z is adjacent to a and antiadjacent to b. Since z has an antineighbour in C_1 it follows that $z \notin C_1$; and since $c \in N_2(a) \cap M_2(b)$, (1) implies that $z \notin N_3(a) \cap M_3(b)$. Consequently $z \in C_2$, and so $\{a, b, z, d\}$ is a square; but then we can add z to A_2, contrary to the maximality of $A_1 \cup A_2$. This proves that there is no square.

Now to complete the proof of (2), suppose that $u, v \in C_1$ are distinct, and $x \in N_2(u) \cap M_2(v)$ and $y \in N_2(v) \cap M_2(u)$. Since $\{u, v, x, y\}$ is not a square (because there are no squares), it follows that $x = y$. Thus $x \in N_2(u) \cap M_2(u)$, so x is semiadjacent to u, and similarly x is semiadjacent to v, which is impossible. This proves (2).

For $i = 1, 2, 3$, if $u, v \in C_i$ we write $u \to v$ if either $M_{i+1}(u) \cap N_{i+1}(v) \neq \emptyset$, or $N_{i-1}(u) \cap M_{i-1}(v) \neq \emptyset$.

(3) *If $u, v \in C_i$ then not both $u \to v$ and $v \to u$. Moreover, if $u, v, w \in C_i$, and $u \to v$ and $v \to w$, then $u \to w$.*

For suppose that $u \to v$. We may assume that $i = 1$, and since $u \to v$ we may assume from the symmetry between C_2, C_3 that $M_2(u) \cap N_2(v) \neq \emptyset$. By (1) $M_3(u) \cap N_3(v) = \emptyset$, and by (2) $M_2(v) \cap N_2(u) = \emptyset$. Consequently $v \not\to u$. This proves the first claim.

For the second, suppose that $u, v, w \in C_1$ and $u \to v$ and $v \to w$. From the symmetry we may assume that there exists $x \in M_2(u) \cap N_2(v)$. Since $w \not\to v$ it follows that $x \notin M_2(w) \cap N_2(v)$, and so x, w are adjacent. Hence $x \in M_2(u) \cap N_2(w)$ and so $u \to w$ as required. This proves (3).

From (3) there is a linear order (say $u_1, \ldots, u_a$) of the members of C_1 such that for $1 \leq i < j \leq a$, every vertex in C_3 adjacent to u_j is strongly adjacent to u_i, and every vertex in C_2 adjacent to u_i is strongly adjacent to u_j. Choose orders $v_1, \ldots, v_b$ of C_2 and $w_1, \ldots, w_c$ of C_3 similarly. Then if we place the vertices of G in a circle, in the order

$$t_2, u_1, \ldots, u_a, t_3, v_1, \ldots, v_b, t_1, w_1, \ldots, w_c, (t_2)$$

this gives a representation of G as a circular interval trigraph. This proves 3.1. ∎

4 Antiprismatic trigraphs

If G is a trigraph, we say $X \subseteq V(G)$ is a *fang* if $|X| = 4$ and at most one pair of vertices in X are strongly adjacent. We say G is *antiprismatic* if no subset of $V(G)$ is a fang or claw. Next we study which antiprismatic trigraphs are quasi-line. Trigraphs with no triad are antiprismatic, and our next results extend 2.1. In [1, 2] we gave a structure theorem describing all antiprismatic trigraphs; but it turns out that so few antiprismatic trigraphs are quasi-line that it is easier not to use that structure theorem, and to prove what we need here from first principles.

Let H be a trigraph with seven vertices $v_1, \ldots, v_7$ and the following adjacencies:

- the pairs $v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_1v_6, v_1v_7, v_3v_7, v_4v_7, v_6v_7$ are strongly adjacent
- v_1, v_3 are semiadjacent, and the adjacency between v_4, v_6 is unspecified, and
- all other pairs are strongly antiadjacent.

We call such a trigraph H a trigraph of *H_7-type*. Such trigraphs are antiprismatic quasi-line trigraphs, but not line trigraphs (because of the semiadjacent pair v_1, v_3), and not circular interval trigraphs; and they will be exceptional in some of the theorems that follow.

We will prove the following:

4.1 *Let G be a slim antiprismatic quasi-line trigraph. Then either G is a line trigraph of a subgraph of K_6, or G is a trigraph of H_7-type, or G is a circular interval trigraph.*

The proof needs several lemmas. We begin with the following, the proof of which is clear:

4.2 *If G is antiprismatic and T is a triad of G and $v \in V(G) \setminus T$ then v is strongly adjacent to two members of T and strongly antiadjacent to the third.*

4.3 *Let H be a graph with six vertices $h_1, \ldots, h_6$ and eight edges, such that (reading subscripts modulo 6) h_i, h_{i+1} are adjacent for $1 \le i \le 6$, and for some i, h_i, h_{i+3} are adjacent, and one of h_{i+1}, h_{i+2} is adjacent to one of h_{i+4}, h_{i+5}. Let G be an antiprismatic quasi-line trigraph, not admitting twins, and containing a line trigraph of H as an induced subtrigraph. Then G is a line trigraph of a subgraph of K_6.*

Proof. For each adjacent pair h_i, h_j of vertices of H with $i < j$, let f_{ij} be the edge of H joining h_i, h_j. Thus H has eight edges, including $f_{12}, f_{23}, f_{34}, f_{45}, f_{56}, f_{16}$ and two others that we do not specify yet in order to preserve the symmetry. Moreover, $E(H) \subseteq V(G)$, and for all $e, f \in E(H)$

- if e, f have a common end in H then they are adjacent in G, and if they have a common end of degree at least three in H, then they are strongly adjacent in G
- if e, f have no common end in H then they are strongly antiadjacent in G.

Let C be the cycle of H formed by the vertices h_1-h_2-$\cdots$-h_6-h_1 in order. For each pair $i, j \in \{1, \ldots, 6\}$ with $i < j$, we say that a vertex $v \in V(G) \setminus E(C)$ is of *ij-type* if v is strongly adjacent to each edge f of C that is incident with h_i or h_j, and strongly antiadjacent to every other edge of C.

(1) *For every vertex $v \in V(G) \setminus E(C)$, there exist distinct i, j such that v is of ij-type and h_i, h_j are not adjacent in C.*

For by 4.2 it follows that v is strongly adjacent to two of f_{12}, f_{34}, f_{56}, and strongly antiadjacent to the third. We may therefore assume that v is strongly adjacent to f_{12}, f_{34} and strongly antiadjacent to f_{56}. Similarly v is strongly adjacent to two of f_{23}, f_{45}, f_{16} and strongly antiadjacent to the third. If v is adjacent to f_{23}, f_{45} then v is of type 24; if it is adjacent to f_{45}, f_{16} then it is of type 14; and if it is adjacent to f_{16}, f_{23} then it is of type 13. This proves (1).

(2) *If $v, v' \in V(G) \setminus E(C)$, with types ij and 14 respectively, then*

- *if $\{i, j\} \cap \{1, 4\} \neq \emptyset$, then v, v' are strongly adjacent, and*
- *if $\{i, j\} \cap \{1, 4\} = \emptyset$, then v, v' are strongly antiadjacent.*

For from the symmetry we may assume that (i, j) is one of $(1, 3), (1, 4), (2, 5), (2, 6)$. In the first two cases it follows that v, v' are strongly adjacent since otherwise $\{f_{16}, v, v', f_{56}\}$ is a claw. In the last two cases it follows that v, v' are strongly antiadjacent since otherwise $\{v', v, f_{23}, f_{56}\}$ is a claw. This proves (2).

(3) *If $v, v' \in V(G) \setminus E(C)$, with types ij and $i'j'$ respectively, and $\{i, j\} \cap \{i', j'\} \neq \emptyset$, then v, v' are strongly adjacent.*

For by (2) we may assume that $(i', j') = (1, 3)$ and (i, j) is one of $(1, 3), (1, 5)$. If $(i, j) = (1, 3)$ then v, v' are strongly adjacent since otherwise $\{f_{16}, v, v', f_{56}\}$ is a claw. Suppose then that $(i, j) = (1, 5)$ and v, v' are antiadjacent. By hypothesis there exists $w \in E(H) \setminus E(C)$ of type $14, 25$ or 36. If w is of type 14, then w is adjacent to both v, v' by (2), and so $\{v, f_{45}, f_{34}, v', f_{12}, w\}$ induces a 5-wheel, a contradiction. From the symmetry we may therefore assume that w has type 25, and hence by

(2) w is adjacent to v and antiadjacent to v'. But then $\{v, w, f_{23}, v', f_{16}, f_{12}\}$ induces a 5-wheel, a contradiction. This proves (3).

(4) *If $v, v' \in V(G) \setminus E(C)$, with types ij and $i'j'$ respectively, then*

- *if $\{i, j\} \cap \{i', j'\} \neq \emptyset$, then v, v' are strongly adjacent, and*
- *if $\{i, j\} \cap \{i', j'\} = \emptyset$, then v, v' are strongly antiadjacent.*

For by (3) we may assume that $\{i, j\} \cap \{i', j'\} = \emptyset$, and by (2) that $(i', j') = (1, 3)$. Suppose that v, v' are adjacent. If $\{i, j\} = \{4, 6\}$ then $\{f_{12}, f_{23}, f_{34}, v, f_{16}, v'\}$ induces a 5-wheel, a contradiction; so from (2) and the symmetry we may assume that $(i, j) = (2, 4)$. By hypothesis there exists $w \in E(H) \setminus E(C)$ of type $14, 25$ or 36. If w is of type 36, then w is adjacent to v' and antiadjacent to v, and $\{w, f_{34}, v, f_{12}, f_{16}, v'\}$ induces a 5-wheel, a contradiction. Thus w is not of type 36, and similarly it is not of type 25; so w is of type 14. By hypothesis, there is an edge x of H incident with one of h_2, h_3 and one of h_5, h_6. From the symmetry we may assume that x is incident with h_2. If x is incident with h_5 then it has type 25, which we already saw was impossible. Thus x has type 26. By (3) x is adjacent to v, and by (2) it is antiadjacent to w. If x is adjacent to v', then $\{x, f_{23}, f_{34}, w, f_{16}, v'\}$ induces a 5-wheel, a contradiction; so x is antiadjacent to v'. But then $\{v, v', x, f_{45}\}$ is a claw, a contradiction. This proves (4).

(5) *If $v \in V(G) \setminus E(C)$ is of type ij, then the two edges of C incident with h_i in C are strongly adjacent in G.*

For from the symmetry we may assume that $(i, j) = (1, 3)$ or $(1, 4)$; and then f_{12}, f_{16} are strongly adjacent, since $\{v, f_{12}, f_{16}, f_{34}\}$ is not a claw. This proves (5).

From (4) it follows that every two members of $V(G) \setminus E(C)$ of the same type are twins; and so all members of $V(G) \setminus E(C)$ are of different types, and therefore (5) implies that G is a line trigraph of a subgraph of K_6. This proves 4.3. ∎

4.4 *Let G be an antiprismatic quasi-line trigraph. Suppose that there are at least two triads, and for some $z \in V(G)$, every triad contains z. Suppose also that there are no twins both different from z, and there is no W-join (P, Q) with $z \notin P \cup Q$. Then either G is a line trigraph of a subgraph of K_6, or G is of H_7-type.*

Proof. Let N be the set of strong neighbours of z, and M the set of antineighbours. Let $\{z, a_i, b_i\}$ $(1 \leq i \leq n)$ be the triads containing z. By hypothesis, $n \geq 2$. Since there is no fang, no two triads have more than one vertex in common, and so $a_1, b_1, \ldots, a_n, b_n$ are all distinct. By 4.2, $\{a_i, b_i\}$ is strongly complete to $M \setminus \{a_i, b_i\}$ for $1 \leq i \leq n$, and z is strongly anticomplete to M.

For all adjacent $u, v \in \{a_1, a_2, b_1, b_2\}$, let $D(uv)$ be the set of members of N adjacent to both u, v. Since every triad contains z and there is no claw, it follows that every vertex in N is adjacent to exactly one of a_1, b_1, and to exactly one of a_2, b_2; and so the four sets $D(a_1a_2), D(a_2b_1), D(b_1b_2), D(b_2a_1)$ are pairwise disjoint and have union N. Since every triad contains z, it follows that for each $x \in M$, the set of vertices in N antiadjacent to x is a strong clique. In particular, the four sets

$$D(a_1a_2) \cup D(a_2b_1), D(a_2b_1) \cup D(b_1b_2), D(b_1b_2) \cup D(b_2a_1), D(b_2a_1) \cup D(a_1a_2)$$

are strong cliques. Since $(\{a_1, a_2\}, \{b_1, b_2\})$ is not an W-join, it follows that $D(a_2b_1) \cup D(b_2a_1) \neq \emptyset$, and similarly $D(a_1a_2) \cup D(b_1b_2) \neq \emptyset$; and we may assume from the symmetry that there exists $d_1 \in D(a_1a_2)$ and $d_2 \in D(a_1b_2)$. Thus d_1, d_2 are strongly adjacent.

Let $X = M \setminus \{a_1, b_1, a_2, b_2\}$. If some vertex $x \in X$ is adjacent to both d_1, d_2, then $\{d_1, a_2, b_1, b_2, d_2, x\}$ induces a 5-wheel, and if some $x \in X$ is antiadjacent to both d_1, d_2 then $\{x, a_2, d_1, d_2, b_2, a_1\}$ induces a 5-wheel, in either case a contradiction. Thus d_1, d_2 have complementary neighbour sets in X (and all their neighbours in X are strong neighbours). Since this holds for all choices of d_1, d_2, we deduce that there is a partition X_1, X_2 of X such that $D(a_1, a_2)$ is strongly complete to X_1 and strongly anticomplete to X_2, and vice versa for $D(a_1b_2)$. By the same argument applied to $D(a_1b_2)$ and $D(b_1b_2)$ it follows that $D(b_1b_2)$ is strongly complete to X_1 and strongly anticomplete to X_2; and by the same argument applied to $D(a_1a_2)$ and $D(a_2b_1)$ it follows that $D(a_2b_1)$ is strongly complete to X_2 and strongly anticomplete to X_1.

We claim that $D(a_1a_2)$ is strongly complete to $D(b_1b_2)$; for if say $p \in D(a_1a_2)$ is antiadjacent to $q \in D(b_1b_1)$ then $\{z, p, a_1, b_2, q, d_2\}$ induces a 5-wheel, a contradiction. Similarly $D(a_1b_2)$ is complete to $D(a_2b_1)$. Thus any two vertices in $D(a_1a_2)$ are twins, and so $D(a_1a_2) = \{d_1\}$, and similarly $D(a_1b_2) = \{d_2\}$ and $|D(b_1b_2)|, |D(a_2b_1)| \leq 1$. Since (X_1, X_2) is not a W-join and there are no twins, it follows that $|X_1|, |X_2| \leq 1$; and in particular $n \leq 3$ and $|V(G)| \leq 11$. Since $\{b_1, b_2, d_2, d_1, a_2, a_1\}$ does not induce a 5-wheel it follows that a_1, b_1 are strongly antiadjacent.

Suppose that $D(b_1b_2) \neq \emptyset$, and let $D(b_1b_2) = \{d_3\}$ say. From the symmetry between d_1, d_3 it follows that a_2, b_2 are strongly antiadjacent. We claim that X_1 is strongly anticomplete to X_2; for if say $x_1 \in X_1$ is adjacent to $x_2 \in X_2$, then $\{x_2, b_1, d_3, d_1, a_1, x_1\}$ induces a 5-wheel, a contradiction. But then G is a line trigraph of a subgraph of K_6 as required.

We may therefore assume that $D(b_1b_2) = \emptyset$, and similarly $D(a_2b_1) = \emptyset$. Since $(X_1 \cup \{a_2\}, X_2 \cup \{b_2\})$ is not a W-join it follows that $X = \emptyset$. If a_2, b_2 are strongly antiadjacent then G is a line trigraph of a subgraph of K_6, and if a_2, b_2 are semiadjacent then G is of H_7-type. This proves 4.4. ∎

In view of 2.1 and 3.1, the next result immediately implies 4.1, the main result of this section.

4.5 *Let G be a slim antiprismatic quasi-line trigraph, such that two triads in G have nonempty intersection. Then either G is a line trigraph of a subgraph of K_6, or G is of H_7-type.*

Proof. Let $z \in V(G)$ belong to at least two triads, and let $\{z, a_i, b_i\}$ $(i = 1, 2)$ be two such triads. Thus a_1, b_1, a_2, b_2 are distinct, and by 4.2, $\{a_1, b_1\}$ is strongly complete to $\{a_2, b_2\}$, and z is strongly anticomplete to $\{a_1, a_2, b_1, b_2\}$.

(1) *If some triad is disjoint from $\{z, a_1, a_2, b_1, b_2\}$ then G is a line trigraph of a subgraph of K_6.*

For suppose that $\{a, b, c\}$ is a triad disjoint from $\{z, a_1, a_2, b_1, b_2\}$. By 4.2 applied to $\{a, b, c\}$ and z, it follows that z is strongly adjacent to two of a, b, c, say a, b, and strongly antiadjacent to c. For $i = 1, 2$, by 4.2 applied to $\{z, a_i, b_i\}$ and c, it follows that c is strongly adjacent to a_i, b_i; and by 4.2 applied to $\{z, a_i, b_i\}$ and a we deduce that a is strongly adjacent to one of a_i, b_i and strongly antiadjacent to the other, say a is strongly adjacent to a_i and strongly antiadjacent to b_i. For $i = 1, 2$, since $\{a_i, a, b, c\}$ is not a claw it follows that a_i, b are strongly antiadjacent; and so by 4.2 applied to $\{z, a_i, b_i\}$ and b it follows that b is strongly adjacent to b_i.

Since $(\{a_1, a_2\}, \{b_1, b_2\})$ is not a W-join, we may assume that some vertex x say is adjacent to a_1 and antiadjacent to a_2. Thus $x \notin \{z, a_1, a_2, b_1, b_2, a, b\}$. By 4.2 applied to $\{z, a_2, b_2\}$ and x it follows that x is strongly adjacent to z, b_2, and strongly antiadjacent to a_2; and by 4.2 applied to $\{z, a_1, b_1\}$ and x, we deduce that x is strongly adjacent to a_1 and strongly antiadjacent to b_1. Now x is strongly adjacent to two of a, b, c and strongly antiadjacent to the third. If x is antiadjacent to c, then $\{x, b_2, c, a_2, a, a_1\}$ induces a 5-wheel, a contradiction; so from the symmetry we may assume that x is strongly antiadjacent to a say, and strongly adjacent to b, c. Thus $\{x, a, b_1\}$ is a triad, and so the pairs a_1b_1, ab, ac are strongly antiadjacent, by 4.2. If b, c are adjacent then $\{a_1, a_2, b_1, b, x, c\}$ induces a 5-wheel, and if a_2, b_2 are adjacent then $\{a_1, a_2, b_1, b, x, b_2\}$ induces a 5-wheel, in either case a contradiction; so bc, a_2b_2 are both strongly antiadjacent. But then the subtrigraph induced on $\{a, z, x, b_2, b_1, a_2, a_1, b\}$ satisfies the hypotheses of 4.3 and so G is a line trigraph of a subgraph of K_6. This proves (1).

By 4.4, we may assume that there is a triad T not containing z, and by (1) we may assume that $b_2 \in T$ say. Thus z is strongly adjacent to the other two members of T, and in particular $a_1, a_2, b_1 \notin T$. Let $T = \{b_2, a_3, b_3\}$ say. By 4.2 we may assume that the pairs a_1a_3, b_1b_3 are strongly adjacent, and a_1b_3, a_3b_1 are strongly antiadjacent. Also, a_2 is strongly adjacent to a_3, b_3, and by three applications of 4.2 it follows that the pairs b_2a_3, b_2b_3, a_2b_2 are strongly antiadjacent. Hence all pairs of vertices within $\{a_1, b_1, a_2, b_2, a_3, b_3, z\}$ are either strongly adjacent or strongly antiadjacent, except possibly for the pairs a_1b_1 and a_3b_3, each of which is either semiadjacent or strongly antiadjacent. Let $W = \{a_1, b_1, a_2, b_2, a_3, b_3, z\}$, and let $M = V(G) \setminus W$. Because W is a union of triads, 4.2 implies that no vertex in M is semiadjacent to a member of W.

(2) *If some vertex is adjacent to both of z, b_2, then G is a line trigraph of a subgraph of K_6.*

For suppose that some x is adjacent to both z, b_2. Consequently $x \in M$, and so x is not semi-adjacent to any member of W. By 4.2, x is antiadjacent to a_2. Suppose that x is adjacent to both b_1, a_3 (and hence antiadjacent to a_1, b_3, by two applications of 4.2). From two applications of 4.2 to $\{x, a_1, b_3\}$ we deduce that a_1b_1 and a_3b_3 are both strongly antiadjacent pairs; but then the subtrigraph induced on $W \cup \{x\}$ satisfies the hypotheses of 4.3 and so G is a line trigraph of a subgraph of K_6. Thus we may assume that x is antiadjacent to at least one of b_1, a_3, and to at least one of a_1, b_3 (by the symmetry taking (a_1, a_3) to (b_1, b_3) and fixing each of a_2, b_2, x, z). Since x is antiadjacent to exactly one of a_1, b_1 and exactly one of a_3, b_3, we may assume (from the same symmetry) that x is antiadjacent to a_1, a_3 and adjacent to b_1, b_3. But then the subtrigraph induced on $W \cup \{x\}$ satisfies the hypotheses of 4.3 and therefore G is a line trigraph of a subgraph of K_6. This proves (2).

For all $w \in W$, let $M(w)$ be the set of all vertices in M that are antiadjacent (and therefore strongly antiadjacent) to w. Because of the triad $\{z, a_2, b_2\}$, every vertex different from z, a_2, b_2 is antiadjacent to exactly one of z, a_2, b_2. By (2), we may therefore assume that $M(a_2) = \emptyset$, and $M(b_2), M(z)$ are disjoint and have union M. Every vertex in $M \setminus M(z)$ is antiadjacent to exactly one of a_1, b_1, and every vertex in $M(z)$ is adjacent to both a_1, b_1; so $M(a_1), M(b_1)$ are disjoint and have union $M(b_2)$, and similarly $M(a_3), M(b_3)$ are disjoint and have union $M(z)$. Thus in summary, M is the union of the four disjoint sets $M(a_1), M(b_1), M(a_3), M(b_3)$; the first two have union $M(b_2)$ and the last two have union $M(z)$. If $M(b_2)$ is not a strong clique, then there is a triad T included in $M(b_2) \cup \{b_2\}$ containing b_2, and the triad $\{z, a_1, b_1\}$ is disjoint from both T and the triad $\{z, a_2, b_2\}$; so there are three triads, exactly one pair of which have nonempty intersection, and the theorem holds by (1). We may therefore assume that $M(b_2)$ is a strong clique, and in particular $M(a_1)$ is strongly complete to $M(b_1)$. Similarly we may assume that $M(a_3)$ is strongly complete to $M(b_3)$. If $p \in M(a_3)$ is adjacent to $q \in M(b_1)$ then $\{p, a_1, a_3, z, b_3, q\}$ induces a 5-wheel, a contradiction; so $M(a_3), M(b_1)$ are strongly anticomplete, and similarly $M(a_1), M(b_3)$ are strongly anticomplete. If some $p \in M(a_3)$ is antiadjacent to some $q \in M(a_1)$, then $\{p, b_1, q, a_3, a_1, a_2\}$ induces a 5-wheel, a contradiction; so $M(a_3)$ is strongly complete to $M(a_1)$ and similarly $M(b_3)$ is strongly complete to $M(b_1)$. Since $(M(a_1) \cup \{b_3\}, M(b_1) \cup \{a_3\})$ is not a W-join and G does not admit twins, it follows that $M(a_1) = M(b_1) = \emptyset$, and similarly $M(a_3) = M(b_3) = \emptyset$. If the pairs a_1b_1 and a_3b_3 are both strongly antiadjacent, then G is a line trigraph of a subgraph of K_6, and otherwise G is of H_7-type. This proves 4.5, and hence completes the proof of 4.1. ∎

5 Spots and stripes

Up to now we have been studying antiprismatic quasi-line trigraphs. This was a digression, and somewhat out of order, since the antiprismatic case is just one of several; but the material was self-contained and we thought it best to treat it separately. Now we return to the main thrust of the paper, proving 1.1. Much of 1.1 follows from two theorems of [4], as we will explain, but first, some more definitions.

Suppose that V_1, V_2 is a partition of $V(G)$ such that V_1, V_2 are nonempty and V_1 is strongly anticomplete to V_2. We call the pair (V_1, V_2) a 0-*join* in G. Thus, G admits a 0-join if and only if it is not connected.

Next, suppose that V_1, V_2 is a partition of $V(G)$, and for $i = 1, 2$ there is a subset $A_i \subseteq V_i$ such that:

- $A_i, V_i \setminus A_i \neq \emptyset$ for $i = 1, 2$;
- $A_1 \cup A_2$ is a strong clique; and
- $V_1 \setminus A_1$ is strongly anticomplete to V_2, and V_1 is strongly anticomplete to $V_2 \setminus A_2$.

In these circumstances, we say that (V_1, V_2) is a 1-*join*. If we replace the first condition above by

- V_1, V_2 are not strongly stable

we call (V_1, V_2) a *pseudo*-1-*join*. If G is connected then every 1-join is a pseudo-1-join.

Next, suppose that V_0, V_1, V_2 is a partition of $V(G)$ (where V_0 may be empty), and for $i = 1, 2$ there are disjoint subsets C_i, D_i of V_i satisfying the following:

- for $i = 1, 2$, C_i, D_i and $V_i \setminus (C_i \cup D_i)$ are all nonempty;
- $V_0 \cup C_1 \cup C_2$ and $V_0 \cup D_1 \cup D_2$ are strong cliques, and V_0 is strongly anticomplete to $V_i \setminus (C_i \cup D_i)$ for $i = 1, 2$; and
- for all $v_1 \in V_1$ and $v_2 \in V_2$, either v_1 is strongly antiadjacent to v_2, or $v_1 \in C_1$ and $v_2 \in C_2$, or $v_1 \in D_1$ and $v_2 \in D_2$.

We call the triple (V_0, V_1, V_2) a *generalized* 2-*join*, and if $V_0 = \emptyset$ we call the pair (V_1, V_2) a 2-*join*. If we replace the first condition above by

- V_1, V_2 are not strongly stable

we call (V_0, V_1, V_2) a *pseudo-2-join*.

Finally, suppose that V_1, V_2, V_3, V_4 is a partition of $V(G)$, satisfying the following:

- $V_1 \neq \emptyset$, and $V_1 \cup V_2, V_1 \cup V_3$ are strong cliques, and V_1 is strongly anticomplete to V_4;
- either $|V_1| \geq 2$, or $V_2 \cup V_3$ is not a strong clique;
- $V_2 \cup V_3 \cup V_4$ is not strongly stable; and
- if $v_2 \in V_2$ and $v_3 \in V_3$ are adjacent then they have the same neighbours in V_4 and neither of them is semiadjacent to any member of V_4.

In these circumstances we call (V_1, V_2, V_3, V_4) a *biclique*.

A vertex v of a trigraph is *simplicial* if $N \cup \{v\}$ is a strong clique, where N is the set of all neighbours of v. Let us say that (G, Z) is a *stripe* if G is a trigraph, and $Z \subseteq V(G)$ is a set of simplicial vertices, such that Z is strongly stable and no vertex has two neighbours in Z. (In [4], we also included the condition that G is claw-free, but let us omit that now.) We call the members of Z the *ends* of the stripe.

A stripe (J, Z) is said to be *unbreakable* if

- J does not admit a 0-join, a pseudo-1-join, a pseudo-2-join or a biclique,
- there are no twins $u, v \in V(J) \setminus Z$,
- there is no W-join (A, B) in J such that $Z \cap A, Z \cap B = \emptyset$, and
- Z is the set of all vertices that are simplicial in J.

In view of theorem 9.1 of [4], in order to prove 1.1 it suffices to show the following:

5.1 *For every unbreakable stripe (J, Z), if J is quasi-line then either*

- $|Z| = 2$ *and (J, Z) is a linear interval stripe, or*
- $|Z| = 1$ *and (J, Z) is a bubble, or*
- $Z = \emptyset$ *and J is a circular interval trigraph.*

We prove this in the following sections. We will eventually need a number of further definitions, and it is convenient to insert them at this point. There are eight classes of trigraphs described in [4], called $\mathcal{S}_0, \dots, \mathcal{S}_7$. To reduce the amount of material we have to copy over from [4], we leave the reader to check that

5.2 *For $i = 1, 2, 4$, if $G \in \mathcal{S}_i$, then G contains a 5-wheel, and therefore is not quasi-line.*

Here are the definitions of the classes $\mathcal{S}_i$ for $i = 0, 3, 5, 6, 7$:

$\mathcal{S}_0$: This is the class of line trigraphs of graphs.

$\mathcal{S}_3$: This is the class of long circular interval trigraphs.

$\mathcal{S}_5$: Let $n \geq 2$. Construct a trigraph H as follows. Its vertex set is the disjoint union of four sets A, B, C and $\{d_1, \ldots, d_5\}$, where $|A| = |B| = |C| = n$, say $A = \{a_1, \ldots, a_n\}, B = \{b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_n\}$. Let $X \subseteq A \cup B \cup C$ with $|X \cap A|, |X \cap B|, |X \cap C| \leq 1$. Adjacency is as follows: A, B, C are strong cliques; for $1 \leq i, j \leq n$, a_i, b_j are adjacent if and only if $i = j$, and c_i is strongly adjacent to a_j if and only if $i \neq j$, and c_i is strongly adjacent to b_j if and only if $i \neq j$. Moreover

- a_i is semiadjacent to c_i for at most one value of $i \in \{1, \ldots, n\}$, and if so then $b_i \in X$
- b_i is semiadjacent to c_i for at most one value of $i \in \{1, \ldots, n\}$, and if so then $a_i \in X$
- a_i is semiadjacent to b_i for at most one value of $i \in \{1, \ldots, n\}$, and if so then $c_i \in X$
- no two of $A \setminus X$, $B \setminus X$, $C \setminus X$ are strongly complete to each other.

Also, d_1 is strongly $A \cup B \cup C$-complete; d_2 is strongly complete to $A \cup B$, and either semiadjacent or strongly adjacent to d_1; d_3 is strongly complete to $A \cup \{d_2\}$; d_4 is strongly complete to $B \cup \{d_2, d_3\}$; d_5 is strongly adjacent to d_3, d_4; and all other pairs are strongly antiadjacent. Let the trigraph just constructed be H, and let $G = H|(V(H) \setminus X)$. Then $\mathcal{S}_5$ is the class of all such trigraphs G.

$\mathcal{S}_6$: Let $n \geq 2$. Construct a trigraph J as follows. Its vertex set is the disjoint union of three sets A', B', C', where $|A'| = |B'| = n+1$ and $|C'| = n$, say $A' = \{a_0, a_1, \ldots, a_n\}, B' = \{b_0, b_1, \ldots, b_n\}$ and $C' = \{c_1, \ldots, c_n\}$. Adjacency is as follows. A', B', C' are strong cliques. For $0 \leq i, j \leq n$ with $(i, j) \neq (0, 0)$, let a_i, b_j be adjacent if and only if $i = j$, and for $1 \leq i \leq n$ and $0 \leq j \leq n$ let c_i, a_j be antiadjacent if and only if $i = j$, and let c_i, b_j be antiadjacent if and only if $i = j$. (There was an error in the definition of $\mathcal{S}_6$ given in [3, 4], corrected here.) a_0, b_0 may be semiadjacent or strongly antiadjacent. All other pairs not specified so far are strongly antiadjacent. Now let $X \subseteq A' \cup B' \cup C' \setminus \{a_0, b_0\}$ with $|C' \setminus X| \geq 2$. Let all adjacent pairs be strongly adjacent except:

- a_i is semiadjacent to c_i for at most one value of $i \in \{1, \ldots, n\}$, and if so then $b_i \in X$
- b_i is semiadjacent to c_i for at most one value of $i \in \{1, \ldots, n\}$, and if so then $a_i \in X$
- a_i is semiadjacent to b_i for at most one value of $i \in \{1, \ldots, n\}$, and if so then $c_i \in X$

Let $G = J \setminus X$. We say that G is *near-antiprismatic*. Let $\mathcal{S}_6$ be the class of all near-antiprismatic trigraphs.

$\mathcal{S}_7$: This is the class of all antiprismatic trigraphs.

For quasi-line trigraphs, we can also eliminate $\mathcal{S}_5$, because of the following.

5.3 *If $G \in \mathcal{S}_5$, then G contains a 5-wheel, and therefore is not quasi-line.*

Proof. Let $A, B, C, d_1, \ldots, d_5, n, X$ etc. be as in the definition of $\mathcal{S}_5$. Let $1 \leq i, j \leq n$ with $i \neq j$. If $a_i, b_j \notin X$, then the subtrigraph induced on $\{d_3, a_i, d_1, b_j, d_4, d_2\}$ is a 5-wheel, a contradiction. Thus X contains one of a_i, b_j, and similarly one of a_j, b_i. Since this holds for all i, j, and since $n \geq 2$ and $|X \cap A|, |X \cap B| \leq 1$, it follows that $n = 2$, and we may assume that $a_2, b_2 \in X$. Since $A \setminus X$, $B \setminus X$ are not strongly complete to each other, it follows that a_1 is semiadjacent to b_1, and so $c_1 \in X$; but then $A \setminus X$ is strongly complete to $C \setminus X$, a contradiction. This proves 5.3. ∎

6 Two-ended stripes

There are fifteen types of stripes described in [4], called $\mathcal{Z}_1, \ldots, \mathcal{Z}_{15}$ (and those we need are defined below). The following is a consequence of the results of [4].

6.1 *Let (J, Z) be an unbreakable claw-free stripe with $|Z| \geq 1$. Then $(J, Z) \in \mathcal{Z}_1 \cup \cdots \cup \mathcal{Z}_{15}$ (and in particular $|Z| \leq 2$).*

Proof. If $V(J)$ is the union of two strong cliques then theorem 10.2 of [4] implies that $(J, Z) \in \mathcal{Z}_1 \cup \cdots \cup \mathcal{Z}_{15}$ as required, so we assume not. By theorem 10.5 of [4], either J is a thickening of an "indecomposable" member of $\mathcal{S}_i$ for some $i \in \{1, \ldots, 7\}$, or J admits a "hex-join". (The meanings of the two terms in quotes are not needed at this point.) In the first case the claim follows from theorem 12.2 of [4]. In the second case theorem 13.1 of [4] implies that $|Z| \leq 2$, and the claim follows from theorems 13.2 and 13.3 of [4]. This proves 6.1. ∎

We leave the reader to verify the next result, which is easy.

6.2 *For $i = 4, 5, 7$, if $(J, Z) \in \mathcal{Z}_i$ then J contains a 5-wheel, and therefore is not quasi-line.*

The main result of this section is the following, which is the first part of 5.1.

6.3 *Every unbreakable quasi-line stripe with at least two ends is a linear interval stripe.*

Proof. Let (G, Z) be an unbreakable quasi-line stripe with $|Z| \geq 2$. By 6.1 it follows that that $(G, Z) \in \mathcal{Z}_i$ for some $i \in \{1, \ldots, 15\}$, and therefore $1 \leq i \leq 5$, since the other classes contain only stripes with one simplicial vertex. By 6.2 it follows that $1 \leq i \leq 3$. Here are the definitions of these three classes:

- $\mathcal{Z}_1$: This is the class of linear interval stripes.
- $\mathcal{Z}_2$: Let $G \in \mathcal{S}_6$, let a_0, b_0 etc. be as in the definition of $\mathcal{S}_6$, with a_0, b_0 strongly antiadjacent, and let $Z = \{a_0, b_0\}$. Then $\mathcal{Z}_2$ is the class of all such (G, Z).
- $\mathcal{Z}_3$: Let H be a graph, and let h_1-h_2-h_3-h_4-h_5 be the vertices of a path of H in order, such that h_1, h_5 both have degree one in H, and every edge of H is incident with one of h_2, h_3, h_4. Let G be obtained from a line trigraph of H by making the edges h_2h_3 and h_3h_4 of H (vertices of G) either semiadjacent or strongly antiadjacent to each other in G. Let $Z = \{h_1h_2, h_4h_5\}$. Then $\mathcal{Z}_3$ is the class of all such (G, Z).

Consequently we may assume that $(G, Z) \in \mathcal{Z}_2 \cup \mathcal{Z}_3$. Suppose first that $(G, Z) \in \mathcal{Z}_2$, and let a_0, b_0, n, X etc. be as in the definition of $\mathcal{S}_6$, with a_0, b_0 strongly antiadjacent, where $Z = \{a_0, b_0\}$. We may assume that for $1 \leq i \leq n$, at most two of $a_i, b_i, c_i \in X$.

Suppose that $|X \cap A| \geq 2$, and $a_1, a_2 \in X$ say. If X also contains b_1, b_2, then it contains neither of c_1, c_2, and they are twins, a contradiction since (G, Z) is unbreakable. Thus one of b_1, b_2 is not in X, and similarly one of c_1, c_2 is not in X. Since for $i = 1, 2$ one of b_i, c_i is not in X, we may assume that $b_1, c_2 \notin X$. Since $(\{b_1, b_2\} \setminus X, \{c_1, c_2\} \setminus X)$ is not a W-join (because (G, Z) is unbreakable), it follows that $b_2, c_1 \in X$. Since a_0 has a neighbour it follows that $n \geq 3$. Suppose that $n = 3$. Then $a_3 \notin X$, and $c_3 \notin X$ (because $|C \setminus X| \geq 2$ from the definition of $\mathcal{S}_6$), and since $(\{a_3\}, \{c_2, c_3\})$ is not a W-join it follows that $b_3 \notin X$, and so c_3, a_3 are strongly antiadjacent. But then c_3 is simplicial in G, contradicting that (G, Z) is unbreakable. Thus $n \geq 4$. If $a_3 \notin X$, then by the same argument with a_2, a_3 exchanged, it follows that X contains exactly one of c_1, c_3, and similarly exactly one of c_2, c_3, which is impossible. Thus $a_3 \notin X$, and similarly $a_3, \ldots, a_n \notin X$. Since $a_3, a_4 \notin X$, the same argument (with A, B exchanged) implies that one of $b_3, b_4 \notin X$, say $b_3 \notin X$. If also $c_3 \notin X$, then the subtrigraph induced on $\{a_3, a_4, c_3, b_1, b_3, c_2\}$ induces a 5-wheel, a contradiction; so $c_3 \in X$. But then $(\{b_1, b_3\}, \{a_3\})$ is a W-join, a contradiction. Thus $|X \cap A| \leq 1$, and similarly $|X \cap B| \leq 1$.

Now suppose that $|X \cap C| \geq 2$, say $c_1, c_2 \in X$. Not both $a_1, a_2 \in X$, and not both b_1, b_2, and yet $(\{a_1, a_2\} \setminus X, \{b_1, b_2\} \setminus X)$ is not a W-join; so X contains exactly one of a_1, a_2, and exactly one of b_1, b_2. Since it contains at most one of a_i, b_i for $i = 1, 2$, we may assume that $a_1, b_2 \notin X$, and $a_2, b_1 \in X$. Since $|C \setminus X| \geq 2$ it follows that $n \geq 4$, and we may assume that $c_3, c_4 \notin X$. But also X contains none of a_3, a_4, b_3, b_4, and $\{a_1, a_3, b_3, b_2, c_3, c_4\}$ induces a 5-wheel, a contradiction. Thus $|X \cap C| \leq 1$, and so $|X| \leq 3$.

Suppose that $n \geq 4$. Since $|X| \leq 3$, we may assume that $a_1, b_1, c_1 \notin X$. Also, since X contains at most one member of each of the three sets $\{a_2, a_3, a_4\}$, $\{b_2, b_3, b_4\}$, $\{c_2, c_3, c_4\}$, and at most two of each of the sets $\{a_i, b_i, c_i\}$ for $i = 2, 3, 4$, we may assume that $a_2, b_3, c_4 \notin X$. But then induces $\{a_1, a_2, c_1, b_3, b_1, c_4\}$ a 5-wheel, a contradiction. Thus $n \leq 3$.

Now $n \geq 2$ since $|C \setminus X| \geq 2$; suppose that $n = 2$. Thus $c_1, c_2 \notin X$. Thus $\{c_1, a_2, b_2\} \setminus X$ and $\{c_2, a_1, b_1\} \setminus X$ are strong cliques. If also c_i is strongly anticomplete to $\{a_i, b_i\} \setminus X$ for $i = 1, 2$, then $(\emptyset, \{c_1, c_2\}, A \cup B \setminus X)$ is a pseudo-2-join, a contradiction. Thus we may assume that $a_1 \notin X$, and c_1, a_1 are semi-adjacent, and so $b_1 \in X$. If $a_2 \in X$ then (G, Z) is a linear interval stripe (in the order $a_0, a_1, c_2, c_1, b_2, b_0$), so we may assume that $a_2 \notin X$. Since G is connected and therefore b_0 has a neighbour, it follows that $b_2 \notin X$. But then $(B, A \cup C)$ is a 1-join, a contradiction. Thus $n = 3$.

Suppose that X has nonempty intersection with $\{a_i, b_i, c_i\}$ for $i = 1, 2, 3$. Then we may assume that $a_1, b_2, c_3 \in X$; but then (G, Z) is a linear interval stripe, with the order

$$a_0, a_2, a_3, c_1, c_2, b_3, b_1, b_0,$$

as required. Thus we may assume that $a_1, b_1, c_1 \notin X$. Suppose that $a_3, c_3 \notin X$ and a_3, c_3 are semiadjacent, and so $b_3 \in X$; thus $b_2 \notin X$, and $\{a_1, a_3, c_1, b_2, b_1, c_3\}$ induces a 5-wheel, a contradiction. Next, suppose that a_3, b_3 are not in X and are semiadjacent, and so $c_3 \in X$, and hence $c_1, c_2 \notin X$; but then $\{a_1, a_3, b_1, b_2, c_1, c_3\}$ induces is a 5-wheel, a contradiction. Thus no two members of $\{a_i, b_i, c_i\} \setminus X$ are semiadjacent, for $i = 2, 3$. But then G is a line trigraph, and

$$(\{a_1\}, \{a_0, a_2, a_3\} \setminus X, \{b_1, c_2, c_3\} \setminus X, \{b_0, b_2, b_3, c_1\} \setminus X)$$

is a biclique, a contradiction.

This completes the argument when $(G, Z) \in \mathcal{Z}_2$; now suppose that $(G, Z) \in \mathcal{Z}_3$. Let H and h_1-h_2-h_3-h_4-h_5 be as in the definition of $\mathcal{Z}_3$. Suppose that some vertex w of H is adjacent to h_2, h_3, h_4. Since $(\{h_2w, h_2h_3\}, \{wh_4, h_3h_4\})$ is not a W-join of G, there is a vertex $w' \neq w, h_2, h_3, h_4$ adjacent to h_3; but then the subtrigraph of G induced on $\{wh_2, wh_4, h_3h_4, w'h_3, h_2h_3, wh_3\}$ is a 5-wheel, a contradiction. Thus there is no such vertex w, and so every vertex of H different from $h_1, \ldots, h_5$ has at most two neighbours in $\{h_2, h_3, h_4\}$.

If some vertex w is adjacent to h_2, h_4 (and therefore not to h_3), then $(\{wh_2, wh_4\}, E(H) \setminus \{wh_2, wh_4\})$ is a pseudo-2-join of G, a contradiction. If there are two vertices w, w' of H both adjacent to h_2, h_3, then $(\{wh_2, w'h_2\}, \{wh_3, w'h_3\})$ is a W-join, a contradiction. Thus at most one vertex of H is adjacent to both h_2, h_3, and similarly at most one to h_3, h_4. But then (G, Z) is a linear interval stripe. This proves 6.3. ∎

7 One-ended stripes

Now we prove an analogous theorem for unbreakable quasi-line stripes (J, Z) with $|Z| = 1$, for the second part of 5.1. First let us make it easier to identify bubbles.

7.1 *Let G be a circular interval trigraph, and let z be a simplicial vertex of G. Then (G, z) is a bubble.*

Proof. The result is clear if G is a strong clique, and so we may assume that some vertex is antiadjacent to z. Let Σ and $F_1, \dots, F_k \subseteq \Sigma$ be as in the definition of circular interval trigraph. Since some vertex is antiadjacent to z, the union of all the sets F_i that contain z is homeomorphic to a closed interval I say. Moreover, since z is simplicial, every two vertices in I are strongly adjacent; and so we may replace all the sets F_i that contain z by I. Thus we may assume that z belongs to F_1 and to none of $F_2, \dots, F_k$. Moreover, since z is simplicial we may assume that no endpoint of F_1 belongs to $V(G)$ (by extending F_1 slightly if it has an endpoint in $V(G)$). But then (G, Z) is a bubble. This proves 7.1. ∎

We must look at several of the classes $\mathcal{Z}_i$, and some of them need "hex-expansion", so we begin by defining this. If A, B, C are strong cliques of a trigraph G, pairwise disjoint and with union $V(G)$, we call (G, A, B, C) a *three-cliqued trigraph*. One type of three-cliqued trigraph of special interest to us is as follows. Let G be a circular interval trigraph, and let Σ be a circle with $V(G) \subseteq \Sigma$, and $F_1, \dots, F_k \subseteq \Sigma$, as in the definition of circular interval trigraph. By a *line* we mean either a subset $X \subseteq V(G)$ with $|X| \leq 1$, or a subset of some F_i homeomorphic to the closed unit interval, with both end-points in $V(G)$ and strongly adjacent. Let L_1, L_2, L_3 be pairwise disjoint lines with $V(G) \subseteq L_1 \cup L_2 \cup L_3$; then $(G, V(G) \cap L_1, V(G) \cap L_2, V(G) \cap L_3)$ is a three-cliqued claw-free trigraph. We call such a three-cliqued trigraph a *trisected circular interval trigraph.* (Note that there are three-cliqued trigraphs (G, A, B, C) with G a circular interval trigraph, that are not trisected. For instance, let G be the graph with vertex set $\{v_1, \dots, v_5\}$ and edge set

$$\{v_1v_2, v_2v_3, v_3v_4, v_1v_4, v_4v_5, v_1v_5\};$$

then G is a circular interval trigraph, but the partition $\{\{v_1, v_4\}, \{v_2, v_3\}n\{v_5\}\}$ into three cliques does not yield a trisection.)

Let (G_i, A_i, B_i, C_i) be a three-cliqued trigraph with $V(G_i) \neq \emptyset$, for $i = 1, 2$. Construct G by taking the disjoint union of G_1 and G_2, and then making

- A_1 strongly complete to $A_2 \cup C_2$ and strongly anticomplete to B_2
- B_1 strongly complete to $A_2 \cup B_2$ and strongly anticomplete to C_2

- C_1 strongly complete to $B_2 \cup C_2$ and strongly anticomplete to A_2.

We say $(G, A_1 \cup A_2, B_1 \cup B_2, C_1 \cup C_2)$ is a *hex-join* of (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2). If G_1, G_2 are claw-free then so is G, but hex-joins do not necessarily preserve being quasi-line.

We will often need the following.

7.2 *Let (G, A, B, C) is a hex-join of (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2). Suppose that*

- *G is quasi-line,*
- *(G_1, A_1, B_1, C_1) is a trisected circular interval trigraph, and G_1 has a triad, and*
- *there are no twins of G both in $V(G_2)$, and there is no W-join (P, Q) of G with $P \cup Q \subseteq V(G_2)$.*

Then (G, A, B, C) is a trisected circular interval trigraph.

Proof. Let $T \subseteq V(G_1)$ be a triad. Let H be the trigraph induced on $T \cup V(G_2)$. Then T is isolated in H, so by 3.1 it follows that H is a circular interval trigraph. Let $V(H) \subseteq \Sigma$ where Σ is a circle, and $V(H)$ is in the appropriate circular order. Let $T = \{t_1, t_2, t_3\}$ where $t_1 \in A_1$, $t_2 \in B_1$ and $t_3 \in C_1$. Let $L_1 \subseteq \Sigma$ be the closed interval of Σ with endpoints t_2, t_3 not containing t_1, and define L_2, L_3 similarly. Since t_2, t_3 are antiadjacent to t_1, it follows that every vertex in L_1 is antiadjacent to t_1, and similarly for $i = 1, 2, 3$ every vertex of L_i is antiadjacent to t_i. Since each vertex of G_2 is antiadjacent to exactly one of t_1, t_2, t_3, we deduce that $V(G_2) \cap L_1 = B_2$, and $V(G_2) \cap L_2 = C_2$, and $V(G_2) \cap L_3 = A_2$. We deduce that (G_2, A_2, B_2, C_2) is a trisected circular interval trigraph. Now the hex-join of the two trisected circular interval trigraphs (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2) is a third trisected circular interval trigraph (to see this, arrange the six cliques in a circle in the order $A_1, A_2, B_1, B_2, C_1, C_2$, in such a way that for $i = 1, 2$ the restriction to $A_i \cup B_i \cup C_i$ gives a representation of G_i as a circular interval trigraph). This proves 7.2. ∎

7.3 *Let $(G, A_1 \cup A_2, B_1 \cup B_2, C_1 \cup C_2)$ be a hex-join of (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2). If G is quasi-line, then*

- *if there exist $a, a' \in A_2, b \in B_2, c \in C_2$ such that a, b, c are pairwise adjacent, and a' is antiadjacent to b, c, then A_1 is strongly complete to B_1*
- *if there exist $a, a' \in A_2, b, b' \in B_2, c \in C_2$ such that the pairs $ab, a'c$ are adjacent, and the pairs $ac, bc, a'b$ are antiadjacent, and b' is adjacent to all of a, a', c, then $C_1 = \emptyset$.*

Proof. For the first statement, suppose that $a_1 \in A_1$ and $b_1 \in B_1$ are antiadjacent; then $\{a_1, c, b, b_1, a', a\}$ induces a 5-wheel. For the second, if $c_1 \in C_1$ then $\{a, a', c, c_1, b, b'\}$ induces a 5-wheel. This proves 7.3. ∎

Let (G, A, B, C) be a three-cliqued trigraph, and let $z \in A$ such that z is strongly anticomplete to $B \cup C$. Let V_1, V_2, V_3 be three disjoint sets of new vertices, and let G' be the trigraph obtained by adding V_1, V_2, V_3 to G with the following adjacencies:

- V_1 and $V_2 \cup V_3$ are strong cliques

- V_1 is strongly complete to $B \cup C$ and strongly anticomplete to A
- V_2 is strongly complete to $C \cup A$ and strongly anticomplete to B
- V_3 is strongly complete to $A \cup B$ and strongly anticomplete to C.

(The adjacency between V_1 and $V_2 \cup V_3$ is unspecified.) It follows that z is a simplicial vertex of G'. We say that (G', z) is a *hex-expansion* of (G, A, B, C). Hex-expansions are thus a special case of hex-joins, and we often need to apply 7.3 to hex-expansions. It is a little tricky to keep track of the symmetry, so for convenience, let us write out some consequences of 7.3 for hex-expansions.

7.4 *Let (G', z) be a hex-expansion of (G, A, B, C), with sets V_1, V_2, V_3 as above.*

- *If there exist $a \in A, b \in B$, and $c, c' \in C$, such that a, b, c are pairwise adjacent, and c' is antiadjacent to a, b, then V_1 is strongly complete to V_2*
- *If there exist $a \in A, b, b' \in B$, and $c \in C$ such that a, b, c are pairwise adjacent, and b' is antiadjacent to a, c, then V_1 is strongly complete to V_3*
- *If there exist $a, a' \in A, b \in B, c \in C$, and $d \in B \cup C \setminus \{b, c\}$, such that the pairs $ab, b'c$ are adjacent, and the pairs bc, ac, ab' are antiadjacent, and d is adjacent to all of a, a', b, c, then $V_1 = \emptyset$.*
- *If there exist $a \in A, b, b' \in B, c \in C$, and $d \in A \cup C \setminus \{a, c\}$, such that the pairs bc, ab' are adjacent, and the pairs $ab, ac, b'c$ are antiadjacent, and d is adjacent to all of a, b, b', c, then $V_2 = \emptyset$.*
- *If there exist $a \in A, b \in B$, $c, c' \in C$, and $d \in A \cup B \setminus \{a, b\}$, such that the pairs ac, bc' are adjacent, and the pairs bc, ab, ac' are antiadjacent, and d is adjacent to all of a, b, c, c', then $V_3 = \emptyset$.*

Proof. Since $(G', C \cup V_1, A \cup V_2, B \cup V_3)$ is the hex-join of (H, V_1, V_2, V_3) and (G, C, A, B), where $H = G'|(V_1 \cup V_2 \cup V_3)$, the first assertion follows from the first assertion of 7.3, and also the third assertion with $d \in B$ follows from the second assertion of 7.3. There are five other ways to view this as a hex-join; for instance, $(G', C \cup V_2, B \cup V_1, A \cup V_3)$ is the hex-join of (H, V_2, V_1, V_3) and (G, C, B, A), and the second statement of 7.3 applied to this yields the fifth assertion of the theorem when $d \in B$. We leave checking the remainder to the reader. This proves 7.4. ∎

The analogue of 6.3 is the following.

7.5 *Every unbreakable quasi-line stripe with one end is a bubble.*

Proof. Let (G, Z) be an unbreakable quasi-line stripe where $|Z| = 1$. By 6.1, Then $(G, Z) \in \mathcal{Z}_1 \cup \cdots \cup \mathcal{Z}_{15}$, and hence belongs to $\mathcal{Z}_i$ for some i with $5 \leq i \leq 15$ since no 1-ended stripes belong to $\mathcal{Z}_i$ for $1 \leq i \leq 4$. By 6.2, $i \neq 5, 7$, and $\mathcal{Z}_6$ is the class of bubbles, so we must check $\mathcal{Z}_i$ for $i = 8, 9, \ldots, 15$. Let $Z = \{z\}$ say.

(1) $(G, Z) \notin \mathcal{Z}_8$.

This follows from 5.3.

(2) *If* $(G, Z) \in \mathcal{Z}_9$ *then* (G, Z) *is a bubble.*

From the definition of $\mathcal{Z}_9$, it follows that G is antiprismatic, with at least one triad, and every triad contains z. Suppose first that there is only one triad. Then this triad is isolated, and by 3.1 it follows that G is a circular interval trigraph; and so 7.1 implies that (G, Z) is a bubble.

Thus we may assume that z belongs to at least two triads. By 4.4, either G is a line trigraph of a subgraph of K_6, or G is of H_7-type. If G is a line trigraph, then since z is its only simplicial vertex, theorem 10.3 of [4] implies that (G, Z) is a bubble. If G is of H_7-type then G admits a generalized 2-join, which is impossible. This proves (2).

(3) $(G, Z) \notin \mathcal{Z}_{10}$.

Suppose that $(G, Z) \in \mathcal{Z}_{10}$. From the definition of $\mathcal{Z}_{10}$, there is a three-cliqued trigraph (H, A, B, C) and a subset $X \subseteq V(H)$ such that (G, Z) is a hex-expansion of $(H \setminus X, A \setminus X, B \setminus X, C)$, satisfying the following:

- $V(H) = \{z, a_1, a_2, b_0, b_1, b_2, b_3, c_1, c_2, d\}$
- $A = \{z, a_1, a_2, d\}$, $B = \{b_0, b_1, b_2, b_3\}$, $C = \{c_1, c_2\}$ and $\{a_1, b_1, c_2\}$ are strong cliques;
- a_2 is strongly adjacent to b_0 and semiadjacent to b_1; b_2, c_2 are semiadjacent; b_2, c_1 are strongly adjacent; b_3, c_1 are either semiadjacent or strongly adjacent; b_0, d are either semiadjacent or strongly adjacent; and all other pairs are strongly antiadjacent
- $X \subseteq \{a_2, b_2, b_3, d\}$ such that either $a_2 \in X$ or $\{b_2, b_3\} \subseteq X$.

Let V_1, V_2, V_3 be as in the definition of hex-expansion; thus, V_1 is strongly complete to $(B \cup C) \setminus X$, and V_2 is strongly complete to $(C \cup A) \setminus X$, and V_3 to $(A \cup B) \setminus X$, and V_2 is strongly complete to V_3. From the first statement of 7.4 applied to a_1, b_1, c_2, c_1 it follows that V_1 is strongly complete to V_2, and from the same applied to a_1, b_1, b_0, c_2 it follows that V_1 is strongly complete to V_3. Moreover V_2 is strongly complete to V_3, so $V_1 \cup V_2 \cup V_3$ is a strong clique.

First suppose that $a_2 \in X$. Since

$$(V_3, \{b_0, z, d\} \setminus X, V(G) \setminus (\{b_0, z, d\} \cup V_3))$$

is not a pseudo-2-join, it follows that $\{b_0, z, d\} \setminus X$ is strongly stable, and so $d \in X$. But then b_0 is simplicial, contradicting that (G, Z) is unbreakable. Thus $a_2 \notin X$, and so $b_2, b_3 \in X$; but then c_1 is simplicial, again a contradiction. This proves (3).

(4) *If* $(G, Z) \in \mathcal{Z}_{11}$ *then* (G, Z) *is a bubble.*

From the definition of $\mathcal{Z}_{11}$, there is a three-cliqued trigraph (H, A, B, C) and a subset X of $V(H)$, such that (G, z) is a hex-expansion of $(H \setminus X, A \setminus X, B \setminus X, C \setminus X)$, and (H, A, B, C) has the following properties.

- $|A| = n+2, |B| = n+1$ and $|C| = n \geq 2$, say $A = \{a_0, a_1, \dots, a_n, z\}, B = \{b_0, b_1, \dots, b_n\}$ and $C = \{c_1, \dots, c_n\}$.
- For $0 \leq i, j \leq n$, a_i, b_j are adjacent if and only if $i = j$; and for $1 \leq i, j \leq n$, c_i, a_j are antiadjacent if and only if $i = j$, and c_i, b_j are antiadjacent if and only if $i = j$.
- All other pairs are strongly antiadjacent.

Moreover, $X \subseteq A \cup B \cup C \setminus \{b_0, z\}$ with $|C \setminus X| \geq 2$. There are no semiadjacent pairs except

- a_i is semiadjacent to c_i for at most one value of $i \in \{1, \dots, n\}$, and if so then $b_i \in X$
- b_i is semiadjacent to c_i for at most one value of $i \in \{1, \dots, n\}$, and if so then $a_i \in X$
- a_i is semiadjacent to b_i for at most one value of $i \in \{1, \dots, n\}$, and if so then $c_i \in X$
- a_0 may be semiadjacent to b_0.

Let V_1, V_2, V_3 be as in the definition of a hex-expansion; thus V_1 is strongly complete to $(B \cup C) \setminus X$, and so on, and V_2 is strongly complete to V_3. We may assume that for $1 \leq i \leq n$, not all three of a_i, b_i, c_i belong to X.

We claim that V_1 is not strongly complete to V_3. For if $a_0 \notin X$ then the claim follows since $(V_3, \{a_0, b_0\}, V(G) \setminus (V_3 \cup \{a_0, b_0\}))$ is not a pseudo-2-join, and if $a_0 \in X$ then the claim follows since b_0 is not simplicial. Thus V_1 is not strongly complete to V_3. If there exists $i \in \{1, \dots, n\}$ such that $a_i, b_i \notin X$, then (since $|C \setminus X| \geq 2$) there exists $j \neq i$ such that $c_j \notin X$, and the quadruple a_i, b_i, b_0, c_j violates the second assertion of 7.4. Thus there is no such i. We may assume that $c_1, c_2 \notin X$. If $a_1, a_2 \in X$, then c_1, c_2 are twins if $b_1, b_2 \in X$ and $(\{c_1, c_2\}, \{b_1, b_2\} \setminus X)$ is a W-join otherwise, in either case a contradiction. So X contains at most one of a_1, a_2, and similarly at most one of b_1, b_2; and since it contains at least one of a_1, b_1, and at least one of a_2, b_2, we may assume that $a_1, b_2 \notin X$, and $a_2, b_1 \in X$. If $c_3 \notin X$ then the same argument applied to c_1, c_3 and to c_2, c_3 shows that X contains exactly one of a_1, a_3 and exactly one of a_2, a_3, which is impossible. Thus $c_3, \dots, c_n \in X$. If $n \geq 3$, and $a_3 \in X$ then $(\{b_2, b_3\}, \{c_2\})$ is a W-join, and if $b_3 \in X$ then $(\{a_1, a_3\}, \{c_1\})$ is a W-join, in either case a contradiction; so $n = 2$. Now the circular order

$$z, a_0, b_0, b_2, c_1, c_2, a_1, z$$

(with a_0 removed if it belongs to X) shows that $(H \setminus X, A \setminus X, B \setminus X, C \setminus X)$ is a circular interval trigraph; and so 7.2 implies that G is a circular interval trigraph and hence (G, z) is a bubble by 7.1. This proves (4).

(5) $(G, Z) \notin \mathcal{Z}_{12}$.

Suppose that $(G, Z) \in \mathcal{Z}_{12}$. From the definition of $\mathcal{Z}_{12}$, there is a three-cliqued trigraph (H, A, B, C) and a subset $X \subseteq A$ such that (G, Z) is a hex-expansion of $(H \setminus X, A \setminus X, B, C)$, satisfying the following:

- $A = \{v_3, v_4, v_5, v_6, v_9, z\}$, $B = \{v_1, v_2\}$, and $C = \{v_7, v_8\}$

- z is strongly anticomplete to $B \cup C$; v_9 is strongly adjacent to v_1, v_8 and strongly antiadjacent to v_2, v_7; v_1 is strongly antiadjacent to v_4, v_5, v_6, v_7, semiadjacent to v_3 and strongly adjacent to v_8; v_2 is strongly antiadjacent to v_5, v_6, v_7, v_8 and strongly adjacent to v_3; v_3, v_4 are strongly antiadjacent to v_7, v_8; v_5 is strongly antiadjacent to v_8; v_6 is semiadjacent to v_8 and strongly adjacent to v_7; and either v_2, v_4 are adjacent or v_5, v_7 are adjacent

- $X \subseteq \{v_3, v_4, v_5, v_6\}$, such that
  - v_2 is not strongly anticomplete to $\{v_3, v_4\} \setminus X$
  - v_7 is not strongly anticomplete to $\{v_5, v_6\} \setminus X$
  - if $X \cap \{v_4, v_5\} = \emptyset$ then v_2 is adjacent to v_4 and v_5 is adjacent to v_7.

Let V_1, V_2, V_3 be as in the definition of a hex-expansion. From the second assertion of 7.4 applied to $\{v_9, v_1, v_2, v_8\}$ it follows that V_1 is complete to V_3, and similarly V_1 is complete to V_2, so $V_1 \cup V_2 \cup V_3$ is a clique. Now

$$(\{v_9\}, (A \setminus (X \cup \{v_9\})) \cup V_2 \cup V_3, \{v_1, v_8\}, V_1 \cup \{v_2, v_7\})$$

is not a biclique; and so there exist $u \in (A \setminus (X \cup \{v_9\})) \cup V_2 \cup V_3$, $v \in \{v_1, v_8\}$ and $w \in V_1 \cup \{v_2, v_7\}$ such that $u, v, w \notin X$, u, v are adjacent, and w is adjacent to one of them and antiadjacent to the other. Now there is a symmetry exchanging v_i with v_{9-i} for $1 \leq i \leq 8$, fixing v_9 and z, exchanging B and C, and exchanging V_2 and V_3. Because of this symmetry we may assume that $v = v_1$. Since v_1 is strongly anticomplete to $\{v_4, v_5, v_6, z\} \cup V_2$, it follows that $u \in \{v_3\} \cup V_3$. If $u = v_3$ (and therefore $v_3 \notin X$) then $w \in V_1$ (because v_1, v_3 are both strongly adjacent to v_2 and strongly antiadjacent to v_7); but then v_3, v_9, v_2, v_8, v_1 contradicts the third assertion of 7.4 since $V_1 \neq \emptyset$. Thus $u \in V_3$, and therefore u, v are both strongly complete to $\{v_2\} \cup V_1$ and strongly anticomplete to v_7; but this is contrary to the existence of w. This proves (5).

(6) *If* $(G, Z) \in \mathcal{Z}_{13}$ *then* (G, Z) *is a bubble.*

From the definition of $\mathcal{Z}_{13}$, (G, Z) is a hex-expansion of a trisected circular interval trigraph in which every vertex is in a triad. From 7.2 we deduce that G is a circular interval trigraph, and so by 7.1 (G, z) is a bubble. This proves (6).

(7) $(G, Z) \notin \mathcal{Z}_{14}$.

Suppose that $(G, Z) \in \mathcal{Z}_{14}$. From the definition of $\mathcal{Z}_{14}$, (G, Z) is a hex-expansion of a three-cliqued trigraph (G_1, A_1, A_2, A_3), and G_1 is a line trigraph of a graph H, satisfying the following.

- There are four vertices v_0, v_1, v_2, v_3 of H, such that v_1, v_2, v_3 are pairwise nonadjacent, v_1 is the only neighbour of v_0, and v_1, v_2, v_3 have degree at least three.

- Every vertex of H different from v_0, v_1, v_2, v_3 is adjacent to both v_2, v_3, and at most one of them is nonadjacent to v_1.

- For $i = 1, 2, 3$, A_i is the set of edges of H incident with v_i, and z is the edge $v_0 v_1$.

Let $V(H) = \{v_0, v_1, \ldots, v_k\}$ where $k \geq 6$, and $v_5, \ldots, v_{k-1}$ are adjacent to all of v_1, v_2, v_3. Let V_1, V_2, V_3 be as in the definition of a hex-expansion. Thus V_2 is strongly complete to V_3. From the second assertion of 7.4 applied to $\{v_1v_4, v_2v_4, v_3v_4, v_2v_5\}$ it follows that V_1 is complete to V_3, and similarly V_1 is complete to V_2. But then G is a line trigraph, and since $V(G)$ is not the union of two strong cliques, this contradicts theorem 10.3 of [4]. This proves (7).

(8) $(G, Z) \notin \mathcal{Z}_{15}$.

Suppose that $(G, Z) \in \mathcal{Z}_{15}$. From the definition of $\mathcal{Z}_{15}$, there is a three-cliqued trigraph (H, A, B, C) and a subset $X \subseteq B \cup C$ such that (G, Z) is a hex-expansion of $(H \setminus X, A, B \setminus X, C \setminus X)$, satisfying the following. (We are correcting an error from [4] here.)

- $V(H) = \{v_1, \ldots, v_8\}$ where $z = v_8$.
- v_i, v_j are strongly adjacent for $1 \leq i < j \leq 6$ with $j - i \leq 2$; the pairs v_1v_5 and v_2v_6 are strongly antiadjacent; v_1, v_6, v_7 are pairwise strongly adjacent, and v_7 is strongly antiadjacent to v_2, v_3, v_4, v_5; v_7, v_8 are strongly adjacent, and v_8 is strongly antiadjacent to $v_1, \ldots, v_6$; the pairs v_1v_4 and v_3v_6 are semiadjacent, and v_2 is antiadjacent to v_5.
- $A = \{v_7, v_8\}$, $B = \{v_1, v_2, v_3\}, C = \{v_4, v_5, v_6\}$, and $X \subseteq \{v_3, v_4\}$.

Let V_1, V_2, V_3 be as in the definition of a hex-expansion. There is a symmetry exchanging v_i with v_{7-i} for $1 \leq i \leq 6$, fixing v_7 and z, exchanging B with C, and exchanging V_2 with V_3. From the first assertion of 7.4 applied to $\{v_1, v_5, v_6, v_7\}$, it follows that V_1 is complete to V_2, and from the symmetry V_1 is complete to V_3. Moreover, by the fourth assertion of 7.4 applied to v_7, v_1, v_3, v_5, v_6, either $v_3 \in X$ or $V_2 = \emptyset$. Suppose that $v_3 \in X$. Since v_5 is not simplicial, it follows that v_2 is semiadjacent to v_5. But then

$$(\{v_6\}, V_1 \cup V_2 \cup \{v_4, v_5\} \setminus X, \{v_1, v_7\}, \{v_2, v_8\} \cup V_3)$$

is a biclique, a contradiction. Thus $v_3 \notin X$, and so $V_2 = \emptyset$. From the symmetry, $V_3 = \emptyset$. But then $(\{v_7, v_8\}, V(G) \setminus \{v_7, v_8\})$ is a 1-join, a contradiction. This proves (8).

From (1)–(8), this proves 7.5. ∎

8 Stripes without ends

In view of 6.3 and 7.5, to complete the proof of 5.1 and hence to prove 1.1, it remains to show the following:

8.1 *If $(G, \emptyset)$ is an unbreakable quasi-line stripe, then G is a circular interval trigraph.*

Proving 8.1 is the goal of the remainder of the paper. We say that a trigraph G *admits a hex-join* if there exist A, B, C such that (G, A, B, C) is the hex-join of two three-cliqued trigraphs. The main theorem of [3] asserts:

8.2 *Let G be a claw-free trigraph. Then either*

- *$G \in \mathcal{S}_0 \cup \cdots \cup \mathcal{S}_7$, or*
- *G admits either twins, or a W-join, or a 0-join, or a 1-join, or a generalized 2-join, or a hex-join.*

We begin with:

8.3 *If $(G, \emptyset)$ is an unbreakable quasi-line stripe, and either G is antiprismatic, or G does not admit a hex-join, then G is a circular interval trigraph.*

Proof. Let $(G, \emptyset)$ be an unbreakable quasi-line stripe, and suppose that G is not a circular interval trigraph. We must show that G is not antiprismatic, and G admits a hex-join. By hypothesis, G does not admit twins, a W-join, a 0-join, a 1-join or a generalized 2-join, and has no simplicial vertex. Since every trigraph of H_7-type admits a generalized 2-join, it follows that G is not of H_7-type. Since G has no simplicial vertex, and $|V(G)| \geq 3$ (since G is not a circular interval trigraph), theorem 10.3 of [4] implies that G is not a line trigraph. Consequently 4.1 implies that G is not antiprismatic.

Suppose that $G \in \mathcal{S}_i$ for some $i \in \{0, \ldots, 7\}$. By 5.2 and 5.3, $i \neq 1, 2, 4, 5$, and we have seen that $i \neq 0, 7$, and $i \neq 3$ by hypothesis. Thus $i = 6$; let a_0, b_0 be as in the definition of $\mathcal{S}_6$. If a_0, b_0 are strongly antiadjacent then they are both simplicial, which is impossible. If a_0, b_0 are semiadjacent, let $V_1 = \{a_0, b_0\}$ and $V_2 = V(G) \setminus V_1$; since V_1, V_2 are not strongly stable, $(\emptyset, V_1, V_2)$ is a pseudo-2-join, a contradiction. This proves that $G \notin \mathcal{S}_i$ for $i \in \{0, \ldots, 7\}$. By 8.2, G admits a hex-join. This proves 8.3. ∎

In view of 8.3, we need to understand the quasi-line trigraphs G such that $(G, \emptyset)$ is an unbreakable quasi-line stripe and G admits a hex-join and is not antiprismatic. To do so, we apply a theorem of [4] describing the structure of all three-cliqued claw-free trigraphs, and we next state that.

Here are some types of three-cliqued claw-free trigraphs.

- Let v_1, v_2, v_3 be distinct nonadjacent vertices of a graph H, such that every edge of H is incident with one of v_1, v_2, v_3. Let v_1, v_2, v_3 all have degree at least three, and let all other vertices of H have degree at least one. Moreover, for all distinct $i, j \in \{1, 2, 3\}$, let there be at most one vertex different from v_1, v_2, v_3 that is adjacent to v_i and not to v_j in H. Let A, B, C be the sets of edges of H incident with v_1, v_2, v_3 respectively, and let G be a line trigraph of H. Then (G, A, B, C) is a three-cliqued claw-free trigraph; let $\mathcal{TC}_1$ be the class of all such three-cliqued trigraphs such that every vertex is in a triad.
- We denote by $\mathcal{TC}_2$ the class of trisected circular interval trigraphs (with notation as usual) with the additional properties that no three of $F_1, \ldots, F_k$ have union Σ and that every vertex is in a triad.
- Let G, J, A', B', C', X be as in the definition of a near-antiprismatic trigraph. Let $A = A' \setminus X$ and define B, C similarly; then (G, A, B, C) is a three-cliqued claw-free trigraph. We denote by $\mathcal{TC}_3$ the class of all such three-cliqued trigraphs with the additional property that every vertex is in a triad.

- Let G be an antiprismatic trigraph and let A, B, C be a partition of $V(G)$ into three strong cliques; then (G, A, B, C) is a three-cliqued claw-free trigraph. We denote the class of all such three-cliqued trigraphs by $\mathcal{TC}_4$. (In [1] we described explicitly all three-cliqued antiprismatic graphs, and their "changeable" edges; and this therefore provides a description of the three-cliqued antiprismatic trigraphs.) Note that in this case there may be vertices that are in no triads.

- $\mathcal{TC}_5$ comprises two classes of trigraphs. First, let H be the trigraph with vertex set $\{v_1, \ldots, v_8\}$ and adjacency as follows: v_i, v_j are strongly adjacent for $1 \leq i < j \leq 6$ with $j - i \leq 2$; the pairs v_1v_5 and v_2v_6 are strongly antiadjacent; v_1, v_6, v_7 are pairwise strongly adjacent, and v_7 is strongly antiadjacent to v_2, v_3, v_4, v_5; v_7, v_8 are strongly adjacent, and v_8 is strongly antiadjacent to $v_1, \ldots, v_6$; the pairs v_1v_4 and v_3v_6 are semiadjacent, and v_2 is antiadjacent to v_5. Let $A = \{v_1, v_2, v_3\}, B = \{v_4, v_5, v_6\}$ and $C = \{v_7, v_8\}$. Let $X \subseteq \{v_3, v_4\}$; then $(H \setminus X, A \setminus X, B \setminus X, C)$ is a three-cliqued claw-free trigraph, and all its vertices are in triads.

- The second class of trigraphs in $\mathcal{TC}_5$ is as follows. Let H be the trigraph with vertex set $\{v_1, \ldots, v_9\}$, and adjacency as follows: the sets $A = \{v_1, v_2\}$, $B = \{v_3, v_4, v_5, v_6, v_9\}$ and $C = \{v_7, v_8\}$ are strong cliques; v_9 is strongly adjacent to v_1, v_8 and strongly antiadjacent to v_2, v_7; v_1 is strongly antiadjacent to v_4, v_5, v_6, v_7, semiadjacent to v_3 and strongly adjacent to v_8; v_2 is strongly antiadjacent to v_5, v_6, v_7, v_8 and strongly adjacent to v_3; v_3, v_4 are strongly antiadjacent to v_7, v_8; v_5 is strongly antiadjacent to v_8; v_6 is semiadjacent to v_8 and strongly adjacent to v_7; and the adjacency between the pairs v_2v_4 and v_5v_7 is arbitrary. Let $X \subseteq \{v_3, v_4, v_5, v_6\}$, such that

  - v_2 is not strongly anticomplete to $\{v_3, v_4\} \setminus X$
  - v_7 is not strongly anticomplete to $\{v_5, v_6\} \setminus X$
  - if $v_4, v_5 \notin X$ then v_2 is adjacent to v_4 and v_5 is adjacent to v_7.

  Then $(H \setminus X, A, B \setminus X, C)$ is a three-cliqued claw-free trigraph. If in addition every vertex is in a triad, we say that $(H \setminus X, A, B \setminus X, C) \in \mathcal{TC}_5$.

If (G, A, B, C) is a three-cliqued trigraph, and H is a thickening of G, let X_v $(v \in V(G))$ be the corresponding strong cliques of H; then $\cup_{v \in A} X_v$ is a strong clique A' say of H, and if we define B', C' from B, C similarly, then (H, A', B', C') is a three-cliqued trigraph, that we say is a *thickening* of (G, A, B, C). If (G, A, B, C) is a three-cliqued trigraph, and $\{P, Q, R\} = \{A, B, C\}$, then (G, P, Q, R) is also a three-cliqued trigraph, and we say it is a *permutation* of (G, A, B, C).

Let $n \geq 0$, and for $1 \leq i \leq n$, let (G_i, A_i, B_i, C_i) be a three-cliqued trigraph, where $G_1, \ldots, G_n$ all have at least one vertex and are pairwise vertex-disjoint. Let $A = A_1 \cup \cdots \cup A_n$, $B = B_1 \cup \cdots \cup B_n$, and $C = C_1 \cup \cdots \cup C_n$, and let G be the trigraph with vertex set $V(G_1) \cup \cdots \cup V(G_n)$ and with adjacency as follows:

- for $1 \leq i \leq n$, $G|V(G_i) = G_i$;

- for $1 \leq i < j \leq n$, A_i is strongly complete to $V(G_j) \setminus B_j$; B_i is strongly complete to $V(G_j) \setminus C_j$; and C_i is strongly complete to $V(G_j) \setminus A_j$; and

- for $1 \leq i < j \leq n$, if $u \in A_i$ and $v \in B_j$ are adjacent then u, v are both in no triads; and the same applies if $u \in B_i$ and $v \in C_j$, and if $u \in C_i$ and $v \in A_j$.

In particular, A, B, C are strong cliques, and so (G, A, B, C) is a three-cliqued trigraph; we call the sequence (G_i, A_i, B_i, C_i) $(i = 1, \ldots, n)$ a *worn hex-chain* for (G, A, B, C). When $n = 2$ we say that (G, A, B, C) is a *worn hex-join* of (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2).

Theorem 4.1 of [4] asserts the following:

8.4 *Every three-cliqued claw-free trigraph admits a worn hex-chain into terms each of which is a thickening of a permutation of a member of one of $\mathcal{TC}_1, \ldots, \mathcal{TC}_5$.*

To complete the proof of 8.1, we need a few more lemmas.

8.5 *Let (G, A, B, C) be a three-cliqued quasi-line trigraph such that $(G, \emptyset)$ is an unbreakable stripe, and such that (G, A, B, C) is a hex-join of (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2). Then (G_1, A_1, B_1, C_1) is not a permutation of a member of $\mathcal{TC}_1$.*

Proof. Suppose it is; thus $(G_1, A_1, B_1, C_1) \in \mathcal{TC}_1$. Choose H, v_1, v_2, v_3 as in the definition of $\mathcal{TC}_1$. Suppose first that some vertex u of H is adjacent to all of v_1, v_2, v_3. Let e_i be the edge uv_i for $i = 1, 2, 3$. Since v_1 has degree at least three, there is an edge f_1 incident with v_1 and not with u; and so $e_1, f_1 \in A$, $e_2 \in B$, $e_3 \in C$, and by the first assertion of 7.3 (with the parts of the hex-join exchanged) it follows that A_2 is strongly complete to C_2. Similarly A_2, B_2, C_2 are pairwise strongly complete, and so G_2 is a strong clique. Since G has no twins, it follows that $|A_2|, |B_2|, |C_2| \leq 1$. Thus G is a line trigraph (if there exists $a_2 \in A_2$, add a_2 to H as an edge joining v_1, v_2, and similarly for B_2, C_2). But this contradicts theorem 10.3 of [4].

Thus no such vertex u exists, and so every vertex different from v_1, v_2, v_3 has degree at most two. Suppose next that some vertex of H has degree one; say u is adjacent only to v_1. Let u' be another neighbour of v_1. If u' also has degree one, then $uv_1, u'v_1$ are twins in G, a contradiction. If u' has degree two in H, let u' be adjacent to v_1, v_2 say; then $(\{uv_1, u'v_1\}, \{u'v_2\})$ is a W-join of G, a contradiction. This proves that every vertex in H different from v_1, v_2, v_3 has degree two. Suppose that u_1, u_2 are distinct vertices of H, both adjacent to both v_1, v_2. Then $(\{u_1v_1, u_2v_1\}, \{u_1v_2, u_2v_2\})$ is a W-join of G, a contradiction. It follows that no two vertices in $V(H) \setminus \{v_1, v_2, v_3\}$ have the same neighbours; but this is impossible since v_1, v_2, v_3 have degree at least three. This proves 8.5. ∎

8.6 *Let (G, A, B, C) be a three-cliqued quasi-line trigraph such that $(G, \emptyset)$ is an unbreakable stripe, and such that (G, A, B, C) is a hex-join of (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2). Suppose that (G_1, A_1, B_1, C_1) is a permutation of a member of $\mathcal{TC}_3$. Then $|V(G_1)| = 6$ and (G_1, A_1, B_1, C_1) is a trisected circular interval trigraph.*

Proof. Suppose (without loss of generality) that $(G_1, A_1, B_1, C_1) \in \mathcal{TC}_3$. Let

$$J, A', B', C', a_0, \ldots, a_n, b_0, \ldots, b_n, c_1, \ldots, c_n, X$$

be as in the definition of near-antiprismatic, such that

$$(G_1, A_1, B_1, C_1) = (J \setminus X, A' \setminus X, B' \setminus X, C' \setminus X).$$

Since $|C' \setminus X| \geq 2$, we may assume that for $1 \leq i \leq n$, not all of a_i, b_i, c_i belong to X (by reducing n by one and removing these three vertices from J).

(1) *$c_1, \ldots, c_n \notin X$.*

For suppose that $c_i \in X$. Since not all of $a_i, b_i, c_i \in X$, we may assume that $a_i \notin X$ say. But every vertex of $J \setminus X$ is in a triad, and yet every triad of J containing a_i also contains c_i, a contradiction. This proves (1).

(2) *X contains at most one of $a_1, \ldots, a_n$ and at most one of $b_1, \ldots, b_n$.*

For suppose that $a_1, a_2 \in X$ say. By (1), $c_1, c_2 \notin X$. If $b_1, b_2 \in X$ then c_1, c_2 are twins of $J \setminus X$ and hence of G, and otherwise $(\{c_1, c_2\}, \{b_1, b_2\} \setminus X)$ is a W-join of $J \setminus X$ and hence of G, a contradiction. This proves (2).

(3) *For $1 \leq i \leq n$, X contains at least one of a_i, b_i.*

For suppose that $a_1, b_1 \notin X$ say. By (1), $c_1, c_2 \notin X$. By three applications of 7.3, to $\{a_0, a_1, b_1, c_2\}$, $\{b_0, a_1, b_1, c_2\}$ and $\{c_1, a_1, b_1, c_2\}$, it follows that A_2, B_2, C_2 are pairwise strongly complete. Since a_0 is not a simplicial vertex of G we deduce that a_0, b_0 are adjacent; but then

$$(A_2, \{a_0, b_0\}, V(G) \setminus (A_2 \cup \{a_0, b_0\}))$$

is a pseudo-2-join of G, a contradiction. This proves (3).

From (1)–(3) it follows that $n = 2$ and we may assume that $X = \{a_1, b_2\}$. But then $J \setminus X$ is a trisected circular interval trigraph; the appropriate circular order is

$$a_0, a_2, c_1, c_2, b_1, b_0.$$

This proves 8.6. ∎

8.7 *Let (G, A, B, C) be a three-cliqued quasi-line trigraph such that $(G, \emptyset)$ is an unbreakable stripe, and such that (G, A, B, C) is a hex-join of (G_1, A_1, B_1, C_1) and (G_2, A_2, B_2, C_2). Then (G_1, A_1, B_1, C_1) is not a permutation of a member of $\mathcal{TC}_5$.*

Proof. Suppose (without loss of generality) that $(G_1, A_1, B_1, C_1) \in \mathcal{TC}_5$. There are two cases in the definition of $\mathcal{TC}_5$. Let $H, v_1, \ldots, v_8, X$ be as in the first case, with

$$(G_1, A_1, B_1, C_1) = (H \setminus X, \{v_1, v_2, v_3\} \setminus X, \{v_4, v_5, v_6\} \setminus X, \{v_7, v_8\}).$$

From 7.3 applied to $\{v_1, v_6, v_7, v_8\}$ it follows that B_2, C_2 are strongly complete. But then v_8 is a simplicial vertex of G, a contradiction.

Now let $H, v_1, \ldots, v_9, X$ be as in the second case of the definition of $\mathcal{TC}_5$, with

$$(G_1, A_1, B_1, C_1) = (H \setminus X, \{v_1, v_2\}, \{v_3, v_4, v_5, v_6, v_9\} \setminus X, \{v_7, v_8\}).$$

From three applications of 7.3, to $\{v_2, v_1, v_8, v_9\}$, $\{v_7, v_1, v_8, v_9\}$ and to $\{v_i, v_1, v_8, v_9\}$ (where $i \in \{3, 4\}$ is chosen so that $v_i \notin X$; this is possible since v_2 is not strongly anticomplete to $\{v_3, v_4\} \setminus X$), we deduce that A_2, B_2, C_2 are pairwise strongly complete. If $C_2 = \emptyset$ then

$$(\{v_9\}, A_2 \cup \{v_1, v_2, v_3, v_4\} \setminus X, B_2 \cup \{v_5, v_6, v_7, v_8\} \setminus X)$$

is a generalized 2-join of G, a contradiction. Thus $C_2 \neq \emptyset$. By the second assertion of 7.3, applied to $\{v_1, v_9, v_6, v_7, v_8\}$, it follows that $v_6 \in X$. Hence $v_5 \notin X$ and v_5, v_7 are adjacent; and so

$$(B_2, \{v_5, v_7\}, V(G) \setminus (B_2 \cup \{v_5, v_7\}))$$

is a pseudo-2-join of G, a contradiction. This proves 8.7.

Finally, we shall need the following, theorem 16.1 of [3]:

8.8 *Let G be a claw-free trigraph, and let B_1, B_2, B_3 be strong cliques in G. Let $Z = B_1 \cup B_2 \cup B_3$. Suppose that:*

- *$Z \neq V(G)$,*
- *there are two triads $T_1, T_2 \subseteq Z$ with $|T_1 \cap T_2| = 2$, and*
- *there is no triad T in G with $|T \cap Z| = 2$.*

Then either

- *there exists $V \subseteq Z$ with $T_1, T_2 \subseteq V$ such that V is a union of triads, and G is a hex-join of $G|V$ and $G|(V(G) \setminus V)$, where $(V \cap B_1, V \cap B_2, V \cap B_3)$ is the corresponding partition of V into strong cliques, or*
- *there are twins in one of B_1, B_2, B_3, both in triads, or*
- *there is a W-join (V_1, V_2) such that V_1 is a subset of one of B_1, B_2, B_3 and V_2 is a subset of another.*

Now we are ready for the proof of 8.1.

Proof of 8.1. Let G be a quasi-line trigraph such that $(G, \emptyset)$ is an unbreakable stripe. We must show that G is a circular interval trigraph. By 8.3, we may assume that G is not antiprismatic, and admits a hex-join.

(1) *There are three cliques A, B, C such that (G, A, B, C) is a hex-join of (G_i, A_i, B_i, C_i) $(i = 1, 2)$ where G_1 is not antiprismatic and every vertex of G_1 is in a triad.*

For since G admits a hex-join, we can choose three cliques A, B, C such that (G, A, B, C) is a hex-join of some (G_i, A_i, B_i, C_i) $(i = 1, 2)$. Since G is not antiprismatic, one of G_1, G_2 is not antiprismatic, say G_1. Let A'_1 be the set of all vertices in A_1 that are in triads, and define B'_1, C'_1 similarly. Let $Z = A'_1 \cup B'_1 \cup C'_1$. Since G_1 is not antiprismatic, there are two triads T_1, T_2 included in Z with two elements in common. If T is a triad with $|T \cap Z| = 2$, and t is its element not in Z, then $t \notin V(G_1)$ from the definition of A'_1, B'_1, C'_1, and $t \notin V(G_2)$ from the definition of a hex-join, a contradiction.

Thus there is no triad T with $|T \cap Z| = 2$. Since G is slim, there are no twins and no W-join in G, and so from 8.8 applied to Z and G, we deduce that there exists $V \subseteq Z$ with $T_1, T_2 \subseteq V$ such that V is a union of triads, and G is a hex-join of $G|V$ and $G|(V(G) \setminus V)$ (with appropriate choices of cliques). This proves (1).

Let us choose A, B, C, G_1, G_2 etc. as in (1) with $|V(G_1)|$ minimum.

(2) (G_1, A_1, B_1, C_1) *does not admit a worn hex-join.*

For suppose that (G_1, A_1, B_1, C_1) is a worn hex-join of (H_1, P_1, Q_1, R_1) and (H_2, P_2, Q_2, R_2) say. The worn hex-join is actually a hex-join since every vertex of G_1 is in triad. One of H_1, H_2 is not antiprismatic, since G_1 is not antiprismatic; and for $i = 1, 2$, every vertex of H_i belongs to a triad of H_i, since the same holds for G_1. Now

$$(H_1, P_1, Q_1, R_1), (H_2, P_2, Q_2, R_2), (G_2, A_2, B_2, C_2)$$

is a hex-chain for (G, A, B, C), and so (G, A, B, C) is the hex-join of (H_1, P_1, Q_1, R_1) and (H_3, P_3, Q_3, R_3), where (H_3, P_3, Q_3, R_3) is the hex-join of (H_2, P_2, Q_2, R_2) and (G_2, A_2, B_2, C_2); so from the minimality of $|V(G_1)|$ it follows that H_1 is antiprismatic. But

$$(H_2, P_2, Q_2, R_2), (G_2, A_2, B_2, C_2), (H_1, Q_1, R_1, P_1)$$

is also a hex-chain for (G, A', B', C') (for some choice of A', B', C'), and so by the same argument H_2 is antiprismatic, a contradiction. This proves (2).

From 8.4 it follows that (G_1, A_1, B_1, C_1) is a thickening of a permutation of some member of one of $\mathcal{TC}_1, \ldots, \mathcal{TC}_5$. Since G is slim, it follows that (G_1, A_1, B_1, C_1) is not a non-trivial thickening of any three-cliqued trigraph; and so (G_1, A_1, B_1, C_1) is a permutation of a member of $\mathcal{TC}_1, \ldots, \mathcal{TC}_5$, say of $\mathcal{TC}_i$. Now $i \neq 4$ since G_1 is not antiprismatic. Suppose that $(G_1, A_1, B_1, C_1) \notin \mathcal{TC}_2$. Then $i \in \{1, 3, 5\}$, contrary to 8.5, 8.6, and 8.7. This proves that $(G_1, A_1, B_1, C_1) \in \mathcal{TC}_2$. Since G_1 is not antiprismatic, there is a triad in G_1. Consequently G is a circular interval trigraph by 7.2. This completes the proof of 8.1, and hence of 1.1. ∎

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