Claw-Free Graphs With Strongly Perfect Complements. Fractional and Integral Version.

Part I. Basic graphs

Maria Chudnovsky*
Department of Industrial Engineering and Operations Research
Columbia University, New York, NY, U.S.A.
mchudnov@columbia.edu

Bernard Ries†
Université Paris Dauphine, Paris, France
bernard.ries@dauphine.fr

Yori Zwols‡
School of Computer Science
McGill University, Montreal, QC, Canada
yori.zwols@mcgill.ca

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Abstract

Strongly perfect graphs have been studied by several authors (*e.g.* Berge and Duchet [1], Ravindra [12], Wang [14]). In a series of two papers, the current paper being the first one, we investigate a fractional relaxation of strong perfection. Motivated by a wireless networking problem, we consider claw-free graphs that are fractionally strongly perfect in the complement. We obtain a forbidden induced subgraph characterization and display graph-theoretic properties of such graphs. It turns out that the forbidden induced subgraphs that characterize claw-free that are fractionally strongly perfect in the complement are precisely the cycle of length 6, all cycles of length at least 8, four particular graphs, and a collection of graphs that are constructed by taking two graphs, each a copy of one of three particular graphs, and joining them in a certain way by a path of arbitrary length. Wang [14] gave a characterization of strongly perfect claw-free graphs. As a corollary of the results in this paper, we obtain a characterization of claw-free graphs whose complements are strongly perfect.

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‡This research was performed while the author was at Columbia University.

1 Introduction

All graphs in this paper are finite and simple. Let G be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and edges, respectively, of G. We denote by G^c the *complement of* G. A *clique* is a set of pairwise adjacent vertices and a *stable set* is a set of pairwise nonadjacent vertices. The *clique number* $\omega(G)$ denotes the size of a maximum cardinality clique in G and the *stability number* $\alpha(G)$ denotes the size of a maximum cardinality stable set in G. Let $\chi(G)$ denote the *chromatic number* of G. G is said to be *perfect* if every induced subgraph G' of G satisfies $\chi(G') = \omega(G')$. For another graph H, we say that G *contains* H *as an induced subgraph* if G has an induced subgraph that is isomorphic to H. The *claw* is a graph with vertex set $\{a_0, a_1, a_2, a_3\}$ and edge set $\{a_0a_1, a_0a_2, a_0a_3\}$. We say that a graph G is *claw-free* if G does not contain the claw as an induced subgraph. We say that G is *connected* if there exists a path between every two $u, v \in V(G)$. A *connected component* of G is a maximal connected subgraph of G. For disjoint sets $A, B \in V(G)$ we say that A is *complete* to B if every vertex in A is adjacent to every vertex in B, and $a \in V(G)$ is complete to B if $\{a\}$ is complete to B.

A graph G is *fractionally co-strongly perfect* if and only if, for every induced subgraph H of G, there exists a function $w : V(H) \to [0, 1]$ such that

$$\sum_{v \in S} w(v) = 1, \text{ for every inclusion-wise maximal stable set } S \text{ of } H. \tag{1}$$

We call a function w that satisfies (1) a *saturating vertex weighting for* H. This paper investigates graphs that are claw-free and that are fractionally co-strongly perfect. We will give a characterization of such graphs in terms of forbidden induced subgraphs.

Motivation

The motivation for studying fractionally co-strongly perfect claw-free graphs is two-fold. Firstly, the class of fractionally co-strongly perfect graphs shows up naturally in an application in wireless networking (see Section 3.2). With this application in mind, the authors and three others characterized in an earlier paper [2] all line graphs that are fractionally co-strongly perfect. Claw-free graphs form a mathematically natural generalization of line graphs, not only because all line graphs are claw-free, but – more importantly – because the structure of claw-free graphs closely resembles that of line graphs. Since characterizing fractionally co-strongly perfect graphs in general seems hard, considering the class of claw-free graphs is a natural step.

Secondly, there is a relationship between co-strongly perfect graphs and so-called strongly perfect graphs, which were first studied by Berge and Duchet [1] in the 1980's. A graph G is *strongly perfect* if every induced subgraph H of G contains a stable set that meets every (inclusion-wise) maximal clique of H. An equivalent definition of strong perfection is: a graph G is strongly perfect if and only if for every induced subgraph H of G, there exists a function $w : V(H) \to \{0, 1\}$ such that $\sum_{v \in K} w(v) = 1$ for every maximal clique K of H. Therefore, fractional strong perfection is, as the name suggests, a fractional relaxation of strong perfection. (The 'co-' part in 'fractional co-strong perfection' refers to the fact that we are interested in this property in the complement.) Strongly perfect graphs are of interest because they form a special class of perfect graphs in the following sense: every perfect graph (and hence each of its induced subgraphs) contains a stable set that meets every maximum cardinality clique (take one color class in an optimal vertex coloring). Strongly perfect graphs satisfy the stronger property that they contain a stable set meeting every inclusion-wise maximal clique. Although a characterization of perfect graphs in terms of excluded induced subgraphs is known [5], no such characterization is known yet for strongly perfect graphs. Wang [14] gave a characterization of claw-free graphs that are strongly perfect. As a corollary of our main theorem, we obtain a characterization of claw-free graphs that are strongly perfect in the complement; see Section 3.1.

Statement of the main results

Before stating our main theorem, we define the following three classes of graphs:

- $\mathcal{F}_1 = \{C_k \mid k = 6 \text{ or } k \geq 8\}$, where C_k is a cycle of length k;
- $\mathcal{F}_2 = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4\}$, where the $\mathcal{G}_i$'s are the graphs drawn in Figure 1(a);
- Let $\mathcal{H} = \{\mathcal{H}_1(k), \mathcal{H}_2(k), \mathcal{H}_3(k) \bigm| k \geq 0\}$, where $\mathcal{H}_i(k)$ is the graph $\mathcal{H}_i$ drawn in Figure 1(b) but whose 'wiggly' edge joining z and x is replaced by an induced k-edge-path. For $i \in \{1,2,3\}$, we call $\mathcal{H}_i(k)$ a *heft of type i with a rope of length k*. We call x the *end of the heft $\mathcal{H}_i(k)$*.

  Now let $i_1, i_2 \in \{1,2,3\}$ and let $k_1, k_2 \geq 0$ be integers. Let $H_1 = \mathcal{H}_{i_1}(k_1)$ and $H_2 = \mathcal{H}_{i_2}(k_2)$, and let x_1, x_2 be the end of heft H_1, H_2, respectively. Construct H from the disjoint union of H_1 and H_2 by deleting x_1 and x_2, and making the neighbors of x_1 complete to the neighbors of x_2. Then H is called a *skipping rope of type (i_1, i_2) of length $k_1 + k_2$*. Let $\mathcal{F}_3$ be the collection of skipping ropes. Figure 2 shows two examples of skipping ropes.

Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$. A graph G is *$\mathcal{F}$-free* if no induced subgraph of G is isomorphic to a graph in $\mathcal{F}$. We say that a clique K is a *dominant clique* in a graph G if every maximal (under inclusion) stable set S in G satisfies $S \cap K \neq \emptyset$. We say that a graph G is *resolved* if at least one of the following is true:

(a) there exists $x \in V(G)$ that is complete to $V(G) \setminus \{x\}$; or

(b) G has a dominant clique; or

(c) G is not perfect and every maximal stable set in G has the same size $k \in \{2,3\}$.

We say that a graph G is *perfectly resolved* if every *connected* induced subgraph of G is resolved. In a series of two papers (the current paper and [4]), we will prove the following theorem:

Theorem 1.1. *Let G be a claw-free graph. Then the following statements are equivalent:*

(i) *G is fractionally co-strongly perfect;*

(ii) *G is $\mathcal{F}$-free;*

(iii) *G is perfectly resolved.*

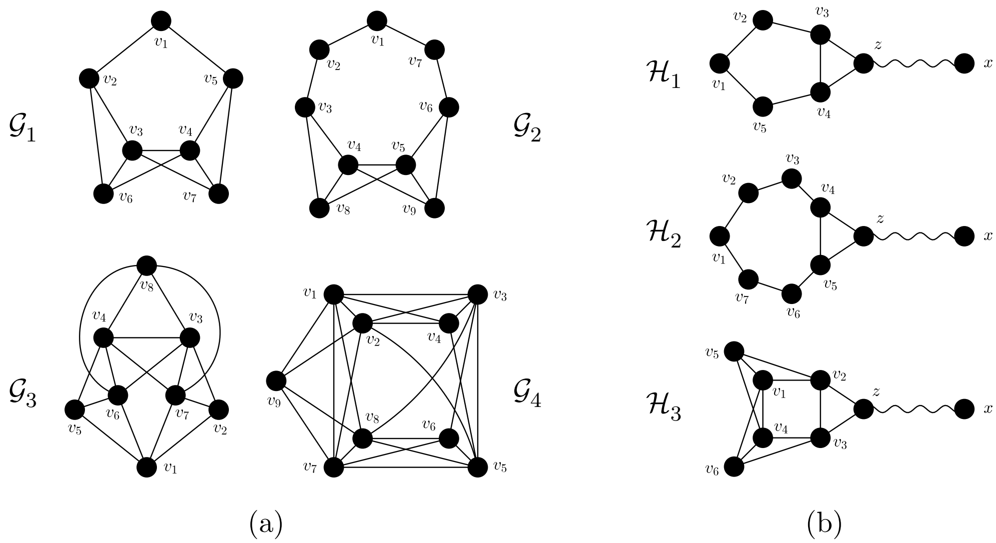

Figure 1: Forbidden induced subgraphs for fractionally co-strongly perfect graphs. (a) The graphs $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$. (b) Hefts $\mathcal{H}$ that are combined to construct skipping ropes.

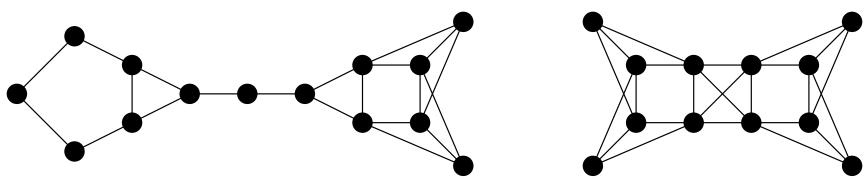

Figure 2: Two examples of skipping ropes. Left: the skipping rope of type $(1, 3)$ of length 3. Right: the skipping rope of type $(3, 3)$ of length 0.

Wang [14] gave a characterization of claw-free graphs that are strongly perfect. Theorem 1.1 allows us to give a characterization of claw-free graphs that are strongly perfect *in the complement*. Specifically, we obtain the following induced subgraph characterization of claw-free graphs that are strongly perfect in the complement:

Theorem 1.2. *Let G be a claw-free graph. G^c is strongly perfect if and only if G is perfect and no induced subgraph of G is isomorphic to $\mathcal{G}_4$, an even hole of length at least six, or a skipping rope of type $(3, 3)$ of length $k \geq 0$.*

Chudnovsky and Seymour [7] proved a structure theorem for claw-free graphs. The theorem roughly states that every claw-free graph is either of a certain 'basic' type or admits a so-called 'strip-structure'. In fact, [7] deals with slightly more general objects called 'claw-free trigraphs'. What is actually meant by 'basic' will be explained in Section 2. The definition of a 'strip-structure' is in [7]. We do not repeat it here because we deal with them in the second paper [4]. The current paper deals with the proof of Theorem 1.1 for the case when G is of the 'basic' type. To summarize, the goal of this paper is to prove the following three results, the last of which is the reason why we are only able to partially prove Theorem 1.1 in this paper:

Theorem 1.3. *If G is fractionally co-strongly perfect, then G is $\mathcal{F}$-free.*

Theorem 1.4. *If G is perfectly resolved, then G is fractionally co-strongly perfect.*

Theorem 1.5. *Every $\mathcal{F}$-free basic claw-free graph G is resolved.*

Notice that in outcome (c) of the definition of a resolved graph, we could drop the requirement that G be imperfect. This extra condition, however, facilitates the proof of Theorem 1.2 and, as it turns out, it will take almost no effort to obtain the condition in the proof of Theorem 1.5.

Organization of this paper

This paper is structured as follows. In Section 2, we will introduce tools that we need throughout the current paper. We suggest skipping Sections 2.2–2.5 at first reading and coming back to them when definitions and results from these sections are needed (which will mainly be in Section 5). In Section 3, we present two applications of Theorem 1.1. The first application is the proof of Theorem 1.2 which gives a characterization of claw-free graphs that are strongly perfect in the complement. The second application lies in wireless networking. We briefly sketch the application and give references for details. Section 4 is devoted to proving Theorem 1.3 and Theorem 1.4. Section 3 and Section 4 are mostly self-contained. The remainder of this paper consists of Section 5 in which we give the proof of Theorem 1.5.

2 Tools

In this section, we introduce definitions, notation and important lemmas that we use throughout the paper. As in [7], it will be helpful to work with "trigraphs" rather than with graphs. We would like to point out that the results in [7] can be stated in terms of graphs as well. Although we originally tried to write this paper using the graph-versions of these results, we quickly realized that whether a graph is resolved can – up to a few exceptions – easily be determined from the underlying trigraph. Therefore, working with trigraphs rather than their graphic thickenings (see Section 2.1) simplifies the analysis considerably.

The purpose of this section is to gather all the tools that are used throughout the current paper and [4] in one place. At first reading, Sections 2.2–2.5 may be skipped. The definitions and results from these sections will not be needed until Section 5.

2.1 Claw-free graphs and trigraphs

For an integer $n \geq 1$, we denote by $[n]$ the set $\{1, 2, \ldots, n\}$. In this section we define terminology for trigraphs. We use this terminology defined for trigraphs in this section for graphs as well. The definitions should be applied to graphs by regarding graphs as trigraphs.

A *trigraph* T consists of a finite set $V(T)$ of vertices, and a map $\theta_T : V(T) \times V(T) \to \{1, 0, -1\}$, satisfying:

- $\theta_T(v, v) = 0$, for all $v \in V(T)$;
- $\theta_T(u, v) = \theta_T(v, u)$, for all distinct $u, v \in V(T)$;
- for all distinct $u, v, w \in V(T)$, at most one of $\theta_T(u, v)$, $\theta_T(u, w)$ equals zero.

We call θ_T the *adjacency function* of T. For distinct $u, v \in V(T)$, we say that u and v are *strongly adjacent* if $\theta_T(u, v) = 1$, *strongly antiadjacent* if $\theta_T(u, v) = -1$, and *semiadjacent* if $\theta_T(u, v) = 0$. We say that u and v are *adjacent* if they are either strongly adjacent or semiadjacent, and *antiadjacent* if they are either strongly antiadjacent or semiadjacent. We denote by $F(T)$ the set of all pairs $\{u, v\}$ such that $u, v \in V(T)$ are distinct and semiadjacent. Thus a trigraph T is a graph if $F(T) = \emptyset$.

We say that u is a *(strong) neighbor* of v if u and v are (strongly) adjacent; u is a *(strong) antineighbor* of v if u and v are (strongly) antiadjacent. For distinct $u, v \in V(T)$ we say that $uv = \{u, v\}$ is an *edge*, a *strong edge*, an *antiedge*, a *strong antiedge*, or a *semiedge* if u and v are adjacent, strongly adjacent, antiadjacent, strongly antiadjacent, or semiadjacent, respectively. For disjoint sets $A, B \subseteq V(T)$, we say that A is *(strongly) complete to* B if every vertex in A is (strongly) adjacent to every vertex in B, and that A is *(strongly) anticomplete to* B if every vertex in A is (strongly) antiadjacent to every vertex in B. We say that A and B are *linked* if every vertex in A has a neighbor in B and every vertex in B has a neighbor in A. For $v \in V(T)$, let $N_T(v)$ denote the set of vertices adjacent to v, and let $N_T[v] = N_T(v) \cup \{v\}$. Whenever it is clear from the context what T is, we drop the subscript and write $N(v) = N_T(v)$ and $N[v] = N_T[v]$. For $X \subseteq V(T)$, we write $N(X) = (\cup_{x \in X} N(x)) \setminus X$ and $N[X] = N(X) \cup X$. We say that a set $K \subseteq V(T)$ is a *(strong) clique* if the vertices in K are pairwise (strongly) adjacent. We say that a set $S \subseteq V(T)$ is a *(strong) stable set* if the vertices in S are pairwise (strongly) antiadjacent.

We say that a trigraph T' is a *thickening of* T if for every $v \in V(T)$ there is a nonempty subset $X_v \subseteq V(T')$, all pairwise disjoint and with union $V(T')$, satisfying the following:

(i) for each $v \in V(T)$, X_v is a strong clique of T';

(ii) if $u, v \in V(T)$ are strongly adjacent in T, then X_u is strongly complete to X_v in T';

(iii) if $u, v \in V(T)$ are strongly antiadjacent in T, then X_u is strongly anticomplete to X_v in T';

(iv) if $u, v \in V(T)$ are semiadjacent in T, then X_u is neither strongly complete nor strongly anticomplete to X_v in T'.

When $F(T') = \emptyset$ then we call T' regarded as a graph a *graphic thickening of* T.

For $X \subseteq V(T)$, we define the trigraph $T|X$ *induced on* X as follows. The vertex set of $T|X$ is X, and the adjacency function of $T|X$ is the restriction of θ_T to X^2. We call $T|X$ an *induced subtrigraph* of T. We define $T \setminus X = T|(V(T) \setminus X)$. We say that a graph G is a *realization of* T if $V(G) = V(T)$ and for distinct $u, v \in V(T)$, u and v are adjacent in G if u and v are strongly

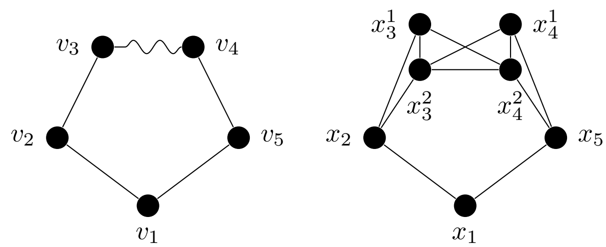

Figure 3: An $\mathcal{F}$-free trigraph T (left) and a graphic thickening of T that is not $\mathcal{F}$-free (right). Here, $X_{v_i} = \{x_i\}$ for $i = 1, 2, 5$ and $X_{v_i} = \{x_i^1, x_i^2\}$ for $i = 3, 4$.

adjacent in T, u and v are nonadjacent in G if u and v are strongly antiadjacent in T, and u and v are either adjacent or nonadjacent in G if u and v are semiadjacent in T. We say that T contains a graph H as a *weakly induced subgraph* if there exists a realization of T that contains H as an induced subgraph. We mention the following easy lemma:

(2.1) *Let T be a trigraph and let H be a graph. If T contains H as a weakly induced subgraph, then every graphic thickening of T contains H as an induced subgraph.*

Proof. Let G be a graphic thickening of T. Since T contains H as a weakly induced subgraph, there exists a realization G' of T that contains H as an induced subgraph. Because every graphic thickening of T contains every realization of T as an induced subgraph, it follows that G contains H as an induced subgraph. This proves (2.1). ■

A stable set S is called a *triad* if $|S| = 3$. T is said to be *claw-free* if T does not contain the claw as a weakly induced subgraph. We state the following trivial result without proof:

(2.2) *Let T be a claw-free trigraph. Then no $v \in V(T)$ is complete to a triad in T.*

A trigraph T is said to be $\mathcal{F}$-free if it does not contain any graph in $\mathcal{F}$ as a weakly induced subgraph. Notice that, by (2.1), if every graphic thickening of a trigraph T is $\mathcal{F}$-free, then T is $\mathcal{F}$-free. The converse, however, is not true: if T is $\mathcal{F}$-free, this does necessarily mean that every graphic thickening of T is $\mathcal{F}$-free. For an example, see Figure 3.

Let $p_1, p_2, \ldots, p_k \in V(T)$ be distinct vertices. We say that $T|\{p_1, p_2, \ldots, p_k\}$ of T is a *weakly induced path (from p_1 to p_k)* in T if, for $i, j \in [k], i < j$, p_i and p_j are adjacent if $j = i + 1$ and antiadjacent otherwise. Let $\{c_1, c_2, \ldots, c_k\} \subseteq V(T)$. We say that $T|\{c_1, c_2, \ldots, c_k\}$ is a *weakly induced cycle (of length k)* in T if for all distinct $i, j \in [k]$, c_i is adjacent to c_j if $|i-j| = 1 \pmod k$, and antiadjacent otherwise. We say that $T|\{c_1, c_2, \ldots, c_k\}$ is a *semihole (of length k)* in T if for all distinct $i, j \in [k]$, c_i is adjacent to c_j if $|i - j| = 1 \pmod k$, and strongly antiadjacent otherwise. A vertex v in a trigraph T is *simplicial* if $N(v)$ is a strong clique. Notice that our definition of a simplicial vertex differs slightly from the definition used in [7], because we allow v to be incident with a semiedge.

Finally, we say that a set $X \subseteq V(T)$ is a *homogeneous set in T* if $|X| \geq 2$ and $\theta_T(x, v) = \theta_T(x', v)$ for all $x, x' \in X$ and all $v \in V(T) \setminus X$. For two vertices $x, y \in V(T)$, we say that x is a *clone of y*

if $\{x, y\}$ is a homogeneous set in T. In that case we say that x and y are *clones*.

2.2 Resolved graphs and trigraphs; finding dominant cliques

We say that a claw-free trigraph T is *resolved* if every $\mathcal{F}$-free graphic thickening of T is resolved. Notice that, by this definition, every trigraph that is *not* $\mathcal{F}$-free is resolved (because such a trigraph has no $\mathcal{F}$-free graphic thickening). Although this seems a bit counterintuitive, we do not particularly care about it, because we are only interested in proving that every $\mathcal{F}$-free trigraph is resolved. (See (5.1)) Also recall that an $\mathcal{F}$-free trigraph may have a graphic thickening that is not $\mathcal{F}$-free (see Figure 3). We state a number of useful lemmas for concluding that a trigraph is resolved. Let T be a trigraph. For a vertex $x \in V(T)$, we say that a stable set $S \subseteq V(T)$ *covers* x if x has a neighbor in S. For a strong clique $K \subseteq V(T)$, we say that a stable set $S \subseteq V(T)$ *covers* K if S covers every vertex in K. We say that a strong clique $K \subseteq V(T)$ is a *dominant clique* if T contains no stable set $S \subseteq V(T) \setminus K$ such that S covers K. It is easy to see that this definition of a dominant clique, when applied to a graph, coincides with our earlier definition of a dominant clique for a graph.

(2.3) *Let T be a trigraph and suppose that K is a dominant clique in T. Then, T is resolved.*

Proof. Let G be a graphic thickening of T. For $v \in V(T)$, let X_v denote the clique in G corresponding to v. We claim that $K' = \bigcup_{z \in K} X_z$ is a dominant clique in G. For suppose not. Then there exists a maximal stable set $S' \subseteq V(G)$ such that $S' \cap K' = \emptyset$. Write $S' = \{s'_1, \dots, s'_p\}$, where $p = |S'|$, and let $s_i \in V(T)$ be such that $s'_i \in X_{s_i}$. Let $S = \{s_1, \dots, s_p\}$. We claim that S covers K, contrary to the fact that K is a dominant clique in T. Since S' is a stable set in G, it follows that S is a stable set in T. Now let $w \in K$ and let $w' \in X_w$. Since S' is maximal and $S' \cap K' = \emptyset$, it follows that w' has a neighbor $s' \in S'$. Let $s \in V(T)$ be such that $s' \in X_s$. It follows that w is adjacent to s. This proves that every $w \in K$ has a neighbor in S and, hence, S covers K, which proves (2.3). ■

Notice that if G is a graphic thickening of some trigraph T and T has no dominant clique, then this does not necessarily imply that G has no dominant clique (consider, for example, a two-vertex trigraph where the two vertices are semiadjacent). The following lemma gives another way of finding a dominant clique:

(2.4) *Let T be a trigraph, let A and B be nonempty disjoint strong cliques in T and suppose that A is strongly anticomplete to $V(T) \setminus (A \cup B)$. Then, T is resolved.*

Proof. Let G be a graphic thickening of T. For $v \in V(T)$, let X_v be the corresponding clique in G. Let $Y = \bigcup_{a \in A} X_a$ and $Z = \bigcup_{b \in B} X_b$. Let $Z' \subseteq Z$ be the set of vertices in Z that are complete to Y. We claim that $K = Y \cup Z'$ is a dominant clique in G. For suppose that S is a maximal stable set in G such that $S \cap K = \emptyset$. First notice that every $y \in Y$ has a neighbor in $(Z \setminus Z') \cap S$, because, if not, then we may add y to S and obtain a larger stable set. In particular, $(Z \setminus Z') \cap S \neq \emptyset$ and, since Z is a clique, $|S \cap (Z \setminus Z')| = 1$. But now the unique vertex z in $(Z \setminus Z') \cap S$ is complete to

Y, contrary to the fact that $z \notin Z'$. This proves (2.4). ■

By letting $|A| = 1$ in (2.4), we obtain the following immediate result that we will use often:

(2.5) *Let T be a trigraph and let $v \in V(T)$ be a simplicial vertex. Then, T is resolved.*

Next, we have a lemma that deals with trigraphs with no triads:

(2.6) *Let T be a trigraph with no triad. Then, T is resolved.*

Proof. Let G be a graphic thickening of T. Since T has no triad, it follows that $\alpha(G) \leq 2$. If some vertex $v \in V(G)$ is complete to $V(G) \setminus \{v\}$, then G is resolved. So we may assume that no such vertex exists. It follows that there is no maximal stable set of size one and, hence, every maximal stable set has size two. If G is imperfect, then G is resolved. So we may assume that G is perfect. From this, since G^c has no triangles, it follows that G^c is bipartite and thus G is the union of two cliques. But now, it follows from (2.4) that G has a dominant clique and, therefore, G is resolved. This proves (2.6). ■

Let T be a trigraph, and suppose that K_1 and K_2 are disjoint nonempty strong cliques. We say that (K_1, K_2) is a *homogeneous pair of cliques* in T if, for $i = 1, 2$, every vertex in $V(T) \setminus (K_1 \cup K_2)$ is either strongly complete or strongly anticomplete to K_i. For notational convenience, for a weakly induced path $P = p_1\text{-}p_2\text{-}\dots\text{-}p_{k-1}\text{-}p_k$, we define the *interior P^* of P* by $P^* = p_2\text{-}p_3\text{-}\cdots\text{-}p_{k-2}\text{-}p_{k-1}$.

(2.7) *Let T be an $\mathcal{F}$-free claw-free trigraph. Let (K_1, K_2) be a homogeneous pair of cliques in T such that K_1 is not strongly complete and not strongly anticomplete to K_2. For $\{i, j\} = \{1, 2\}$, let $N_i = N(K_i) \setminus N[K_j]$ and $M = V(T) \setminus (N[K_1] \cup N[K_2])$. If there exists a weakly induced path P between antiadjacent $v_1 \in N_1$ and $v_2 \in N_2$ such that $V(P^*) \subseteq M$ and $|V(P)| \geq 3$, then T is resolved.*

Proof. Let G be an $\mathcal{F}$-free graphic thickening of T. For $v \in V(T)$, let X_v denote the corresponding clique in G. Let $K_1' = \bigcup_{v \in K_1} X_v$ and define K_2', N_1', N_2', M' analogously. Let $Z' = (N(K_1') \cap N(K_2')) \setminus (K_1' \cup K_2')$. Since (K_1, K_2) is a homogeneous pair of cliques, it follows that, for $\{i, j\} = \{1, 2\}$, N_i' is complete to K_i' and anticomplete to K_j', and Z' is complete to $K_1' \cup K_2'$. Hence, from the fact that K_1' is not anticomplete to K_2' and the fact that G is claw-free, it follows that N_1' and N_2' are cliques. Z' is anticomplete to M', because if $z \in Z'$ has a neighbor $u \in M'$, then let $a \in K_1', b \in K_2'$ be nonadjacent and observe that z is complete to the triad $\{a, b, u\}$, contrary to (2.2). We start with the following claim.

(i) *Suppose that there exist $a_1, a_2 \in K_1'$, $b \in K_2'$ such that b is adjacent to a_1 and nonadjacent to a_2. Let $x_1 \in N_1', x_2 \in N_2'$ be nonadjacent such that there is an induced path Q between x_1 and x_2 that satisfies $V(Q^*) \subseteq M'$. Then $|V(Q)| \in \{3, 5\}$ and Z' is complete to N_1'.*

Since $b\text{-}a_1\text{-}x_1\text{-}Q^*\text{-}x_2\text{-}b$ is an induced cycle of length $|V(Q)| + 2$ and G contains no induced cycle of length 6 or at least 8, it follows that $|V(Q)| \in \{3, 5\}$. We may assume that $Z' \neq \emptyset$, otherwise we are done. We first claim that Z' is complete to x_1. For suppose that $z \in Z'$ is nonadjacent to x_1. If z is nonadjacent to x_2, then z-a_2-x_1-Q^*-x_2-b-z is an induced cycle of length $|V(Q)| + 3 \in \{6, 8\}$, a contradiction. Therefore, z is adjacent to x_2. But now, $G|(V(Q) \cup \{a_1, a_2, b, z\})$ is isomorphic to $\mathcal{G}_1$ if $|V(Q)| = 3$ or $\mathcal{G}_2$ if $|V(Q)| = 5$, a contradiction. This proves that Z' is complete to x_1.

Now let $p \in N_1'$ and suppose that p is nonadjacent to some $z \in Z'$. Let $u \in V(Q)$ be the unique neighbor of x_1 in Q. Because x_1 is complete to $\{p, u, z\}$, it follows from (2.2) that $\{p, u, z\}$ is not a triad and hence p is adjacent to u. If p is nonadjacent to x_2, then possibly by shortcutting Q, there is a path between nonadjacent p and x_2, and it follows from the previous argument that Z' is complete to p, a contradiction. It follows that p is adjacent to x_2. If $|V(Q)| = 5$, then u is nonadjacent to x_2 and hence p is complete to the triad $\{a_1, x_2, u\}$, contrary to (2.2). It follows that $|V(Q)| = 3$. Now, if z is nonadjacent to x_2, then $G|\{z, x_1, p, x_2, b, a_2, u\}$ is isomorphic to $\mathcal{G}_1$. Thus, z is adjacent to x_2. But now, $G|\{a_1, b, x_2, u, x_1, p, z, a_2\}$ is isomorphic to $\mathcal{G}_3$. This proves **(i)**. □

Let $P = p_1$-p_2-$\dots$-p_{k-1}-p_k be a weakly induced path between antiadjacent $p_1 \in N_1$ and $p_2 \in N_2$ such that $V(P^*) \subseteq M$ and $|V(P)| \geq 3$. For $i \in [k]$, let $p_i' \in X_{p_i}$ such that p_1'-$\dots$-p_k' is an induced path in G. It follows that $p_1' \in N_1'$, $p_k' \in N_2'$, and $V((P')^*) \subseteq M'$. We claim the following:

(ii) *Z' is a clique.*

Because K_1' is not complete and not anticomplete to K_2', we may assume from the symmetry that there exist $a_1, a_2 \in K_1'$ and $b \in K_2'$ such that b is adjacent to a_1 and nonadjacent to a_2. It follows from **(i)** that Z' is complete to p_1'. Let $u \in V(P')$ be the unique neighbor of p_1' in P'. If $z_1, z_2 \in Z'$ are nonadjacent, then p_1' is complete to the triad $\{z_1, z_2, u\}$, contrary to (2.2). This proves **(ii)**. □

The last claim deals with an easy case:

(iii) *If some vertex in K_1' is complete to K_2', then the lemma holds.*

Suppose that $a_1 \in K_1'$ is complete to K_2'. First observe that no vertex in K_1' has both a neighbor and a nonneighbor in K_2', because if $a_2 \in K_1'$ has a neighbor $b_1 \in K_2'$ and a nonneighbor $b_2 \in K_2'$, then $G|(V(P') \cup \{a_1, a_2, b_1, b_2\})$ is isomorphic to $\mathcal{G}_1$ if $|V(P')| = 3$ and to $\mathcal{G}_2$ if $|V(P')| = 5$. It follows that every vertex in K_1' is either complete or anticomplete to K_2'. Since K_1' is not complete to K_2', it follows that there exists $a_2 \in K_1'$ that is is anticomplete to K_2'. Now it follows from **(i)** that Z' is complete to N_1'. Thus, a_2 is a simplicial vertex and the lemma holds by (2.5). This proves **(iii)**. □

It follows from **(iii)** and the symmetry that we may assume that, for $\{i, j\} = \{1, 2\}$, no vertex in K_i' is complete to K_j'. Thus, it follows from **(i)** and the fact that K_1' is not complete and not anticomplete to K_2' that Z' is complete to $N_1' \cup N_2'$. We claim that $K = K_1' \cup Z' \cup N_1'$ is a dominant clique. For suppose not. Then there exists a maximal stable set S in G such that $S \cap K = \emptyset$. Let $a \in K_1'$. Since $N(a) \subseteq K \cup K_2'$, it follows that a has a neighbor in $S \cap K_2'$, because otherwise we may add a to S and obtain a larger stable set. In particular, $S \cap K'_2 \neq \emptyset$ and, since K'_2 is a clique, $|S \cap K'_2| = 1$. But now, the unique vertex v in $S \cap K'_2$ is complete to K'_1, a contradiction. This proves that K is a dominant clique, thus proving (2.7). ■

We note the following special case of (2.7), in which the two strong cliques of the homogeneous pair of cliques have cardinality one:

(2.8) *Let T be an $\mathcal{F}$-free claw-free trigraph and suppose that T contains a weakly induced cycle c_1-c_2-...-c_k-c_1 with $k \geq 5$ and such that $c_1c_2 \in F(T)$. Then, T is resolved.*

Proof. Since $c_1c_2 \in F(T)$, it follows from the definition of a trigraph that, for $i \in [2]$, every vertex in $V(T) \setminus \{c_1, c_2\}$ is either strongly adjacent or strongly antiadjacent to c_i. Thus, $(\{c_1\}, \{c_2\})$ is a homogeneous pair of cliques in T. Moreover, c_3-...-c_k is a weakly induced path that meets the conditions of (2.7). Thus, T is resolved by (2.7). This proves (2.8). ■

The following lemma states that we may assume that trigraphs do not have strongly adjacent clones.

(2.9) *Let T be a trigraph and suppose that $v, w \in V(T)$ are strongly adjacent clones. If $T \setminus v$ is resolved, then T is resolved.*

Proof. First notice that it follows from the definitions of trigraphs and clones that v and w only have strong neighbors and strong antineighbors. Let G be an $\mathcal{F}$-free graphic thickening of T, and for all $u \in V(T)$ let X_u be the clique in G corresponding to u. Since $T \setminus v$ is resolved, we have that $G \setminus X_v$ is resolved, and thus there are three possibilities. First, suppose that $G \setminus X_v$ contains a vertex z that is complete to $V(G \setminus X_v) \setminus \{z\}$. Since v and w are clones, it follows that z is complete to X_v, and hence z is complete to $V(G) \setminus \{z\}$. Therefore, G is resolved. Next, suppose that $G \setminus X_v$ has a dominant clique K. First notice that since K is a dominant clique, K is also an inclusion-wise maximal clique in $G \setminus X_v$. Indeed, if $K \subsetneq K'$ where K' is a clique in $G \setminus X_v$, then any maximal stable set S in $G \setminus X_v$ that contains a vertex from $K' \setminus K$ satisfies $S \cap K = \emptyset$, contrary to the definition of a dominant clique. From this, it follows that either $X_w \subseteq K$ or $X_w \cap K = \emptyset$. Let $K' = K \cup \{X_v\}$ if $X_w \subseteq K$, and let $K' = K$ otherwise. We claim that K' is a dominant clique in G. For suppose there exists a maximal stable set S such that $S \cap K' = \emptyset$. If $X_v \cap S = \emptyset$, then clearly, S is a maximal stable set in $G \setminus X_v$ with $S \cap K = \emptyset$, contrary to the fact that K is a dominant clique in $G \setminus X_v$. Therefore, $X_v \cap S \neq \emptyset$ and hence $K' = K$. Since v and w are clones in T, the set $S' = (S \setminus \{X_v\}) \cup \{w'\}$, where $w' \in X_w$, is a stable set in $G \setminus X_v$. But now S' is a maximal stable set in $G \setminus X_v$ with $S' \cap K = \emptyset$, contrary to the fact that K is a dominant clique in $G \setminus X_v$. This proves that K' is a dominant clique and therefore G is resolved. So we may assume that $G \setminus X_v$ is not perfect and there exists $k \in \{2, 3\}$ such that every maximal stable set in $G \setminus X_v$ has size k. It follows that G is not perfect. Since $G \setminus X_v$ and $G \setminus X_w$ are isomorphic and every maximal stable set in G is either contained in $V(G \setminus X_v)$ or in $V(G \setminus X_w)$, it follows that every maximal stable set in G has size k, and therefore G is resolved. This proves that every graphic thickening of T is resolved and, thus, T is resolved, completing the proof of (2.9). ■

2.3 Classes of trigraphs

Let us define some classes of trigraphs:

- **Line trigraphs**. Let H be a graph, and let T be a trigraph with $V(T) = E(H)$. We say that T is a *line trigraph* of H if for all distinct $e, f \in E(H)$:
  - if e, f have a common end in H then they are adjacent in T, and if they have a common end of degree at least three in H, then they are strongly adjacent in T;
  - if e, f have no common end in H then they are strongly antiadjacent in T.
- **Trigraphs from the icosahedron**. The icosahedron is the unique planar graph with twelve vertices all of degree five. Let it have vertices $v_0, v_1, \ldots, v_{11}$ where for $1 \leq i \leq 10$, v_i is adjacent to v_{i+1}, v_{i+2} (reading subscripts modulo 10), and v_0 is adjacent to v_1, v_3, v_5, v_7, v_9, and v_{11} is adjacent to $v_2, v_4, v_6, v_8, v_{10}$. Let this graph be T_0, regarded as a trigraph. Let T_1 be obtained from T_0 by deleting v_{11}. Let T_2 be obtained from T_1 by deleting v_{10}, and possibly by making v_1 semiadjacent to v_4, or making v_6 semiadjacent to v_9, or both. Then each of T_0, T_1, and the several possibilities for T_2 is a *trigraph from the icosahedron*.
- **Long circular interval trigraphs**. Let Σ be a circle, and let $F_1, \ldots, F_k \subseteq \Sigma$ be homeomorphic to the interval $[0, 1]$, such that no two of $F_1, \ldots, F_k$ share an endpoint, and no three of them have union Σ. Now let $V \subseteq \Sigma$ be finite, and let T be a trigraph with vertex set V in which, for distinct $u, v \in V$,
  - if $u, v \in F_i$ for some i then u, v are adjacent, and if also at least one of u, v belongs to the interior of F_i then u, v are strongly adjacent;
  - if there is no i such that $u, v \in F_i$ then u, v are strongly antiadjacent.

  Such a trigraph T is called a *long circular interval trigraph*. If, in addition, $\bigcup_{i=1}^{k} F_i \neq \Sigma$, then T is called a *linear interval trigraph*.
- **Antiprismatic trigraphs**. Let T be a trigraph such that for every $X \subseteq V(T)$ with $|X| = 4$, $T|X$ is not a claw and there are at least two pairs of vertices in X that are strongly adjacent in T. In particular, it follows that, if $u, v \in V(T)$ are semiadjacent, then either
  - neither of u, v is in a triad; or
  - there exists $w \in V(T)$ such that $\{u, v, w\}$ is a triad, but there is no other triad that contains u or v.

  Then, T is called an *antiprismatic trigraph*.

We will use the following structural result from [7]. We would like to point out that the current paper deals only with the trigraphs mentioned in outcomes (a)-(d) of Theorem 2.10 and, thus, the reader does not need to know what a nontrivial strip-structure is.

Theorem 2.10. (**7.2** in [7]) *Let G be a connected claw-free graph. Then, either G admits a nontrivial strip-structure, or G is the graphic thickening of one of the following trigraphs:*

(a) *a trigraph of the icosahedron, or*

(b) *an antiprismatic trigraph, or*

(c) *a long circular interval trigraph, or*

(d) *a trigraph that is the union of three strong cliques.*

We say that a claw-free trigraph T is *basic* if T satisfies one of the outcomes (a)-(d) of Theorem 2.10. Analogously, a claw-free graph G is said to be *basic* if G is a graphic thickening of a basic claw-free trigraph.

2.4 Three-cliqued claw-free trigraphs

Let T be a trigraph such that $V(T) = A \cup B \cup C$ and A, B, C are strong cliques. Then (T, A, B, C) is called a *three-cliqued trigraph*. We define the following types of three-cliqued claw-free trigraphs:

$\mathcal{TC}_1$: **A type of line trigraph**. Let v_1, v_2, v_3 be distinct pairwise nonadjacent vertices of a graph H, such that every edge of H is incident with (exactly) one of v_1, v_2, v_3. Let v_1, v_2, v_3 all have degree at least three, and let all other vertices of H have degree at least one. Moreover, for all distinct $i, j \in [3]$, let there be at most one vertex different from v_1, v_2, v_3 that is adjacent to v_i and not to v_j in H. Let A, B, C be the sets of edges of H incident with v_1, v_2, v_3 respectively, and let T be a line trigraph of H. Then (G, A, B, C) is a three-cliqued claw-free trigraph. Let $\mathcal{TC}_1$ be the class of all such three-cliqued trigraphs such that every vertex is in a triad.

$\mathcal{TC}_2$: **Long circular interval trigraphs**. Let T be a long circular interval trigraph, and let Σ be a circle with $V(T) \subseteq \Sigma$, and $F_1, \ldots, F_k \subseteq \Sigma$, as in the definition of long circular interval trigraph. By a *line* we mean either a subset $X \subseteq V(T)$ with $|X| = 1$, or a subset of some F_i homeomorphic to the closed unit interval, with both end-points in $V(T)$. Let L_1, L_2, L_3 be pairwise disjoint lines with $V(T) \subseteq L_1 \cup L_2 \cup L_3$. Then $(T, V(T) \cap L_1, V(T) \cap L_2, V(T) \cap L_3)$ is a three-cliqued claw-free trigraph. We denote by $\mathcal{TC}_2$ the class of such three-cliqued trigraphs with the additional property that every vertex is in a triad.

$\mathcal{TC}_3$: **Near-antiprismatic trigraphs** . Let $n \geq 2$. Construct a trigraph T as follows. Its vertex set is the disjoint union of three sets A, B, C, where $|A| = |B| = n + 1$ and $|C| = n$, say $A = \{a_0, a_1, \ldots, a_n\}$, $B = \{b_0, b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_n\}$. Adjacency is as follows. A, B, C are strong cliques. For $0 \leq i, j \leq n$ with $(i, j) \neq (0, 0)$, let a_i, b_j be adjacent if and only if $i = j$, and for $1 \leq i \leq n$ and $0 \leq j \leq n$ let c_i be adjacent to a_j, b_j if and only if $i \neq j \neq 0$. a_0, b_0 may be semiadjacent or strongly antiadjacent. All other pairs not specified so far are strongly antiadjacent. Now let $X \subseteq A \cup B \cup C \setminus \{a_0, b_0\}$ with $|C \setminus X| \geq 2$. Let all adjacent pairs be strongly adjacent except:

- a_i is semiadjacent to c_i for at most one value of $i \in [n]$, and if so then $b_i \in X$;
- b_i is semiadjacent to c_i for at most one value of $i \in [n]$, and if so then $a_i \in X$;

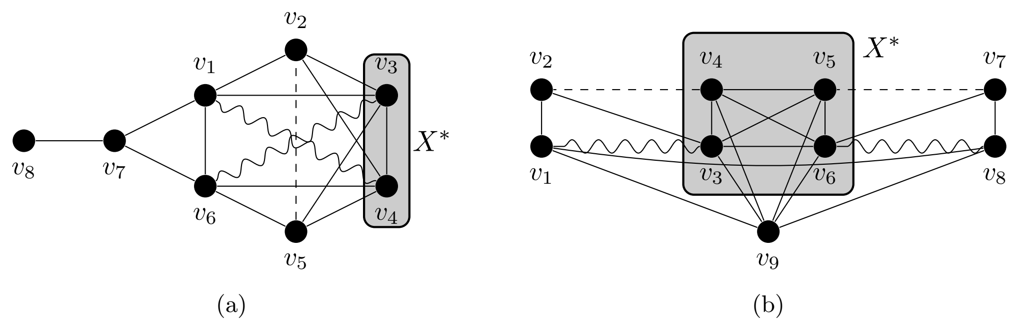

Figure 4: Sporadic exceptions types 1 (left) and 2 (right). The curly lines represent semiedges, and the dashed lines represent arbitrary adjacencies, except that v_2 and v_5 are not strongly adjacent in (a). Also, under some restrictions, vertices from the sets X^* may be deleted.

- a_i is semiadjacent to b_i for at most one value of $i \in [n]$, and if so then $c_i \in X$.

Let the trigraph just constructed be T. Then $T' = T \setminus X$ is a *near-antiprismatic trigraph*. Let $A' = A \setminus X$ and define B', C' similarly; then (T', A', B', C') is a three-cliqued trigraph. We denote by $\mathcal{TC}_3$ the class of all such three-cliqued trigraphs with the additional property that every vertex is in a triad.

$\mathcal{TC}_4$: **Antiprismatic trigraphs**. Let T be an antiprismatic trigraph and let A, B, C be a partition of $V(T)$ into three strong cliques; then (T, A, B, C) is a three-cliqued claw-free trigraph. We denote the class of all such three-cliqued trigraphs by $\mathcal{TC}_4$. Note that in this case there may be vertices that are in no triads.

$\mathcal{TC}_5$: **Sporadic exceptions**. There are two types of sporadic exceptions: (See Figure 4)

(1) Let T be the trigraph with vertex set $\{v_1, \dots, v_8\}$ and adjacency as follows: v_i, v_j are strongly adjacent for $1 \le i < j \le 6$ with $j - i \le 2$; the pairs v_1v_5 and v_2v_6 are strongly antiadjacent; v_1, v_6, v_7 are pairwise strongly adjacent, and v_7 is strongly antiadjacent to v_2, v_3, v_4, v_5; v_7, v_8 are strongly adjacent, and v_8 is strongly antiadjacent to $v_1, \dots, v_6$; the pairs v_1v_4 and v_3v_6 are semiadjacent, and v_2 is antiadjacent to v_5. Let $A = \{v_1, v_2, v_3\}$, $B = \{v_4, v_5, v_6\}$ and $C = \{v_7, v_8\}$. Let $X \subseteq \{v_3, v_4\}$; then $(T \setminus X, A \setminus X, B \setminus X, C)$ is a three-cliqued trigraph, and all its vertices are in triads.

(2) Let T be the trigraph with vertex set $\{v_1, \dots, v_9\}$, and adjacency as follows: the sets $A = \{v_1, v_2\}$, $B = \{v_3, v_4, v_5, v_6, v_9\}$ and $C = \{v_7, v_8\}$ are strong cliques; v_9 is strongly adjacent to v_1, v_8 and strongly antiadjacent to v_2, v_7; v_1 is strongly antiadjacent to v_4, v_5, v_6, v_7, semiadjacent to v_3 and strongly adjacent to v_8; v_2 is strongly antiadjacent to v_5, v_6, v_7, v_8 and strongly adjacent to v_3; v_3, v_4 are strongly antiadjacent to v_7, v_8; v_5 is strongly antiadjacent to v_8; v_6 is semiadjacent to v_8 and strongly adjacent to v_7; and the adjacency between the pairs v_2v_4 and v_5v_7 is arbitrary. Let $X \subseteq \{v_3, v_4, v_5, v_6\}$, such that

- v_2 is not strongly anticomplete to $\{v_3, v_4\} \setminus X$;
- v_7 is not strongly anticomplete to $\{v_5, v_6\} \setminus X$;
- if $v_4, v_5 \notin X$ then v_2 is adjacent to v_4 and v_5 is adjacent to v_7.

Then $(T \setminus X, A, B \setminus X, C)$ is a three-cliqued trigraph.

We denote by $\mathcal{TC}_5$ the class of such three-cliqued trigraphs (given by one of these two constructions) with the additional property that every vertex is in a triad.

We say that a three-cliqued trigraph (T, A, B, C) is *basic* if $(T, A, B, C) \in \bigcup_{i=1}^{5} \mathcal{TC}_i$. If (T, A, B, C) is a three-cliqued trigraph, and $\{A', B', C'\} = \{A, B, C\}$, then (T, A', B', C') is also a three-cliqued trigraph, that we say is a *permutation* of (T, A, B, C). Let $n \geq 0$, and for $1 \leq i \leq n$, let (T_i, A_i, B_i, C_i) be a three-cliqued trigraph, where $V(T_1), \dots, V(T_n)$ are all nonempty and pairwise vertex-disjoint. Let $A = A_1 \cup \dots \cup A_n$, $B = B_1 \cup \dots \cup B_n$, and $C = C_1 \cup \dots \cup C_n$, and let T be the trigraph with vertex set $V(T_1) \cup \dots \cup V(T_n)$ and with adjacency as follows:

- for $1 \leq i \leq n$, $T|V(T_i) = T_i$;
- for $1 \leq i < j \leq n$, A_i is strongly complete to $V(T_j) \setminus B_j$; B_i is strongly complete to $V(T_j) \setminus C_j$; and C_i is strongly complete to $V(T_j) \setminus A_j$; and
- for $1 \leq i < j \leq n$, if $u \in A_i$ and $v \in B_j$ are adjacent then u, v are both in no triads; and the same applies if $u \in B_i$ and $v \in C_j$, and if $u \in C_i$ and $v \in A_j$.

In particular, A, B, C are strong cliques, and so (T, A, B, C) is a three-cliqued trigraph; we call the sequence (T_i, A_i, B_i, C_i), $i \in [n]$, a *worn hex-chain* for (T, A, B, C). When $n = 2$, we say that (T, A, B, C) is a *worn hex-join* of (T_1, A_1, B_1, C_1) and (T_2, A_2, B_2, C_2). Note also that every triad of T is a triad of one of $T_1, \dots, T_n$, and if each T_i is claw-free then so is T. If we replace the third condition above by the strengthening

- for $1 \leq i < j \leq n$, the pairs (A_i, B_j), (B_i, C_j) and (C_i, A_j) are strongly anticomplete,

then we call the sequence a *hex-chain* for (T, A, B, C). When $n = 2$, (T, A, B, C) is a *hex-join* of (T_1, A_1, B_1, C_1) and (T_2, A_2, B_2, C_2). We will use the following theorem, which is a corollary of **4.1** in [7].

Theorem 2.11. *Every claw-free graph that is a graphic thickening of a three-cliqued trigraph is a graphic thickening of a trigraph that admits a worn hex-chain into terms, each of which is a permutation of a basic three-cliqued trigraph.*

2.5 Properties of linear interval trigraphs and long circular interval trigraphs

A graph G is said to be a *long circular interval graph* if G, regarded as a trigraph, is a long circular interval trigraph. We use a characterization of long circular interval graphs that was given by **1.1** in [6]. We need some more definitions. A *net* is a graph with six vertices $a_1, a_2, a_3, b_1, b_2, b_3$, such that $\{a_1, a_2, a_3\}$ is a clique and a_i, b_i are adjacent for $i = 1, 2, 3$, and all other pairs are nonadjacent. An *antinet* is the complement graph of a net. A $(1,1,1)$-prism is a graph with six vertices $a_1, a_2, a_3, b_1, b_2, b_3$, such that $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are cliques, and a_i, b_i are adjacent for $i = 1, 2, 3$, and all other pairs are nonadjacent. Let T be a trigraph. A *center* for a weakly induced cycle C is a vertex in $V(T) \setminus V(C)$ that is complete to $V(C)$ and a weakly induced cycle C

is *dominating* in T if every vertex in $V(T) \setminus V(C)$ has a neighbor in $V(C)$. Since every realization of a long circular interval trigraph is a long circular interval graph, the following lemma is a straightforward corollary of **1.1** in [6].

(2.12) *Let T be a long circular interval trigraph. Then, T does not contain a claw, net, antinet or $(1,1,1)$-prism as a weakly induced subgraph, and every weakly induced cycle of length at least four is dominating and has no center.*

(Notice that, although **1.1** in [6] gives necessary and sufficient conditions, the reverse implication of (2.12) is not true.) Recall that a linear interval trigraph is a special case of a circular interval trigraph. The following lemma gives a necessary and sufficient condition for a long circular interval graph to be a linear interval trigraph:

(2.13) *Let T be a long circular interval trigraph. Then T is a linear interval graph if and only if T has no semihole of length at least four.*

Proof. To prove the 'only if' direction, let T be a linear interval trigraph and suppose that T contains a semihole C with $k = |V(C)| \geq 4$. It follows from the definition of a linear interval trigraph that there exists a linear ordering $(\leq, V(T))$ such that, for all distinct $x, y, z \in V(T)$, it holds that if x and y are adjacent and $x < z < y$, then z is strongly adjacent to x and y. Let $c_1, c_2, \ldots, c_k$ be the vertices of C in order. It follows from the totality of the order that, for all distinct $c_i, c_j \in V(C)$, we have either $c_i < c_j$ or $c_i > c_j$. Clearly, we cannot have $c_i > c_{i+1}$ for all $i \in [k]$ or $c_i < c_{i+1}$ for all $i \in [k]$ (where subscript arithmetic is modulo k). Hence, there exists $i \in [k]$ such that either $c_{i+1} > c_i > c_{i+2}$ or $c_{i+1} > c_{i+2} > c_i$. But, since c_{i+1} is adjacent to c_i and c_{i+2}, this implies that c_i is strongly adjacent to c_{i+2}, a contradiction.

For the 'if' direction, let T be a long circular interval trigraph and suppose that T is not a linear interval trigraph. We will show that T has a semihole of length at least four. Let $\Sigma, F_1, \ldots, F_k$ be as in the definition of T. Notice that the choice of k and $\{F_i\}$ is not unique. Choose k minimal and choose $\{F_i\}$ such that the length of F_i ($i \in [k]$) is minimal. This implies that $|F_i \cap V(T)| \geq 2$ for all $i \in [k]$. Let $v_1, \ldots, v_n$ be the vertices of T in clockwise order on Σ. Let $v_i, v_j \in F_l$, $i \neq j$. We say that v_i is a *clockwise neighbor of* v_j if $v_{j+1}, \ldots, v_{i-1} \in F_l$ (subscripts are taken modulo n). Let $N^+(v_i)$ denote the set of clockwise neighbors of v_i, for all $i \in \{1, \ldots, n\}$. We first claim that:

($*$) *For every $v_i \in V(T)$, $|N^+(v_i)| \geq 1$.*

Suppose that there exists $v_j \in V(T)$ that has no clockwise neighbor. Now consider the interval $I = (v_j, v_{j+1}) \subseteq \Sigma$. We claim that $F_i \cap I = \emptyset$ for all $i \in [k]$. For suppose that there exists $i \in [k]$ such that $F_i \cap I \neq \emptyset$. If $v_j \in F_i$, then $v_{j+1} \notin F_i$ and hence, we can replace F_i by $F_i \setminus I$ without changing the graph, contrary to the assumption that F_i is chosen minimal. Thus, $v_j \notin F_i$ and, similarly, $v_{j+1} \notin F_i$. Since $F_i \cap I \neq \emptyset$ and F_i is homeomorphic to the interval $[0,1]$, this implies that $V(T) \cap F_i = \emptyset$, contradicting the minimality of k. Thus, $I \cap \bigcup_{i=1}^{k} F_i = \emptyset$, contrary to our assumption that T is not a linear interval trigraph. This proves ($*$). □

Now let $\mathcal{C}$ be the set of cycles $C = c_1\text{-}c_2\text{-}\dots\text{-}c_p\text{-}c_1$ in T such that $c_{i+1} \in N^+(c_i)$ for all $i \in [p]$. It follows from (∗) that $\mathcal{C} \neq \emptyset$. Let $C \in \mathcal{C}$ and let $F'_1, \dots, F'_{|V(C)|} \in \{F_1, \dots, F_k\}$ be such that $\{c_i, c_{i+1}\} \subset F'_i$ for all $i \in [k]$. Clearly, $\Sigma = \bigcup_{j=1}^{|V(C)|} F'_i$. Therefore, by the definition of a long circular interval graph, we have $|V(C)| \geq 4$ for all $C \in \mathcal{C}$. Now choose $C^* \in \mathcal{C}$ with $|V(C^*)|$ minimum. Since $|V(C^*)|$ is minimum, C^* is a semihole. Moreover, $|V(C^*)| \geq 4$. This proves (2.13). ■

3 Applications

In this section, we give two applications of Theorem 1.1, assuming its validity.

3.1 Claw-free graphs that are strongly perfect in the complement

Wang [14] gave a characterization of claw-free graphs that are strongly perfect. Theorem 1.1 allows us to give a characterization of claw-free graphs whose complement is strongly perfect:

Theorem 1.2. *Let G be a claw-free graph. G^c is strongly perfect if and only if G is perfect and no induced subgraph of G is isomorphic to $\mathcal{G}_4$, an even hole of length at least six, or a skipping rope of type $(3,3)$ of length $k \geq 0$.*

Proof of Theorem 1.2. For the 'only-if' direction, let G be a claw-free graph such that G^c is strongly perfect. Since G^c is strongly perfect, G is fractionally co-strongly perfect. Therefore, it follows from Theorem 1.1 that G is $\mathcal{F}$-free and, in particular, no induced subgraph of G is isomorphic to $\mathcal{G}_4$, an even cycle of length at least six, or a skipping rope of type $(3,3)$ of length $k \geq 0$. Moreover, it follows from Theorem 5 of [1] applied to G^c that G is perfect. This proves the 'only-if' direction.

For the 'if' direction, let G be a perfect claw-free graph such that no induced subgraph of G is isomorphic to $\mathcal{G}_4$, an even cycle of length at least six, or a skipping rope of type $(3,3)$ of length $k \geq 0$. Since G is perfect, by the strong perfect graph theorem [5], G has no odd hole or odd antihole of length at least five as an induced subgraph. Because all graphs in $\mathcal{F}$ other than $\mathcal{G}_4$, the even holes and the skipping ropes of type $(3,3)$ contain an induced cycle of length five or length seven, it follows that G is $\mathcal{F}$-free and hence, by Theorem 1.1, G is perfectly resolved. Now recall that a graph G^c is strongly perfect if and only if every induced subgraph of G has a dominant clique. We note that every disconnected induced subgraph of G has a dominant clique if and only if one of its components has a dominant clique. Therefore, it suffices to show that every connected induced subgraph of G has a dominant clique. So suppose to the contrary that G has a minimal connected induced subgraph H such that H has no dominant clique. Since G is perfectly resolved, it follows that H is resolved. Since H is perfect and H has no dominant clique, it follows that H has a vertex x that is complete to $V(H) \setminus \{x\}$. Since H is minimal, $H - x$ has a dominant clique K. But this implies that $K \cup \{x\}$ is a dominant clique in H, a contradiction. This proves Theorem 1.2. ■

Theorem 1.2 states that if a claw-free graph is perfect and it is fractionally co-strongly perfect, then it is integrally co-strongly perfect. We conjecture that this is true in general:

Conjecture 3.1. *If G is perfect and fractionally strongly perfect, then G is strongly perfect.*

3.2 Scheduling in Wireless Networking

Consider a wireless communication network $H = (V, E)$, in which V is the set of nodes (*i.e.* transmitters and receivers), and $E \subseteq \{ij : i, j \in V, i \neq j\}$ is a set of connections representing pairs of nodes between which data flow can occur. Next, consider a so-called *interference graph G of H*, whose vertices are the edges of H and in which two edges $e, e' \in E$ are adjacent if they are not allowed to send data simultaneously because of interference constraints. At each node of the network packets are created over time and these packets must be transmitted to their destination (*i.e.* an adjacent node).

Following the model of [3, 8, 10, 13], assume that time is slotted and that packets are of equal size, each packet requiring one time slot of service across a link. A queue is associated with each edge in the network, representing the packets waiting to be transmitted on this link. A *scheduling algorithm* selects a set of edges to activate at each time slot, and transmits packets on those edges. Since they must not interfere, the selected edges most form a stable set in the interference graph G. A scheduling algorithm is called *stable on G* if, informally speaking, the sizes of the queues do not grow to infinity under the algorithm. It was shown in [13] that the Maximum Weight Stable Set algorithm (MWSS) that selects the stable set in the interference graph that corresponds to the links in the network with the largest total queue sizes at each slot is stable for every interference graph G. Although this MWSS algorithm is stable, it is not a tractable algorithm in many situations because it needs centralized computing of an optimal solution. Hence, there has been an increasing interest in simple and potentially distributed algorithms. One example of such an algorithm is known as the Greedy Maximal Scheduling (GMS) algorithm [9, 11]. This algorithm greedily selects the set of served links according to the queue lengths at these links (*i.e.* greedily selects a maximal weight stable set in the interference graph). A drawback of using this algorithm is that, in general, the resulting schedule is not necessarily optimal. However, [8] gave the following sufficient condition on interference graphs on which the GMS algorithm is stable:

Theorem. (Dimakis and Walrand [8]) *Let G be a fractionally co-strongly perfect graph. Then GMS is stable on G.*

In [2], the authors and three others characterized all line graphs that are fractionally co-strongly perfect. Here we generalize this result to claw-free graphs. Thus, Theorem 1.1 describes all claw-free graphs on which GMS is stable.

4 Proofs of Theorem 1.3 and Theorem 1.4

In this section, we give the proofs of Theorem 1.3 and Theorem 1.4.

4.1 Fractionally co-strongly perfect graphs are $\mathcal{F}$-free

We first prove a result on saturating vertex weightings in graphs that display a certain symmetry:

(4.1) *Let G be a graph that has a saturating vertex weighting. Let $\phi : V(G) \to V(G)$ be an automorphism for G. Then there exists a saturating vertex weighting $\bar{w}$ such that $\bar{w}(x) = \bar{w}(\phi(x))$ for every $x \in V(G)$.*

Proof. Suppose that w is a saturating vertex weighting for G. Let $\phi^1 = \phi$ and for $k \geq 2$, let $\phi^k = \phi^{k-1} \circ \phi$. Since a set $S \subseteq V(G)$ is stable if and only if $\phi^k(S)$ is stable, it follows that $w \circ \phi^k$ is a saturating vertex weighting for G. Let $K \geq 1$ be such that $V(G) = \phi^K(V(G))$ and consider the function $\bar{w} = \frac{1}{K}\sum_{i=0}^{K-1} w \circ \phi^i$. Since $\bar{w}$ is a convex combination of solutions to the system of linear equations (1), it follows that $\bar{w}$ is a solution to (1) and, therefore, $\bar{w}$ is a saturating vertex weighting. Now observe that $\bar{w} = \bar{w} \circ \phi$. This proves (4.1). ■

Next, we need the following technical result. For a connected graph G, we say that $X \subseteq V(G)$ is a *clique cutset* if X is a clique and $G \setminus X$ is disconnected.

(4.2) *Let G be a graph, let X be a clique cutset in G, let B be a connected component of $G \setminus X$ and let $G' = G \setminus V(B)$. Suppose that for every $x \in X$, $G|(V(B) \cup \{x\})$ is a heft with end x and $N(x) \cap V(B) = N(x') \cap V(B)$ for all $x, x' \in X$. Suppose in addition that there exists a maximal stable set in G' that does not meet X. Then, every saturating vertex weighting w for G satisfies $w(v) = 0$ for all $v \in V(B)$.*

Proof. Let $x \in X$, $i \in [3]$, and $k \geq 0$ be such that $B' = G|(V(B) \cup \{x\})$ is isomorphic to the heft $\mathcal{H}_i(k)$. We prove the lemma for the case when $i = 1$ only, as the other two cases are analogous. Let $P = p_1\text{-}p_2\text{-}\dots\text{-}p_k = x$ be the rope of B' and let $v_1, v_2, \dots, v_5$ be the other vertices of B', labeled as in Figure 1(b). We use induction on k. Let w be a saturating vertex weighting for G. By (4.1), we may assume that $w(v_2) = w(v_5)$ and $w(v_3) = w(v_4)$. First suppose that $k = 0$. Let S be a maximal stable set in G' such that $x \in S$. Let $S_1 = S \cup \{v_2, v_5\}$ and let $S_2 = S \cup \{v_1\}$. Since w is a saturating vertex weighting and S_1 and S_2 are maximal stable sets with $S_1 \setminus S_2 = \{v_2, v_5\}$ and $S_2 \setminus S_1 = \{v_1\}$, it follows that $w(v_1) = w(v_2) + w(v_5) = 2w(v_5)$. Now let S' be a maximal stable set in G such that $v_3 \in S$. Clearly, either $v_1 \in S'$ or $v_5 \in S'$. Let $S'_1 = (S' \setminus \{v_1\}) \cup \{v_5\}$ and $S'_2 = (S' \setminus \{v_5\}) \cup \{v_1\}$. Since w is a saturating vertex weighting and S'_1 and S'_2 are maximal stable sets with $S'_1 \setminus S'_2 = \{v_5\}$ and $S'_2 \setminus S'_1 = \{v_1\}$, it follows that $w(v_5) = w(v_1)$. Combining this with the equality found above, it follows that $w(v_1) = 2w(v_1)$ and hence that $w(v_1) = w(v_2) = w(v_5) = 0$. Finally, let S'' be a maximal stable set in G' such that S'' does not meet X. Let $S''_1 = S'' \cup \{v_3, v_5\}$ and $S''_2 = S'' \cup \{v_2, v_5\}$. Since w is a saturating vertex weighting and S''_1 and S''_2 are maximal stable sets with $S''_1 \setminus S''_2 = \{v_3\}$ and $S''_2 \setminus S''_1 = \{v_2\}$, it follows that $w(v_3) = w(v_2) = 0$ and, hence, $w(v_4) = 0$. This proves the claim for $k = 0$.

Next, suppose that $k \geq 1$ and let y be the unique neighbor of x in $V(B)$. Since $\{y\}$ is a clique cutset, B is isomorphic to the heft $\mathcal{H}_1(k-1)$, and there clearly exists a maximal stable set in $G|(V(G') \cup \{y\})$ that does not meet y. Now it follows from the induction hypothesis that $w(v) = 0$

for all $v \in V(B) \setminus \{y\}$ and therefore it suffices to show that $w(y) = 0$. Let S be a maximal stable set in G' such that $S \cap X = \emptyset$. Let S_1 be a maximal stable set in B such that $y \in S_1$ and let S_2 be a maximal stable set in B such that $y \notin S_2$. Since $S \cup S_1$ and $S \cup S_2$ are maximal stable sets, it follows that

$$\sum_{v \in S_2} w(v) = \sum_{v \in S_1} w(v) = w(y) + \sum_{v \in S_1 \setminus \{y\}} w(v).$$

Now, since $\sum_{v \in S_1 \setminus \{y\}} w(v) = \sum_{v \in S_2} w(v) = 0$, it follows that $w(y) = 0$. This proves (4.2). ■

This puts us in a position to prove Theorem 1.3, the statement of which we repeat for clarity:

Theorem 1.3. *Let G be a fractionally co-strongly perfect graph. Then G is $\mathcal{F}$-free.*

Proof. It suffices to show that no graph in $\mathcal{F}$ is fractionally co-strongly perfect. First, let $H \in \mathcal{F}_1$, *i.e.* H is a cycle of length $n = 2k$, $k \geq 3$ or of length $n = 2k+1, k \geq 4$. Suppose that there exists a saturating vertex weighting w for H. It follows from (4.1) that there exists $c \in [0,1]$ such that $w : V(H) \to [0,1]$ with $w(v) = c$ for every $v \in V(H)$. Let $v_1, v_2, \dots, v_n$ be the vertices of H in order. Since $\{v_2, v_4, \dots, v_{2k}\}$ is a maximal stable set of cardinality k, it follows that $c = \frac{1}{k}$. Now let $S = \{v_1\} \cup \bigcup_{i=2}^{k-1} \{v_{2i}\}$ if k is even and let $S = \{v_1, v_4\} \cup \bigcup_{i=3}^{k-1} \{v_{2i+1}\}$ if k is odd. Then S is a maximal stable set, but $\sum_{v \in S} w(v) = \frac{|S|}{k} = \frac{k-1}{k} < 1$, a contradiction.

Next, suppose that there exists a saturating vertex weighting w for $\mathcal{G}_1$. It follows from (4.1) and the fact that the graph is symmetric along the vertical axis that we may assume that $w(v_2) = w(v_5)$, $w(v_3) = w(v_4)$ and $w(v_6) = w(v_7)$. Since $\{v_2, v_5\}$ is a maximal stable set, it follows that $w(v_2) = w(v_5) = \frac{1}{2}$. By considering the maximal stable sets $\{v_2, v_4\}$ and $\{v_2, v_7\}$, it follows that $w(v_4) = w(v_7) = \frac{1}{2}$ and therefore $w(v_3) = w(v_6) = \frac{1}{2}$. By considering the maximal stable set $\{v_1, v_3\}$ we obtain $w(v_1) = \frac{1}{2}$. But now, $S = \{v_1, v_6, v_7\}$ is a maximal stable set with $\sum_{v \in S} w(v) = \frac{3}{2} \neq 1$, a contradiction. This proves that $\mathcal{G}_1$ is not fractionally co-strongly perfect. The proofs for the graphs $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$ are analogous.

Finally, consider any $H \in \mathcal{F}_3$. It follows that H is a skipping rope. Let (H_1, k_1), (H_2, k_2), x_1, and x_2 be as in the definition of a skipping rope. By applying (4.2) to each of the two hefts, it follows that every saturating vertex weighting w satisfies $w(v) = 0$ for all $v \in V(H)$, clearly contradicting the fact such w is a saturating vertex weighting. Hence, H has no saturating vertex weighting and therefore H is not fractionally co-strongly perfect. This proves Theorem 1.3. ■

4.2 Perfectly resolved claw-free graphs are fractionally co-strongly perfect

The next step is to show that perfectly resolved graphs are fractionally co-strongly perfect. We start with a simple lemma:

(4.3) *A graph G is fractionally co-strongly perfect if and only if every connected component of G is fractionally co-strongly perfect.*

Proof. The 'only-if' direction follows immediately from the definition of fractional co-strongly perfection. For the 'if' direction, let H be an induced subgraph of G. Let $C_1, C_2, \dots, C_q$ be the connected components of G and, for $i \in [q]$, let $H_i = G|(V(H) \cap V(C_i))$. From the symmetry, we may assume that $V(H_1) \neq \emptyset$. Since C_1 is fractionally co-strongly perfect, so is H_1 and, hence, there exists $w_1 : V(H_1) \to [0,1]$ such that $\sum_{v \in T} w_1(v) = 1$ for every maximal stable set T of H_1. Now define $w : V(H) \to [0,1]$ by $w(u) = w_1(u)$ for all $u \in V(H_1)$ and $w(v) = 0$ for all $v \in V(H) \setminus V(H_1)$. Let S be a maximal stable set S of H. Since $S \cap V(H_1) \neq \emptyset$ is a maximal stable set in H_1, it follows that $\sum_{v \in S} w(v) = \sum_{v \in S \cap V(H_1)} w(v) = 1$, thus proving (4.3). ■

This lemma enables us to prove Theorem 1.4, the statement of which we repeat for clarity:

Theorem 1.4. *Let G be a claw-free graph. If G is perfectly resolved, then G is fractionally co-strongly perfect.*

Proof. Let G' be an induced subgraph of G. We argue by induction on $|V(G')|$. It follows from (4.3) that we may assume that G' is connected. It suffices to show that G' has a saturating vertex weighting. Since G is perfectly resolved, G' is resolved. It follows that either there exists $x \in V(G')$ such that x is complete to $V(G') \setminus \{x\}$, or G' has a dominant clique, or G' is not perfect and there exists $k \in \{2,3\}$ such that every maximal stable set in G' has size k. First, suppose that there exists $x \in V(G')$ such that x is complete to $V(G') \setminus \{x\}$. It follows from the inductive hypothesis that $G' \setminus \{x\}$ has a saturating vertex weighting w_0. Define $w : V(G') \to [0,1]$ by setting $w(x) = 1$ and $w(v) = w_0(v)$ for all $v \in V(G') \setminus \{x\}$. It is not hard to see that this is a saturating vertex weighting for G' and the claim holds. Next, suppose that G' has a dominant clique K. Define $w : V(G') \to [0,1]$ by $w(v) = 1$ if $v \in K$ and $w(v) = 0$ otherwise. This is clearly a saturating vertex weighting for G' and, hence, the claim holds. Finally, suppose that G' is not perfect and there exists k such that every maximal stable set in G' has cardinality k. Now $w : V(G') \to [0,1]$ defined by $w(v) = 1/k$ for all $v \in V(G')$ is clearly a saturating vertex weighting for G'. Therefore, the claim holds. This proves Theorem 1.4. ■

5 $\mathcal{F}$-free basic claw-free graphs are perfectly resolved

In this section, the goal is to prove Theorem 1.5 using the structure theorem for claw-free graphs, Theorem 2.10. In fact, we prove the following:

(5.1) *Every $\mathcal{F}$-free basic claw-free trigraph is resolved.*

Since an $\mathcal{F}$-free claw-free trigraph T is resolved if and only if every $\mathcal{F}$-free graphic thickening of T is resolved, Theorem 1.5 is an immediate corollary of (5.1). We prove (5.1) by dealing with the outcomes of Theorem 2.10 separately. We first make the following easy observation concerning trigraphs from the icosahedron. (See Section 2.3 for the definition of a trigraph from the icosahedron.)

(5.2) *No trigraph from the icosahedron is $\mathcal{F}$-free.*

Proof. Let T be a trigraph from the icosahedron and let $v_1, v_2, \dots, v_9$ be as in the definition of T. Then, v_1-v_3-v_5-v_6-v_8-v_9-v_1 is a weakly induced cycle of length six in T, and thus T is not $\mathcal{F}$-free. This proves (5.2). ■

We will deal with the remaining outcomes of (5.1), namely antiprismatic trigraphs, circular interval trigraphs, and trigraphs that are the union of three cliques, in Section 5.1, Section 5.2, and Section 5.3, respectively.

5.1 $\mathcal{F}$-free antiprismatic trigraphs

The following lemma deals with $\mathcal{F}$-free antiprismatic trigraphs. (See Section 2.3 for the definition of an antiprismatic trigraph.)

(5.3) *Every $\mathcal{F}$-free antiprismatic trigraph is resolved.*

Proof. Let T be an $\mathcal{F}$-free antiprismatic trigraph. If T contains no triad, then T is resolved by (2.6). Thus, we may assume that T contains a triad $\{a_1, a_2, a_3\}$. Let B_1 be the vertices that are complete to $\{a_2, a_3\}$, B_2 the vertices that are complete to $\{a_1, a_3\}$, and B_3 the vertices that are complete to $\{a_1, a_2\}$. Since T is antiprismatic, it follows that $V(T) = \{a_1, a_2, a_3\} \cup B_1 \cup B_2 \cup B_3$. We may assume that T is not resolved. We give the proof using a number of claims.

(i) *For distinct $i, j \in [3]$, a_i is strongly antiadjacent to a_j and $B_i \cup B_j$ is not a strong clique.*

We may assume that $i = 1$, $j = 2$. First suppose that $a_1a_2 \in F(T)$. If $b_1, b_1' \in B_1$ are antiadjacent, then a_2 is complete to the triad $\{a_1, b_1, b_1'\}$, contrary to (2.2). Thus, B_1 is a strong clique and, by the symmetry, B_2 is a strong clique. If B_1 is strongly complete to B_2, then a_3 is a simplicial vertex, contrary to (2.5). Thus, there exist antiadjacent $b_1 \in B_1$ and $b_2 \in B_2$. But now, (2.8) applied to a_1-a_2-b_1-a_3-b_2-a_1 implies that T is resolved, a contradiction. This proves that $a_1a_2 \notin F(T)$, and thus a_1 is strongly antiadjacent to a_2. Now suppose that $B_1 \cup B_2$ is a strong clique. Then, a_3 is a simplicial vertex, contrary to (2.5). This proves **(i)**. □

(ii) *Let $i, j \in [3]$ be distinct. Let $x_1, x_2 \in B_i$ be antiadjacent. Then, B_j can be partitioned into sets $B_j(x_1)$, $B_j(x_2)$ such that, for $\{k, l\} = \{1, 2\}$, x_k is strongly complete to $B_j(x_k)$ and strongly anticomplete to $B_j(x_l)$.*

From the symmetry, we may assume that $i = 1$ and $j = 2$. If x_1 and x_2 have a common neighbor $z \in B_2$, then z is complete to the triad $\{a_1, x_1, x_2\}$, a contradiction. If x_1 and x_2 have a common antineighbor $z' \in B_2$, then a_3 is complete to the triad $\{x_1, x_2, z'\}$, a contradiction. Thus, x_1 and x_2 have no common neighbor and no common antineighbor in B_2. It follows that for every $z \in B_2$, one of x_1, x_2 is strongly adjacent to z, and the other is strongly antiadjacent to z. This proves **(ii)**. □

(iii) *There is no triad $\{b_1, b_2, b_3\}$ with $b_i \in B_i$ for $i = 1, 2, 3$.*

Suppose that $\{b_1, b_2, b_3\}$ is a triad with $b_i \in B_i$. Then a_1-b_3-a_2-b_1-a_3-b_2-a_1 is a weakly induced cycle of length six, a contradiction. This proves **(iii)**. □

(iv) *B_1, B_2, B_3 are all nonempty strong cliques.*

First suppose for a contradiction that, for $i = 1, 2$, there exist antiadjacent $p_i, q_i \in B_i$. It follows from **(ii)** that we may assume that p_1 is strongly adjacent to p_2 and strongly antiadjacent to q_2, and q_1 is strongly adjacent to q_2 and strongly antiadjacent to p_2. Now, a_2-p_1-p_2-a_1-q_2-q_1-a_2 is a weakly induced cycle of length six, a contradiction. This proves that at most one of B_1, B_2, B_3 is not a strong clique.

Next, suppose that $B_1 = \emptyset$. Since at most one of B_2, B_3 is not a strong clique, we may assume that B_2 is a strong clique. But now $B_1 \cup B_2$ is a strong clique, contrary to **(i)**. This proves that B_1, B_2 and B_3 are all nonempty.

We may assume that B_1 is not a strong clique, because otherwise the claim holds. It follows that B_2 and B_3 are strong cliques. Let $x, y \in B_1$ be antiadjacent. For $i = 2, 3$, let $B_i(x) \subseteq B_i$ and $B_i(y) \subseteq B_i$ be as in **(ii)** applied to x, y, B_1, and B_i. It follows from **(iii)** that $B_2(x)$ is strongly complete to $B_3(x)$ and $B_2(y)$ is strongly complete to $B_3(y)$. Hence, from **(i)** and the symmetry, we may assume that there exist antiadjacent $x_2 \in B_2(x)$ and $y_3 \in B_3(y)$. If there exists $x_3 \in B_3(x)$, then $T|\{x_2, x_3, y_3, y, a_3, x, a_1\}$ contains $\mathcal{G}_1$ as a weakly induced subgraph, a contradiction. This proves that $B_3(x) = \emptyset$ and, by the symmetry, that $B_2(y) = \emptyset$. Observe that this implies that $B_2(x) = B_2$ and $B_3(y) = B_3$.

So we may assume that for every two antiadjacent $x', y' \in B_1$, one of x', y' is strongly complete to B_2 and strongly anticomplete to B_3, and the other is strongly complete to B_3 and strongly anticomplete to B_2. Since $B_2, B_3 \neq \emptyset$, it follows that the complement of $T|B_1$ contains no odd cycles, and thus B_1 is the union of two strong cliques. For $i = 2, 3$, let $Z_i \subseteq B_1$ be the set of vertices in B_1 that have an antineighbor in B_1 and that are strongly complete to B_i. It follows that Z_2 and Z_3 are strong cliques. Let $Z^* = B_1 \setminus (Z_2 \cup Z_3)$. By definition, Z^* is a strong clique and Z^* is strongly complete to $Z_1 \cup Z_2$.

Now observe (Z_2, Z_3) is a homogeneous pair of strong cliques. It follows from **(i)** that there exist antiadjacent $b_2 \in B_2$ and $b_3 \in B_3$. But now, by (2.7) applied to (Z_2, Z_3) and the weakly induced path b_2-a_1-b_3, it follows that T is resolved, a contradiction. This proves **(iv)**. □

(v) *Let $\{i, j, k\} = \{1, 2, 3\}$. Let $b_i \in B_i$ and $b_j \in B_j$ be antiadjacent. Then, at least one of b_i, b_j is strongly complete to B_k.*

We may assume that $i = 1$, $j = 2$, $k = 3$. Suppose that b_1 has an antineighbor $x \in B_3$ and b_2 has a antineighbor $y \in B_3$. It follows from **(iii)** that $x \neq y$ and that x is strongly adjacent to b_2 and y is strongly adjacent to b_1. It follows from **(iv)** that x is strongly adjacent to y. Now, $T|\{a_3, b_1, x, y, b_2, a_1, a_2\}$ contains $\mathcal{G}_1$ as a weakly induced subgraph, a contradiction. This proves **(v)**. □

(vi) *Let $\{i, j, k\} = \{1, 2, 3\}$. Then, no vertex in B_i has antineighbors in both B_j and B_k.*

We may assume that $i = 1$, $j = 2$, $k = 3$. Suppose that $b_1 \in B_1$ has antineighbors $b_2 \in B_2$ and $b_3 \in B_3$. It follows from **(iii)** that b_2 is strongly adjacent to b_3. It follows from **(v)** that b_2 is strongly complete to B_3 and b_3 is strongly complete to B_2. From **(i)**, there exist antiadjacent $b'_2 \in B_2$ and $b'_3 \in B_3$. It follows that $\{b_2, b_3\} \cap \{b'_2, b'_3\} = \emptyset$. It follows from **(v)** that b'_2, b'_3 are both strongly complete to B_1. Now $T|\{b_3, b_2, a_3, b_1, a_2, b'_2, b'_3, a_1\}$ contains $\mathcal{G}_3$ as a weakly induced subgraph, a contradiction. This proves **(vi)**. □

It follows from **(i)** that for $i = 1, 2, 3$, there exist $x_i, y_i \in B_i$ such that the pairs x_1y_2, x_2y_3, x_3y_1 are antiadjacent. It follows from **(iv)** and **(vi)** that $x_i \neq y_i$ for $i = 1, 2, 3$ and all pairs among $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ except the aforementioned are strongly adjacent. Now, $T|\{a_1, x_1, y_1, a_2, x_2, y_2, a_3, x_3, y_3\}$ contains $\mathcal{G}_4$ as a weakly induced subgraph, a contradiction. This proves (5.3). ■

5.2 $\mathcal{F}$-free long circular interval trigraphs

In this section, we prove that $\mathcal{F}$-free long circular interval trigraphs are resolved. We start with the following easy result, which shows that we may assume that the long circular interval trigraphs that we are dealing with in this section are really long circular interval trigraphs and not linear interval trigraphs. (See Section 2.3 and Section 2.5 for definitions and basic results on linear interval trigraphs and long circular interval trigraphs.)

(5.4) *Every linear interval trigraph is resolved.*

Proof. Let T be a linear interval trigraph. Thus, we may order the vertices of T as $v_1, v_2, \ldots, v_n$ such that for $i < j$, if v_i is adjacent to v_j, then v_k is strongly adjacent to v_l for all $i < k \leq l \leq j$. It follows that $N(v_1)$ is a strong clique and hence that v_1 is a simplicial vertex in T. Thus, T is resolved by (2.5). This proves (5.4). ■

In handling long circular interval trigraphs, it turns out to be convenient to make a distinction depending on the existence of a semihole of length at least five in the trigraph. Section 5.2.1 deals with the case where the trigraph contains no semihole of length at least five. It will turn out that there are two types of such trigraphs, namely ones that have a structure that is similar to the complement of a 7-cycle, and ones that have a structure that is similar to a 4-cycle with certain attachments. Section 5.2.2 deals with the remaining case where the trigraph does contain such semihole. In this case, the trigraph has a structure that is similar to either a 5-cycle or a 7-cycle, with certain attachments.

5.2.1 Long circular interval trigraphs with no long semiholes

Let $\bar{C}_7$ be a graph that is the complement of a 7-cycle. We say that a trigraph T is *of the $\bar{C}_7$ type* if $V(T)$ can be partitioned into seven nonempty strong cliques $W_1, \ldots, W_7$ such that for all $i \in [7]$,

W_i is strongly complete to W_{i+1}, W_i is complete to W_{i+2}, W_i is strongly anticomplete to W_{i+3} (where subscript arithmetic is modulo 7). We first look at long circular interval trigraphs with no long semiholes that contain $\bar{C}_7$ as a weakly induced subgraph.

(5.5) *Let T be a long circular interval trigraph with no semihole of length at least five. If T contains $\bar{C}_7$ as a weakly induced subgraph, then T is of the $\bar{C}_7$ type.*

Proof. Let $W_1, W_2, \ldots, W_7 \subseteq V(T)$ be such that for all $i \neq j$ (with subscript arithmetic modulo 7), W_i is a nonempty clique, $W_i \cap W_j = \emptyset$, W_i is complete to $W_{i+1} \cup W_{i+2}$, W_i is anticomplete to $W_{i+3} \cup W_{i+4}$, and $\bigcup_{i=1}^{7} W_i$ is maximal. The cliques W_i, $i \in [7]$, exist since T contains $\bar{C}_7$ as a weakly induced subgraph. We start with some claims:

(i) *For $i \in [7]$, W_i is strongly anticomplete to $W_{i+3} \cup W_{i+4}$.*

Without loss of generality we may assume that $i = 1$, and from the symmetry it follows that it is enough to show that W_1 is strongly anticomplete to W_4. So suppose that there exists a vertex $x \in W_1$ which is semiadjacent to some vertex $y \in W_4$. From the definition of a trigraph, it follows that x is strongly complete to $W_6 \cup W_7$ and strongly anticomplete to W_5, and y is strongly complete to $W_5 \cup W_6$ and strongly anticomplete to W_7. But now any vertex $z \in W_6$ is complete to the semihole $\{x, u, v, y\}$, where $u \in W_7$ and $v \in W_5$, which contradicts (2.12). □

The following claim states that many edges in $W_1 \cup W_2 \cup \cdots \cup W_7$ are in fact strong edges.

(ii) *For $i \in [7]$, $W_i \cup W_{i+1}$ is a strong clique.*

Suppose that $w, w' \in W_i \cup W_{i+1}$ are antiadjacent. Let $w_{i+2} \in W_{i+2}$ and $w_{i+4} \in W_{i+4}$. Then, w_{i+2} is anticomplete to the triad $\{w, w', w_{i+4}\}$, contrary to (2.2). This proves **(ii)**. □

(iii) *Suppose that x has a neighbor in W_i. Then, up to symmetry,*

(a) *x is complete to at least one of W_{i-1}, W_{i+1}; and*

(b) *x is complete to at least one of W_{i-1}, W_{i+2}; and*

(c) *x is complete to at least one of W_{i-2}, W_{i+2}.*

Let y_i be a neighbor of x in W_i. Suppose that x has a strong antineighbor $y_{i-1} \in W_{i-1}$ and a strong antineighbor $y_{i+1} \in W_{i+1}$. If x has an antineighbor $y_{i-2} \in W_{i-2}$, then y_i is complete to the triad $\{x, y_{i+1}, y_{i-2}\}$, a contradiction. Thus x is complete to W_{i-2}. From the symmetry, it follows that x is complete to W_{i+2}. Let $y_{i-2} \in W_{i-2}$ and $y_{i+2} \in W_{i+2}$. Now x-y_{i-2}-y_{i-1}-y_{i+1}-y_{i+2}-x is a semihole of length five, a contradiction. This proves part (a). Next suppose that x has a strong antineighbor $y_{i-1} \in W_{i-1}$ and a strong antineighbor $y_{i+2} \in W_{i+2}$. Then y_i is complete to the triad $\{y_{i-1}, y_{i+2}, x\}$, a contradiction. This proves part (b). Finally suppose that x has a strong antineighbor $y_{i-2} \in W_{i-2}$ and a strong antineighbor $y_{i+2} \in W_{i+2}$. Then y_i is complete to the triad $\{y_{i-2}, y_{i+2}, x\}$, a contradiction. This proves part (c), thus completing the proof of **(iii)**. □

We claim that $V(T) = \bigcup_{i=1}^{7} W_i$. For suppose not. Then there exists $x \in V(T) \setminus \bigcup_{i=1}^{7} W_i$ with a neighbor in $\bigcup_{i=1}^{7} W_i$. Because $T|(\bigcup_{i=1}^{7} W_i)$ contains a semihole of length four, it follows from (2.12) that x has a neighbor in some set W_i. It follows from **(iii)** that, for some $i \in [7]$, x is complete to $W_i \cup W_{i+1}$. From the symmetry, we may assume that x is complete to $W_1 \cup W_2$. Now it follows from **(iii)** that x is complete to at least one of W_3, W_7. We may assume that x is complete to W_3. Finally, it follows from **(iii)** that x is complete to at least one of W_4, W_7. We may assume that x is complete to W_4. If x has a neighbor $y_6 \in W_6$, then let $y_1 \in W_1, y_2 \in W_2, y_4 \in W_4$ and observe that y_1-y_2-y_4-y_6-y_1 is a semihole of length four and x is complete to it, contrary to (2.12). This proves that x is strongly anticomplete to W_6,

(iv) *x is complete to exactly one of W_5, W_7 and strongly anticomplete to the other.*

Suppose that x has both a strong antineighbor $y_5 \in W_5$ and a strong antineighbor $y_7 \in W_7$. Then y_7-y_5-y_4-x-y_1-y_7, where $y_1 \in W_1$, is a semihole of length five, a contradiction. This proves that x is complete to one of W_5, W_7. Finally, suppose that x has a neighbor $y_5 \in W_5$ and a neighbor $y_7 \in W_7$. Let $y_2 \in W_2$ and $y_3 \in W_3$. Then x is a center for the semihole y_2-y_7-y_5-y_3-y_2, contrary to (2.12). This proves **(iv)**. □

From **(iv)**, we may assume that x is complete to W_5 and strongly anticomplete to $W_6 \cup W_7$. But now we may add x to W_3 and obtain a larger structure, a contradiction. This proves that $V(T) = \bigcup_{i=1}^{7} W_i$. Now it follows from the definition of $W_1, \dots, W_7$ and from **(ii)** that T is a trigraph of the $\bar{C}_7$ type. This proves (5.5). ■

The previous statement shows that if a long circular interval trigraph with no long semiholes contains $\bar{C}_7$ as a weakly induced subgraph, then it basically looks like $\bar{C}_7$. The following shows that such trigraphs have no triads, hence that they are resolved by (2.6):

(5.6) *Let T be a trigraph of the $\bar{C}_7$ type. Then T contains no triad.*

Proof. Let $W_1, W_2, \dots, W_7 \subseteq V(T)$ be as in the definition of a trigraph of the $\bar{C}_7$ type. Now suppose that T has a stable set $\{s_1, s_2, s_3\}$. Since W_i is a strong clique and W_i is strongly complete to W_{i+1}, it follows that for $j \neq k$, s_j and s_k are not in consecutive sets. Therefore, from the symmetry, we may assume that $s_1 \in W_1$, $s_2 \in W_3$, and $s_3 \in W_6$. It follows that s_1 is semiadjacent to both s_2 and s_3, a contradiction. This proves (5.6). ■

So, we may exclude $\bar{C}_7$ and concentrate on what happens otherwise. Let T be a trigraph. Let A_1, A_2, A_3, A_4, $B_1, B_2, B_3, B_4 \subseteq V(T)$ be disjoint strong cliques such that, for $i \in [4]$, (with subscript arithmetic modulo 4)

(1) $A_1, \dots, A_4$ are nonempty, and
(2) if $i \in \{1, 3\}$, then A_i is complete to A_{i+1}, and if $i \in \{2, 4\}$, then A_i and A_{i+1} are linked, and
(3) A_i is strongly anticomplete to A_{i+2}, and
(4) B_i is strongly complete to $A_i \cup A_{i+1}$ and strongly anticomplete to $A_{i+2} \cup A_{i+3}$, and
(5) B_i is strongly anticomplete to B_j for $i \neq j$, and

(6) if $B_i \neq \emptyset$, then A_i is complete to A_{i+1}, and

(7) no vertex in A_i has antineighbors in both A_{i-1} and A_{i+1}.

We call such $(A_1, \ldots, A_4, B_1, \ldots, B_4)$ a *C_4-structure* in T. If, for T, there exists a C_4-structure $(A_1, \ldots, A_4, B_1, \ldots, B_4)$ such that $V(T) = A_1 \cup \ldots \cup A_4 \cup B_1 \cup \ldots \cup B_4$, then we say that T *admits a C_4-structure*. The following lemma states that if a long circular interval trigraph T with no long semiholes does not contain $\bar{C}_7$ as a weakly induced subgraph, then T is either a linear interval trigraph, or T admits a C_4-structure:

(5.7) *Let T be a long circular interval trigraph that has no semihole of length at least five. Then, either*

(1) *T is a linear interval trigraph, or*

(2) *T is of the $\bar{C}_7$ type, or*

(3) *T admits a C_4-structure.*

Proof. In view of outcome (1), we may assume that T is not a linear interval trigraph. Hence, by (2.13) and the fact that T has no semiholes of length at least five, T has a semihole of length four. Next, in view of outcome (2) and (5.5), we may assume that T has no weakly induced $\bar{C}_7$. Let $A_1, A_2, A_3, A_4 \subseteq V(T)$ be cliques in T such that:

(a) A_1 is complete to A_2 and A_3 is complete to A_4, and,

(b) A_1 is strongly anticomplete to A_3, and A_2 is anticomplete to A_4, and

(c) A_2 and A_3 are linked, and A_1 and A_4 are linked.

We may choose A_1, A_2, A_3, A_4 with maximal union. We call such quadruple a *structure*. Since T contains a semihole of length four, we may assume that $A_i \neq \emptyset$ for all $i \in [4]$. Let $A = \bigcup_{i=1}^{4} A_i$. Let u_1-u_2-u_3-u_4-u_1 with $u_i \in A_i$ be a semihole.

(i) *Let $v \in V(T) \setminus A$. Then there exists $i \in [4]$ such that v is strongly complete to $A_i \cup A_{i+1}$.*

Since T is claw-free, it follows from (2.12) that v is adjacent to at least two consecutive vertices of u_1-u_2-u_3-u_4-u_1. Let k be such that v is adjacent to u_k and u_{k+1}. We may assume that $k \in \{1, 2\}$. First suppose that $k = 1$. Since no vertex is complete to u_1-u_2-u_3-u_4-u_1, we may assume that v is strongly antiadjacent to u_3. Since T is claw-free and u_2 is complete to A_1, it follows that v is strongly complete to A_1. If v is complete to A_4, then the claim holds, so we may assume that v has a strong antineighbor $a_4 \in A_4$. Let $a_1 \in A_1$ be a neighbor of a_4. Since a_1 is complete to A_2 and T is claw-free, it follows that v is strongly complete to A_2, as desired. So we may assume that $k = 2$ and v is strongly anticomplete to $\{u_1, u_4\}$. Suppose that v has an antineighbor $a_2 \in A_2$. Then a_2 is strongly antiadjacent to u_3, because otherwise u_3 is complete to the triad $\{u_4, v, a_2\}$, a contradiction. Since A_2 is linked to A_3, there exists a vertex $a_3 \in A_3$ such that a_2 is adjacent to a_3. Now a_3 is adjacent to u_2, because otherwise $T|\{u_1, a_2, u_2, u_4, a_3, u_3\}$ is a weakly induced $(1,1,1)$-prism, contrary to (2.12). This implies that v is adjacent to a_3, because otherwise u_2 is complete to the triad $\{u_1, v, a_3\}$. But now a_3 is complete to the triad $\{u_4, v, a_2\}$, a contradiction. Thus v is strongly complete to A_2 and, from the symmetry, v is also strongly complete to A_3. This proves **(i)**. □

(ii) *Suppose that, for some $i \in [4]$, $v \in V(T) \setminus A$ is strongly complete to $A_i \cup A_{i+1}$. Then v is strongly anticomplete to $A_{i+2} \cup A_{i+3}$.*

From the symmetry, we may assume that $i \in \{1, 2\}$. For $j = i + 2, i + 3$, let $Z_j = N(v) \cap A_j$ and let $Y_j = A_j \setminus Z_j$.

First suppose that both Z_{i+2} and Z_{i+3} are nonempty. Because no vertex is complete to a semihole of length four by (2.12), it follows that Z_{i+2} is strongly anticomplete to Z_{i+3}. It follows that $i = 2$. Now let $x_4 \in Z_4$ and $x_1 \in Z_1$. Since A_4 and A_1 are linked, x_4 has a neighbor $y_1 \in Y_1$ and x_1 has a neighbor $y_4 \in Y_4$. Since x_4 is complete to $\{v, y_4, y_1\}$, the latter is not a triad and hence it follows that y_4 is adjacent to y_1. But now $T|\{x_4, u_2, y_4, v, y_1, u_3, x_1\}$ contains $\bar{C}_7$ as a weakly induced subgraph, a contradiction.

So we may assume that at least one of Z_{i+2}, Z_{i+3} is empty. If both are empty, then v is strongly anticomplete to $A_{i+2} \cup A_{i+3}$ and the claim holds. Therefore, from the symmetry, we may assume that $Z_{i+2} \neq \emptyset$ and $Z_{i+3} = \emptyset$. If $i = 1$, then we may add v to A_2 and obtain a larger structure, a contradiction. If $i = 2$ and $Y_{i+2} = \emptyset$, then we may add v to A_3 and obtain a larger structure, a contradiction. Hence, we may assume that $i = 2$ and $Y_4 \neq \emptyset$.

Now suppose that $a_2 \in A_2$ and $a_3 \in A_3$ are strongly antiadjacent. Let $q_1 \in A_1$ and $y_4 \in Y_4$ be adjacent. Then a_2-v-a_3-y_4-q_1-a_2 is a semihole of length five, a contradiction. This proves that A_2 is complete to A_3.

We claim that for every $a_1 \in A_1$, $x_4 \in Z_4$ and $y_4 \in Y_4$, a_1 is either complete or strongly anticomplete to $\{x_4, y_4\}$. For suppose not. If a_1 is adjacent to x_4 and strongly antiadjacent to y_4, then x_4 is complete to the triad $\{a_1, y_4, v\}$, a contradiction. So we may assume that a_1 is adjacent to y_4 and strongly antiadjacent to x_4. But now, v-x_4-y_4-a_1-u_2-v is a semihole of length five, a contradiction. This proves the claim.

Since Z_4 and Y_4 are both nonempty, it follows that every vertex in A_1 is either complete or anticomplete to A_4. Since every vertex in A_1 has a neighbor in A_4, this implies that A_1 is complete to A_4. But now, letting $A'_1 = A_2$, $A'_2 = A_3 \cup \{v\}$, $A'_3 = A_4$ and $A'_4 = A_1$, we obtain a larger structure, a contradiction. This proves **(ii)**. □

For $i \in [4]$, let B_i be the vertices that are strongly complete to $A_i \cup A_{i+1}$. It follows from **(ii)** that B_i is strongly anticomplete to $A_{i+2} \cup A_{i+3}$. It follows from **(i)** that $V(T) = A_1 \cup \cdots \cup A_4 \cup B_1 \cup \cdots \cup B_4$. The next few claims state some properties of the sets $A_1, \ldots, A_4, B_1, \ldots, B_4$.

(iii) *For $i \in [4]$, no vertex in A_i has both an antineighbor in A_{i+1} and an antineighbor in A_{i-1}.*

Suppose that $a_i \in A_i$ has nonneighbors $a_{i+1} \in A_{i+1}$ and $a_{i-1} \in A_{i-1}$. From the symmetry, we may assume that $i = 1$. Since A_1 is complete to A_2, it follows that a_1 and a_2 are semiadjacent and hence that a_1 and a_4 are strongly antiadjacent. Now let $a'_1 \in A_1$ be a neighbor of a_4. Since A_1 is complete to A_2, it follows that a'_1 is adjacent to a_2. Now a'_1 is complete to the triad $\{a_1, a_2, a_4\}$, a contradiction. This proves **(iii)**. □

(iv) *For $i, j \in [4]$, B_i is strongly anticomplete to B_j for $j \neq i$.*

Let $i \in \{1, 3\}$. If $b_i \in B_i$ is adjacent to $b_{i+1} \in B_{i+1}$, then b_i-b_{i+1}-u_{i+2}-u_{i+3}-u_i-b_i is a semihole of length five, a contradiction. If $b_i \in B_i$ is adjacent to $b_{i+2} \in B_{i+2}$, then $T|\{u_i, u_{i+1}, u_{i+2}, u_{i+3}, b_i, b_{i+1}\}$ contains a weakly induced $(1,1,1)$-prism, contrary to (2.12). Thus, it follows from the symmetry that B_i is strongly anticomplete to B_j for $j \neq i$. This proves **(iv)**. □

(v) *For $i \in [4]$, if $B_i \neq \emptyset$, then A_i is complete to A_{i+1}.*

This is trivial if $i = 1, 3$. So from the symmetry we may assume that $i = 2$. If $a_2 \in A_2$ and $a_3 \in A_3$ are nonadjacent, then for any vertex $b_2 \in B_2$, a_2-b_2-a_3-u_4-u_1-a_2 is a semihole of length five, a contradiction. This proves **(v)**. □

We claim that T admits a C_4-structure. We already noted that $A_1, \ldots, A_4, B_1, \ldots, B_4$ is a partition of $V(T)$. Properties (1)-(7) in the definition of a C_4-structure follow from the definition of $A_1, \ldots, A_4, B_1, \ldots, B_4$ and **(iii)**, **(iv)**, and **(v)**. This proves (5.7). ■

We are now ready to prove the first main result of this subsection.

(5.8) *Every $\mathcal{F}$-free long circular interval trigraph with no semihole of length at least five is resolved.*

Proof. Let T be long circular interval trigraph with no semihole of length at least five. It follows from (5.7) that either T is a linear interval trigraph, or T is of the $\bar{C}_7$ type, or T admits a C_4-structure. If T is a linear interval trigraph, then the lemma holds by (5.4). If T is of the $\bar{C}_7$ type, then the lemma holds by (5.6). Therefore, we may assume that T admits a C_4-structure. Let $A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4$ be as in the definition of a C_4-structure. We may assume that T is not resolved.

(i) *If, for some $i \in [4]$, $B_i \neq \emptyset$, then A_i is not strongly complete to A_{i+1}.*

Suppose that $B_i \neq \emptyset$ and A_i is strongly complete to A_{i+1}. Then any vertex in B_i is a simplicial vertex and hence T is resolved by (2.5), a contradiction. This proves **(i)**. □

(ii) *If, for some $i \in [4]$, $B_i \neq \emptyset$, then A_{i+2} is strongly complete to A_{i+3}.*

Let i be such that $B_i \neq \emptyset$, let $b_i \in B_i$, and suppose that there exist two antiadjacent vertices $x \in A_{i+2}$ and $y \in A_{i+3}$. It follows from **(i)** that there exist antiadjacent $a_i \in A_i$ and $a_{i+1} \in A_{i+1}$. It follows from property (7) of a C_4-structure that a_i is strongly complete to A_{i+3} and a_{i+1} is strongly complete to A_{i+2}. If x is semiadjacent to y, then a_i-b_i-a_{i+1}-x-y-a_i is a weakly induced cycle of length five and $xy \in F(T)$ and, thus, T is resolved by (2.8), a contradiction. Thus, x is strongly antiadjacent to y. Now let $x' \in A_{i+2}$ be a neighbor of y and let $y' \in A_{i+3}$ be a neighbor of x. If x' and y' are adjacent, then $T|\{b_i, a_{i+1}, x', y', a_i, x, y\}$ contains $\mathcal{G}_1$ as a weakly induced subgraph, a contradiction. Thus x' and y' are strongly antiadjacent.

We claim that no vertex in A_{i+3} is complete to $\{x, x'\}$. For suppose that such vertex $z \in A_{i+3}$ exists. Then, $T|\{b_i, a_{i+1}, x', z, a_i, x, y\}$ contains $\mathcal{G}_1$ as a weakly induced subgraph, a contradiction. Hence, no vertex in A_{i+3} is complete to $\{x, x'\}$ and, in particular, every vertex in A_{i+3} has an antineighbor in A_{i+2}. Thus, property (7) of a C_4 structure implies that A_{i+3} is strongly complete to A_i. By the symmetry, A_{i+2} is strongly complete to A_{i+1}. But now, (A_{i+2}, A_{i+3}) is a homogeneous pair of cliques and a_i-b_i-a_{i+1} is a weakly induced path between their respective neighborhoods, and hence T is resolved by (2.7). This proves **(ii)**. □

(iii) *For each $i \in [4]$, at least one of B_i, B_{i+1} is empty.*

Suppose that for some $i \in [4]$, B_i and B_{i+1} are both nonempty. By **(i)**, there exist antiadjacent $a_i \in A_i$ and $a_{i+1} \in A_{i+1}$ and antiadjacent $a'_{i+1} \in A_{i+1}$ and $a'_{i+2} \in A_{i+2}$. It follows from property (7) of a C_4 structure that $a_{i+1} \neq a'_{i+1}$ and in particular a_{i+1} is strongly adjacent to a'_{i+2} and a'_{i+1} is strongly adjacent to a_i. Let $a_{i+3} \in A_{i+3}$ be a common strong neighbor of a_i and a_{i+2}. Such a_{i+3} exists since from **(ii)** it follows that A_{i+2} is strongly complete to A_{i+3} and A_{i+3} is strongly complete to A_i. But now $T|\{a_{i+3}, a_i, a'_{i+1}, a_{i+1}, a'_{i+2}, b_i, b_{i+1}\}$ contains $\mathcal{G}_1$ as a weakly induced subgraph, a contradiction. This proves **(iii)**. □

First suppose that $B_i = \emptyset$ for all $i \in [4]$. Then, it follows from property (7) of a C_4 structure that T does not contain a triad and, thus, T is resolved by (2.6). Hence, we may assume that $B_i \neq \emptyset$ for some $i \in [4]$. From **(i)**, **(ii)**, and **(iii)**, it follows that $B_j = \emptyset$ for all $j \neq i$. It follows from **(ii)** that A_{i+2} is strongly complete to A_{i+3}. But now, T has no triad and hence T is resolved by (2.6). This proves (5.8). ■

5.2.2 Long circular interval trigraphs with long semiholes

Lemma (5.8) deals with long circular interval trigraphs with no long semiholes. The following lemmas deal with the remaining case. The first lemma is an attachment lemma that describes how vertices can attach to a semihole in a long circular interval trigraph. We need some more definitions first. Let T be a trigraph and let C be a semihole of length k in T. Suppose that the vertices of C are ordered, so that $C = c_1$-c_2-$\dots$-c_k-c_1. Let $x \in V(T) \setminus V(C)$. Let $i \in [k]$. We say that x is a *hat of type i for C* if x is strongly complete to $\{c_i, c_{i+1}\}$ and strongly anticomplete to $V(C) \setminus \{c_i, c_{i+1}\}$. We say that x is a *clone of type i for C* if x is complete to $\{c_{i-1}, c_{i+1}\}$, strongly adjacent to c_i, and strongly anticomplete to $V(C) \setminus \{c_{i-1}, c_i, c_{i+1}\}$. Finally, we say that x is a *star of type i for C* if x is strongly antiadjacent to c_i and complete to $\{c_{i-1}, c_{i+1}\}$, and strongly complete to $V(C) \setminus \{c_{i-1}, c_i, c_{i+1}\}$.

(5.9) *Let T be an $\mathcal{F}$-free long circular interval trigraph. Let C be a semihole of length $k \geq 5$. Then, $k \in \{5, 7\}$, and every $x \in V(T) \setminus V(C)$ is either a hat, or a clone, or a star of type i for C, for some $i \in [k]$. Moreover, if x is a star for C, then $k = 5$.*

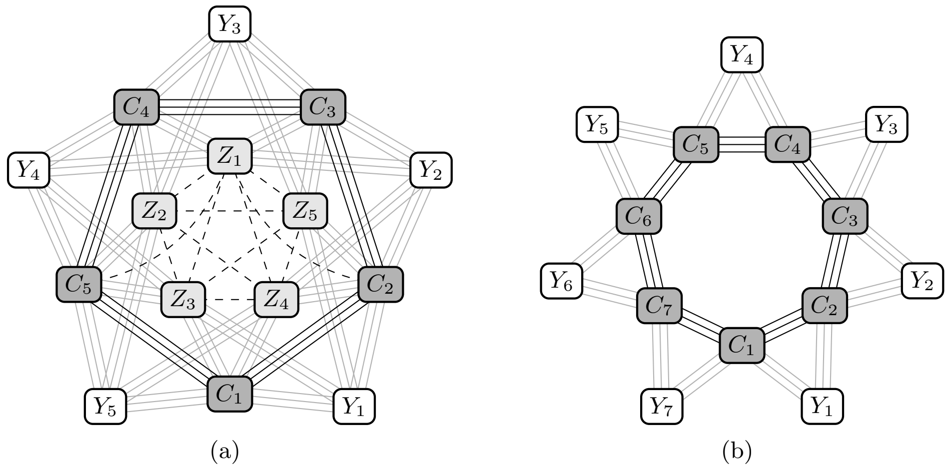

Figure 5: The structure of an $\mathcal{F}$-free long circular interval trigraph with a semihole of length five or seven; see (5.10) and (5.11). The circle represent strong cliques, the triple edges between circles indicate that the corresponding cliques are strongly complete to each other, and the dashed edges represent arbitrary adjacencies.

Proof. Let $C = c_1\text{-}c_2\text{-}\dots\text{-}c_k\text{-}c_1$. Since T is $\mathcal{F}$-free it follows that $k \in \{5, 7\}$. We first observe that:

($*$) if x is adjacent to c_i, then x is strongly adjacent to at least one of c_{i-1}, c_{i+1}, because otherwise $\{x, c_{i-1}, c_{i+1}\}$ is a triad and c_i is complete to it.

It follows from (2.12) that C is dominating and has no center, and therefore x has at least one neighbor and one strong antineighbor in $V(C)$. We may assume that x is adjacent to c_1 and strongly antiadjacent to c_2. It follows from ($*$) that x is strongly adjacent to c_k. First suppose that x is adjacent to c_3. Then, by ($*$), x is strongly adjacent to c_4. If $k = 5$, then, x is a star of type 2, and the claim holds. So we may assume that $k = 7$. x is strongly antiadjacent to c_5 because otherwise x is complete to the triad $\{c_1, c_3, c_5\}$. Thus, by the symmetry, x is strongly antiadjacent to c_6. But now $C' = x\text{-}c_4\text{-}c_5\text{-}c_6\text{-}c_7\text{-}x$ is a semihole and c_2 has no neighbors in $V(C')$, contrary to (2.12). So we may assume that x is strongly antiadjacent to c_3. If $k = 5$, then, x is a clone of type 5 if x is adjacent to c_4 and x is a hat of type 5 if x is strongly antiadjacent to c_4 (x is strongly adjacent to c_1 by ($*$) in this case). Thus we may assume that $k = 7$. Suppose that x is adjacent to c_4. x is strongly antiadjacent to c_6, because otherwise x is complete to the triad $\{c_1, c_4, c_6\}$, a contradiction. But now $c_1\text{-}c_2\text{-}c_3\text{-}c_4\text{-}x\text{-}c_1$ is a nondominating semihole and c_6 has no neighbor in it, contrary to (2.12). This proves that x is strongly antiadjacent to c_4. If x is adjacent to c_5, then $c_1\text{-}c_2\text{-}c_3\text{-}c_4\text{-}c_5\text{-}x\text{-}c_1$ is a weakly induced cycle of length six, a contradiction. Therefore, x is strongly antiadjacent to c_5. Now, x is a clone of type 7 if x is adjacent to c_6 and x is a hat of type 7 if x is strongly antiadjacent to c_6. This proves (5.9). ■

Next, we have two lemmas that describe the structure of an $\mathcal{F}$-free long circular interval trigraph that contains a semihole of length five and seven, respectively. (See Figure 5)

(5.10) *Let T be an $\mathcal{F}$-free long circular interval trigraph. Assume that T has a semihole of length five and no semihole of length seven. Then, $V(T)$ can be partitioned into 15 strong cliques $C_1, \dots, C_5$, $Y_1, \dots, Y_5$, $Z_1, \dots, Z_5$ such that for all $i, j \in [5]$, (subscript arithmetic is modulo 5)*

(1-a) *C_i is complete to C_{i+1} and strongly anticomplete to C_j with $j \notin \{i-1, i, i+1\}$,*

(1-b) *Y_i is strongly complete to $C_i \cup C_{i+1}$ and strongly anticomplete to C_j with $j \notin \{i, i+1\}$.*

(1-c) *Z_i is strongly complete to $C_{i+2} \cup C_{i+3}$, strongly anticomplete to C_i, and every vertex in Z_i is strongly complete to one of C_{i+1}, C_{i+4} and has a neighbor in the other,*

(1-d) *if $i \neq j$, then Y_i is strongly anticomplete to Y_j.*

(1-e) *Y_i is strongly complete to $Z_{i+2} \cup Z_{i+4}$, and strongly anticomplete to $Z_i \cup Z_{i+1} \cup Z_{i+3}$,*

Moreover, if there exists $y \in Y_i$, then:

(2) *$C_i \cup C_{i+1} \cup Z_{i+2} \cup Z_{i+4}$ is a strong clique.*

Proof. Let $C_1, \dots, C_5$ be cliques that satisfy property (1-a), and let $C = \bigcup_{i=1}^{5} C_i$ be maximal. Let $Y_1, \dots, Y_5 \subseteq V(T) \setminus C$ be cliques that satisfy property (1-b), and let $Y = \bigcup_{i=1}^{5} Y_i$ be maximal. Let $Z_1, \dots, Z_5 \subseteq V(T) \setminus (C \cup Y)$ be cliques that satisfy property (1-c), and let $Z = \bigcup_{i=1}^{5} Z_i$ be maximal. It follows from the fact that T has a semihole of length five that $C_i \neq \emptyset$ for $i \in [5]$. Furthermore, we claim that each C_i, Y_i, and Z_i is a strong clique. This follows immediately from (1-a) and (1-b) for C_i and Y_i. For Z_i, let $z, z' \in Z_i$. From the symmetry and (1-c), we may assume that z is strongly complete to C_{i+1}. It follows from (1-c) that z' has a neighbor $c_{i+1} \in C_{i+1}$. Let $c_i \in C_i$. Now, since c_{i+1} is complete to $\{z, z', c_i\}$, it follows because T is claw-free that z and z' are strongly adjacent. Thus, Z_i is a strong clique for all $i \in [5]$.

We claim that $V(T) = C \cup Y \cup Z$. So suppose for a contradiction that there exists $x \in V(T) \setminus (C \cup Y \cup Z)$. In what follows, we say that $F = f_1\text{-}f_2\text{-}\dots\text{-}f_5\text{-}f_1$ is an *aligned semihole in C* if $f_i \in C_i$ for all $i \in [5]$. It follows from (5.9) that, for every aligned semihole in C, x is either a star, a clone, or a hat. First suppose that x is star of type i, say, for some aligned semihole $F = f_1\text{-}f_2\text{-}\dots\text{-}f_5\text{-}f_1$ in C. From the symmetry, we may assume that $i = 1$. By rerouting F, it follows from the fact that T is claw-free that x is strongly complete to $C_3 \cup C_4$, and from (5.9) that x is strongly anticomplete to C_1. We claim that x is strongly complete to at least one of C_2, C_5. For suppose that x has antineighbors $c_2 \in C_2$ and $c_5 \in C_5$. Then, $T|(V(F) \cup \{c_2, c_5, x\})$ contains $\mathcal{G}_1$ as a weakly induced subgraph, a contradiction. By the maximality of Z_i, this means that $x \in Z_i$, a contradiction. So we may assume that x is not a star for any aligned semihole in C. Next, suppose that x is a clone of type i, say, for some aligned semihole $F = f_1\text{-}f_2\text{-}\dots\text{-}f_5\text{-}f_1$ in C. From the symmetry, we may assume that $i = 1$. By rerouting F, it follows from the fact that T is claw-free that x is strongly complete to C_1, and from (5.9) that x is strongly anticomplete to $C_3 \cup C_4$. We claim that x is complete to C_2. For suppose that x has a strong antineighbor $c_2' \in C_2$. Then, $c_2' \neq f_2$ and $T|(V(F) \cup \{c_2', x\})$ contains $\mathcal{G}_1$ as a weakly induced subgraph, a contradiction. Thus, x is complete to C_2 and, from the symmetry, to C_5. But now, by the maximality of C_1, $x \in C_1$, a contradiction. So we may assume that x is not a clone for any aligned semihole in C. It follows that x is a hat for every aligned semihole in C. Choose any aligned semihole $F = f_1\text{-}f_2\text{-}\dots\text{-}f_5\text{-}f_1$ in C. We may assume that x is a hat of type 1 for C. By rerouting F, it follows from (5.9) that

x is strongly complete to $C_1 \cup C_2$ and strongly anticomplete to $C_3 \cup C_4 \cup C_5$. Therefore, by the maximality of Y_1, $x \in Y_1$, a contradiction. This proves that $V(T) = C \cup Y \cup Z$.

The following claim proves property (1-d):

(i) *If $i \neq j$, then Y_i is strongly anticomplete to Y_j.*

Let $c_j \in C_j$ with $j \in [5]$. If there exist adjacent $y_i \in Y_i$ and $y_{i+1} \in Y_{i+1}$ for some i, then y_i-y_{i+1}-c_{i+2}-c_{i+3}-c_{i+4}-c_i-y_i, is a weakly induced cycle of length six, a contradiction. If there exist adjacent $y_i \in Y_i$ and $y_{i+2} \in Y_{i+2}$ for some i, then y_i-y_{i+2}-c_{i+2}-c_{i+1}-y_i is a weakly induced cycle and c_{i+4} has no neighbor in it, contrary to (2.12). By the symmetry, this proves **(i)**. □

The following claim proves property (1-e):

(ii) *Y_i is strongly complete to $Z_{i+2} \cup Z_{i+4}$ and strongly anticomplete to $Z_i \cup Z_{i+1} \cup Z_{i+3}$.*

Let $y \in Y_i$. Suppose that y is adjacent to $z \in Z_i \cup Z_{i+1} \cup Z_{i+3}$. It follows from the definition of Z_j ($j = i, i+1, i+3$) that z has neighbors $c_{i+2} \in C_{i+2}$ and $c_{i+4} \in C_{i+4}$. But now, z is complete to the triad $\{y, c_{i+2}, c_{i+4}\}$, a contradiction. This proves that Y_i is strongly anticomplete to $Z_i \cup Z_{i+1} \cup Z_{i+3}$. Next, suppose that y is antiadjacent to $z' \in Z_{i+2} \cup Z_{i+4}$. From the symmetry, we may assume that $z' \in Z_{i+2}$. But now, let $c_{i+1} \in C_{i+1}$ be a neighbor of z' (such a neighbor exists because of (1-c)). Now, c_{i+1} is complete to the triad $\{c_{i+2}, z', y\}$, a contradiction. This proves that Y_i is strongly complete to $Z_{i+2} \cup Z_{i+4}$. This proves **(ii)**. □

The following claim proves property (2):

(iii) *If $Y_i \neq \emptyset$, then $C_i \cup C_{i+1} \cup Z_{i+2} \cup Z_{i+4}$ is a strong clique.*

Let $y \in Y_i$. For $j \in [5]$, let $c_j \in C_j$. C_i is strongly complete to C_{i+1} because if there exist antiadjacent $c'_i \in C_i$ and $c'_{i+1} \in C_{i+1}$, then c'_i-y-c'_{i+1}-c_{i+2}-c_{i+3}-c_{i+4}-c'_i is a weakly induced cycle of length six, a contradiction. It follows from the definition of Z_{i+2} that C_i is strongly complete to Z_{i+2}. C_i is strongly complete to Z_{i+4}, because if there exist antiadjacent $c'_i \in C_i$ and $z_{i+4} \in Z_{i+4}$, then $T|\{c'_i, c_{i+1}, \dots, c_{i+4}, y, z_{i+4}\}$ contains $\mathcal{G}_1$ as a weakly induced subgraph. From the symmetry, it follows that C_{i+1} is strongly complete to $Z_{i+2} \cup Z_{i+4}$. Finally, suppose that there exist antiadjacent $z_{i+2} \in Z_{i+2}$ and $z_{i+4} \in Z_{i+4}$. Let $c'_{i+3} \in C_{i+3}$ be a neighbor of z_{i+2}. If c'_{i+3} is antiadjacent to z_{i+4}, then $T|\{c_i, c_{i+2}, c'_{i+3}, c_{i+4}, z_{i+2}, z_{i+4}, y\}$ contains $\mathcal{G}_1$ as a weakly induced subgraph. Thus, c'_{i+3} is adjacent to z_{i+4}. But now, $T|\{c_i, c_{i+1}, c_{i+2}, c'_{i+3}, c_{i+4}, z_{i+2}, z_{i+4}, y\}$ contains $\mathcal{G}_3$ as a weakly induced subgraph, a contradiction. This proves that Z_{i+2} is strongly complete to Z_{i+4}. Now **(iii)** follows from the symmetry. □

This proves (5.10). ■

(5.11) *Let T be an $\mathcal{F}$-free long circular interval trigraph. Assume that T has a semihole of length seven. Then, $V(T)$ can be partitioned into 14 strong cliques $C_1, \dots, C_7$, $Y_1, \dots, Y_7$ such that*

- (a) *C_i is complete to C_{i+1} and strongly anticomplete to C_j with $j \notin \{i-1, i, i+1\}$,*
- (b) *Y_i is strongly complete to $C_i \cup C_{i+1}$ and strongly anticomplete to C_j with $j \notin \{i, i+1\}$,*

(c) Y_i *is strongly anticomplete to* Y_j *for* $i \neq j$.

Proof. Let $C_1, \ldots, C_7$ be cliques such that C_i is complete to C_{i+1} and strongly anticomplete to C_j with $j \notin \{i-1, i, i+1\}$, and let $C = \bigcup_{i=1}^{7} C_i$ be maximal. For $i = 1, \ldots, 7$, let Y_i be the vertices in $V(T) \setminus C$ that are strongly complete to $C_i \cup C_{i+1}$ and strongly anticomplete to C_j with $j \notin \{i, i+1\}$, and let $Y = \bigcup_{i=1}^{7} Y_i$. It follows from the fact that T has a semihole of length seven that $C_i \neq \emptyset$ for $i \in [7]$. Furthermore, since T is claw-free it follows that each C_i and each Y_i is a strong clique.

We claim that $V(T) = C \cup Y$. For suppose for a contradiction that there exists $x \in V(T) \setminus (C \cup Y)$. In what follows, we say that $F = f_1\text{-}f_2\text{-}\ldots\text{-}f_7\text{-}f_1$ is an *aligned semihole in* C if $f_i \in C_i$ for all $i \in [7]$. It follows from (5.9) that, for every aligned semihole F in C, x is either a hat or a clone for F. First suppose that x is a hat of type i, say, for some aligned semihole $F = f_1\text{-}f_2\text{-}\ldots\text{-}f_7\text{-}f_1$ in C. From the symmetry, we may assume that $i = 1$. We claim that x is strongly anticomplete to C_3. For suppose that x has a neighbor $c_3 \in C_3$. Then, $T|(V(F) \cup \{x, c_3\}$ contains $\mathcal{G}_2$ as a weakly induced subgraph, a contradiction. Therefore, x is strongly anticomplete to C_3, and by symmetry x is strongly anticomplete to C_7. By rerouting F, it follows from (5.9) that x is strongly anticomplete to $C_4 \cup C_5 \cup C_6$. Next, again by rerouting F, it follows from (5.9) that x is strongly complete to $C_1 \cup C_2$ and, by the maximality of Y_1, $x \in Y_1$, a contradiction. So we may assume that x is not a hat for any aligned semihole in C. Now let $F = f_1\text{-}f_2\text{-}\ldots\text{-}f_7\text{-}f_1$ be an aligned semihole in C. It follows that x is a clone of type i, say, for F. We may assume that $i = 1$. By rerouting F, it follows that x is complete to $C_2 \cup C_7$, strongly complete to C_1, and strongly anticomplete to $C_3 \cup C_4 \cup C_5 \cup C_6$. Therefore, by the maximality of C_i, $x \in C_i$, a contradiction. This proves that $V(T) = C \cup Y$.

Now suppose that $y_i \in Y_i$ and $y_j \in Y_j$ ($i \neq j$) are adjacent. Suppose that $j = i + 1$. Let $c_j \in C_j$ for all $j \in [7]$. Then, $T|(V(C) \cup \{y_i, y_j\})$ contains a weakly induced cycle of length eight, a contradiction. Thus, $j \notin \{i+1, i-1\}$. We may assume that $i = 1$ and $2 < j < 5$. Now, $y_i\text{-}c_{i+1}\text{-}c_{i+2}\text{-}\ldots\text{-}c_j\text{-}y_j\text{-}y_i$ is a semihole of length at least 4 and c_7 has no neighbor in it, contrary to (2.12). This proves that Y_i is strongly anticomplete to Y_j for $i \neq j$, thus completing the proof of (5.11). ■

This allows us to deal with long circular interval trigraphs that contain a long semihole:

(5.12) *Every $\mathcal{F}$-free long circular interval trigraph that has a semihole of length at least five is resolved.*

Proof. Let T be an $\mathcal{F}$-free long circular interval trigraph. From (5.4), we may assume that T is not a linear interval trigraph. By (2.8), we may assume that for every semihole in T of length five or more, all adjacent pairs are in fact strongly adjacent.

First suppose that T has a semihole of length seven. Then, let $C_1, \ldots, C_7$, $Y_1, \ldots, Y_7$ be as in (5.11). Since the edges of every semihole in T of length seven are strong edges, it follows that C_i is strongly adjacent to C_{i+1} for all $i \in [7]$. If there exists $y \in Y_i$ for some $i \in [7]$, then it follows that y is a simplicial vertex in T and hence T is resolved by (2.5). So we may assume that $Y_i = \emptyset$ for all $i \in [7]$. From (2.9), we may assume that T has no strongly adjacent clones. It follows that T is a cycle of length seven and, thus, every graphic thickening G of T is imperfect and all maximal stable sets in G have size three. Thus, T is resolved because every graphic thickening of T is resolved.

So we may assume that T has a semihole of length five and no semihole of length seven. Then, let $C_1, \ldots, C_5$, $Y_1, \ldots, Y_5$, $Z_1, \ldots, Z_5$ be as in (5.10) and let $C = \bigcup_{i=1}^{5} C_i$ and $Z = \bigcup_{i=1}^{5} Z_i$. Since the edges of every semihole in T of length five are strong edges, it follows that C_i is strongly adjacent to C_{i+1} for all $i \in [5]$. Suppose first that $Y_i \neq \emptyset$ for some i. Let $y_i \in Y_i$. It follows from (5.10) that $N[y_i] = Y_i \cup C_i \cup C_{i+1} \cup Z_{i+2} \cup Z_{i+4}$ and $Y_i \cup C_i \cup C_{i+1} \cup Z_{i+2} \cup Z_{i+4}$ is a strong clique. Hence, y_i is a simplicial vertex in T and, thus, T is resolved by (2.5). So may assume that $Y_i = \emptyset$ for all $i \in [5]$.

If T has no triad, then T is resolved by (2.6). Therefore, we may assume that T has a triad $S = \{s_1, s_2, s_3\}$. First suppose that $|S \cap Z| = 3$. From the symmetry, we may assume that $s_1 \in Z_1$, $s_2 \in Z_2$ and $s_3 \in Z_3 \cup Z_4$. It follows from the definition of Z_i that $Z_1 \cup Z_2$ is complete to C_4. Suppose first that $s_3 \in Z_3$. Let $c_4 \in C_4$ be a neighbor of s_3. Now, c_4 is complete to S, a contradiction. It follows that $s_3 \in Z_4$. From the symmetry, we may assume that Z_4 is complete to C_3. Let $c_3 \in C_3$ be a neighbor of s_2. It follows that c_3 is complete to S, a contradiction. Next, suppose that $|S \cap Z| = 2$ and hence $|S \cap C| = 1$. We may assume that $s_1 \in C_1$. It follows from (5.10) that C_1 is complete to $Z_3 \cup Z_4$. Hence, from the symmetry, we may assume that $s_2 \in Z_1 \cup Z_2$ and $s_3 \in Z_5$. First suppose that $s_2 \in Z_1$. Let $c_2 \in C_2$ be a neighbor of s_2. Then c_2 is complete to S, a contradiction. It follow that $s_2 \in Z_2$. Let $c_3 \in C_3$ be a neighbor of s_2, and let $c_4 \in C_4$ be a neighbor of s_3. Now, $T|\{s_1, c_2, c_3, c_4, c_5, s_2, s_3\}$, where $c_2 \in C_2$ and $c_5 \in C_5$, contains $\mathcal{G}_1$ as a weakly induced subgraph, a contradiction. Therefore, since $T|C$ contains no triad, it follows that $|S \cap Z| = 1$ and $|S \cap C| = 2$. From the symmetry, we may assume that $s_1 \in C_1$ and $s_2 \in C_3$. Because C_1 is strongly complete to $Z_3 \cup Z_4$, and C_3 is strongly complete to $Z_1 \cup Z_5$, it follows that $s_3 \in Z_2$. But this contradicts the fact that Z_2 is strongly complete to at least one of C_1, C_3. This proves (5.12). ■

The previous two lemmas imply the main result of this section:

(5.13) *Every $\mathcal{F}$-free long circular interval trigraph is resolved.*

Proof. Let T be a $\mathcal{F}$-free long circular interval trigraph. If T is a linear interval trigraph, then it follows from (5.4) that T is resolved. If T has a semihole of length at least five, then T is resolved by (5.12). Therefore, we may assume that T has no semihole of length at least five and, thus, the result follows from (5.8). This proves (5.13). ■

5.3 $\mathcal{F}$-free three-cliqued trigraphs

In this section, we deal with three-cliqued claw-free trigraphs. The approach is as follows. Theorem 2.11 states that every three-cliqued claw-free trigraph either lies in $\mathcal{TC}_1 \cup \mathcal{TC}_2 \cup \ldots \cup \mathcal{TC}_5$, or admits a worn hex-chain of trigraphs in $\mathcal{TC}_1 \cup \mathcal{TC}_2 \cup \ldots \cup \mathcal{TC}_5$. (See Section 2.4 for the definitions of the classes $\mathcal{TC}_1, \ldots, \mathcal{TC}_5$.) We first show that in the context of $\mathcal{F}$-free three-cliqued claw-free trigraphs, it suffices to consider only the basic three-cliqued claw-free trigraphs, and basic three-cliqued claw-free trigraphs that are hex-joined with a strong clique. After having stated and proved this result, we will go through the remaining cases and conclude that $\mathcal{F}$-free three-cliqued claw-free trigraphs are resolved.

A three-cliqued claw-free trigraph (T, A, B, C) is called *very basic* if $(T, A, B, C) \in \mathcal{TC}_1 \cup \mathcal{TC}_2 \cup \mathcal{TC}_3 \cup \mathcal{TC}_5$. We start with the following lemma, which states that it suffices to consider three-cliqued claw-free trigraphs that are very basic, or that are a hex-join of a very basic three-cliqued claw-free trigraph and a strong clique.

(5.14) *Let (T, A, B, C) be an $\mathcal{F}$-free three-cliqued claw-free trigraph. Then, either T is resolved or (T, A, B, C) is*

(a) *a very basic three-cliqued claw-free trigraph, or*

(b) *a trigraph that is the hex-join of a very basic three-cliqued claw-free trigraph and a strong clique.*

Proof. We may assume that (T, A, B, C) is not very basic. Thus, (T, A, B, C) admits a worn hex-chain. We may assume that T is not resolved. We start with two claims about worn hex-joins.

(i) *Suppose that (T, A, B, C) is a worn hex-join of two three-cliqued claw-free trigraphs (T_1, A_1, B_1, C_1) and (T_2, A_2, B_2, C_2). Then, at least one of T_1, T_2 does not contain a triad.*

Suppose that for $i = 1, 2$, T_i contains a triad $\{a_i, b_i, c_i\}$. From the symmetry and the fact that A_i, B_i, C_i are strong cliques, we may assume that for $i = 1, 2$, $a_i \in A_i$, $b_i \in B_i$ and $c_i \in C_i$. But now a_1-a_2-b_1-b_2-c_1-c_2-a_1 is a weakly induced cycle of length six in T, a contradiction. This proves **(i)**. ▫

(ii) *A worn hex-chain of antiprismatic three-cliqued claw-free trigraphs is antiprismatic.*

Since a worn hex-chain can be constructed by iteratively hex-joining two trigraphs, it suffices to show the lemma for worn hex-joins. So, for $i = 1, 2$, let (T_i, A_i, B_i, C_i), be an antiprismatic three-cliqued claw-free trigraph and consider the worn hex-join T' of (T_1, A_1, B_1, C_1) and (T_2, A_2, B_2, C_2). In order to show that T' is antiprismatic, it suffices to show that for every triad S in T', every vertex $v \in V(T') \setminus S$ has at least two strong neighbors in S. So let S be a triad in T'. From the symmetry, we may assume that S has at least one vertex in T_1. From the definition of a worn hex-join, and the fact that A_1, B_1, C_1 are strong cliques, it follows that $S = \{a, b, c\}$ with $a \in A_1, b \in B_1, c \in C_1$. Now let $v \in V(T') \setminus S$. If $v \in V(T_1)$, then it follows from the fact that T_1 is antiprismatic that v has at least two strong neighbors in S. So we may assume that $v \in V(T_2)$, and from the symmetry we may assume that $v \in A_2$. Now v is strongly complete to $A_1 \cup B_1$, and hence v is strongly adjacent to a and b. This proves **(ii)**. ▫

First, notice that every very basic three-cliqued claw-free trigraph contains a triad. Hence, it follows from **(i)**, Theorem 2.11 and the symmetry that we may assume that (T, A, B, C) admits a worn hex-chain into terms, at most one of which is a basic three-cliqued claw-free trigraph, and whose other terms are three-cliqued claw-free trigraphs with no triad (and, in particular, they are antiprismatic). If all terms are antiprismatic three-cliqued claw-free trigraphs, then T is antiprismatic by **(ii)** and thus the lemma holds by (5.3). So we may assume that exactly one of the terms is a very basic three-cliqued claw-free trigraph. Notice that a worn hex-chain of antiprismatic three-cliqued claw-free trigraphs is an antiprismatic three-cliqued claw-free trigraph. Possibly by taking together all terms that are antiprismatic three-cliqued claw-free trigraphs, it follows that T is a worn hex-join of a very basic three-cliqued claw-free trigraph L, and an antiprismatic three-cliqued claw-free trigraph R that contains no triad. Since every vertex of a very basic three-cliqued claw-free trigraph is in a triad, it follows that T is not only a worn hex-join, but in fact a hex-join of a very basic three-cliqued claw-free trigraph $L = (L_1, L_2, L_3)$, and an antiprismatic three-cliqued claw-free trigraph $R = (R_1, R_2, R_3)$ that contains no triad. We may assume that for $\{i, j, k\} = \{1, 2, 3\}$, R_i is strongly anticomplete to L_i and strongly complete to $L_j \cup L_k$.

(iii) *For $i = 1, 2, 3$, L_i is not strongly anticomplete to $L \setminus L_i$.*

It suffices to show this for $i = 2$. Suppose that L_2 is strongly anticomplete to $L \setminus L_2$. First suppose that L_1 is strongly anticomplete to L_3. Then L is a disjoint union of strong cliques and, by (2.9) applied to L, we may assume that L is a triad, and thus that L is antiprismatic, a contradiction. Hence, L_1 is not strongly anticomplete to L_3. Let $l_2 \in L_2$. Since l_2 is not simplicial, there exist antiadjacent $r_1 \in R_1$ and $r_3 \in R_3$. Now (L_1, L_3) is a homogeneous pair of cliques in T such that L_1 is neither strongly complete nor strongly anticomplete to L_3, and r_1-l_2-r_3 is a weakly induced path that contradicts (2.7). This proves **(iii)**. □

(iv) *Suppose that there exist antiadjacent $r_1 \in R_1$ and $r_2 \in R_2$. Then,*

(iv-a) *there is no weakly induced path x_1-x_2-x_3-x_4-x_5 with $x_1 \in L_2$, $x_2, x_3 \in L_1$ and $x_4, x_5 \in L_3$, or with $x_1 \in L_1$, $x_2, x_3 \in L_2$ and $x_4, x_5 \in L_3$;*

(iv-b) *there is no triad $\{l_1, l_2, l_3\}$ with $l_i \in L_i$ such that l_1 and l_2 are semiadjacent;*

(iv-c) *if $l_1 \in L_1$ is adjacent to $l_3 \in L_3$, and $l_2 \in L_2$ is in a triad with l_1, then l_2 is strongly anticomplete to L_1.*

For part **(iv-a)**, suppose that there exist such $r_1, r_2, x_1, \ldots, x_5$. Then, $T|\{x_1, r_1, x_4, r_2, x_2, x_3, x_5\}$ contains $\mathcal{G}_1$ as a weakly induced subgraph, a contradiction. This proves **(iv-a)**.

For part **(iv-b)**, suppose that such l_1, l_2, l_3 exist. Then, l_1-l_2-r_1-l_3-r_2-l_1 is a weakly induced cycle of length five that contradicts (2.8). This proves **(iv-b)**.

For part **(iv-c)**, let $l_1 \in L_1$ and $l_3 \in L_3$ be adjacent, and let $l_2 \in L_2$ be in a triad with l_1. Suppose first that $\{l_1, l_2, l_3\}$ is a triad. It follows that l_1 is semiadjacent to l_3 and l_2 is strongly antiadjacent to l_1 and l_3. We may assume that l_2 has a neighbor $l'_1 \in L_1$ because otherwise **(iv-c)** holds. Because l'_1 is not complete to the triad $\{l_1, l_2, l_3\}$, it follows that l'_1 is strongly antiadjacent to l_3. But now, l_1-l_3-r_1-l_2-l'_1-l_1 is a weakly induced cycle in T that contradicts (2.8). This proves that $\{l_1, l_2, l_3\}$ is not a triad.

Let $\{l_1, l_2, l'_3\}$ be a triad. It follows that $l'_3 \neq l_3$. It follows from **(iv-b)** that $l_1 l_2$ is not a semiedge and thus l_1 is strongly antiadjacent to l_2. Since l_3 is not complete to $\{l_1, l_2, l'_3\}$, it follows that l_3 is strongly antiadjacent to l_2. Because $\{l_1, l_2, l_3\}$ is not a triad, it follows that l_1 is strongly adjacent to l_3. Let $l'_1 \in L_1$ be a nonneighbor of l_3 (l'_1 exists because l_3 is in a triad). Suppose first that l'_1 is adjacent to l_2. Because l'_1 is not complete to $\{l_1, l_2, l'_3\}$, l'_1

is strongly antiadjacent to l'_3. But now l_2-l'_1-l_1-l_3-l'_3 is a weakly induced path contradicting **(iv-a)**. This proves that l'_1 is strongly antiadjacent to l_2. We may assume that l_2 has a neighbor $l''_1 \in L_1$. Because l''_1 is not complete to $\{l'_1, l_2, l_3\}$ and not complete to $\{l_1, l_2, l'_3\}$, it follows that l''_1 is strongly anticomplete to $\{l_3, l'_3\}$. But now l_2-l''_1-l_1-l_3-l'_3 is a weakly induced path that contradicts **(iv-a)**. This proves **(iv-c)**, thus completing the proof of **(iv)**. □

(v) *At least one of the pairs* (R_1, R_2), (R_2, R_3), (R_1, R_3) *is strongly complete.*

We first claim that every vertex of R_2 is strongly complete to at least one of R_1, R_3. For suppose that there exists $r_2 \in R_2$ with antineighbors $r_1 \in R_1$ and $r_3 \in R_3$. Since R contains no triad, it follows that r_1 is strongly adjacent to r_3. It follows from **(iii)** that L_2 is not strongly anticomplete to $L_1 \cup L_3$ and thus, from the symmetry, we may assume that there exist adjacent $l_1 \in L_1$ and $l_2 \in L_2$. Let $\{l'_1, l_2, l_3\}$ be a triad containing l_2. If $l'_1 = l_1$, then it follows that l_1 is semiadjacent to l_2, thus contradicting **(iv-b)**. Thus, $l_1 \neq l'_1$. Since l_1 is not complete to the triad $\{l'_1, l_2, l_3\}$, it follows that l_1 is strongly antiadjacent to l_3. But now $T|\{l_3, r_2, l_1, r_3, r_1, l'_1, l_2\}$ contains $\mathcal{G}_1$ as a weakly induced subgraph. This proves the claim. Notice that by symmetry it follows that for $\{i, j, k\} = \{1, 2, 3\}$, every vertex of R_i is strongly complete to at least one of R_j, R_k.

Suppose that there exist antiadjacent pairs (r_1, r'_2), (r_2, r'_3), (r'_1, r_3) with $r_i, r'_i \in R_i$. It follows from our previous claim that $r_i \neq r'_i$ for $i = 1, 2, 3$, and all pairs except (r_1, r'_2), (r_2, r'_3), (r'_1, r_3) are strongly adjacent. Let $\{l_1, l_2, l_3\}$ with $l_i \in L_i$ be a triad. Now, $T|\{l_1, l_2, l_3, r_1, r'_1, r_2, r'_2, r_3, r'_3\}$ contains $\mathcal{G}_4$ as a weakly induced subgraph, a contradiction. This proves **(v)**. □

By **(v)**, we may assume that R_1 is strongly complete to R_3. We may assume that R is not a strong clique and thus we may assume that there exist antiadjacent $r_1 \in R_1$ and $r_2 \in R_2$.

(vi) *No vertex in* L_1 *has both a neighbor in* L_2 *and a neighbor in* L_3.

Suppose that $l_1 \in L_1$ has a neighbor $l_3 \in L_3$. Let $l_2 \in L_2$ be in a triad with l_1. By **(iv-c)**, l_2 is strongly anticomplete to L_1. Since l_2 is not simplicial, l_2 has a neighbor in L_3. Now, from the symmetry between L_1 and L_2 and by **(iv-c)**, it follows that l_1 is strongly anticomplete to L_2. This proves **(vi)**. □

We may assume that $K = R_1 \cup L_2 \cup R_3$ is not a dominant clique in T. Thus, there exists a stable set $S \subseteq (V(T) \setminus K)$ that covers K. First suppose that $S \cap R_2 \neq \emptyset$. Then, since R_2 is strongly complete to $L_1 \cup L_3$, it follows that $S \subseteq R_2$. But now, S does not cover L_2, a contradiction. Therefore, $S \cap R_2 = \emptyset$. It follows that $S \subseteq L_1 \cup L_3$. Suppose next that $S \subseteq L_1$. Let l_1 be the unique vertex in S, and let $\{l_1, l_2, l_3\}$ be a triad. Clearly, $\{l_1, l_2, l_3\}$ is a larger stable set than S, a contradiction. From this and from the symmetry, it follows that $S = \{l_1, l_3\}$ with $l_1 \in L_1$ and $l_3 \in L_3$.

Let $z \in L_2$. By the maximality of S, it follows that l_1 and l_3 are not both antiadjacent to z. This proves that for every $z \in L_2$, z is strongly adjacent to at least one of l_1, l_3.

Let $l_2, l'_2 \in L_2$ be antineighbors of l_1, l_3, respectively. Notice that l_2, l'_2 exist since each vertex in L is in a triad. It follows by the previous argument that $l_2 \neq l'_2$, l_1 is strongly adjacent to l'_2, and

l_3 is strongly adjacent to l_2. Let $l'_3 \in L_3$ be an antineighbor of l_2. It follows from **(vi)** that l'_3 is strongly antiadjacent to l_1. Because l'_2 is not complete the triad $\{l_1, l_2, l'_3\}$, it follows that l'_2 is strongly antiadjacent to l'_3. But now, l_1-l'_2-l_2-l_3-l'_3 is a weakly induced path that contradicts **(iv-a)**. Thus K is a dominant clique, a contradiction. This proves that R is a strong clique, and hence this proves (5.14). ∎

Recall that the $\mathcal{F}$-free three-cliqued claw-free trigraphs that remain open after (5.14) are the very basic three-cliqued claw-free trigraphs, and the hex-joins of very basic three-cliqued claw-free trigraphs with a strong clique. The next few lemmas deal with these cases. We start with three-cliqued claw-free trigraphs where the part that is very basic is a type of line trigraph.

(5.15) *No three-cliqued claw-free trigraph in $\mathcal{TC}_1$ is $\mathcal{F}$-free.*

Proof. Let $(T, L_1, L_2, L_3) \in \mathcal{TC}_1$. Let H, v_1, v_2, v_3 be as in the definition of $\mathcal{TC}_1$ with respect to T. First observe that if H contains a cycle of length six (not necessarily induced), then, by the definition of a line trigraph, T contains a weakly induced cycle of length six, and thus the lemma holds. So we may assume now that H does not contain any cycle of length six.

For $i = 1, 2, 3$, let W_i be the vertices of $V(H) \setminus \{v_1, v_2, v_3\}$ that are complete to $\{v_1, v_2, v_3\} \setminus \{v_i\}$ and nonadjacent to v_i, and let Z be the vertices that are complete to $\{v_1, v_2, v_3\}$. It follows from the definition of $\mathcal{TC}_1$ that $|W_i| \le 1$ for all i. Also, if $|Z| \ge 3$, say $z_1, z_2, z_3 \in Z$, then $H|\{z_1, z_2, z_3, v_1, v_2, v_3\}$ contains a cycle of length six, a contradiction. Thus, we may assume that $|Z| \le 2$.

If W_1, W_2, W_3 are all nonempty, say $w_i \in W_i$ for $i = 1, 2, 3$, then $H|\{v_1, v_2, v_3, w_1, w_2, w_3\}$ contains a cycle of length six, a contradiction. By symmetry, we may assume that $W_2 = \emptyset$. Now, from the fact that $|W_3| \le 1$, $|Z| \le 2$, and $\deg_H(v_1) \ge 3$, it follows that $|W_3| = 1$ and $|Z| = 2$. From the symmetry, it follows that $|W_1| = 1$. Let $W_i = \{w_i\}$ for $i = 1, 3$ and $Z = \{z_1, z_2\}$. But now, $H|\{v_1, v_2, v_3, w_1, w_3, z_1\}$ contains a cycle of length six, a contradiction. This proves (5.15). ∎

Next, we deal with three-cliqued claw-free trigraphs where the part that is very basic is a long circular interval trigraph. We first prove the following lemma.

(5.16) *Every $(T, L_1, L_2, L_3) \in \mathcal{TC}_2$ is either a linear interval trigraph or contains a semihole of length at least five.*

Proof. Suppose that T has no induced semihole of length at least five. It follows from (5.7) and the definition of $\mathcal{TC}_2$ that either T is a linear interval trigraph, or T is of the $\bar{C}_7$ type, or T admits a C_4-structure. If T is a linear interval trigraph, then we are done. If T is of the $\bar{C}_7$ type, then it follows from (5.6) that T has no triad, a contradiction. So we may assume that T admits a C_4-structure $(A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4)$. Recall that every vertex in T is in a triad and that T contains no four pairwise antiadjacent vertices.

(i) *For $i \in [4]$, if $a_i \in A_i$ is strongly complete to A_{i+1}, then $B_{i+1} \neq \emptyset$.*

Let $i \in [4]$, let $a_i \in A_i$ be strongly complete to A_{i+1}, and suppose that $B_{i+1} = \emptyset$. Let $S = \{a_i, s_1, s_2\}$ be a triad in T. Since a_i is strongly complete to $A_{i+1} \cup B_i \cup B_{i+4}$, and $B_{i+1} = \emptyset$, it follows that $\{s_1, s_2\} \subseteq A_{i+2} \cup A_{i+3} \cup B_{i+2}$. First suppose that $s_1 \in A_{i+2}$. Because S is a triad and A_{i+2} is strongly complete to B_{i+2}, it follows that $s_2 \in A_{i+3}$. But now, $s_2 \in A_{i+3}$ has a nonneighbor in both A_i and A_{i+2}, a contradiction. Thus, we may assume that $S \cap A_{i+2} = \emptyset$. It follows that we may assume that $s_1 \in A_{i+3}$ and $s_2 \in B_{i+2}$. But this contradicts the fact that A_{i+3} is strongly complete to B_{i+2}. This proves **(i)**. □

First suppose that for all $i \in [4]$, A_i is strongly complete to A_{i+1}. Then, it follows from **(i)** that $B_i \neq \emptyset$ for all $i \in [4]$. But now, $\{b_1, b_2, b_3, b_4\}$ is a set of four pairwise antiadjacent vertices, a contradiction. Thus, we may assume that, for some $i \in [4]$, there exist antiadjacent $a_i \in A_i$ and $a_i \in A_{i+1}$. It follows from the definition of a C_4-structure that a_i is strongly complete to A_{i+3} and a_{i+1} is strongly complete to A_{i+2}. Thus, it follows from **(i)** applied to a_i and A_{i+3} that there exists $b_{i+2} \in B_{i+2}$. If there exist semiadjacent $a_{i+2} \in A_{i+2}$ and $a_{i+3} \in A_{i+3}$, then it follows from the symmetry that there exists $b_i \in B_i$, but now a_i-b_i-a_{i+1}-a_{i+2}-b_{i+2}-a_{i+3}-a_i is a weakly induced cycle of length six, a contradiction. Therefore, A_{i+2} is strongly complete to A_{i+3}. Thus, it follows from **(i)** applied to A_{i+2} and A_{i+3} that there exists $b_{i+3} \in B_{i+3}$ and, symmetrically, there exists $b_{i+1} \in B_{i+1}$. Since T has no weakly induced cycle of length six, it follows that at least one of the pairs (A_{i+1}, A_{i+2}) and (A_i, A_{i+3}) is strongly complete. We may assume that A_{i+1} is strongly complete to A_{i+2}. Now, it follows from **(i)** that there exists $b_i \in B_i$. But now, $\{b_1, b_2, b_3, b_4\}$ is a set of four pairwise antiadjacent vertices, a contradiction. This proves (5.16). ■

This enables us to deal with three-cliqued claw-free trigraphs where the part that is very basic is a long circular interval trigraph.

(5.17) *Let T be an $\mathcal{F}$-free trigraph that is a hex-join of $(T_1, L_1, L_2, L_3) \in \mathcal{TC}_2$ and (T_2, R_1, R_2, R_3), where $R_1 \cup R_2 \cup R_3$ is a strong clique. Then T is resolved.*

Proof. It follows from (2.9) that we may assume that $|R_i| \leq 1$ for $i = 1, 2, 3$. Next, we note that if $|R_1 \cup R_2 \cup R_3| < 3$, then T is a long circular interval trigraph, and the lemma holds by (5.13). So we may assume that $|R_i| = 1$ for $i = 1, 2, 3$. Let $R_i = \{r_i\}$, for $i = 1, 2, 3$. It follows from (5.16) that either T_1 is a linear interval trigraph or T_1 has a semihole of length at least five. To avoid confusion, recall that L_1 is strongly complete to $R_1 \cup R_3$ and strongly anticomplete to R_2, L_2 is strongly complete to $R_1 \cup R_2$ and strongly anticomplete to R_3, and L_3 is strongly complete to $R_2 \cup R_3$ and strongly anticomplete to R_1.

Let us treat the case when T_1 is a linear interval trigraph first.

(i) *If T_1 is a linear interval trigraph, then T is resolved.*

Since T_1 is a linear interval graph, there exists a linear ordering $(\leq, V(T_1))$ such that, for all distinct $x, y, z \in V(T_1)$, it holds that if x and y are adjacent and $x < z < y$, then z is strongly adjacent to x and y. From the symmetry and from the definition of $\mathcal{TC}_2$, it follows that we may assume that $l_1 < l_2 < l_3$ for all $l_i \in L_i$, $i \in [3]$. Notice that if there exist adjacent $l_1 \in L_1$

and $l_3 \in L_3$, then, by the definition of a linear interval trigraph, L_2 is strongly complete to $L_1 \cup L_3$. However, since T_1 contains a triad and this triad hits each of L_1, L_2, L_3, we have that L_2 is not strongly complete to $L_1 \cup L_3$ and, therefore, L_1 is strongly anticomplete to L_3. We may assume that, for $i \in [3]$, every $l_i \in L_i$ has a neighbor in $L_1 \cup L_2 \cup L_3 \setminus L_i$ because otherwise l_i is a simplicial vertex and we are done by (2.5). If $R_3 = \emptyset$, then $N(L_1) \subseteq R_1 \cup L_2$, which is a strong clique, and hence T is resolved by (2.4). Thus, we may assume that there exists $r_3 \in R_3$.

First suppose that some $l_1 \in L_1$ and $l_2 \in L_2$ are semiadjacent. Since every vertex in T_1 is in a triad, there exists $l_3 \in L_3$ that is antiadjacent to l_2. Let $l_2' \in L_2$ be a neighbor of l_3. Since l_2' is not complete to $\{l_1, l_2, l_3\}$ (otherwise it forms a claw), it follows that l_2' is strongly antiadjacent to l_1. Now, l_1-l_2-l_2'-l_3-r_3-l_1 is a weakly induced cycle of length five with one semiedge and, hence, T is resolved by (2.8). Thus, we may assume that there are no semiedges between L_1 and L_2.

We claim that $K = L_1 \cup R_1 \cup R_3$ is a dominant clique. For suppose not. Then there exists a stable set S in T that covers K. Since $R_2 \cup L_3$ is strongly anticomplete to L_1, it follows that S contains a vertex $l_2 \in L_2$ that is strongly complete to L_1, contrary to the fact that l_2 is in a triad in T_1. Thus, K is a dominant clique and T is resolved. This proves **(i)**. □

In view of **(i)**, we may now assume that T_1 contains a semihole of length at least five. It follows from the fact that T_1 is a three-cliqued claw-free trigraph that T_1 has no semihole of length seven. Thus, since T_1 is $\mathcal{F}$-free, it follows that T_1 contains a semihole of length five. Let $C_1, \dots, C_5$, $Y_1, \dots, Y_5$, $Z_1, \dots, Z_5$ be as in (5.10). If there are semiadjacent $c_i \in C_i$ and $c_{i+1} \in C_{i+1}$, then it follows from (2.8) that T is resolved. So we may assume that C_i is strongly complete to C_{i+1} for all $i \in [5]$. If $Y_i = \emptyset$ for all i, then it follows from the proof of (5.12) that T has no triad, a contradiction. So from the symmetry we may assume that $Y_1 \neq \emptyset$. Recall that (T_1, L_1, L_2, L_3) is a three-cliqued claw-free trigraph. The following claim shows how $C_1, \dots, C_5$, and Y_1 relate to the three cliques L_1, L_2, L_3.

(ii) *Up to symmetry, $Y_1 \cup C_1 \cup C_2 \subseteq L_1$, $C_3 \subseteq L_2$, $C_4 \subseteq L_2 \cup L_3$, and $C_5 \subseteq L_3$.*

Let $y_1 \in Y_1$. We may assume that $y_1 \in L_1$. Since L_1, L_2 and L_3 are strong cliques, it follows from the symmetry that we may assume that $C_3 \subseteq L_2$, $C_5 \subseteq L_3$, and $C_4 \subseteq L_2 \cup L_3$. Therefore, it follows that $Y_1 \subseteq L_1$. Now, let $c_4 \in C_4$. From the symmetry, we may assume that $c_4 \in L_2$. It follows that $C_2 \subseteq L_1$. We claim that $C_1 \subseteq L_1$. For suppose not. Then, since L_2 is a strong clique, it follows that there exists $c_1 \in C_1$ such that $c_1 \in L_3$. For $i = 2, 3, 5$, let $c_i \in C_i$. Now, $T|\{c_1, c_2, c_3, c_4, c_5, y_1, r_3\}$ is weakly isomorphic to $\mathcal{G}_1$, a contradiction. Thus, $C_1 \subseteq L_1$ and **(ii)** holds. □

It follows from **(ii)** that we may assume that $Y_1 \cup C_1 \cup C_2 \subseteq L_1$, $C_3 \subseteq L_2$, $C_4 \subseteq L_2 \cup L_3$, and $C_5 \subseteq L_3$ Let $y_1 \in Y_1$. We claim that y_1 is a simplicial vertex in T. It follows from (5.10) that $N[y_1] = Y_1 \cup C_1 \cup C_2 \cup Z_3 \cup Z_5 \cup \{r_2, r_3\}$ and $N[Y_1] \setminus \{r_2, r_3\}$ is a strong clique. From this, and from the symmetry, it suffices to show that $Y_1 \cup Z_3$ is strongly complete to $\{r_2, r_3\}$. Since $C_1 \cup C_2 \subseteq L_1$, it follows immediately from the definition of a hex-join that $C_1 \cup C_2$ is strongly complete to $\{r_2, r_3\}$. So let $z_3 \in Z_3$. Let $c_j \in C_j$ for $j \in [5]$. If z_3 is antiadjacent to r_2, then c_2 is complete to the triad $\{c_3, r_2, z_3\}$, a contradiction. Thus, z_3 is strongly adjacent to r_2. Now suppose that z_3 is antiadjacent to r_3. If r_3 is adjacent to c_4, then $T|\{c_1, c_2, \ldots, c_5, z_3, r_3, y_1\}$ contains $\mathcal{G}_3$ as a weakly induced subgraph, a contradiction. If r_3 is antiadjacent to c_4, then $T|\{c_1, c_3, c_4, c_5, z_3, r_3, y_1\}$ contains $\mathcal{G}_1$ as a weakly induced subgraph, a contradiction. This proves that Z_3 is strongly complete to $\{r_2, r_3\}$ and, from the symmetry, that Z_5 is strongly complete to $\{r_2, r_3\}$ Thus, $N[y_1]$ is a strong clique, hence y_1 is a simplicial vertex in T and the lemma holds by (2.5). This proves (5.17). ■

The next lemma deals with three-cliqued claw-free trigraphs where the part that is very basic is a near-antiprismatic trigraph.

(5.18) *Let T be an $\mathcal{F}$-free trigraph that is a hex-join of $(T_1, L_1, L_2, L_3) \in \mathcal{TC}_3$ and (T_2, R_1, R_2, R_3), where $R_1 \cup R_2 \cup R_3$ is strong clique. Then T is resolved.*

Proof. Let $(T_1, L_1, L_2, L_3) \in \mathcal{TC}_3$ and let a_0, b_0, A, B, C, X, n be as in the definition of a near-antiprismatic trigraph. Notice that $L_1 = A \setminus X, L_2 = B \setminus X, L_3 = C \setminus X$. If a_0 is strongly antiadjacent to b_0, then $N(a_0) = L_1 \cup (R_2 \cup R_3)$, hence a_0 is a simplicial vertex and the lemma holds by (2.5). So we may assume that a_0 is semiadjacent to b_0. First suppose that there exist antiadjacent $a_i \in L_1$ and $b_j \in L_2$, for $i, j \leq n$ and $i \neq j$. Because $|L_3| \geq 2$, it follows that both a_i and b_j have a neighbor in L_3. Therefore, there exists an shortest weakly induced path P from a_i to b_j with interior in L_3. Now, (2.8) applied to a_0-a_i-P^*-b_j-b_0-a_0 implies that T is resolved.

Thus, we may assume that L_1 is strongly complete to L_2. It follows from the definition of $\mathcal{TC}_3$ that $L_1 = \{a_1\}$, $L_2 = \{b_1\}$, and hence that $n = 2$ and $L_3 = \{c_1, c_2\}$. Moreover, c_1 is strongly anticomplete to $\{a_1, b_1\}$. Therefore, $N(c_1) = \{c_2\} \cup R_1 \cup R_2$, which is a strong clique. Thus, c_1 is a simplicial vertex and T is resolved by (2.5). This proves (5.18). ■

Finally, we deal with trigraphs where the part that is very basic is a sporadic exception.

(5.19) *Let T be an $\mathcal{F}$-free trigraph T that is a hex-join of $(T_1, L_1, L_2, L_3) \in \mathcal{TC}_5$ and (T_2, R_1, R_2, R_3), where $R_1 \cup R_2 \cup R_3$ is strong clique. Then T is resolved.*

Proof. First suppose that T_1 is of the first type of sporadic trigraphs. Let $v_1, \ldots, v_8, A, B, C, X$ be as in the definition of T_1. Observe that $L_1 = A \setminus X, L_2 = B \setminus X, L_3 = C$. It follows from the definition of T_1 and a hex-join that $N(v_8) = \{v_7\} \cup R_1 \cup R_2$ is a strong clique. Therefore, v_8 is a simplicial vertex in T and hence T is resolved by (2.5).

So we may assume that T_1 is of the second type of sporadic trigraphs. Let $v_1, \ldots, v_9$ be as in the definition of T_1. Let $j \in \{3, 4\}$ be largest such that v_2 is adjacent to v_j and let $k \in \{5, 6\}$ be smallest such that v_7 is adjacent to v_k. Such j, k exist by the fact that v_2 is not strongly anticomplete to $\{v_3, v_4\} \setminus X$ and v_7 is not strongly anticomplete to $\{v_5, v_6\} \setminus X$. But now v_1-v_2-v_j-v_k-v_7-v_8-v_1 is a weakly induced cycle of length six in T, a contradiction. This proves (5.19). ■

This allows us to prove that $\mathcal{F}$-free three-cliqued claw-free trigraphs are resolved:

(5.20) *Every $\mathcal{F}$-free three-cliqued claw-free trigraph is resolved.*

Proof. Let (T, A, B, C) be a three-cliqued claw-free trigraph. It follows from (5.14) that either T is resolved and the lemma holds, or (T, A, B, C) is very basic, or (T, A, B, C) is a hex-join of a very basic trigraph and a strong clique. We may assume that the former outcome does not hold. If (T, A, B, C) is very basic, we set $(T', A', B', C') = (T, A, B, C)$. Otherwise, let (T', A', B', C') be such that T is a hex-join of a very basic trigraph (T', A', B', C') and a strong clique. Since $(T', A', B', C') \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_5$, the lemma follows from (5.15), (5.17), (5.18) and (5.19). This proves (5.20). ■

5.4 Proof of (5.1)

(5.1). *Every $\mathcal{F}$-free basic claw-free trigraph is resolved.*

Proof. Let T be an $\mathcal{F}$-free basic claw-free trigraph. It follows that T is either a trigraph from the icosahedron, or an antiprismatic trigraph, or a long circular interval trigraph, or a trigraph that is the union of three strong cliques. It follows from (5.2) that T is not a trigraph from the icosahedron. If T is an antiprismatic trigraph, a long circular interval trigraph, or a trigraph that is the union of three strong cliques, then it follows from (5.3), (5.13), (5.20), respectively, that T is resolved. This proves (5.1). ■

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