Partial characterizations of clique-perfect graphs I: subclasses of claw-free graphs

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Abstract

A *clique-transversal* of a graph G is a subset of vertices that meets all the cliques of G. A *clique-independent set* is a collection of pairwise vertex-disjoint cliques. The *clique-transversal number* and *clique-independence number* of G are the sizes of a minimum clique-transversal and a maximum clique-independent set of G, respectively. A graph G is *clique-perfect* if these two numbers are equal for every induced subgraph of G. The list of minimal forbidden induced subgraphs for the class of clique-perfect graphs is not known. In this paper, we present a partial result in this direction, that is, we characterize clique-perfect graphs by a restricted list of forbidden induced subgraphs when the graph belongs to two different subclasses of claw-free graphs.

Key words: Claw-free graphs, clique-perfect graphs, hereditary clique-Helly graphs, line graphs, perfect graphs.

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1 Introduction

Let G be a graph, with vertex set $V(G)$ and edge set $E(G)$. Denote by $\overline{G}$, the complement of G. Given two graphs G and G' we say that G' is *smaller* than G if $|V(G)'| < |V(G)|$, and that G *contains* G' if G' is isomorphic to an induced subgraph of G. When we need to refer to the non-induced subgraph containment relation, we will say so explicitly. A *claw* is the graph isomorphic to $K_{1,3}$. A graph is *claw-free* if it does not contain a claw. The *line graph* $L(G)$ of G is the intersection graph of the edges of G. A graph F is a *line graph* if there exists a graph H such that $L(H) = F$. Clearly, line graphs are a subclass of claw-free graphs.

The neighborhood of a vertex v is the set $N(v)$ consisting of all the vertices which are adjacent to v. The closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. A vertex v of G is *universal* if $N[v] = V(G)$. Two vertices v and w are *twins* if $N[v] = N[w]$; and u *dominates* v if $N(v) \subseteq N[u]$.

A *complete set* or just a *complete* of G is a subset of vertices pairwise adjacent. (In particular, an empty set is a complete set.) We denote by K_n the graph induced by a complete set of size n. A *clique* is a complete set not properly contained in any other. We may also use the term clique to refer to the corresponding complete subgraph. Let X and Y be two sets of vertices of G. We say that X is *complete to* Y if every vertex in X is adjacent to every vertex in Y, and that X is *anticomplete to* Y if no vertex of X is adjacent to a vertex of Y. A *stable set* in a graph G is a subset of pairwise non-adjacent vertices of G. The *stability number* $\alpha(G)$ is the cardinality of a maximum stable set of G.

A complete of three vertices is called a *triangle*, and a stable set of three vertices is called a *triad*. Let A be a set of vertices of G, and v a vertex of G not in A. Then v is A-*complete* if it is adjacent to every vertex in A, and A-*anticomplete* if it has no neighbor in A.

A vertex is called *simplicial* if its neighbors induce a complete, and *singular* if its non-neighbors induce a complete. Equivalently, a vertex is singular if it is in no stable set of size three. The *core* of G is the subgraph induced by G on the set of non-singular vertices.

Let G be a graph and X be a subset of vertices of G. Denote by $G|X$ the subgraph of G induced by X and by $G \setminus X$ the subgraph of G induced by $V(G) \setminus X$. X is *connected*, if there is no partition of X into two non-empty sets Y and Z, such that no edge has one end in Y and the other one in Z. In this case the graph $G|X$ is also connected. X is *anticonnected* if it is connected in $\overline{G}$. In this case the graph $G|X$ is also anticonnected.

The set X is a *cutset* if $G \setminus X$ has more connected components than G. Let G be a connected graph, X a cutset of G, and M_1, M_2 a partition of $V(G) \setminus X$ such that M_1, M_2 are non-empty and M_1 is anticomplete to M_2 in G. In this case we say that $G = M_1 + M_2 + X$, and $M_i + X$ denote $G|(M_i \cup X)$, for $i = 1, 2$. When $X = \{v\}$, we simplify the notation to $M_1 + M_2 + v$ and $M_i + v$, respectively.

Let X be a cutset of G. If $X = \{v\}$ we say that v is a *cutpoint*. If X contains a vertex adjacent in G to every other vertex of X and to no vertex of $G \setminus X$, it is called a *star cutset*. If X is complete, it is called a *clique cutset*. A clique cutset X is *internal* if $G = M_1 + M_2 + X$ and each M_i contains at least two vertices that are not twins.

Let G be a graph and H a subgraph of G (not necessarily induced). The graph H is a *clique subgraph* of G if every clique of H is a clique of G.

A *clique cover* of a graph G is a subset of cliques covering all the vertices of G. The *clique-covering number* $k(G)$ is the cardinality of a minimum clique cover of G. The *chromatic number* of a graph G is the smallest number of colors that can be assigned to the vertices of G in such a way that no two adjacent vertices receive the same color, and is denoted by $\chi(G)$. An obvious lower bound is the maximum cardinality of the cliques of G, the *clique number* of G, denoted by $\omega(G)$.

A graph G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G. Perfect graphs are interesting from the algorithmic point of view, see [18]. While determining the clique-covering number, the independence number, the chromatic number and the clique number of a graph are NP-complete problems, they are solvable in polynomial time for perfect graphs [19].

The *clique graph* $K(G)$ of G is the intersection graph of the cliques of G. A graph G is *K-perfect* if $K(G)$ is perfect.

A graph is *bipartite* if its vertex set can be partitioned into two stable sets. Bipartite graphs are perfect.

A *hole* is a chordless cycle of length at least 4. An *antihole* is the complement of a hole. A hole or antihole is said to be *odd* if it consists of an odd number of vertices (and, equivalently, edges). A hole of length n is denoted by C_n.

A graph is *chordal* if it does not contain a hole as an induced subgraph. Chordal graphs can be recognized in polynomial time [25].

An *r-sun*, $r \geq 3$, is a chordal graph of $2r$ vertices whose vertex set can be partitioned into two sets: $W = \{w_1, \ldots, w_r\}$ and $U = \{u_1, \ldots, u_r\}$, such that W is a stable set and for each i and j, w_j is adjacent to u_i if and only if $i = j$

or $i \equiv j+1 \pmod{r}$. An r-sun is said to be *odd* if r is odd.

A graph is *balanced* if its vertex-clique incidence matrix is balanced. A 0-1 matrix is balanced if it does not contain the incidence matrix of an odd cycle as a submatrix.

A family of sets S is said to satisfy the *Helly property* if every subfamily of it, consisting of pairwise intersecting sets, has a common element. A graph is *clique-Helly* (CH) if its cliques satisfy the Helly property, and it is *hereditary clique-Helly* (HCH) if H is clique-Helly for every induced subgraph H of G.

A *clique-transversal* of a graph G is a subset of vertices that meets all the cliques of G. A *clique-independent set* is a collection of pairwise vertex-disjoint cliques. The *clique-transversal number* and *clique-independence number* of G, denoted by $\tau_C(G)$ and $\alpha_C(G)$, are the sizes of a minimum clique-transversal and a maximum clique-independent set of G, respectively. It is easy to see that $\tau_C(G) \geq \alpha_C(G)$ for any graph G. A graph G is *clique-perfect* if $\tau_C(H) = \alpha_C(H)$ for every induced subgraph H of G. Clique-perfect graphs have been implicitly studied in [1,3,7,5,8,16,20,21]. The terminology "clique-perfect" has been introduced in [20]. There are two main open problems concerning this class of graphs:

- find all minimal forbidden induced subgraphs for the class of clique-perfect graphs, and
- is there a polynomial time recognition algorithm for this class of graphs?

In this paper, we present some results related to these problems. We characterize clique-perfect graphs by forbidden subgraphs when the graph belongs to a certain class. Both classes studied are subclasses of claw-free graphs: line graphs and HCH claw-free graphs. As corollaries of these partial characterizations, we can immediately deduce polynomial time algorithms to recognize clique-perfect graphs in these classes of graphs.

A preliminary version of this paper appeared in [4].

2 Preliminaries

It has been proved recently that perfect graphs can be characterized by two families of minimal forbidden induced subgraphs [10] and recognized in polynomial time [9].

Theorem 1 (Strong Perfect Graph Theorem) [10] *Let G be a graph. Then the following are equivalent:*

(i) *no induced subgraph of G is an odd hole or an odd antihole.*
(ii) *G is perfect.*

On the other hand, the problem of recognition of clique-perfect chordal graphs can be reduced to the recognition of balanced graphs, which is solvable in polynomial time [6,15].

Theorem 2 [21] *Let G be a chordal graph. Then the following are equivalent:*

(i) *G does not contain odd suns.*
(ii) *G is balanced.*
(iii) *G is clique-perfect.*

Next we define the family of the so called "generalized suns" [5]. Let G be a graph and C be a cycle of G not necessarily induced. An edge of C is *non proper* (or *improper*) if it forms a triangle with some vertex of C. An *r-generalized sun*, $r \geq 3$, is a graph G whose vertex set can be partitioned into two sets: a cycle C of r vertices, with all its non proper edges $\{e_j\}_{j \in J}$ (J is permitted be an empty set) and a stable set $U = \{u_j\}_{j \in J}$, such that for each $j \in J$, u_j is adjacent only to the endpoints of e_j. An r-generalized sun is said to be *odd* if r is odd. Clearly odd holes and odd suns are odd generalized suns.

Theorem 3 [5] *Odd generalized suns and antiholes of length $t = 1, 2$ mod 3 ($t \geq 5$) are not clique-perfect.*

Unfortunately, odd generalized suns are not necessary minimal (with respect to taking induced subgraphs) and besides there are other minimal non-clique-perfect graphs, for example the following family of graphs. Define the graph S_k, $k \geq 2$, as follows: $V(S_k) = \{v_1, \ldots, v_{2k}, v, v', w, w'\}$ where $v_1, \ldots, v_{2k}$ induce a path; v is adjacent to v', v_1, v_2 and v_{2k}; v' is adjacent to v, v_1, v_{2k-1} and v_{2k}; w is adjacent only to v_1 and v_2; and w' is adjacent only to v_{2k-1} and v_{2k} (Figure 1).

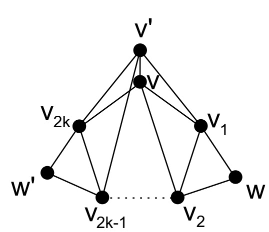

Fig. 1. The graph S_k.

At this time we do not know whether the list of all such forbidden graphs has a nice description. However, if we restrict our attention to certain classes of graphs (that can be described by forbidding certain induced subgraphs), we can describe all the minimal forbidden induced subgraphs.

Hereditary clique-Helly graphs are of particular interest because in this case it follows from [5] that if $K(H)$ is perfect for every induced subgraph H of G, then G is clique-perfect (the converse is not necessarily true). On the other hand, the class of hereditary clique-Helly graphs can be characterized by forbidden induced subgraphs.

Theorem 4 [23] *A graph G is hereditary clique-Helly if and only if it does not contain the graphs of Figure 2 as induced subgraphs.*

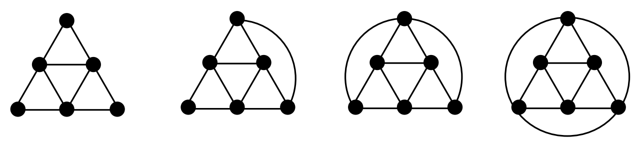

Fig. 2. Forbidden induced subgraphs for hereditary clique-Helly graphs: (left to right) 3-sun (or 0-pyramid), 1-pyramid, 2-pyramid and 3-pyramid.

One of our main results in this paper is a characterization of clique-perfect HCH claw-free graphs by induced subgraphs. To prove this characterization we use a recent structure theorem for claw-free graphs [12]. In order to state that theorem we need to introduce some definitions.

A graph G is *prismatic* if for every triangle T of G, every vertex of G not in T has a unique neighbor in T. A graph G is *antiprismatic* if its complement graph $\overline{G}$ is prismatic.

Construct a graph G as follows. Take a circle C, and let $V(G)$ be a finite set of points of C. Take a set of intervals from C (an *interval* means a proper subset of C homeomorphic to $[0,1]$) such that there are not three intervals covering C; and say that $u, v \in V(G)$ are adjacent in G if the set of points $\{u, v\}$ of C is a subset of one of the intervals. Such a graph is called *circular interval graph*. When the set of intervals does not cover C, the graph is called *linear interval graph*.

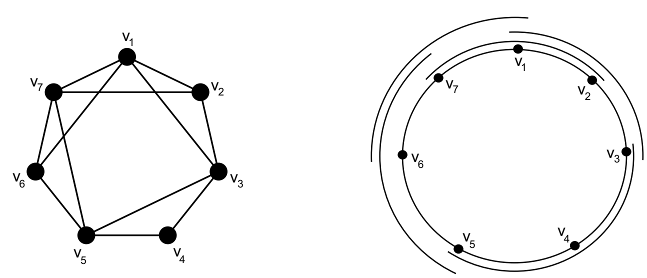

Fig. 3. Example of a circular interval graph and its circular interval representation.

Let G be a graph and A, B be disjoint subsets of $V(G)$. The pair (A, B) is called a *homogeneous pair* in G if for every vertex $v \in V(G) \setminus (A \cup B)$, v is either A-complete or A-anticomplete and either B-complete or B-anticomplete. If, in addition, B is empty, then A is called a *homogeneous set*. Let (A, B) be a homogeneous pair, such that A, B are both completes, and A is neither complete nor anticomplete to B. In these circumstances the pair (A, B) is called a *W-join*. Note that there is no requirement that $A \cup B \neq V(G)$. The pair (A, B) is *non-dominating* if some vertex of $G \setminus (A \cup B)$ has no neighbor in $A \cup B$, and it is *coherent* if the set of all $(A \cup B)$-complete vertices in $V(G) \setminus (A \cup B)$ is a complete.

Next, suppose that V_1, V_2 is a partition of $V(G)$ such that V_1, V_2 are non-empty and there are no edges between V_1 and V_2. The pair (V_1, V_2) is called a *0-join* in G. Thus G admits a 0-join if and only if it is not connected.

Next, suppose that V_1, V_2 is a partition of $V(G)$, and for $i = 1, 2$ there is a subset $A_i \subseteq V_i$ such that:

- for $i = 1, 2$, A_i is a complete, and A_i, $V_i \setminus A_i$ are both non-empty
- A_1 is complete to A_2
- every edge between V_1 and V_2 is between A_1 and A_2.

In these circumstances, the pair (V_1, V_2) is a *1-join*.

Now, suppose that V_0, V_1, V_2 are disjoint subsets with union $V(G)$, and for $i = 1, 2$ there are subsets A_i, B_i of V_i satisfying the following:

- for $i = 1, 2$, A_i, B_i are completes, $A_i \cap B_i = \emptyset$, and A_i, B_i and $V_i \setminus (A_i \cup B_i)$ are all non-empty
- A_1 is complete to A_2, and B_1 is complete to B_2, and there are no other edges between V_1 and V_2
- V_0 is a complete, and for $i = 1, 2$, V_0 is complete to $A_i \cup B_i$ and anticomplete to $V_i \setminus (A_i \cup B_i)$.

The triple (V_0, V_1, V_2) is called a *generalized 2-join*, and if $V_0 = \emptyset$, the pair (V_1, V_2) is called a 2-join. This is closely related to, but not the same as, what has been called a 2-join in other papers, like [9].

The last decomposition is the following. Let (V_1, V_2) be a partition of $V(G)$, such that for $i = 1, 2$ there are completes A_i, B_i, $C_i \subseteq V_i$ with the following properties:

- For $i = 1, 2$ the sets A_i, B_i, C_i are pairwise disjoint and have union V_i
- V_1 is complete to V_2 except that there are no edges between A_1 and A_2, between B_1 and B_2, and between C_1 and C_2
- V_1, V_2 are both non-empty.

In these circumstances it is said that G is a *hex-join* of $G|V_1$ and $G|V_2$. Note that if G is expressible as a hex-join as above, then the sets $A_1 \cup B_2$, $B_1 \cup C_2$ and $C_1 \cup A_2$ are three completes with union $V(G)$, and consequently no graph G with $\alpha(G) > 3$ is expressible as a hex-join.

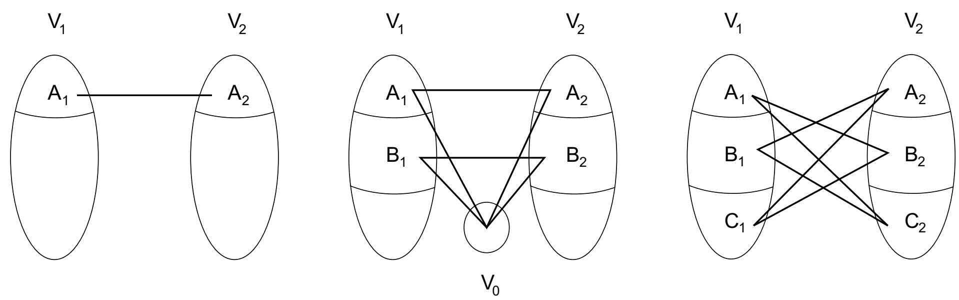

Fig. 4. Scheme for 1-join, 2-join and hex-join.

Now, define classes $\mathcal{S}_0, \dots, \mathcal{S}_6$ as follows.

- $\mathcal{S}_0$ is the class of all line graphs.

- The *icosahedron* is the unique planar graph with twelve vertices all of degree five. For $0 \leq k \leq 3$, $icosa(-k)$ denotes the graph obtained from the icosahedron by deleting k pairwise adjacent vertices. A graph $G \in \mathcal{S}_1$ if G is isomorphic to $icosa(0)$, $icosa(-1)$ or $icosa(-2)$. As it can be seen in Figure 5, all of them contain odd holes.

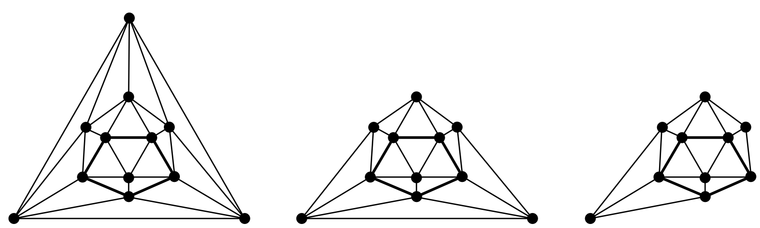

Fig. 5. Graphs $icosa(0)$, $icosa(-1)$ and $icosa(-2)$.

- Let H_1 be the graph with vertex set $\{v_1, \dots, v_{13}\}$, with adjacency as follows: $v_1 v_2 \dots v_6 v_1$ is a hole in G of length 6; v_7 is adjacent to v_1, v_2; v_8 is adjacent to v_4, v_5 and possibly to v_7; v_9 is adjacent to v_6, v_1, v_2, v_3; v_{10} is adjacent to v_3, v_4, v_5, v_6, v_9; v_{11} is adjacent to v_3, v_4, v_6, v_1, v_9, v_{10}; v_{12} is adjacent to v_2, v_3, v_5, v_6, v_9, v_{10}; and v_{13} is adjacent to v_1, v_2, v_4, v_5, v_7, v_8. A graph $G \in \mathcal{S}_2$ if G is isomorphic to $H_1 \setminus X$, where $X \subseteq \{v_{11}, v_{12}, v_{13}\}$. Please note that vertices $v_3 v_4 v_5 v_6 v_9 v_3$ induce a hole of length five in G.

- $\mathcal{S}_3$ is the class of all circular interval graphs.

- Let H_2 be the graph with seven vertices $h_0, \dots, h_6$, in which $h_1, \dots, h_6$ are pairwise adjacent and h_0 is adjacent to h_1. Let H_3 be the graph obtained from the line graph $L(H_2)$ of H_2 by adding one new vertex, adjacent precisely to the members of $V(L(H_2)) = E(H_2)$ that are not incident with h_1 in H_2. Then H_3 is claw-free. Let $\mathcal{S}_4$ be the class of all graphs isomorphic to induced subgraphs of H_3. Note that the vertices of H_3 corresponding to the members of $E(H_2)$ that are incident with h_1 in H_2, form a complete in H_3.

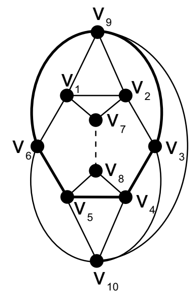

Fig. 6. Graph $H_1 \setminus \{v_{11}, v_{12}, v_{13}\}$. Every graph in $\mathcal{S}_2$ contains it as an induced subgraph.

So every graph in $\mathcal{S}_4$ is either a line graph or it has a singular vertex.

- Let $n \geq 0$. Let $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$, $C = \{c_1, \ldots, c_n\}$ be three completes, pairwise disjoint. For $1 \leq i, j \leq n$, let a_i, b_j be adjacent if and only if $i = j$, and let c_i be adjacent to a_j, b_j if and only if $i \neq j$. Let d_1, d_2, d_3, d_4, d_5 be five more vertices, where d_1 is $(A \cup B \cup C)$-complete; d_2 is complete to $A \cup B \cup \{d_1\}$; d_3 is complete to $A \cup \{d_2\}$; d_4 is complete to $B \cup \{d_2, d_3\}$; d_5 is adjacent to d_3, d_4; and there are no more edges. Let the graph just constructed be H_4. A graph $G \in \mathcal{S}_5$ if (for some n) G is isomorphic to $H_4 \setminus X$ for some $X \subseteq A \cup B \cup C$. Note that vertex d_1 is adjacent to all the vertices but the triangle formed by d_3, d_4 and d_5, so it is a singular vertex in G (Figure 7).

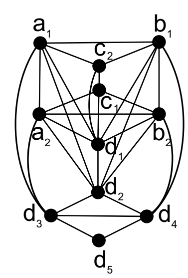

Fig. 7. Graph H_4, for $n = 2$.

- Let $n \geq 0$. Let $A = \{a_0, \ldots, a_n\}$, $B = \{b_0, \ldots, b_n\}$, $C = \{c_1, \ldots, c_n\}$ be three completes, pairwise disjoint. For $0 \leq i, j \leq n$, let a_i, b_j be adjacent if and only if $i = j > 0$, and for $1 \leq i \leq n$ and $0 \leq j \leq n$ let c_i be adjacent to a_j, b_j if and only if $i \neq j \neq 0$. Let the graph just constructed be H_5. A graph $G \in \mathcal{S}_6$ if (for some n) G is isomorphic to $H_5 \setminus X$ for some $X \subseteq A \cup B \cup C$, and then G is said to be 2-*simplicial of antihat type* (Figure 8).

The structure theorem in [12] is the following:

Theorem 5 *Let G be a claw-free graph. Then either $G \in \mathcal{S}_0 \cup \cdots \cup \mathcal{S}_6$, or G admits twins, or a non-dominating W-join, or a coherent W-join, or a* 0*-join,*

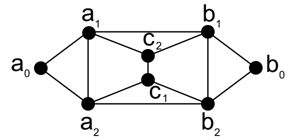

Fig. 8. Graph H_5, for $n = 2$.

or a 1*-join, or a generalized* 2*-join, or a hex-join, or G is antiprismatic.*

In the proofs in this paper we will mention some special graphs, shown in Figure 9, and we will use the following results on perfect graphs, cutsets and clique graphs (some of the results bellow are immediate, and in these cases we do not give a proof or a reference; we state these in order to make it more convenient to refer to them in the future.).

Lemma 6 *Let G be a graph and v be a simplicial vertex of G. Then G is perfect if and only if $G \setminus \{v\}$ is.*

Theorem 7 [2] *Let G be a graph and X be a clique cutset of G, such that $G = M_1 + M_2 + X$. Then the graph G is perfect if and only if the graphs $M_1 + X$ and $M_2 + X$ are.*

This theorem due to Berge was generalized by Chvátal for star cutsets.

Theorem 8 [13] *Let G be a graph and X be a star cutset of G, such that $G = M_1 + M_2 + X$. Then the graph G is perfect if and only if the graphs $M_1 + X$ and $M_2 + X$ are.*

Let P be an induced path of a graph G. The *length* of P is the number of edges in P. The *parity* of P is the parity of its length . We say that P is *even* if its length is even, and *odd* otherwise.

Theorem 9 *Let G be a perfect graph and let $e = v_1v_2$ be an edge of G. Assume that $\{v_1, v_2\}$ is a cutset in G. Assume also that no vertex of G is a common neighbor of v_1 and v_2. Then $G \setminus e$ is perfect.*

PROOF. Since G is perfect, it is enough to check that there is no odd hole or antihole in $G \setminus e$ using both v_1 and v_2. Suppose such a hole or an antihole exists, denote it by A. Since no vertex of G is a common neighbor of v_1, v_2, it follows that A is not an antihole. So A is a hole, and let A_1, A_2 be the two subpaths of A joining v_1 and v_2. Then both A_1, A_2 have length at least three, and one of them, say A_1, is even. But then $G|V(A_1)$ is an odd hole, a contradiction. $\Box$

Theorem 10 [14] *Let G be a graph and let U be a homogeneous set in G. Let G' be the graph obtained from G by deleting all but one vertex of U. Then G*

is perfect if and only if both G' and $G|U$ are.

This, together with Theorem 8, implies the following:

Theorem 11 *Let G be a graph, and let $u, v \in V(G)$ such that u dominates v. Then G is perfect if and only if both $G \setminus \{u\}$ and $G \setminus \{v\}$ are.*

PROOF. The "only-if" part is clear, so it is enough to prove that if $G \setminus \{u\}$ and $G \setminus \{v\}$ are perfect, then so is G. If $\{u, v\}$ is a homogeneous set in G, the result holds by Theorem 10. Otherwise, since u properly dominates v, it follows that $N(v) \cup \{u\}$ is a star cutset in G. By Theorem 8, if $G \setminus \{v\}$ and $G|(N[v] \cup \{u\})$ are perfect, then so is G. Since $\{u, v\}$ is a homogeneous set in $G|(N[v] \cup \{u\})$, Theorem 10 implies that if $G|(N[v] \setminus \{u\})$ is perfect, then so is $G|(N[v] \cup \{u\})$. But now, since $G|(N[v] \setminus \{u\})$ is an induced subgraph of $G \setminus \{u\}$, the result follows. $\square$

Theorem 12 [11] *Let G be a claw-free graph admitting an internal clique cutset. Then G is either a linear interval graph or G is the 3-sun, or G admits twins, or a 0-join, or a 1-join, or a coherent W-join.*

Lemma 13 *Let G be a graph and H a clique subgraph of G. Then $K(H)$ is an induced subgraph of $K(G)$.*

Lemma 14 *If G admits twins u, v, then $K(G) = K(G \setminus \{v\})$.*

Lemma 15 *If G is disconnected, then so is $K(G)$, and G is K-perfect if and only if each connected component is.*

Theorem 16 [24] *Let G be a claw-free graph with no induced 3-fan, 4-wheel or odd hole. Then $K(G)$ is bipartite.*

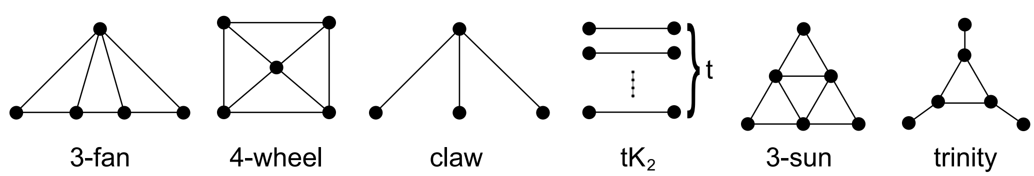

Fig. 9. Some graphs mentioned in the paper.

3 Partial characterizations

We say that a graph is *interesting* if no induced subgraph of it is an odd generalized sun or an antihole of length greater than 5 and equal to $1, 2$ *mod* 3.

Our two main results are the following.

Theorem 17 *Let G be a line graph. Then G is clique perfect if and only if no induced subgraph of G is an odd hole or a 3-sun.*

Theorem 18 *Let G be an HCH claw-free graph. Then G is clique perfect if and only if no induced subgraph of G is an odd hole or an antihole of length seven.*

We observe the following:

Proposition 19 *Let S be an odd generalized r-sun, and assume that S is claw-free. Then either S is an odd hole or $r = 3$.*

PROOF. As in the definition of a generalized sun, let C be a cycle of S, and let $U = V(S) \setminus V(C)$ be a stable set, such that every vertex of U is complete to both ends of exactly one non-proper edge of C and has no other neighbor in $V(C)$. We may assume that S is not an odd hole, and so C has at least one non-proper edge. Let c_1c_2 be a non-proper edge of C, let $c_3 \in V(C) \setminus \{c_1, c_2\}$ be such that $\{c_1, c_2, c_3\}$ is a triangle, and let u be the vertex of U adjacent to c_1 and c_2. We may assume $r > 3$, and therefore, possibly with c_1 and c_2 switched, c_1 has a neighbor c_2' in C, different from c_2 and c_3. Since $\{c_1, u, c_3, c_2'\}$ does not induce a claw in S, it follows that c_2' is adjacent to c_3, and therefore c_1c_2' is another non-proper edge of S. Let u' be the vertex of U adjacent to c_1 and c_2'. Then $\{c_1, u, u', c_3\}$ is a claw, a contradiction. $\square$

Let us call a class of graphs $\mathcal{C}$ *hereditary* if for every $G \in \mathcal{C}$, every induced subgraph of G also belongs to $\mathcal{C}$. The following is a useful fact about hereditary clique-Helly graphs:

Proposition 20 *Let $\mathcal{L}$ be a hereditary graph class, which is HCH and such that every interesting graph in $\mathcal{L}$ is K-perfect. Then every interesting graph in $\mathcal{L}$ is clique-perfect.*

PROOF. Let G be an interesting graph in $\mathcal{L}$. Let H be an induced subgraph of G. Since $\mathcal{L}$ is hereditary, H is an interesting graph in $\mathcal{L}$, so it is K-perfect. Since $\mathcal{L}$ is a HCH class, H is clique-Helly and then $\alpha_C(H) = \alpha(K(H)) = k(K(H)) = \tau_C(H)$ [5], and the result follows. $\square$

3.1 Line graphs

First, we prove that interesting line graphs are K-perfect.

Proposition 21 *A line graph is interesting if and only if it has no induced subgraph isomorphic to an odd hole or a 3-sun.*

PROOF. Since no line graph contains an antihole of length at least seven, and every line graph is claw-free, the result follows from Proposition 19. This proves Proposition 21. □

Theorem 22 *If G is an interesting line graph, then $K(G)$ is perfect.*

PROOF. Let $G = L(H)$. By Lemma 15, we may assume H is connected. If H is bipartite then $G = K(H)$ and $K(G) = K^2(H)$ is an induced subgraph of H [17], so it is bipartite and hence perfect.

If H is not bipartite, all the odd cycles of H are triangles, otherwise G has an odd hole (the line graph of a subgraph of H is an induced subgraph of $L(H)$).

A *trinity* is the complement of the 3-sun, and its line graph is also the 3-sun. Therefore H does not contain a trinity as a subgraph, for otherwise G contains a 3-sun as an induced subgraph.

The proof is by induction on $|V(G)|$. The theorem holds for the graph with one vertex, and in each case we will reduce the K-perfection of G to the K-perfection of some proper induced subgraphs of G. Since every induced subgraph of an interesting line graph is also an interesting line graph, the result will then follow from the inductive hypothesis.

Suppose H contains a triangle $T = \{v_1, v_2, v_3\}$, and let $e_{ij} = v_i v_j$ be the edges of T. We start by looking at paths joining v_1, v_2 and v_3 in the graph $H_T = H \setminus \{e_{12}, e_{23}, e_{31}\}$. Suppose that v_1 and v_2 are connected by a path P in H_T such that $v_3 \notin P$. If P has length at least 3, then either $P + e_{12}$ or $P + e_{23} + e_{31}$ is an odd cycle of length at least 5 in H (which implies an odd hole in G). So the length of P must be 2. Suppose now that two pairs of vertices of T, say $\{v_1, v_2\}$ and $\{v_2, v_3\}$, are connected in H_T through vertices outside T, say w and w', respectively. If $w \neq w'$, then $v_1 w v_2 w' v_3 v_1$ is a cycle of length 5 in H, a contradiction. So $w = w'$ and H contains a complete set of size four.

The proof now breaks into two cases, depending of whether H contains a complete set of size four. Note that if H does not contain a complete set of size four, then for every triangle T of H, at most one pair of vertices of T is joined in H_T by a path not using the third vertex of T.

<u>Case 1</u>: H contains a complete set of size four.

Let K be a complete set of size four in G. Every vertex outside of K is adjacent to at most one of the vertices of K, otherwise H contains cycle of length five as a subgraph. If two vertices v, v' of K have different neighbors w, w', respectively, outside of K, then H contains a trinity as a subgraph. So at most one vertex v of K has neighbors outside of K.

If all the edges of H are those joining two vertices of K and those incident with v, then $K(L(H))$ is the complement of $4K_2$, and so it is perfect (it is the complement of a bipartite graph).

Otherwise, let k_v be the vertex of $K(G)$ corresponding to the clique of G formed by the edges of H incident with v. Then k_v is a cutpoint of $K(G)$. Moreover, $K(G) = M_1 + M_2 + k_v$, where $M_1 + k_v$ is the clique graph of the line graph of K_4 (the complement of $4K_2$, a perfect graph) and $M_2 + k_v$ is the clique graph of the line graph of $H \setminus \{z, z'\}$, with z and z' vertices of the K_4 different from v, so $M_2 + k_v$ is perfect by the inductive hypothesis. By Theorem 7, since $M_1 + k_v$ and $M_2 + k_v$ are perfect, so is $K(G)$.

Case 2: H does not contain a complete set of size four, and hence, for every triangle T of H, at most one pair of vertices of T is joined in H_T by a path not using the third vertex of T.

First note that G has two kinds of cliques: those formed by the vertices of G corresponding to the edges of H with a common endpoint v (we will denote by k_v the vertex of $K(G)$ corresponding to such a clique) and those formed by three vertices corresponding to the three edges of a triangle T of H (we will denote by k_T the vertex of $K(G)$ corresponding to such a clique).

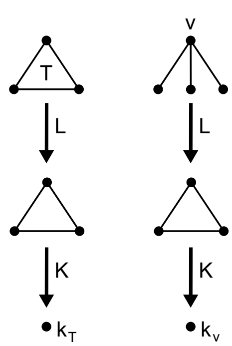

Fig. 10. Cliques of $K(L(H))$.

Let $T_1 = \{v_1, v_2, w_1\}$ be a triangle of H, and, without loss of generality, suppose that there is no path from w_1 to $\{v_1, v_2\}$ in the graph obtained from H by removing the edges of T_1.

Let $W = w_1, \ldots, w_s$ be the set of common neighbors of v_1 and v_2. Then W is a stable set of H, because H does not contain a complete set of size four. Let T_i be the triangle of H formed by v_1, v_2, w_i. Let $A_1 = N(v_1) \setminus (W \cup \{v_2\})$ and

$A_2 = N(v_2) \setminus (W \cup \{v_1\})$. Note that A_1 and A_2 are disjoint.

Case 2.1: $|W| \geq 2$ and $|W \cup A_1 \cup A_2| \geq 3$.

We note that in this case W is anticomplete to $A_1 \cup A_2$, for if $w \in W$ is adjacent to $a \in A_1$, say, then for the triangle $\{v_1, v_2, w\}$, there is a path from v_1 to w through a, and a path from v_1 to v_2 though $w' \in W \setminus \{w\}$, a contradiction. Next we observe that all the vertices in W are adjacent only to v_1 and v_2, otherwise H contains a trinity as a subgraph. In $K(G)$, each k_{T_i} is a simplicial vertex, because $N[k_{T_i}] = \{k_{T_1}, \ldots, k_{T_s}, k_{v_1}, k_{v_2}\}$ is a complete set in $K(G)$. So, by Lemma 6, $K(G)$ is perfect if and only if $K(G) \setminus k_{T_s}$ is perfect. And $K(G) \setminus \{k_{T_s}\} = K(L(H \setminus \{w_s\}))$ because $s \geq 2$, hence it is perfect by the inductive hypothesis.

Case 2.2: $|W| = 2$, A_1 and A_2 are empty.

We claim that there is no path from w_1 to w_2 in $H \setminus \{v_1, v_2\}$. Suppose such a path P exists. Since W is a stable set, P has at least one internal vertex. But now either $v_1 w_1 P w_2 v_1$ or $v_1 w_1 P w_2 v_2 v_1$ is an odd cycle of length at least five in H, a contradiction. This proves the claim.

Let $B_1 = N(w_1) \setminus \{v_1, v_2\}$ and $B_2 = N(w_2) \setminus \{v_1, v_2\}$. If $B_1 = B_2 = \emptyset$, then, since H is connected, $V(H) = \{v_1, v_2, w_1, w_2\}$, and therefore $K(G)$ is perfect. So we may assume that B_1 is non-empty, say. Then the graph $H \setminus \{w_1\}$ is disconnected. Let H_2 be the component of $H \setminus \{w_1\}$ containing w_2 and let $H_1 = H \setminus (V(H_2) \cup \{w_1\})$. Then $B_1 \subseteq V(H_1)$. It follows that $\{k_{T_1}\}$ is a clique cutset of $K(G)$. Moreover, $K(G) = M_1 + M_2 + \{k_{T_1}\}$, and $M_i + \{k_{T_1}\} = K(L(H_i))$. The graphs $L(H_1)$ and $L(H_2)$ are induced subgraphs of G, so by the inductive hypothesis they are K-perfect, and so it follows from Theorem 7 that $K(G)$ is perfect.

Case 2.3: $|W| = 1$, A_1 and A_2 are empty.

The vertices of G corresponding to the edges $w_1 v_1$ and $w_1 v_2$ are twins in G. So $K(G) = K(L(H \setminus w_1 v_1))$.

Case 2.4: $|W| = 1$, A_1 and A_2 are non-empty.

In this case, w_1 has no neighbor in $A_1 \cup A_2$, because there is no path from w_1 to $\{v_1, v_2\}$ in H_{T_1}. Therefore w_1 is adjacent only to v_1 and v_2, otherwise H contains a trinity as a subgraph. In $K(G)$, k_{T_1} is a simplicial vertex, because $N[k_{T_1}] = \{k_{T_1}, k_{v_1}, k_{v_2}\}$ is a complete in $K(G)$. So, by Lemma 6, $K(G)$ is perfect if and only if $K(G) \setminus \{k_{T_1}\}$ is perfect; and $K(G) \setminus \{k_{T_1}\} = K(L(H \setminus \{w_1\}))$ is perfect by the inductive hypothesis.

Case 2.5: $|W| = 1$, and exactly one of A_1 or A_2 is empty.

Renaming the vertices (v_1 or v_2, respectively, playing the role of w_1), we can reduce this case either to Case 2.3 or to Case 2.4. $\square$

Theorem 17 is an immediate corollary of the following:

Theorem 23 *Let G be a line graph. Then the following are equivalent:*

(i) *no induced subgraph of G is and odd hole, or a 3-sun.*
(ii) *G is clique-perfect.*
(iii) *G is perfect and it does not contain a 3-sun.*

PROOF. The equivalence between (i) and (iii) is a corollary of Theorem 1, because line graphs do not contain antiholes $\overline{C_n}$ with $n \geq 7$ as induced subgraphs. From Theorem 3 it follows that (ii) implies (i).

It therefore suffices to prove that (i) implies (ii). This proof is again by induction on $|V(G)|$. The class of line graphs with no odd holes or induced 3-suns is hereditary, so we only have to prove that for every graph in this class τ_C equals to α_C. By Theorem 22 and Proposition 21, every such graph is K-perfect. So, by Proposition 20, an interesting HCH line graph is clique-perfect. Let $G = L(H)$ and suppose that G is not HCH. Then G contains a 0-,1-,2- or 3-pyramid. as an induced subgraph.

A 0-pyramid is a 3-sun. A 2-pyramid is not a line graph, and therefore is not an induced subgraph of G.

Assume first that H contains a complete set of size four, say K. By Lemma 15 we may assume H is connected. We analyze how vertices of $V(H) \setminus K$ attach to K. If a vertex v is adjacent to two different vertices of K, then H contains an odd cycle as a subgraph and G contains an odd hole. If two different vertices v, w are adjacent to two different vertices of K, then H contains a trinity as a subgraph and so G contains a 3-sun as an induced subgraph. These cases can be seen in Figure 11.

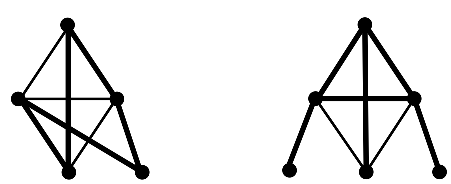

Fig. 11. How the remaining vertices of H can be attached to the K_4.

So only one of the four vertices x_1, x_2, x_3, x_4 of K may have neighbors in $H \setminus K$, say x_1. Let v, w, z_1, z_2, z_3 and z_4 be the vertices of G corresponding to the edges x_1x_2, x_3x_4, x_1x_3, x_1x_4, x_2x_4 and x_2x_3 of H, respectively. The vertex w is adjacent in G only to z_1, z_2, z_3 and z_4, which induce a hole of length 4 and are adjacent also to v. So $G \setminus \{w\}$ is a clique subgraph of G (every clique of $G \setminus \{w\}$ is a clique of G). On the other hand, since x_2 has no neighbors in $H \setminus K$, all the neighbors of v are vertices corresponding to edges of H containing x_1, and they are a complete in G. This situation can be seen in Figure 12.

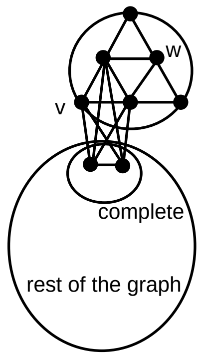

Fig. 12. Structure of G when H has a K_4.

By the inductive hypothesis, $G \setminus \{w\}$ is clique-perfect. Let A be a maximum clique-independent set and T be a minimum clique-transversal of $G \setminus \{w\}$. By maximality and by the structure of G, A has exactly one clique containing v. Adding w, four new cliques appear, each one disjoint from a different one of the four cliques containing v, and adding w to T we have a clique-transversal of G, so $\alpha_C(G) = \alpha_C(G \setminus \{w\}) + 1 = \tau_C(G \setminus \{w\}) + 1 = \tau_C(G)$. So we may assume that H contains no complete set of size four.

Since if G contains a 3-pyramid as an induced subgraph, then H contains a complete set of size four, it follows that the only remaining case is when G contains a 1-pyramid. Since G contains a 1-pyramid, H contains as a subgraph a graph on five vertices $v_1, \dots, v_5$ where v_1 is adjacent to v_2, v_3 and v_4, v_2 is adjacent to v_3 and v_4, and v_3 is adjacent to v_5 (Figure 13). Moreover, v_3 and v_4 are not adjacent because H does not contain a complete set of size four, v_1 and v_2 are not adjacent to v_5, otherwise H contains an odd cycle as a subgraph, and v_1 and v_2 do not have other neighbors, otherwise H contains a trinity as a subgraph. Then v_1 and v_2 form a cutset in H, because if there is a path $v_3 P v_4$ in $H \setminus \{v_1, v_2\}$, then either $v_3 P v_4 v_1 v_3$ or $v_3 P v_4 v_1 v_2 v_3$ is an odd cycle in H.

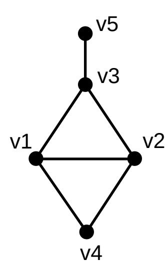

Fig. 13. Subgraph of H when H contains no K_4 and G contains a 1-pyramid.

Let $w_1, \ldots, w_5$ be the vertices of G corresponding to the edges v_1v_3, v_2v_3, v_1v_4, v_2v_4 and v_1v_2 of H, respectively. Then $w_1w_2w_4w_3w_1$ is a hole of length four in G, w_5 is adjacent only to $w_1, \ldots, w_4$ and $w_1, \ldots, w_5$ is a cutset of G. The remaining neighbors of w_1 or w_2 are adjacent to both w_1 and w_2, and form a non-empty complete in G (they are the vertices corresponding to the edges of H containing v_3 and not v_1 or v_2, and there exists at least one such edge, namely the edge v_3v_5). Similarly, the neighbors of w_3 or w_4 are adjacent to both w_3 and w_4, and form a (possibly empty) complete in G. The structure of G in this case can be seen in Figure 14.

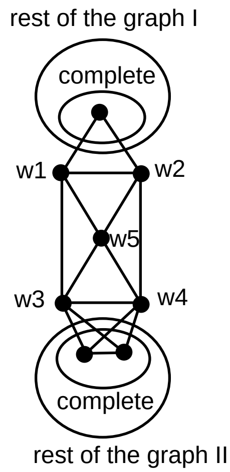

Fig. 14. Structure of G when H has no K_4.

We show that $\alpha_C(G) = \alpha_C(G')$ and $\tau_C(G) = \tau_C(G')$, where G' is the line graph of the graph H', obtained from H by deleting the edges $v2v3$ and $v1v4$. So $G' = G \setminus \{w_2, w_3\}$.

Since every clique transversal of G' either contains w_5, or contains both w_1 and w_4, it follows that every clique transversal of G' is a clique transversal of G. On the other hand, starting with a clique transversal T of G and replacing the vertices w_2 and w_3 by w_1 and w_4 respectively, if w_2 or w_3 belong to T, produces a clique transversal of G'. Therefore $\tau_C(G) = \tau_C(G')$.

We claim that there is a maximum clique-independent set not containing either of the cliques $\{w_1, w_3, w_5\}$, $\{w_2, w_4, w_5\}$. Suppose the claim is false. Let I be a clique independent set, we may assume I contains the clique $\{w_1, w_3, w_5\}$. Then I does not contain any other clique containing w_1 or w_5; and since the only clique containing w_2 and not w_1 is $\{w_1, w_2, w_5\}$, it follows that every clique in I is disjoint from $\{w_1, w_2, w_5\}$. But now the set obtained from I by removing the clique $\{w_1, w_3, w_5\}$ and adding the clique $\{w_1, w_2, w_5\}$ has a the desired property. This proves the claim.

Let I a maximum clique independent set of G not containing either of the cliques $\{w_1, w_3, w_5\}$, $\{w_2, w_4, w_5\}$. Let I' be a set of cliques of G', obtained from I by replacing the clique $\{w_1, w_2, w_5\}$ by $\{w_1, w_5\}$ if $\{w_1, w_2, w_5\} \in I$, and the clique $\{w_3, w_4, w_5\}$ by $\{w_4, w_5\}$ if $\{w_3, w_4, w_5\} \in I$. On the other hand, clearly every clique independent set of G' gives rise to a clique independent set of G, and therefore $\alpha_C(G) = \alpha_C(G')$.

But now, since G' is a proper induced subgraph of G, it follows inductively that $\alpha_c(G') = \tau_C(G')$, and therefore $\alpha_c(G') = \tau_C(G')$. This completes the proof of Theorem 23. $\square$

The recognition problem for line graphs can be solved in polynomial time [22]. By the theorem above, the recognition of clique-perfect line graphs can be reduced to the recognition of perfect graphs with no 3-sun, which is solvable in polynomial time [9].

3.2 Hereditary clique-Helly claw-free graphs

We will use Proposition 20 to prove the characterization for HCH claw-free graphs, so first we will prove the following.

Theorem 24 *Let G be an interesting HCH claw-free graph. Then $K(G)$ is perfect.*

Proposition 25 *No HCH graph contains an antihole of length at least eight. An HCH claw-free graph is interesting if and only if it does not contain an odd hole or an antihole of length seven.*

PROOF. Since by Theorem 4 an HCH graph contains no induced subgraph isomorphic to one of the graph of Figure 2, it follows that no HCH graph contains a 3-sun as an induced subgraph. Since every antihole of length at least eight contains a 2-pyramid, it follows that no HCH graph contains an antihole of length at least eight. Finally, since by Proposition 19, every claw-free odd generalized sun is either an odd hole or a 3-sun, it follows that an HCH claw-free graph is interesting if an only if it contains no odd hole and no antihole of length seven. This proves Proposition 25. $\square$

In the remainder of this section we use Theorem 5 to prove that every interesting HCH claw-free is K-perfect. The proof is by induction on $|V(G)|$.

3.2.1 Circular Interval Graphs

First we prove that clique graph of interesting HCH circular interval graphs are perfect.

Lemma 26 *Let G be a circular interval graph. Then $K(G)$ is an induced subgraph of G.*

PROOF. Let G be a circular interval graph with vertices $v_1, \ldots, v_n$ in clockwise order, say. We define a homomorphism v from $V(K(G))$ to $V(G)$ (meaning that for two distinct vertices $a, b \in V(K(G))$, $v(a) \neq v(b)$; and a is adjacent to b if and only if $v(a)$ is adjacent to $v(b)$). For every clique M of G, since no three intervals in the definition of a circular interval graph cover the circle, $M = \{v_i, \ldots, v_{i+t}\}$ (where the indices are taken mod n). In this case we say that v_i is the *first vertex* of M. We define $v(M) = v_i$. Since v_i is the first vertex of a unique clique, it follows that $v(M) \neq v(M')$ if M and M' are distinct cliques of G. It remains to show that $v(M)$ is adjacent to $v(M')$ if and only if $M \cap M' \neq \emptyset$. If M and M' intersect at a vertex v_k, then the clockwise order of $v(M)$, $v(M')$ and v_k is either $v(M)$, $v(M')$, v_k or $v(M')$, $v(M)$, v_k and in both cases $v(M)$ and $v(M')$ are adjacent. On the other hand, if there are two cliques such that $v(M)$ and $v(M')$ are adjacent, we may assume $v(M)$ appears first clockwise in the circular interval which contains both $v(M)$ and $v(M')$. Then since $v(M)$ is the first vertex of the clique M, it follows that $v(M')$ belongs to M, so M and M' intersect. □

Proposition 27 *Let G be an HCH interesting circular interval graph. Then $K(G)$ is perfect.*

PROOF. By Lemma 26, $K(G)$ is an induced subgraph of G. Since G is HCH and interesting, it contains no odd hole and no antihole of length at least seven, and therefore it is perfect by Theorem 1.

3.2.2 Decompositions

Now we show that if an interesting HCH claw-free graph admits one of the decompositions of Theorem 5, then either it is K-perfect or we can reduce the problem to a smaller one.

Theorem 28 *Let G be an interesting HCH claw-free graph. If G admits a 1-join, then $K(G)$ has a cutpoint v, $K(G) = H_1 + H_2 + v$, and $H_i + v$ is the clique graph of a smaller interesting HCH claw-free graph.*

PROOF. Since G admits a 1-join, it follows that $V(G)$ is the disjoint union of two non-empty sets V_1 and V_2, each V_i contains a complete M_i, such that $M_1 \cup M_2$ is a complete and there are no other edges from V_1 to V_2. So $M_1 \cup M_2$ is a clique in G. Let v be the vertex of K(G) corresponding to $M_1 \cup M_2$. Every other clique of G is either contained in V_1 or in V_2, and no clique of the first type intersects a clique of the second type. So v is a cutpoint of $K(G)$, and $K(G) = H_1 + H_2 + v$. Let G_i be the graph obtained from $G|V_i$ by adding a vertex v_i complete to M_i and with no other neighbors in G_i. Then G_i is isomorphic to an induced subgraph of G, so it is interesting, HCH and claw-free, and for $i = 1, 2$, $H_i + v$ is isomorphic to $K(G_i)$ (where the vertex v is mapped to the vertex of $K(G_i)$ corresponding to the clique $M_i \cup \{v_i\}$ of G_i). This proves Theorem 28. $\square$

Theorem 29 *Let G be an interesting HCH claw-free graph. If G admits a generalized 2-join and no twins, 0-join or 1-join, then there exist two clique graphs of smaller interesting HCH claw-free graphs, H_1 and H_2, such that if H_1 and H_2 are perfect, then so is $K(G)$.*

PROOF. Since G admits a generalized 2-join, it follows that $V(G)$ is the disjoint union of three sets V_0, V_1 and V_2, for $i = 1, 2$ each V_i contains two completes A_i, B_i such that A_i, B_i and $V_i \setminus (A_i \cup B_i)$ are all non-empty, $A_1 \cup A_2 \cup V_0$ and $B_1 \cup B_2 \cup V_0$ are completes and there are no other edges from V_1 to V_2 or from V_0 to $V_1 \cup V_2$. Since G admits no twins, it follows that $|V_0| \leq 1$.

So $A_1 \cup A_2 \cup V_0$ and $B_1 \cup B_2 \cup V_0$ are cliques of G, and they correspond to vertices w_1, w_2 of $K(G)$. Every other clique of G is either contained in V_1 or in V_2, and no clique of the first type intersects a clique of the second type. So $\{w_1, w_2\}$ is a cutset in $K(G)$.

If V_0 is non-empty, then w_1 is adjacent to w_2 and $\{w_1, w_2\}$ is a clique cutset in $K(G)$. Let $V_0 = \{v_0\}$. Now $K(G) = M_1 + M_2 + \{w_1, w_2\}$, where, for $i = 1, 2$, $H_i = M_i + \{w_1, w_2\}$ is the clique graph of the subgraph of G induced by $V_i \cup \{v_0\}$. By Theorem 7, $K(G)$ is perfect if and only if H_1 and H_2 are. So we may assume that V_0 is empty, and therefore w_1 is non-adjacent to w_2.

We start with the following easy observation

(*) Let S be a graph which is either a claw, or an odd hole, or $\overline{C_7}$, or a 0-,1-,2-, or 3-pyramid, and suppose there exists a vertex $s \in V(S)$, whose neighborhood is the union of two non-empty completes with no edges between them. Then S is and odd hole.

Since G admits no 0-join or 1-join, for $i = 1, 2$ there exist a_i in A_i and b_i in B_i joined by an induced path with interior in $V_i \setminus (A_i \cup B_i)$. (The *interior* of a path are the vertices different from the endpoints; the interior may be empty, if a_i and b_i are adjacent.)

Then, since G contains no odd hole, for every a_i in A_i and b_i in B_i, all induced paths from a_1 to b_1 with interior in $V_1 \setminus (A_1 \cup B_1)$ and all induced paths from a_2 to b_2 with interior in $V_2 \setminus (A_2 \cup B_2)$ have the same parity.

<u>Case 1</u>: This parity is even.

Note that in this case A_i is anticomplete to B_i. Let H be the graph obtained from $K(G)$ by adding the edge w_1w_2. Since A_i is anticomplete to B_i, there is no clique in G intersecting both $A_1 \cup A_2$ and $B_1 \cup B_2$. So w_1 and w_2 have no common neighbor in $K(G)$. By Theorem 9, if H is perfect then $K(G)$ is.

Construct graphs G_i with vertex set $V_i \cup \{v_i\}$, where $G_i|V_i = G|V_i$ and v_i is complete to $A_i \cup B_i$ and has no other neighbors in G_i. Now, $H = M_1 + M_2 + \{w_1, w_2\}$, with $M_i + \{w_1, w_2\} = K(G_i)$, and $\{w_1, w_2\}$ is a clique cutset in H. By Theorem 7, it follows that if $K(G_1)$ and $K(G_2)$ are perfect then H is perfect and thus $K(G)$ is perfect.

We claim that for $i = 1, 2$ the graphs G_i are claw-free, HCH and interesting. Suppose that G_1, say, is not. So G_1 contains an induced subgraph S isomorphic to a claw, an odd hole, $\overline{C_7}$, or a 0-,1-,2- or 3-pyramid. If $V(S)$ does not contain v_1, then S is isomorphic to an induced subgraph of G, a contradiction. If $V(S)$ contains v_1 but has empty intersection with A_1 or B_1, say B_1, then S is isomorphic to an induced subgraph of G, obtained by replacing v_1 by any vertex of A_2, a contradiction. So $V(S)$ meets both A_1 and B_1, and therefore the neighborhood of v_1 in S can be partitioned into two non-empty completes A_S, B_S, such that A_S is anticomplete to B_S. By (*), S is an odd hole. Let $a_1 \in A_1$ and $b_1 \in B_1$ be the neighbors of v_1 in S. Then $S \setminus \{v_1\}$ is an induced odd path from a_1 to b_1 with interior in $V_1 \setminus (A_1 \cup B_1)$, a contradiction.

<u>Case 2</u>: This parity is odd.

Let H be the graph obtained from $K(G)$ by adding a vertex w adjacent only to w_1 and w_2. Since $K(G)$ is an induced subgraph of H, if H is perfect, so is $K(G)$. Construct graphs G_i with vertex set $V_i + \{v_{A,i}, v_{B,i}\}$, where $G_i|V_i = G|V_i$, $v_{A,i}$ is complete to A_i, $v_{B,i}$ is complete to B_i, $v_{A,i}$ is adjacent to $v_{B,i}$, and there are no other edges in G_i. Now, $\{w_1, w_2, w\}$ is a star cutset in H, and $H = M_1 + M_2 + \{w_1, w_2, w\}$, with $M_i + \{w_1, w_2, w\} = K(G_i)$. By Theorem 8, it follows that if $K(G_1)$ and $K(G_2)$ are perfect then H is perfect and thus $K(G)$ is perfect.

We claim that both G_i are claw-free, interesting and HCH. Suppose that G_1 contains an induced subgraph S isomorphic to a claw, an odd hole, $\overline{C_7}$, or a 0-,1-,2-,or 3-pyramid.

If $V(S)$ does not contain $v_{A,1}$ or $v_{B,1}$, say $v_{B,1}$, then S is isomorphic to an induced subgraph of G, obtained by replacing $v_{A,1}$ by any vertex of A_2, a contradiction. If $V(S)$ contains $v_{A,1}$ and $v_{B,1}$ but has empty intersection with A_1 or B_1, say B_1, then S is isomorphic to an induced subgraph of G, obtained by replacing $v_{A,1}$ and $v_{B,1}$ by two adjacent vertices a_2, c_2 of V_2 such that $a_2 \in A_2$ and $c_2 \in V_2 \setminus A_2$ (such a pair of vertices exist because there is at least one path from A_2 to B_2 in G), a contradiction. So $V(S)$ meets both A_1 and B_1, and the neighborhood of $v_{A,1}$ in S can be partitioned into two non-empty completes with no edges between them, namely $A_S = A_1 \cap V(S)$ and $\{v_{B,1}\}$. By (*) S is an odd hole. Let $a_1 \in A_1$ and $b_1 \in B_1$ be the neighbors of $v_{A,1}$ and $v_{B,1}$ in $V(S) \cap V_1$, respectively. Then $S \setminus \{v_{A,1}, v_{B,1}\}$ is an induced even path from a_1 to b_1 with interior in $V_1 \setminus (A_1 \cup B_1)$, a contradiction. This concludes the proof of Theorem 29. $\square$

Lemma 30 *Let G be an HCH graph such that $\overline{G}$ is a bipartite graph. Then $K(G)$ is perfect.*

PROOF. In this proof we use the vertices of $K(G)$ and the cliques of G interchangeably. By Theorem 1, if $K(G)$ is not perfect then it contains an odd hole or an odd antihole.

Let A, B be two disjoint completes of G such that $A \cup B = V(G)$. If there exists a vertex v of G adjacent to every other vertex in G, then v belongs to every clique of G and $K(G)$ is a complete graph, and therefore perfect. So we may assume that no vertex of A is complete to B and no vertex of B is complete to A. Then A and B are cliques of G, and every other clique of G meets both A and B. The degrees of A and B in $K(G)$ is $|V(K(G))| - 1$, so they cannot be part of an odd hole or an odd antihole in $K(G)$.

It is therefore enough to show that there is no odd hole or antihole in the graph obtained from $K(G)$ by deleting the vertices A and B. We prove a stronger statement, namely that there is no induced path of length two in this graph. Since every hole and antihole of length at least five contains a two edge path, the result follows.

Suppose for a contradiction that there are three cliques X, Y and Z in G, each meeting both A and B, and such that X is disjoint from Z, and both $X \cap Y$ and $Y \cap Z$ are non-empty. From the symmetry we may assume that $X \cap Y$ contains a vertex $a_{xy} \in A$.

Suppose first that there is a vertex $a_{yz} \in A \cap Y \cap Z$. Let b_y be a vertex in $Y \cap B$. Since no vertex of B is complete to A, there is a vertex a in A non-adjacent to b_y. Since a_{yz} does not belong to X, there is a vertex b_x in X non-adjacent to a_{yz}, and since A is a complete, b_x belongs to B. Analogously, since a_{xy} does not belong to Z, there is a vertex b_z in $B \cap Z$ non-adjacent to a_{xy}. But now $\{a_{xy},\, a_{yz},\, b_y,\, b_z,\, b_x,\, a\}$ induce a 1-, 2- or 3-pyramid, a contradiction.

So $A \cap Y \cap Z$ is empty, and therefore $B \cap Y \cap Z$ is non-empty, and, by the argument of the previous paragraph with A and B exchanged, $B \cap X \cap Y$ is empty. Choose b_{yz} in $B \cap Y \cap Z$. Choose a_z in $Z \cap A$, then $a_z \notin X \cup Y$. Since a_z does not belong to X, there is a vertex $b_x \in X$ non-adjacent to a_z, and since A is a complete, b_x is in B. Since b_{yz} does not belong to X and B is a complete, there is a vertex $a_x \in A \cap X$ non-adjacent to b_{yz}; and since a_{xy} does not belong to Z and A is a complete, there is a vertex $b_z \in B \cap Z$ non-adjacent to a_{xy}. But now $\{a_z, a_{xy}, b_{yz}, a_x, b_x, b_z\}$ induces a 2- or a 3-pyramid, a contradiction. This proves Lemma 30. $\square$

Theorem 31 *Let G be a connected interesting HCH claw-free graph, and suppose G admit no twins. Assume that G admits a coherent or a non-dominating W-join (A, B). Then either $K(G)$ is perfect, or there exist induced subgraphs $G_1, \ldots, G_k$ of G, each smaller than G, such that if $K(G_i)$ is perfect for every $i = 1, \ldots, k$, then $K(G)$ is perfect.*

PROOF. Choose a coherent or non-dominating W-join (A, B) with $A \cup B$ minimal. Let C be the vertices complete to A and anticomplete to B, D be the vertices complete to B and anticomplete to A, E be the vertices complete to $A \cup B$, and F be the vertices anticomplete to $A \cup B$. Since the W-join (A, B) is either coherent or non-dominating, it follows that either E is a complete, or F is non-empty.

31.1 $A \cup C$, $B \cup D$ are both completes, and E is anticomplete to F.

Suppose not. Assume first that there exist two nonadjacent vertices c_1, c_2 in C. Choose a in A and b in B such that a is adjacent to b, now $\{a, c_1, c_2, b\}$ is a claw, a contradiction. So C is a complete, and since A is a complete, it follows that $A \cup C$ is a complete. From the symmetry it follows that $B \cup D$ is a complete.

Next assume that there are two adjacent vertices e in E and f in F. Choose a in A and b in B such that a is not adjacent to b. Then $\{e, a, b, f\}$ is a claw, a contradiction. This proves 31.1.

Let E_1 be a clique of $G|E$. Let $\mathcal{L}$ be the set of all cliques of $G|(A \cup B)$. Let

$$U = \{E_1 \cup L \;\;:\;\; L \in \mathcal{L} \;\; \textit{and} \;\; L \neq A, B\}.$$

Since E is anticomplete to F, and every member of U meets both A and B, it follows that the members of U are cliques of G.

31.2 We may assume that $|U| \geq 2$.

Suppose $|U| \leq 1$. Since in G there is at least one edge between A and B, it follows that there is a unique clique L in $G|(A \cup B)$ meeting both A and B, and $|U| = 1$. Let $A' = A \cap L$, $B' = B \cap L$. Then A' is complete to B', $A \setminus A'$ is anticomplete to B and $B \setminus B'$ is anticomplete to A. Since G does not admit twins, each of A', $A \setminus A'$, B', $B \setminus B'$ has size at most 1, and by the minimality of $A \cup B$ at most one of $A \setminus A'$, $B \setminus B'$ is non-empty. By the symmetry, we may assume that $B \setminus B'$ is empty and $|A'| = |B'| = |A \setminus A'| = 1$. Let $A' = \{a_1\}$, $B' = \{b_1\}$ and $A \setminus A' = \{a_2\}$.

If $K(G \setminus \{a_2\}) = K(G)$ then the theorem holds, so we may assume not. Therefore there exists a subset E' of E such that $M = A \cup E'$ is a clique of G. It follows, in particular, that no vertex of C is complete to E.

Assume first that E is a complete, consider the cliques $M_1 = \{a_1, b_1\} \cup E$ and $M_2 = \{a_1, a_2\} \cup E$ of G. Since every clique of G containing a_2 also contains a_1, it follows that every clique of G that has a non-empty intersection with M_2, meets M_1. Therefore the vertex w_1 of $K(G)$, corresponding to M_1, dominates the vertex w_2 of $K(G)$, corresponding to M_2. Since $K(G) \setminus \{w_1\}$ is an induced subgraph of $K(G \setminus \{a_1\})$ and $K(G) \setminus \{w_2\} = K(G \setminus \{a_2\})$, by Theorem 11, $K(G)$ is perfect if $K(G \setminus \{a_1\})$ and $K(G \setminus \{a_2\})$ are, and the theorem holds. So we may assume that E is not a complete.

Next we claim that D is empty. Since E is not a compelete, there are two non-adjacent vertices e_1, e_2 in E, and let d in D. If d is non-adjacent to both of e_1 and e_2, then $\{b_1, e_1, e_2, d\}$ is a claw, a contradiction. But then, $\{b_1, e_1, e_2, d, a_1, a_2\}$ induces a 1- or 2-pyramid, a contradiction. This proves that D is empty.

Since D is empty, every clique disjoint from F contains the vertex a_1, and, since every clique containing a vertex of F is disjoint from A, B and E, it follows that the vertices of $K(G)$ corresponding to the cliques $\{a_1, b_1\} \cup E'$, with E' a clique of $G|E$, are simplicial in $K(G)$. By Lemma 6, $K(G)$ is perfect if and only if $K(G \setminus \{b_1\})$ is. This proves 31.2.

31.3 We may assume that no vertex of B is complete to A, and no vertex of A is complete to B.

Suppose there is a vertex $b \in B$ complete to A. Since A is not complete to B, there is a vertex $b' \in B \setminus \{b\}$. By 31.2, $|A| > 1$. But now $(A, B \setminus \{b\})$ is a coherent or non-dominating W-join in G, contrary to the minimality of $A \cup B$. This proves 31.3.

In view of 31.2 and 31.3, we henceforth assume that $|U| \geq 2$, no vertex of A is complete to B, and no vertex of B is complete to A.

31.4 E is a complete.

Since no vertex of B is complete to A, and there is at least one edge between A and B, there is a vertex $a_1 \in A$ with a neighbor b_1 and a non-neighbor b_2 in B. Since b_1 is not complete to A, there is a vertex $a_2 \in A$, non-adjacent to b_1. Since A, B are both cliques, a_1 is adjacent to a_2 and b_1 to b_2. If there exist two non-adjacent vertices e_1 and e_2 in E, now $\{a_1, a_2, b_1, b_2, e_1, e_2\}$ induces a 2- or a 3-pyramid in G, a contradiction. This proves 31.4.

31.5 Every vertex of $K(G) \setminus U$ with a neighbor in U is complete to U.

Throughout the proof of 31.5 we use cliques of G and vertices of $K(G)$ interchangeably.

It follows from 31.4 that $E_1 = E$. Let w be a vertex of $K(G) \setminus U$ with a neighbor in U. Since w has a neighbor in U, it follows that w meets one of A, B, E. If w meets E, then w is complete to U and the result follows. If w includes one of A, B, then since every member of U meets each of A, B, we again deduce that w is complete to U and the result follows. So we may assume that w is disjoint from E, and the sets $w \cap (A \cup B)$, $A \setminus \{w\}$, and $B \setminus \{w\}$ are all non-empty.

Assume first that w meets both A and B. Since w is a clique of G, $C \cup F$ is anticomplete to B and $D \cup F$ is anticomplete to B, it follows that $w \subseteq A \cup B \cup E$. But now, since w is a clique, it follows that w includes E and w belongs to U, a contradiction. So we may assume that w is disjoint from at least one of A and B.

By the symmetry we may assume that w is disjoint from B, and therefore w meets A. Since $F \cup D$ is anticomplete to A, it follows that w is a subset of $A \cup C \cup E$, and since w is a clique, w includes A, a contradiction. This proves 31.5.

31.6 U is a homogeneous set in $K(G)$ and the graph $K(G)|U$ is perfect.

It follows from 31.5 that U is a homogeneous set in $K(G)$. The graph $K(G)|U$ is isomorphic to the graph obtained from $K(G|(A \cup B \cup E))$ by deleting the vertices corresponding to the cliques $A \cup E$ and $B \cup E$. Since $\overline{G|(A \cup B \cup E)}$ is bipartite, it follows from Theorem 30 that $K(G)|U$ is perfect. This proves 31.6.

Choose $u \in U$.

31.7 If there exist $a_1, a_2 \in A$ and $b_1, b_2 \in B$, such that a_1 is adjacent to b_1 and not to b_2, and a_2 is adjacent to b_2 and not to b_1, then either $K(G)$ is perfect, or there is an induced subgraph G' of G, such that $K(G) \setminus (U \setminus \{u\}) = K(G')$.

If there exist non-adjacent $c \in C$ and $e \in E$, then $\{a_1, a_2, e, c, b_1, b_2\}$ induces a 1-pyramid, a contradiction, so C is complete to E, and similarly D is complete to E. By 31.4, E is a complete. Since G admits no twins, $|E| \leq 1$. If $C \cup D$ is empty, then, since G is connected, F is empty, and G is the complement of a bipartite graph. By Lemma 30, $K(G)$ is perfect. So we may assume that C is non-empty, and in particular, $A \cup E$ is not a clique of G. But now $K(G) \setminus (U \setminus \{u\}) = K(G \setminus ((A \cup B) \setminus \{a_1, b_1, b_2\}))$. This proves 31.7.

To finish the proof, let $a_1 \in A$ and $b_1 \in B$ be adjacent. By 31.3, there exist a vertex $b_2 \in B$, non-adjacent to a_1 and a vertex $a_2 \in A$ non-adjacent to b_1. If a_2 is adjacent to b_2, then the theorem follows from 31.6, 31.7 and Theorem 10. So we may assume that a_2 is non-adjacent to b_2. Let $G' = G \setminus ((A \cup B) \setminus \{a_1, b_1, a_2, b_2\})$. We deduce from 31.2 that G' is smaller than G. Moreover, G' is an induced subgraph of G. But $K(G) \setminus (U \setminus \{u\}) = K(G')$, and, together with 31.6 and Theorem 10, this implies that the theorem holds. This proves Theorem 31. □

Theorem 32 *Let G be an interesting HCH claw-free graph. Suppose G admits a hex-join and no twins and every vertex of G is in a triad. Then $G = C_6$.*

PROOF. Since G admits a hex-join, there exist six completes A_1, A_2, A_3, B_1, B_2, B_3 in G such that A_i is anticomplete to B_i and complete to B_j for i different from j; $A_1 \cup A_2 \cup A_3$ and $B_1 \cup B_2 \cup B_3$ are non-empty; and $V(G) = A_1 \cup A_2 \cup A_3 \cup B_1 \cup B_2 \cup B_3$. Since every vertex of G is in a stable set of size three and no stable set of size three meets both $A_1 \cup A_2 \cup A_3$ and $B_1 \cup B_2 \cup B_3$, it follows that A_i, B_i are all non-empty.

Suppose there is an edge $a_1 a_2'$ with a_1 in A_1 and a_2' in A_2. Since every vertex is a stable set of size three, there exists a stable set $\{b_1, b_2, b_3\}$ with b_i in B_i and a stable set $\{a_1, a_2, a_3\}$ with a_i in A_i. Since G is interesting, $a_1 a_2' b_1 a_3 b_2 a_1$ is not a hole in G, so a_2' is adjacent to a_3. But now $\{a_2', a_1, a_2, a_3\}$ is a claw in G, a contradiction. So A_1 is anticomplete to A_2, A_3. Since the vertices of A_1 are not twins in G, it follows that $|A_1| = 1$. From the symmetry, $|B_i| = |A_i| = 1$ for all i, and $G = C_6$. This proves Theorem 32. □

Theorem 33 *Let G be an interesting HCH graph. Assume that G admits no twins and no coherent or non-dominating W-join, and contains no stable set of size three. Then $K(G)$ is perfect.*

PROOF. We may assume G contains either a 4-wheel or a 3-fan, otherwise, by Theorem 16, $K(G)$ is bipartite.

Case 1: G contains a 4-wheel. Let $a_1a_2a_3a_4a_1$ be a hole and let c be adjacent to all a_i. We claim every vertex in G is adjacent to c. Suppose v is non-adjacent to c. Then since G contains no stable set of size three, from the symmetry we may assume v is adjacent to a_1, a_2. But now $\{a_1, a_2, a_3, a_4, c, v\}$ induces a 1-,2-, or 3-pyramid, a contradiction. So every clique in G contains c, then $K(G)$ is a complete graph and the result follows. This proves Case 1.

Case2: G contains a 3-fan and no 4-wheel.

Let $A_1, .., A_k$ be anticonnected sets in G, pairwise complete to each other, with $k > 2$, $|A_1| > 1$, and subject that with maximal union, say A. (Such sets exist because there is a 3-fan. Let $a_1a_2a_3a_4$ be a path and let c be adjacent to all a_i. Then $A_1 = \{a_1, a_3\}$, $A_2 = \{a_2\}$, $A_3 = \{c\}$ make a family of sets with the desired properties.)

Suppose $|A_2| > 1$. Then, since A_1, A_2 are both anticonnected, each of A_1, A_2 contains a non-edge, say a_ib_i. Choose a_3 in A_3. Now $\{a_1, a_2, b_1, b_2, a_3\}$ is a 4-wheel, a contradiction. So for $2 \leq i \leq k$, $|A_i| = 1$, and let $A_i = \{a_i\}$.

(*) No vertex in $V(G) \setminus A$ is complete to more than one of $A_1, \dots, A_k$.

Let v be a vertex in $V(G) \setminus A$ and define $I = \{i : 1 \leq i \leq k$ and v is complete to $A_i\}$ and $J = \{j : 1 \leq j \leq k$ and v has a non-neighbor in $A_j\}$. Suppose $|I| > 1$. Define $A'_t = A_t$ for $t \in I$ and $A'_J = \bigcup_{j \in J} A_j \cup \{v\}$. Then $\{A'_i\}_{i \in I}, A'_J$ is a collection of at least three anticonnected sets, pairwise complete to each other, but their union is a proper superset of A, contrary to the maximality of A. This proves (*).

(**) There is no C_4 in A_1.

Otherwise, G contains a 4-wheel with center a_2, a contradiction. This proves (**).

Since $|A_1| > 1$ and A_1 is anticonnected, A_1 contains a non-edge, and so, since there is no stable set of size three in G, every vertex of $V(G) \setminus A$ has a neighbor in A_1. Let $A' = A \setminus A_1$. If no vertex of $V(G) \setminus A$ has a neighbor in A', then the vertices of A' are twins, a contradiction.

So there exists v in $V(G) \setminus A$ with a neighbor in A_1 and a neighbor a' in A'. By (*) v has a non-neighbor a'' in A'. If v has two non-adjacent neighbors in A_1, say x, y then $xvya''x$ is a 4-hole and a' is complete to it, so G contains a 4-wheel, a contradiction. So the neighbors of v in A_1 are a complete. Since G has no stable set of size three, the non-neighbors of v in A_1 are a complete. Thus

$G|A_1$ is complement bipartite, and since it is anticonnected the bipartition is unique, say X, Y, both X and Y are non-empty, and every vertex of $V(G) \setminus A$ with a neighbor in A' is either complete to X and anticomplete to Y, or complete to Y and anticomplete to X. Let X' be the vertices with a neighbor in A' and complete to X, Y' be the vertices with a neighbor in A' and complete to Y. Then, $X' \cup Y'$ is non-empty, and since there is no stable set of size three in G, X', Y' are both completes.

For $i = 2, \ldots, k$ let X_i be the vertices of X' adjacent to a_i, and let Y_i be defined similarly. By (*), $A_i \cap A_j = \emptyset$ for $i \neq j$, and the same holds for B_i, B_j. If there is an edge from X to Y then there is no edge from X_i to Y_i, or else G contains a 4-wheel with center a_i. Let Z be the vertices of G with no neighbor in A'. Then, since G contains no triad, Z is a complete.

33.1 Every vertex in Z is complete to $X' \cup Y'$ and to one of X, Y.

If some vertex z in Z has a non-neighbor x_2 in X_2, then z, x_2, a_3 is a stable set of size three, a contradiction, so Z is complete to X', and similarly Y'. Next suppose some vertex z in Z has a non-neighbor x in X and a non-neighbor y in Y. Then x is adjacent to y, and there is an odd antipath Q from x to y in $X \cup Y$. Thus $xQyzx$ is an antihole, so Q has length 1 mod 3. But then Q has length at least 4, and so $X \cup Y$ contains a C_4, contrary to (**). This proves 33.1.

Let Z_x be the vertices of Z complete to X, and let $Z_y = Z \setminus Z_x$.

33.2 $k \leq 4$ and $X' = X_i$, $Y' = Y_j$ for some i different from j.

Suppose both X_2, X_3 are non-empty, choose x_2 in X_2 and x_3 in X_3. Then $a_2x_2x_3a_3a_2$ is a hole of length four, and every x in X is complete to it, so G contains a 4-wheel, a contradiction. So we may assume that $X' = X_2$ and, similarly, $Y' = Y_j$ for some j. If Y_2 is non-empty, then since x_2, y_2, a_3 is not a stable set of size three, x_2 is adjacent to y_2. Since A_1 is anticonnected, there exist non-adjacent vertices $x \in X$ and $y \in Y$. But now $xx_2y_2ya_3x$ is a hole of length five, a contradiction. So Y_2 is empty and therefore i is different from j, say $j = 3$. Since a_4, a_5 are not twins, $k \leq 4$. This proves 33.2.

By 33.2 we may assume that $X' = X_2$, $Y' = Y_3$. Let M_1 be the vertices in X with a neighbor in Z_y, $M_2 = X \setminus M_1$. Let N_1 be the vertices in Y with a neighbor in Z_x, $N_2 = X \setminus N_1$.

33.3 If Z, X', Y' are all non-empty then the theorem holds.

We may assume Z_x is non-empty. Since $a_2x_2zy_3a_3a_2$ (where $z \in Z$, $x_2 \in X_2$ and $y_3 \in Y_3$) is not a hole of length five, X_2 is complete to Y_3. Suppose z in Z_x has a neighbor y in Y. Since A_1 is anticonnected, y has a non-neighbor

x in X. But now $a_3za_2y_3xyx_2a_3$ (with x_2 in X_2 and y_3 in Y_3) is an antihole of length seven, a contradiction. So Z_x is anticomplete to Y. Choose z in Z_x and non-adjacent x in X and y in Y. Then zxa_2yy_3z is a hole of length five, a contradiction. This proves 33.3.

33.4 If Z is empty then the theorem holds.

The pairs (X, Y) and (X_2, Y_3) are coherent homogeneous pairs, and since G does not admit twins or a coherent W-join, all four of these sets have size ≤ 1. Every vertex of G is adjacent to a_3, except the vertex x_2 of X_2, if X' is non-empty. So every clique of G contains either a_3 or x_2, and therefore $K(G)$ is perfect (it is either a complete graph, or the complement of a bipartite graph). This proves 33.4.

In view of 33.4, we henceforth assume that $Z \neq \emptyset$. By 33.3 we may assume X' is empty, and so Y' is non-empty. By 33.2 we may assume $Y' = Y_3$. Since the vertices of Y_3 are not twins, $Y_3 = \{y_3\}$.

33.5 Z is complete to Y.

Suppose not. Choose z in Z, with a non-neighbor y is in Y. Then z in Z_x. Since A_1 is anticonnected, y has a non-neighbor x in X. But now zxa_2yy_3z is a hole of length five, a contradiction. This proves 33.5.

Let M be the set of vertices in X with a neighbor in Z. Suppose some z in Z has adjacent neighbors x in X and y in Y. Then xya_3 is a triangle, z is adjacent to x, y and not to a_3; y_3 is adjacent to a_3, y and not to x. Choose a non-neighbor x' of y in X. Then x' is adjacent to a_3, x. But now the graph induced by $\{x, x', y, y_3, a_3, z\}$ is a 1- or 2-pyramid, a contradiction. This proves that M is anticomplete to Y. Now (Z, M) is a coherent homogeneous pair, and the same for $(X \setminus M, Y)$. Since G admits no twins and no coherent W-join, all four of these sets have size ≤ 1. Also, since a_2 and a_4 are not twins, $k = 3$. Let $Z = \{z\}$. Every vertex of G different from z is adjacent to a_3. So every clique of G contains either a_3 or z, and then $K(G)$ is perfect (it is the complement of a bipartite graph). This completes the proof of Theorem 33. $\square$

Theorem 34 *Let G be an interesting HCH claw-free graph, and suppose that G is connected, does not admit a coherent or non-dominating W-join, a 1-join or twins. If G contains a stable set of size three and a singular vertex, then $K(G)$ is perfect.*

PROOF. The proof is by induction on $|V(G)|$. Assume that for every smaller graph G' satisfying the hypotheses of the theorem, $K(G')$ is perfect. Let v be a singular vertex in G with maximum number of neighbors. Let A be the set of neighbors of v and B be the set of its non-neighbors. Since v is singular, B is a complete.

Since G contains a stable set of size three, and every such set meets both A and B (because B is a clique, and G is claw-free), there exist vertices in B that are non singular. Let U be the set of all such vertices.

34.1 If U is anticomplete to A then $K(G)$ is perfect.

Let $V = B \setminus U$, so every vertex of V is singular, and since G is connected, V is non-empty. Let a_1, a_2 be two non-adjacent vertices in A. If $b \in V$ is non-adjacent to both a_1, a_2, then $\{b, a_1, a_2\}$ is a stable set of size three, and if b is adjacent to both a_1, a_2 then $\{b, a_1, a_2, u\}$ is a claw for every $u \in U$; in both cases we get a contradiction. So every vertex in V is adjacent to exactly one of a_1, a_2. Suppose there exist v_1, v_2 in V with v_i adjacent to a_i. Then $v_1 v_2 a_2 v a_1 v_1$ is a hole of length five, a contradiction. So one of a_1, a_2 is anticomplete to V, and therefore the other one is complete to V. Let A_1 be the vertices in A complete to V, A_2 be the vertices in A anticomplete to V and $A_3 = A \setminus (A_1 \cup A_2)$. It follows from the previous argument that $A_1 \cup A_3$ and $A_2 \cup A_3$ are both completes. If A_3 is non-empty, then $|V| > 1$ and (A_3, V) is a coherent W-join, a contradiction. So we may assume A_3 is empty. Now (A_1, A_2) is a coherent homogeneous pair, and all the vertices of each of U, V are twins. So all these sets have size at most 1 and $K(G)$ is the clique graph of an induced subgraph of a 4-edge path, and hence perfect. This proves 34.1.

So we may assume that there exists a non-singular vertex u in B with a neighbor in A. Let M be the set of neighbors of u in A, N the set of non-neighbors. Since u is non-singular, N contains two non-adjacent vertices x, y. Choose m in M. If m is adjacent to both x, y then $\{m, x, y, u\}$ is a claw. If m is non-adjacent to both x, y then $\{v, x, y, m\}$ is a claw. So every vertex in M is adjacent to exactly one of x, y. So there is no complement of an odd cycle in $G|N$, and therefore the complement of $G|N$ is bipartite and N is the union of two completes.

Let M_1 be the vertices in M adjacent to x, M_2 those adjacent to y, then $M_1 \cup M_2 = M$ and $M_1 \cap M_2 = \emptyset$.

If there exists m_1 in M_1 and m_2 in M_2 such that m_1 is adjacent to m_2, then the graph induced by $\{m_1, m_2, v, x, y, u\}$ is 3-sun, a contradiction. So there are no edges between M_1 and M_2, M_1 is anticomplete to y and M_2 is anticomplete to x. Since $\{v, m, m', y\}$ is not a claw for m, m' in M_1, it follows that M_1 is a complete, and the same holds for M_2.

Case 1: M_1 and M_2 are both non-empty.

Since A contains no stable set of size three (for otherwise there would be a claw in G), every vertex in N is complete to one of M_1, M_2. Let N_3 be the vertices complete to $M_1 \cup M_2$, N_1 the vertices of $N \setminus N_3$ complete to M_1 and N_2 vertices of $N \setminus N_3$ complete to M_2. So $x \in N_1$ and $y \in N_2$. Since $\{m, n, n', u\}$ is not a claw for m in M_1 and n, n' in $N_1 \cup N_3$, it follows that $N_1 \cup N_3$ is a complete. Similarly $N_2 \cup N_3$ is a complete. Suppose N_3 is non-empty, and choose $n \in N_3$. Then n is complete to $(A \cup \{v\}) \setminus \{n\}$, and therefore is singular (for its non-neighbors are a subset of B); and by the choice of v, n and v are twins. Since G admits no twins, it follows that N_3 is empty. Suppose some n_1 in N_1 is adjacent to n_2 in N_2. Choose m_1' in M_1 non-adjacent to n_2 and m_2' in M_2 non-adjacent to n_1. Then $m_1'n_1n_2m_2'um_1'$ is a hole of length five, a contradiction. So N_1 is anticomplete to N_2. Suppose n_1 in N_1 has a neighbor m_2' in M_2. Then $\{m_2', n_1, y, u\}$ is a claw, a contradiction. So N_1 is anticomplete to M_2, and, similarly, N_2 is anticomplete to M_1.

For $i = 1, 2$ choose m_i' in M_i, and assume that m_i' has a non-neighbor b_i in B. If m_1' and m_2' have a common non-neighbor $b \in B$, then $\{u, m_1', m_2', b\}$ is a claw, a contradiction. So there are two vertices b_1 and b_2 in B such that b_1 is non-adjacent to m_1' and adjacent to m_2', and b_2 is non-adjacent to m_2' and adjacent to m_1'. But then $m_1'b_2b_1m_2'vm_1'$ is a hole of length five, again a contradiction. So, exchanging M_1 and M_2 if necessary, we may assume that M_1 is complete to B, and since G admits no twins, $|M_1| = 1$, say $M_1 = \{m_1\}$.

Let b be a vertex of B with a neighbor in N_1. We claim that b is complete to M_2 and anticomplete to N_2. For if b has a non-neighbor m_2 in M_2, then $n_1bum_2vn_1$ is a hole of length five; and if b has a neighbor n_2 in N_2, then $\{b, n_1, n_2, u\}$ is a claw; in both cases a contradiction. This proves the claim.

So every vertex of B is either anticomplete to N_1, or complete to M_2 and anticomplete to N_2. Let B_1 be the set of vertices of B with a neighbor in N_1. Then (B_1, N_1) is a non-dominating homogeneous pair, and since G does not admit a non-dominating W-join or twins, it follows that $|B_1| \leq 1$ and $|N_1| = 1$, say $N_1 = \{n_1\}$.

Assume that B_1 is non-empty, let $B_1 = \{b_1\}$. Let $B_2 = B \setminus B_1$. We claim that in this case B_2 is complete to M_2. If b_2 in B_2 has a non-neighbor m_2 in M_2, then $b_2 \neq b_1$ and $\{b_1, n_1, m_2, b_2\}$ is a claw, a contradiction. This proves the claim. But now the vertices of M_2 are all twins, and since G does not admit twins, $|M_2| = 1$. Moreover, (B_2, N_2) is a non-dominating homogeneous pair, and since G does not admit a non-dominating W-join or twins, it follows that $|B_2| = |N_2| = 1$, so $B_2 = \{u\}$ and $N_2 = \{n_2\}$. But now every clique of G contains either v or b_1, and hence $K(G)$ is the complement of a bipartite graph, and therefore perfect. This finishes the case when B_1 is non-empty.

If B_1 is empty, $(B, M_2 \cup N_2)$ is a non-dominating homogeneous pair, and since G does not admit a non-dominating W-join or twins, it follows that $|B| = |M_2 \cup N_2| = 1$, a contradiction because both M_2 and N_2 are non-empty. This finishes the case when both M_1 and M_2 are non-empty.

Case 2: One of M_1, M_2 is empty.

We may assume that M_2 is empty, and so M is complete to x and anticomplete to y. Let N_1 be the set of vertices in N complete to M, N_2 the set of vertices in N that are anticomplete to M and let $N_3 = N \setminus (N_1 \cup N_2)$.

We claim that $N_1 \cup N_3$ and $N_2 \cup N_3$ are both completes. Choose two different vertices n_3 in $N_3 \cup N_1$ and n_1 in N_1, and let m be a neighbor of n_3 in M. Since $\{m, u, n_1, n_3\}$ is not a claw, n_1 is adjacent to n_3; and therefore N_1 is a complete and N_1 is complete to N_3. Next, choose two different vertices n_3 in $N_3 \cup N_2$ and n_2 in N_2, and let m be a non-neighbor of n_3 in M. Since $\{v, m, n_2, n_3\}$ is not a claw, n_2 is adjacent to n_3; and therefore N_2 is a complete and N_2 is complete to N_3. Finally, suppose there exist two non-adjacent vertices n_3 and n_3' in N_3. Since $\{m, u, n_3, n_3'\}$ is not a claw for any $m \in M$, it follows that no vertex of M is adjacent to both n_3 and n_3'. Let m be a neighbor of n_3 in M and m' be a neighbor of n_3' in M. Then m is non-adjacent to n_3' and m' is non-adjacent to n_3, and the graph induced by $\{v, m, m', u, n_3, n_3'\}$ is a 3-sun, a contradiction. So N_3 is a complete. This proves the claim. Since there exist two non-adjacent vertices in N, both N_1 and N_2 are non-empty.

34.2 Let b in B adjacent to n_3 in N_3 and to m in M. Then n_3 is non-adjacent to m.

Suppose they are adjacent. Let m' be a non-neighbor of n_3 in M, and let n_2 be in N_2. Then n_3mv is a triangle, b is adjacent to n_3, m; n_2 is adjacent to v and n_3; m' is adjacent to v and m, and this is a 0-, 1- or 2-pyramid, a contradiction. This proves 34.2.

34.3 Every vertex in N_1 has a non-neighbor in N_2.

Suppose some vertex n_1 of N_1 is complete to N_2. Then the set of non-neighbors of n_1 is included in B, and therefore n_1 is singular; and it is complete to $A \setminus \{n_1\}$. From the choice of v, n_1 has no neighbor in B, but now n_1 and v are twins, a contradiction. This proves 34.3.

34.4 M is complete to B.

Let B_1 be the set of vertices in B that are complete to M. Suppose there exists b_2 in $B \setminus B_1$, and let m be a non-neighbor of b_2 in M.

34.4.1 $|N_2| = 1$, N_2 is anticomplete to B, and consequently all stable sets of size three using u share a vertex in A.

Let n be in N_2. Since nb_2umvn is not a hole of length five, it follows that n is non-adjacent to b_2, and the same holds for every vertex of $B \setminus B_1$. So n is anticomplete to $B \setminus B_1$. Since $\{b_1, b_2, m, n\}$ is not a claw for $b_1 \in B_1$, it follows that n is anticomplete to B_1, and the same holds for every vertex of N_2. Therefore N_2 is anticomplete to B. But now $\{v\} \cup N_1 \cup N_3$ is a clique cutset separating N_2 from $M \cup B$. By Theorem 12, G is either a linear interval graph or G is the 3-sun, or G admits twins, or a 0-join, or a 1-join, or a coherent W-join, or it is not an internal clique cutset; and it follows from the hypotheses of the theorem and from Theorem 27, that we may assume that the last alternative holds, and $|N_2| = 1$, say $N_2 = \{n_2\}$. Now, since M, B and $N_1 \cup N_3$ are all completes, it follows that n_2 belongs to every stable set of size three using u. This proves 34.4.1.

34.4.2 N_1 is anticomplete to n_2.

Follows from 34.3.

34.4.3 We may assume that every vertex of B has a neighbor in A.

Suppose not. Let b be a vertex of B anticomplete to A.

We claim that in this case $K(G)$ is perfect if and only if $K(G \setminus \{b\})$ is. Since every vertex of $G \setminus B$ has a non-neighbor in B, B is a clique of G. b is a simplicial vertex and B is the only clique containing b. Let v_B be the vertex of $K(G)$ corresponding to B. There are two possibilities: either $B \setminus \{b\}$ is a clique of $G \setminus \{b\}$, and then $K(G \setminus \{b\}) = K(G)$, or there is a vertex m_B in A complete to $B \setminus \{b\}$ in G, and then $K(G \setminus \{b\}) = K(G) \setminus \{v_B\}$. The vertex m_B belongs to M because, in particular, it is adjacent to u. We claim that every clique of G different from B and having non-empty intersection with B contains the vertex m_B. Otherwise, there is a clique of G containing a vertex of B, say b_3, and a vertex a of A non-adjacent to m_B. But now $\{b_3, b, m_B, a\}$ is a claw, a contradiction. Thus v_B is simplicial in $K(G)$, and Lemma 6 completes the proof of the claim. But now, since $K(G \setminus \{b\})$ is perfect, so is $K(G)$. This proves 34.4.3.

We henceforth assume that every vertex of B has a neighbor in A.

34.4.4 Let $b \in B$ be a vertex non-adjacent to some $n_3 \in N_3$; and let m be in M. Then n_3 is adjacent to m.

Suppose not. Then b is in a stable set of size three $\{b, n_3, m\}$ and b has a neighbor in A; and by 34.4.1 applied to b instead of u, $\{b, n_2\} \cup N_1$ does not contain a stable set of size three. So b is complete to N_1. But now $\{n_1, b, m, n_3\}$

is a claw for every $n_1 \in N_1$, a contradiction. This proves 34.4.4.

34.4.5 B is anticomplete to N_3.

Suppose a vertex $b \in B$ has a neighbor $n \in N_3$. By the definition of N_3, n has a neighbor m in M. By 34.2, m is non-adjacent to b. By 34.4.4 n is adjacent to m. But now $\{n, n_2, b, m\}$ is a claw, a contradiction. This proves 34.4.5.

Now $M \cup N_1$ is a clique cutset separating $\{v\} \cup N_2 \cup N_3$ from B. Since $|B| > 1$ and $|\{v\} \cup N_2 \cup N_3| > 1$, it follows from Theorem 12, that G is a linear interval graph, and therefore $K(G)$ is perfect by Theorem 27. This completes the proof of 34.4.

By 34.4, for every non-singular vertex in B, the set of its neighbors in A is complete to B.

34.5 B is anticomplete to N_3.

Suppose some vertex b in B has a neighbor n_3 is N_3. By the definition of N_3, n_3 has a neighbor in M, and this contradicts 34.2. This proves 34.5.

34.6 N_3 is empty and $|M| = 1$.

If N_3 is non-empty then $|M| > 1$ and (N_3, M) is a coherent homogeneous pair. So N_3 is empty, but now the vertices of M are twins, so $|M| = 1$. This proves 34.6.

It follows from 34.6 that every singular vertex in B has at most one neighbor in A, and since M is complete to B and has size 1, every singular vertex in B is complete to M and anticomplete to $A \setminus M$. Therefore the vertices of U are all twins, and since G admits no twins, $U = \{u\}$. Let $B_2 = B \setminus U$.

34.7 B_2 is non-empty.

Otherwise (N_1, N_2) is a coherent homogeneous pair, so each of them has size 1 and $K(G)$ is a three-edge path. This proves 34.7.

34.8 If n_1 in N_1 is non-adjacent to n_2 in N_2, then every b in B_2 is adjacent to exactly one of n_1, n_2.

Let b_2 in B_2. Since b_2 in B_2 is singular, b_2 is adjacent to at least one of n_1, n_2. Since $\{b_2, n_1, n_2, u\}$ is not a claw, b_2 is non-adjacent to at least one of n_1, n_2. This proves 34.8.

34.9 No vertex of N_1 has a neighbor and a non-neighbor in B_2.

Suppose n_1 in N_1 has a neighbor b_1 in B_2 and a non-neighbor b_2 in B_2. By 34.3 n_1 has a non-neighbor n_2 in N_2. By 34.8 n_2 is adjacent to b_2 and not to b_1. But now $b_1n_1vn_2b_2b_1$ is a hole of length five, a contradiction. This proves 34.9.

Let N_{11} be the vertices of N_1 complete to B_2, $N_{12} = N_1 \setminus N_{11}$. So N_{12} is anticomplete to B. It follows from 34.8 every vertex of N_2 is either complete to N_{11} or to N_{12}. Let N_{22} be the set of vertices in N_2 with a non-neighbor in N_{11}. Then N_{22} is complete to N_{12}. Let N_{21} be the vertices in N_2 with a non-neighbor in N_{12}. Then N_{21} is complete to N_{11}. Let $N_{23} = N_2 \setminus (N_{21} \cup N_{22})$. So N_{23} is complete to N_1. By 34.8 B_2 is anticomplete to N_{22} and complete to N_{21}. Now (B_2, N_{23}) is a coherent homogeneous pair, and all the vertices of $N_{11}, N_{12}, N_{22}, N_{21}$ are twins, so all these sets have size at most 1.

Now, every clique of G contains either v or b_2, so $K(G)$ is the complement of a bipartite graph, and hence it is perfect. This completes the proof of Theorem 34. □

3.2.3 Basic classes

Finally we show that if an interesting HCH claw-free graph belongs to one of the basic classes of Theorem 5, then its clique graph is perfect.

Theorem 35 *If G is interesting HCH, antiprismatic and every vertex of G is in a triad, then $K(G)$ is perfect.*

PROOF. We prove that G contains no 4-wheel or 3-fan, and then, by Theorem 16, $K(G)$ is bipartite.

Suppose G contains a 4-wheel. Let $a_1a_2a_3a_4a_1$ be a hole and let c be adjacent to all a_i. Since every vertex is in a triad, there are two vertices c_1, c_2 different from a_1, a_2, a_3, a_4 such that $\{c, c_1, c_2\}$ is a stable set. Since G is antiprismatic, every other vertex in G is adjacent exactly to two of $\{c, c_1, c_2\}$. In particular, each a_i is adjacent either to c_1 or to c_2. If two consecutive vertices of the hole, for instance a_1, a_2, are adjacent to the same c_j, then $\{a_1,\ a_3,\ a_2,\ a_4,\ c,\ c_j\}$ induces a 1-,2- or 3-pyramid, a contradiction because G is HCH. So, without loss of generality, we may assume that a_1 and a_3 are adjacent to c_1 and not to c_2, while a_2 and a_4 are adjacent to c_2, and not to c_1. But then $\{a_1, a_3, c_2\}$ is a claw, a contradiction. This proves that G does not contain a 4-wheel.

Suppose now that G contains a 3-fan. Let $a_1a_2a_3a_4$ be an induced path and let c be adjacent to all a_i. Since every vertex is in a triad, there are two vertices c_1, c_2 different from a_1, a_2, a_3, a_4 such that $\{c, c_1, c_2\}$ is a stable set. Since G is antiprismatic, each a_i is adjacent either to c_1 or to c_2. If a_2 and

a_3, are adjacent to the same c_j, then $\{a_1,\ a_3,\ a_2,\ a_4,\ c,\ c_j\}$ induces a 0-,1- or 2-pyramid, a contradiction because G is HCH. So, without loss of generality, we may assume that a_2 is adjacent to c_1 and not c_2, while a_3 is adjacent to c_2 and not c_1. Since $\{a_3, a_2, c_2, a_4\}$ is not a claw, a_4 is adjacent to c_2, and, analogously, a_1 is adjacent to c_1. By the same argument applied to the 3-fan induced by the path $a_2ca_4c_2$ and the vertex a_3, there is a vertex d adjacent to a_4 and c_2 but not adjacent to a_2, c or a_3, and so $d \notin \{a_1, a_2, a_3, a_4, c, c_1, c_2\}$ (see Figure 15).

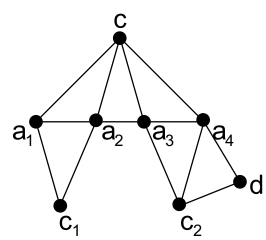

Fig. 15. Situation for the second part of the proof of Theorem 35.

Since $c_1a_2a_2a_4dc_1$ is not a hole of length five, d is non-adjacent to c_1. Thus c_1, c and d form a triad, but the vertex c_2 is adjacent only to one of them, a contradiction because G is antiprismatic. This concludes the proof of Theorem 35. □

Theorem 36 *Let $G \in \mathcal{S}_6$ be a connected interesting HCH graph such that every vertex of G is in a triad. Then $K(G)$ is perfect.*

PROOF. Let A, B and C be the sets of vertices of the graph H_5 in the definition of the class $\mathcal{S}_6$, and let A_G, B_G and C_G be those sets intersected with $V(G)$. We remind the reader that $a_0 \in A_G$ and $b_0 \in B_G$ by the definition of $\mathcal{S}_6$. Every triad in G is of the form $\{a_i, b_j, c_k\}$, since A_G, B_G and C_G are complete sets. Moreover, either $i = j = 0$ or $k = i$ and $j = 0$ or $k = j$ and $i = 0$. Since every vertex of G is in a triad, it follows that A_G, B_G and C_G are non-empty and if $i \neq 0$ and $a_i \in A_G$, then $c_i \in C_G$. Analogously, if $i \neq 0$ and $b_i \in A_G$, then $c_i \in C_G$. Let $I_A = \{i > 0 : a_i \in A_G\}$, $I_B = \{i > 0 : b_i \in B_G\}$ and $I_C = \{i > 0 : c_i \in C_G\}$. Then $I_A \cup I_B \subseteq I_C$.

Assume first that $I_C \setminus (I_A \cup I_B)$ is non-empty. Since the set $C' = \{c_i\ :\ i \in C \setminus (I_A \cup I_B)\}$ is complete to $V(G) \setminus (C' \cup \{a_0, b_0\})$, and the only cliques containing a_0 or b_0 are A_G and B_G, respectively, it follows that every pair of cliques of G, except for the pair A_G, B_G, has non-empty intersection. Thus $V(K(G))$ is the union of a stable set and a complete. On the other hand, if A is an odd hole or antihole, there is no partition of the vertex set of A into a complete and a stable set. Therefore $K(G)$ contains no odd hole or antihole, and hence $K(G)$ is perfect by Theorem 1.

So we may assume that $I_A \cup I_B = I_C$. If $|I_A \cup I_B| \geq 3$, we may assume by switching A and B if necessary that $1, 2 \in I_A$, and then the graph induced by $\{a_1, a_2, c_1, c_2, c_3, a_0\}$ is a 1-pyramid, a contradiction because G is HCH. On the other hand, since G is connected, both I_A and I_B are non-empty and $|I_A \cup I_B| \geq 2$. So, without loss of generality, we consider three cases: $I_A = I_B = \{1, 2\}$; $I_A = \{1, 2\}$ and $I_B = \{2\}$; $I_A = \{1\}$ and $I_B = \{2\}$. Graphs obtained in each case are depicted in Figure 16, with their corresponding clique graphs, which are all perfect. That concludes this proof. □

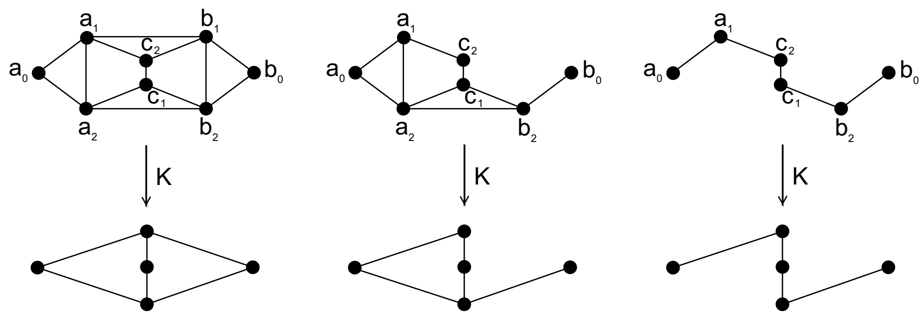

Fig. 16. Last three cases for the proof of Theorem 36.

3.3 Proof of Theorem 18

Proof of Theorem 24. Let G be an interesting HCH claw-free graph. The proof is by induction on $|V(G)|$, using the decomposition of Theorem 5. Assume that for every smaller interesting HCH claw-free G', $K(G')$ is perfect. We show that $K(G)$ is perfect.

If G admits twins, then $K(G)$ is perfect by Lemma 14, and if G is not connected, then $K(G)$ is perfect by Lemma 15. If G is connected, admits a 1-join and no twins, then $K(G)$ is perfect by Theorem 28 and Lemma 7. If G admits no twins, 0- or 1-joins, but admits a 2-join, then $K(G)$ is perfect by Theorem 29. If G admits a coherent or non-dominating W-join and no twins, then $K(G)$ is perfect by Theorem 31. If G contains a singular vertex, then $K(G)$ is perfect by Theorems 33 and 34. So we may assume not. If G admits a hex-join and no twins, then by Theorem 32 $G = K(G) = C_6$, and therefore $K(G)$ is perfect.

So we may assume that G admits none of the decompositions of the previous paragraph, and by Theorem 5, G is antiprismatic, or belongs to $\mathcal{S}_0 \cup \cdots \cup \mathcal{S}_6$.

If $G \in \mathcal{S}_0$, then $K(G)$ is perfect by Theorem 22. The graphs $icosa(-2)$, $icosa(-1)$ and $icosa(0)$ contain holes of length five, and therefore are not interesting, so $G \notin \mathcal{S}_1$. $G \notin \mathcal{S}_2$, because vertices v_3, v_4, v_5, v_6, v_9 induce a hole of length five in H_1 (Figure 6). If $G \in \mathcal{S}_3$, then by Proposition 27, $K(G)$ is perfect. If $G \in \mathcal{S}_4$ then, since G does not contain a singular vertex, G is a line graph and $K(G)$ is perfect by Theorem 22. $G \notin \mathcal{S}_5$, because the vertex d_1 in the definition of the class $\mathcal{S}_5$ is singular. If $G \in \mathcal{S}_6$, then $K(G)$ is perfect by Theorem 36, and finally, if G is antiprismatic, then $K(G)$ is perfect by Theorem 35. This completes the proof of Theorem 24. □

Theorem 18 is an immediate corollary of the following:

Theorem 37 *Let G be claw-free and assume that G is HCH. Then the following are equivalent:*

(i) *no induced subgraph of G is an odd hole, or $\overline{C_7}$.*
(ii) *G is clique-perfect.*
(iii) *G is perfect.*

PROOF. The equivalence between (i) and (iii) is a corollary of Theorem 1, because by Proposition 25 HCH graphs contain no antiholes of length at least eight. From Theorem 3 it follows that (ii) implies (i). Finally, by Theorem 24 and Propositions 20 and 25, we deduce that (i) implies (ii), and this completes the proof. □

The recognition of clique-perfect HCH claw-free graphs can be reduced to the recognition of perfect graphs, which is solvable in polynomial time [9].

3.4 Summary

These results allow us to formulate partial characterizations of clique-perfect graphs by forbidden subgraphs, as is shown in Table 1.

Graph classes	**Forbidden subgraphs**	**Reference**
HCH claw-free graphs	odd holes $\overline{C_7}$	Thm 18
Line graphs	odd holes 3-sun	Thm 17

Table 1
Forbidden induced subgraphs for clique-perfect graphs in each studied class.

Note that in both cases all the forbidden induced subgraphs are minimal.

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