CLIQUES IN THE UNION OF GRAPHS

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Abstract. Let B and R be two simple graphs with vertex set V, and let $G(B, R)$ be the simple graph with vertex set V, in which two vertices are adjacent if they are adjacent in at least one of B and R. For $X \subseteq V$, we denote by $B|X$ the subgraph of B induced by X; let $R|X$ and $G(B, R)|X$ be defined similarly. A *clique* in a graph is a set of pairwise adjacent vertices. A subset $U \subseteq V$ is *obedient* if U is the union of a clique of B and a clique of R. Our first result is that if B has no induced cycles of length four, and R has no induced cycles of length four or five, then every clique of $G(B, R)$ is obedient. This strengthens a previous result of the second author, stating the same when B has no induced C_4 and R is chordal.

The *clique number* of a graph is the size of its maximum clique. We say that the pair (B, R) is *additive* if for every $X \subseteq V$, the sum of the clique numbers of $B|X$ and $R|X$ is at least the clique number of $G(B, R)|X$. Our second result is a sufficient condition for additivity of pairs of graphs.

1. Introduction

Throughout this paper B (for "Blue") and R (for "Red") are two graphs (identified with their edge sets) on the same vertex set V. If in this setting $B \cup R$ is a clique, then it is not necessarily the case that V is the union of a clique in B and a clique in R. It is not even always true that $\omega(B) + \omega(R) \geq |V|$ (where, as usual, $\omega(G)$ is the maximal size of a clique in G). For example, if B is the random graph and R is its complement, then $\omega(B) = \omega(R) = O(\log |V|)$. If B is the disjoint union of k cliques of size k each, and R is its complement, then $|V| = k^2$ while $\omega(B) = \omega(R) = k$, and in this case B does not contain an induced C_4 and R does not contain induced cycles of length larger than 4 (but it does contain induced C_4s). In this paper we study sufficient conditions on B and R for the above two properties to hold. Namely, we shall find conditions implying that a clique in $G(B, R)$ is the vertex union of a clique in B and a clique in R, and conditions implying that the sum of the clique numbers of B and of R is the clique number of their union.

For B and R as above, we denote the union of B and R by $G(B, R)$, so two vertices in V are adjacent in $G(B, R)$ if they are adjacent in at least one of B and R.

Definition 1.1. *We say that a subset $U \subseteq V$ is* obedient *if there exists an R-clique X and a B-clique Y such that $U = X \cup Y$. We say that it is* size obedient *if $|U| \leq \omega(B|U) + \omega(R|U)$.*

In the above definition of obedience we may clearly assume that $X \cap Y = \emptyset$. We then say that the pair (X, Y) is a *good partition* of U.

The original motivation for our study came from a theorem of Tardos, on 2-intervals. A 2*-interval* is the union of 2 intervals, each on a separate line. In [5] Tardos proved the following:

Theorem 1.2. *If F is a finite family of 2-intervals, sharing the same pair of lines, then $\tau(F) \leq 2\nu(F)$. Moreover, if $\nu(F) = k$ then there exist k points on the first line and k points on the second line, that together meet all 2-intervals in F.*

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In the language of union of graphs, Theorem 1.2 is:

Theorem 1.3. *Let B and R be two interval graphs on the same vertex set V, and let k be the maximal size of a stable set in $G(B,R)$. Then there exist k cliques $C_1,\dots,C_k$ in G and k cliques $C_{k+1},\dots,C_{2k}$ in R such that $V=\bigcup_{i=1}^{2k} C_i$.*

This result is non-trivial even for $k=1$. In [1], the second author of this paper generalized the result of Tardos to chordal graphs. The case $k=1$ of this result is:

Theorem 1.4. *Let B and R be two chordal graphs on the same vertex set V. If $G(B,R)$ is a complete graph, then V is obedient.*

The methods used in [5] and [1] are topological. However, for $k=1$ a combinatorial proof is known. This proof yields the stronger result, for whose formulation we need some further definitions. A *stable set* of G is a clique of G^c, the complement of G. For a subset X of $V(G)$, the graph $G|X$ is the subgraph of G induced by X. For a graph H, we say that G *contains* H if some induced subgraph of G is isomorphic to H. If G does not contain H, then G is H*-free*. If $\mathcal{H}$ is a family of graphs, then G is $\mathcal{H}$*-free* if G is H-free for every $H\in\mathcal{H}$.

Let $k\geq 3$ be an integer. We denote by C_k the cycle on k vertices. For a graph G, if $v_1,\dots,v_k\in V(G)$ are distinct vertices such that $v_iv_j\in E(G)$ if and only if $|i-j|=1$ or $|i-j|=k-1$, we say that $v_1-\ldots-v_k-v_1$ *is a* C_k *in* G. A graph is *chordal* if no induced subgraph of it is a cycle on at least four vertices.

The combinatorial proof of Theorem 1.4 yields that it suffices to assume that one of the graphs is chordal, and the other is C_4-free. The main result in the first part of the paper is a further strengthening of this fact:

Theorem 1.5. *Let B and R be two graphs on the same vertex set V, such that B is $\{C_4,C_5\}$-free, and R is C_4-free. If $G(B,R)$ is a complete graph, then V is obedient.*

A graph is called *split* if its vertex set can be split into a clique and an indpendent set. Theorem 1.5 is a generalization of the well-known characterization of split graphs [2]: a graph is split if and only if no induced subgraph of it is a cycle on four or five vertices, or a pair of disjoint edges (the complement of a 4-cycle). .

A corollary of Theorem 1.5 is:

Theorem 1.6. *Let B and R be two graphs on the same vertex set V, such that B is $\{C_4,C_5\}$-free, and R is C_4-free. Then $\omega(G(B,R))\leq\omega(B)+\omega(R)$.*

Here is a symmetric formulation of Theorem 1.6:

Theorem 1.7. *Let B and R be two graphs with vertex set V, and suppose that some clique in $G(B,R)$ is not obedient. Then either*

- *one of B,R contains C_4, or*
- *both B and R contain C_5.*

We remark that neither conclusion of Theorem 1.7 is redundant, namely each may occur while the other does not. Let $B|X$ be isomorphic to C_4 for some $X\subseteq V$, and $R|X=B^c|X$; then X is a clique in $G(B,R)$, and yet X cannot be expressed as the union of a clique of B and a clique of R. Similarly, let $B|X$ be isomorphic to C_5 for some $X\subseteq V$, and $R|X=B^c|X$ (and thus $R|X$ is also isomorphic to C_5); then again X is a clique in $G(B,R)$, and yet X cannot be expressed as the union of a clique of B and a clique of R. Thus Theorem 1.7 provides an answer to the question: for which pairs (B,R) every clique of $G(B,R)$ is obedient?

Our second goal is to give sufficient conditions concerning $\omega(G(B.R))$.

Definition 1.8. *The pair* (B, R) *is* additive *if for every* $X \subseteq V$,

$$\omega(B|X) + \omega(R|X) \geq \omega(G(B, R)|X).$$

The following is immediate:

Theorem 1.9. *Let* B *and* R *be two graphs with vertex set* V*. The pair* (B, R) *is additive if and only if every clique in* $G(B, R)$ *is size obedient.*

Note that if $B|X$ is isomorphic to C_4 for some $X \subseteq V$, and $R|X = B^c|X$, then $\omega(B|X) = \omega(R|X) = 2$, and thus

$$\omega(B|X) + \omega(R|X) = |X|.$$

So, additivity does not imply obedience, and our goal here is to modify the first conclusion of 1.7, in order to obtain a characterization of additive pairs.

For a graph G and two disjoint subsets X and Y of $V(G)$, we say that X is G*-complete* (G*-anticomplete*) to Y if every vertex of X is adjacent (non-adjacent) to every vertex of Y. If $|X| = 1$, say $X = \{x\}$, we write "x is G-complete (G-anticomplete) to Y" instead of "$\{x\}$ is G-complete (G-anticomplete) to Y". When there is no risk of confusion, we write "complete" ("anticomplete") instead of "G-complete" ("G-anticomplete").

Let $\mathcal{F}$ be the family of graphs with vertex set $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ where $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are cliques, a_i is non-adjacent to b_i for $i \in \{1, 2, 3\}$, and the remaining adjacencies are arbitrary.

Let P_0 be the graph with vertex set $\{a_1, a_3, a_3, b_1, b_2, b_3, c\}$ where

- $\{a_1, a_2, a_3\}$ is a clique,
- $\{b_1, b_2, b_3\}$ is a stable set,
- for $i \in \{1, 2, 3\}$, b_i is non-adjacent to a_i, and complete to $\{a_1, a_2, a_3\} \setminus \{a_i\}$,
- c is adjacent to b_1, and has no other neighbors in P_0.

Let P_1 be the graph obtained from P_0 by adding the edge cb_2, and let P_2 be the graph obtained from P_1 by adding the edge cb_3. Let $\mathcal{P} = \{P_0, P_1, P_2\}$.

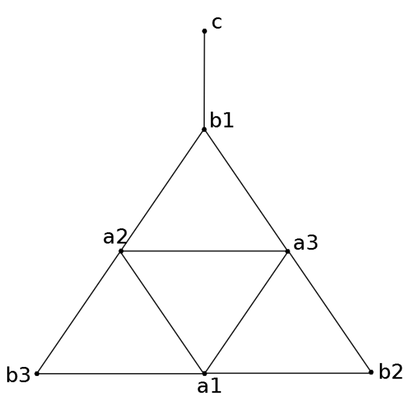

FIGURE 1. P_0

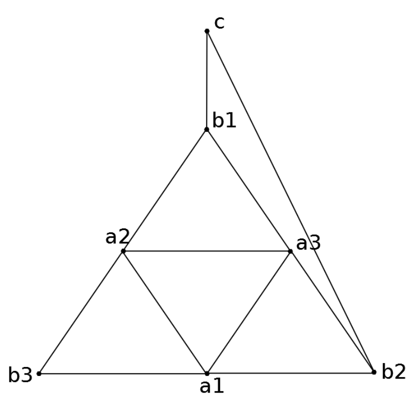

FIGURE 2. P_1

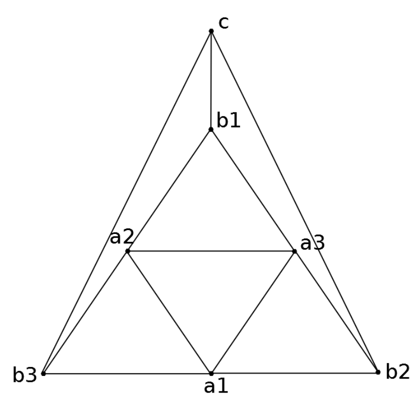

FIGURE 3. P_2

We can now state our second result.

Theorem 1.10. *Let* B *and* R *be two graphs with vertex set* V*. Then at least one of the following holds:*

(1) the pair (B, R) *is additive*

(2) one of B, R contains a member of $\mathcal{F}$
(3) both B and R contain C_5
(4) both B and R contain P_0^c
(5) B contains P_0^c, and R contains a member of $\mathcal{P}$
(6) R contains P_0^c, and B contains a member of $\mathcal{P}$.

Let us show that, similarly to the state of affairs in Theorem 1.7, all conclusions of Theorem 1.10 are necessary. Taking B to be a member of $\mathcal{F}$ (or C_5), and taking $R = B^c$, we construct a pair that is not additive, and that satisfies only (1) (or only (2)). Next, let $B = P_0^c$, and let R be the graph obtained from $P_0 = B^c$ by adding the edge ca_1; then (B, R) is not additive, and it only satisfies (3). Finally, let $B = P_0^c$, and let R be the graph obtained from B^c by adding none, one or both of the edges cb_2 and cb_3; then the pair (B, R) is not additive, and it only satisfies (4). Clearly, (5) is just (4) with the roles of R and B reversed.

2. Pairs of chordal - C_4-free graphs

We start with a simple proof of Theorem 1.5 in the case that B is chordal.

Theorem 2.1. *Let B, R be two graphs on the same vertex set V, such that B is chordal and R is C_4-free. If $G(B, R)$ is a complete graph, then V is obedient.*

Proof. By a theorem of [3], a graph is chordal if and only if it is the intersection graph of a collection of subtrees of some tree. Let T be a tree, and let $(t_v : v \in V)$ be subtrees of T such that $uv \in E(B)$ if and only of $V(t_u) \cap V(t_v) \neq \emptyset$. For $x \in V(T)$ and an edge $e = xy \in E(T)$ containing it, write $V_x = \{v \in V : x \in V(t_v)\}$ and $V_e = \{v \in V : e \in E(t_v)\}$. Note that V_x is a B-clique. Taking again $e = xy$, let V_{xe} denote the set of all $v \in V$ such that t_v is contained in the connected component of $T - x$ containing y. Similarly, let V_{ex} denote the set of all $v \in V$ such that t_v is contained in the connected component of $T - e$ containing x. We also define $V_{e\bar{x}} = V \setminus (V_e \cup V_{ex}) = V_{ey}$ and $V_{x\bar{e}} = V \setminus (V_x \cup V_{xe}) = \bigcup_{x \in e' \neq e} V_{xe'}$. Note that $V_{e\bar{x}} = V_{xe}$. We write $x \rightsquigarrow e$ if $V_{x\bar{e}}$ is an R-clique and write $e \rightsquigarrow x$ if $V_{e\bar{x}}$ is an R-clique.

Assertion 2.1.1. *Every vertex $x \in V(T)$ belongs to some edge $e \in E(T)$ such that $x \rightsquigarrow e$. For every edge $e = xy \in E(T)$ either $e \rightsquigarrow x$ or $e \rightsquigarrow y$.*

Proof. Let $e = xy \in E(T)$ and assume for contradiction that neither V_{ex} nor V_{ey} is an R-clique. Let $a, b \in V_{ex}$ and $c, d \in V_{ey}$ satisfy $ab, cd \notin E(R)$. Then $a - c - b - d - a$ is a C_4 in R.

Similarly, let $x \in V(T)$ and let $e_1, \ldots, e_d$ be the edges containing it. By the same argument as above, all but at most one of the sets $V_{xe_1}, \ldots, V_{xe_d}$ are cliques in R. Therefore $x \rightsquigarrow e_i$ for some $i = 1, \ldots, d$. $\square$

Using Assertion 2.1.1 we can choose a vertex $x_0 \in V(T)$ and construct a walk $x_0 \rightsquigarrow e_1 \rightsquigarrow x_1 \rightsquigarrow e_2 \rightsquigarrow x_2 \rightsquigarrow e_3 \rightsquigarrow x_3 \rightsquigarrow \ldots$. Since T is finite and has no cycles, the walk must turn back at some stage, i.e., there exist some vertex $x \in V(T)$ and edge $e = xy \in E(T)$ such that $x \rightsquigarrow e \rightsquigarrow x$. This means that $V_{x\bar{e}}$ and $V_{e\bar{x}} = V_{xe}$ are R-cliques. Since $V_{x\bar{e}}$ is R-complete to V_{xe}, the union $D = V_{x\bar{e}} \cup V_{xe} = V \setminus V_x$ is also an R-clique. Taking the B-clique $C = V_x$ we get $V = C \cup D$. $\square$

3. Proof of Theorem 1.7

Let $B \setminus R$ be the graph with vertex set V, such that two vertices are adjacent in $B \setminus R$ if and only if they are adjacent in B and non-adjacent in R. The graph $R \setminus B$ is defined similarly.

Let G be a graph. A set $C \subset V(G)$ is a *cutset* if there exist disjoint $P, Q \subseteq V(G)$ such that $V(G) \setminus C = P \cup Q$, and P is anticomplete to Q in G. We say that C is a *clique cutset* if it is a cutset, and C is a clique of G.

We start with the following easy observation:

Lemma 3.1. *Let B, R be C_4-free graphs with vertex set V, let C be a cutset of $B \setminus R$, and let P, Q be as in the definition of a cutset. If $G(B,R)$ is a complete graph then the following hold:*

(1) One of P and Q is an R-clique.
(2) $N_R(c) \cap Q$ is an R-clique for every $c \in C$ with $(N_B(c) \setminus N_R(c)) \cap P \neq \emptyset$.

(Here and below, if v is a vertex in a graph G we denote by $N_G(v)$ the set of neighbors of v in G.)

Proof. Since $G(B,R)$ is a complete graph, it follows that P is R-compelte to Q. If (1) fails, then there exist vertices $x, y \in P$ and $u, v \in Q$ such that xy and uv are edge of $B \setminus R$. But now $x - y - u - v - x$ is a C_4 in R, a contradiction. If (2) fails, then there exist $x \in P$ and $u, v \in Q$ such that xc, uv are edges of $B \setminus R$, and so $x - u - c - v - x$ is a C_4 in R, again a contradiction. This proves the theorem. □

A *weak clique cutset in B* is a clique C of B that is a cutset in $B \setminus R$. Note that a clique cutset of B is in particular a weak clique cutset, but the converse need not be true. Weak clique cutsets are useful to us because of the following:

Lemma 3.2. *Let B and R be C_4-free graphs with vertex set V, and assume that every proper subset $U \subset V$ is obedient. Assume also that $G(B,R)$ is a complete graph. If there is a weak clique cutset in B (or in R), then V is obedient.*

Proof. Let C, P, Q be as in the definition of a weak clique cutset in B, and assume that C is minimal with these properties. Since $G(B,R)$ is a complete graph, it follows that P is R-complete to Q. The minimality of C implies that for every $c \in C$ we have $(N_B(c) \setminus N_R(c)) \cap P \neq \emptyset$, for otherwise we could move c to Q. Similarly, $(N_B(c) \setminus N_R(c)) \cap Q \neq \emptyset$. By Lemma 3.1(1), we may assume that P is an R-clique. By the induction hypothesis $V \setminus P$ is obedient, namely there exist $X, Y \subseteq V$ such that $X \cup Y = V \setminus P$, X is a B-clique, and Y is an R-clique; let X and Y be chosen with $Y \cap C$ minimal.

We may assume that $(X, Y \cup P)$ is not a good partition of V, for otherwise the theorem holds. This implies that there exists $p \in P$ such that $(N_B(p) \setminus N_R(p)) \cap Y \neq \emptyset$. Let $Z = (N_B(p) \setminus N_R(p)) \cap Y$. Since P is R-complete to Q, it follows that $Z \subseteq C$. Choose $z \in Z$ with $(N_R(z) \setminus N_B(z)) \cap (Q \cap X)$ minimal. Let $N = (N_R(z) \setminus N_B(z)) \cap (Q \cap X)$. Since $z \in Y$, and Y is an R-clique, we deduce that z is R-compelte to $N \cup Y$. Therefore, Lemma 3.1(2) implies that $N \cup (Y \cap Q)$ is an R-clique.

By the minimality of $Y \cap C$, the pair $(X \setminus N) \cup \{z\}, (Y \setminus \{z\}) \cup N$ is not a good partition of $V \setminus P$. Since C is a B-clique, so is $(X \setminus N) \cup \{z\}$. This implies that $(Y \setminus \{z\}) \cup N$ is not an R-clique, and since Y and N are R-cliques there exists $y \in (C \cap Y) \setminus \{z\}$ and $n \in N$ such that $yn \in E(B \setminus R)$. Since $p - y - z - n - p$ is not a C_4 in R, it follows that $y \in Z$. By the choice of z, there exists $m \in Q \cap X$ such that $my \in E(R \setminus B)$, and $mz \in E(B)$. But now $y - z - m - n - y$ is an induced C_4 in B, a contradiction. This proves Lemma 3.2. □

Remark 3.3. Since a chordal graph is either complete or admits a clique cutset (say, the set of neighbors of a simplicial vertex), Lemma 3.2 yields another proof of Theorem 2.1.

Lemma 3.4. *Let B and R be C_4-free graphs with vertex set V, and assume that every proper subset U of V is obedient. Assume also that B is C_5-free. Let $v_1 - v_2 - v_3 - v_4 - v_5 - v_1$ be a C_5 in R, where $v_1v_2, v_3v_4 \in E(R \setminus B)$. Then V is obedient.*

Proof. Since $v_1 - v_2 - v_3 - v_4 - v_5 - v_1$ is a C_5 in R, it follows that all edges of the cycle $v_1 - v_3 - v_5 - v_2 - v_4 - v_1$ belong to $E(B \setminus R)$. Then $v_2v_3 \in E(R \setminus B)$, since otherwise $v_1 - v_4 - v_2 - v_3 - v_1$ is a C_4 in B. Since B is C_5-free, it follows that at least one of the edges v_1v_5, v_4v_5 belongs to $E(B)$. By symmetry we may assume that $v_1v_5 \in E(B)$. Since $v_1 - v_4 - v_2 - v_5 - v_1$ is not an induced C_4 in B, it follows that $v_4v_5 \in E(B)$.

We now observe that there is a unique good partition of $\{v_1, .., v_5\}$, namely the B-clique $\{v_4, v_5, v_1\}$ and the R-clique $\{v_2, v_3\}$. Let $u \in V \setminus \{v_1, \ldots, v_5\}$. Note that the set $\{u, v_1, \ldots, v_5\}$ is obedient. This can be seen by exhaustive search. Another way to see this is to note that $G|\{(u, v_1, \dots, v_5\}$ is not isomorphic to C_6 and does not contain C_4 or C_5, and is therefore chordal, so the set $\{u, v_1, \dots, v_5\}$ is obedient by Lemma 2.1. This implies that every vertex of $V \setminus \{v_1, .., v_5\}$ is either R-complete to $\{v_2, v_3\}$ or B-complete to $\{v_1, v_4, v_5\}$.

Let A be the set of vertices in $V \setminus \{v_1, .., v_5\}$ that are not R-complete to $\{v_2, v_3\}$. Then A is B-complete to $\{v_1, v_4, v_5\}$. We claim that A is a B-clique. Assume for a contradiction that there exist two distinct vertices u, v in A such that $uv \notin E(B)$. Then each of u and v is not R-complete to $\{v_2, v_3\}$. By symmetry we may assume that $uv_2 \notin E(R)$. If $vv_2 \notin E(R)$ then $v - v_2 - u - v_1 - v$ is a C_4 in B, a contradiction. Thus $vv_2 \in E(R)$, and consequently $vv_3 \notin E(R)$. Exchanging the roles of v_2 and v_3 we deduce that $uv_3 \in E(R)$. But now $u - v - v_2 - v_3 - u$ is a C_4 in R, a contradiction. This proves the claim that A is a B-clique.

Since no vertex of A is R-complete to $\{v_2, v_3\}$, it follows that A is B-complete to $\{v_1, v_4, v_5\}$, and so the claim of the previous paragraph implies that $A \cup \{v_1, v_4, v_5\}$ is a B-clique. Let $Z = V \setminus (A \cup \{v_1, v_4, v_5\})$. If $Z = \{v_2, v_3\}$ then $A \cup \{v_1, v_4, v_5\}, Z$ is a good partition of V, and so V is obedient. Thus we may assume that $Z \setminus \{v_2, v_3\} \neq \emptyset$. Since $\{v_2, v_3\}$ is R-complete to $Z \setminus \{v_2, v_3\}$, it follows that $A \cup \{v_1, v_4, v_5\}$ is a weak clique cutset in B, and hence V is obedient by Lemma 3.2. This proves Lemma 3.4. ∎

We can now prove Theorem 1.7, which we restate:

Theorem 3.5. *Let B and R be two graphs with vertex set V, and suppose that some clique of $G(B, R)$ is not obedient. Then either*

- *one of B, R contains C_4, or*
- *both B and R contain C_5.*

Proof. Suppose that the theorem is false, and let B, R be graphs with vertex set V, such that some clique of $G(B, R)$ is not obedient, and

- both B and R are C_4-free, and
- at least one of B and R is C_5-free.

We may assume that subject to these conditions V, B, R and $G(B, R)$ are chosen with $|V|$ minimum. Then $G(B, R)$ is a complete graph, and so every pair of vertices of V is an edge of at least one of B, R. The minimality of $|V|$ implies that V is not obedient, but U is obedient for every proper subset U of V.

Let C be a cutset of $B \setminus R$, and let P, Q be as in the definition of a cutset. Then P is R-compelte to Q. We may assume that C is chosen so that the number of edges in $(R \setminus B)|C$ is minimum. We may also assume that C is a minimal cutset of $B \setminus R$, which implies that for every $c \in C$, the sets $N_{B\setminus R}(c) \cap P$ and $N_{B\setminus R}(c) \cap Q$ are both non-empty.

By Lemma 3.1(1) we may assume that P is an R-clique. Let Z be the set of vertices of C that have an $R \setminus B$-neighbor in C. By Lemma 3.2 $Z \neq \emptyset$. Let $z \in Z$ be with $N_{B\setminus R}(z) \cap P$ minimal (which is the same as saying that $N_R(z) \cap P$ is maximal). Let $N = N_{B\setminus R}(z) \cap P$. Let $y \in Z$ satisfy $zy \in E(R \setminus B)$.

Assertion 3.5.1. *y is $B \setminus R$-complete to N.*

Proof. Assume that there exists a vertex $n \in N$ such that $yn \in E(R)$. By the minimality property of z, there exists a vertex $m \in N_{B\setminus R}(y) \setminus N_{B\setminus R}(z)$. Then $y - z - m - n - y$ is a C_4 in R, a contradiction. ∎

Assertion 3.5.2. *N is B-complete to C.*

Proof. Suppose that there exist $x \in C$ and $n \in N$ such that $xn \in R \setminus B$. Since $z - x - y - n - z$ is not a C_4 in B, and since, by Assertion 3.5.1, zn, yn belong to $E(B)$ (in fact, to $E(B \setminus R)$), it follows that at least one of xz, xy is in $E(R \setminus B)$. This implies, in particular, that $x \in Z$. By the minimality of z, the fact that $xn \in E(R)$ implies that there exists $p \in P$ such that $xp \in E(B \setminus R)$ and $zp \in E(R)$. Since $x - n - p - z - x$ is not a C_4 in R, it follows that $xz \in E(B \setminus R)$ (recall that $pn \in E(R)$ since P is an R-clique). Therefore, since $x - y - n - z - x$ is not a C_4 in B, we deduce that $xy \in E(R \setminus B)$. Since $n - p - y - x - n$ is not a C_4 in R, it follows that $yp \in E(B \setminus R)$.

Summarizing the information in the last paragraph, we see that $p - n - x - y - z - p$ is a C_5 in R, containing two disjoint edges $nx, yz \in E(R \setminus B)$. Consequently, B is C_5-free, and so Lemma 3.4 implies that V is obedient, a contradiction. $\square$

Assertion 3.5.3. *$P = N$.*

Proof. Suppose not. Then $(C \setminus \{z\}) \cup N$ is a cutset in $B \setminus R$ (since $P \setminus N$ is R-complete to $Q \cup \{z\}$). Now, since N is B-complete to X, this contradicts the minimality of the number of edges of $(R \setminus B)|C$. $\square$

By the minimality of z, and the fact that N is B-complete to C, Assertion 3.5.3 implies:

Assertion 3.5.4. *P is $B \setminus R$-complete to Z, and P is B-complete to C.*

Assertion 3.5.5. *P is a B-clique.*

Proof. Suppose not, and let $u, v \in P$ such that $uv \in E(R \setminus B)$. Then $u - z - v - y - u$ is a C_4 in B, a contradiction. $\square$

So far we focused mainly on the edges with ends in $C \cup P$. We now switch our attention to the edges with ends in $C \cup Q$. Let $z' \in Z$ be such that $N' = N_{B \setminus R}(z') \cap Q$ is minimal. Let $y' \in Z$ satisfy $y'z' \in E(R \setminus B)$.

Let $c \in P$. Let (S, T) be a good partition of $V \setminus \{c\}$ where S is a B-clique and T is an R-clique. Since Z is not a B-clique, $Z \cap T \neq \emptyset$. Since P is $B \setminus R$-complete to Z, this implies that $P \setminus \{c\} \subseteq S$. Let M be the set of vertices $m \in Q \cap S$ such that $cm \in E(R \setminus B)$.

Assertion 3.5.6. *$M \cap N' = \emptyset$, and therefore z' is R-complete to M.*

Proof. It is enough to prove that $M \cap N' = \emptyset$. Assume that there exists an element $n \in M \cap N'$. Since $n - z' - c - y' - n$ is not a C_4 in B, it follows that $y'n \in E(R \setminus B)$. By the minimality property of z' it follows that there exists $q \in Q \setminus N'$ such that $y'q \in E(B \setminus R)$ and $z'q \in E(R)$. Since $z' - y' - n - q - z'$ is not a C_4 in R, it follows that $qn \in E(B \setminus R)$. By Assertion 3.5.3, $cy', cz' \in E(B \setminus R)$. Now $n - c - q - z' - y' - n$ is a C_5 in R. Therefore B is C_5-free. Since $cn, y'z' \in E(R \setminus B)$, Lemma 3.4 implies that V is obedient, a contradiction. $\square$

Assertion 3.5.7. *$M \cup (T \cap Q)$ is an R-clique, then V is obedient.*

Proof. Suppose that $a, b \in M \cup (T \cap Q)$ and $ab \in E(B \setminus R)$. Since T is an R-clique, we may assume that $a \in M$, and hence by Assertion 3.5.6 $az' \in R$. Since Z is $B \setminus R$-compelte to P, 3.1 (2) implies that $N_R(z') \cap Q$ is an R-clique, and therefore $b \in T \cap Q$, and z is $z'b \in E(B \setminus R)$. Consequently, $z' \in S$. Since $y'z' \in E(R \setminus B)$, we deduce that $y' \in T$. Since both y' and b are in T, it follows that $y'b \in E(R)$. Since $y' - b - a - c - y'$ is not a C_4 in R, it follows that $ay' \in E(B \setminus R)$. Now $c - b - y' - z' - a - c$ is a C_5 in R, and so B is C_5-free. Since $ca, y'z' \in E(R \setminus B)$, Lemma 3.4 implies that V is obedient, a contradiction. $\square$

Let W be the set of vertices in $C \cap T$ that are R-complete to M. Since by Assertion 3.5.7 $M \cup (T \cap Q)$ is an R-clique, it follows that $W \cup M \cup (T \cap Q)$ is an R-clique. Let $U = (T \cap C) \setminus W$.

Assertion 3.5.8. *U is B-complete to $S \setminus M$.*

Proof. Suppose $u \in U$ has an $R \setminus B$-neighbor s in $S \setminus M$. Since $u \in U$, there exists $m \in M$ such that um is in $E(B \setminus R)$. Now $u - c - s - m - u$ is a C_4 in B ($uc \in E(B)$ by Assertion 3.5.4; $cs \in E(B)$ because $s \notin M$; and $sm \in E(B)$ because $s, m \in S$), a contradiction. $\square$

We claim that $D = W \cup M \cup (T \cap Q)$ is a weak clique cutset in R. By Assertion 3.5.7, $W \cup M \cup (T \cap Q)$ is an R-clique. By Assertion 3.5.8, $U \cup \{c\}$ is B-complete to $S \setminus M$. Now since $P \setminus \{c\} \subseteq S$, and $C \setminus (U \cup W) \subseteq S$, it follows that $V(G) \setminus D \subseteq (S \setminus M) \cup U \cup \{c\}$, and the claim follows. But then V is obedient by Lemma 3.2, a contradiction. This proves Theorem 3.5. $\square$

4. Proof of Theorem 1.10

In this section we prove Theorem 1.10. Let $L = \omega(R)$ and $K = \omega(B)$. Suppose that Theorem 1.10 is false, and let B and R be two graphs with vertex set V such that the pair (B, R) is not additive, and

- both B, R are $\mathcal{F}$-free, and
- at least one of B and R is C_5-free, and
- at least one of B and R is P_0^c-free, and
- B is P_0^c-free or R is $\mathcal{P}$-free, and
- R is P_0^c-free, or B is $\mathcal{P}$-free, and
- B and R are chosen with $|V|$ minimum subject to the conditions above.

Let $|V| = n$. By Lemma 1.9, the minimality of $|V|$ implies that $G(B, R)$ is a complete graph with vertex set V, and $K + L < n$. Consequently, neither of B, R is a complete graph, and so, since every pair of vertices of V is adjacent in $G(B, R)$, we deduce that $K \geq 2$, and $L \geq 2$.

Lemma 4.1. $n \geq 6$

Proof. Suppose $n \leq 5$. Since both $K \geq 2$, and $L \geq 2$, and $K + L < n$, it follows that $|V| = 5$, and $K = L = 2$. But then both B and R are isomorphic to C_5, a contradiction. This proves Lemma 4.1. $\square$

Lemma 4.2. *$K + L = n - 1$ and for every $v \in V$, $\omega(B \setminus v) = K$ and $\omega(R \setminus v) = L$.*

Proof. Let $v \in V$. Then it follows from the minimality of $|V|$, that

$$n - 1 \leq \omega(B \setminus v) + \omega(R \setminus v) \leq K + L \leq n - 1 \ .$$

Thus all the inequlities above must be equalities, namely $K + L = n - 1$, $\omega(B \setminus v) = K$ and $\omega(R \setminus v) = L$.

This proves Lemma 4.2. $\square$

We assume without loss of generality that $K \geq L$ and hence by Lemmas 4.1 and 4.2 we have $K \geq 3$.

For a graph G and two disjoint subsets X and Y of $V(G)$ with $|X| = |Y|$, we say that X is *matched* to Y if there is a matching $e_1, \ldots, e_{|X|}$ of G, so that for all $i \in \{1, \ldots, |X|\}$, the edge e_i has one end in X and the other in Y.

Lemma 4.3. *Let K_1, K_2 be B-cliques of size K. Then $K_1 \setminus K_2$ and $K_2 \setminus K_1$ are matched in $R \setminus B$.*

Proof. Suppose not. Let $k = |K_1 \setminus K_2| = |K_2 \setminus K_1|$. Then by Hall's Theorem [4], there exist $Y \subseteq K_1 \setminus K_2$ and $Z \subseteq K_2 \setminus K_1$ such that $|Z| > k - |Y|$, and Y is $R \setminus B$-anticomplete to Z. Since $G(B, R)$ is a complete graph, it follows that Y is B-complete to Z. But then $(K_1 \cap K_2) \cup Y \cup Z$ is a clique of size at least $K + 1$ in B, contrary to the definition of K. This proves Theorem 4.3. $\square$

Lemma 4.3 implies the following:

Lemma 4.4. *Let K_1, K_2 be cliques of size K in B. Then $|K_1 \setminus K_2| \leq 2$.*

Proof. Suppose $|K_1 \setminus K_2| \geq 3$, and let $a_1, a_2, a_3 \in K_1 \setminus K_2$ be all distinct. By Lemma 4.3, there exist $b_1, b_2, b_3 \in K_2 \setminus K_1$, all distinct, such that the sets $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are matched in $R \setminus B$. But then $B|\{a_1, a_2, a_3, b_1, b_2, b_2\}$ is a member of $\mathcal{F}$, a contradiction. This proves Lemma 4.4. □

In view of Theorem 4.2, for every $v \in V$ there exists a clique K_v of size K in $B \setminus v$.

Lemma 4.5. *There exist $u, w \in V$ such that $|K_u \setminus K_w| = 2$.*

Proof. Let $v \in V$. Since $K \geq 2$, there exist distinct vertices $u, w \in K_v$. By Lemma 4.4, we may assume that $|K_v \setminus K_u| = 1$, where $K_v \setminus K_u = \{u\}$. Let x be the unique vertex of $K_u \setminus K_v$ (possibly $x = v$). Similarly, we may assume that $|K_v \setminus K_w| = 1$, and $K_v \setminus K_w = \{w\}$. Let y be the unique vertex of $K_w \setminus K_v$ (again, possibly $y = v$). By Lemma 4.3 ux is an edge $R \setminus B$, and so u is non-adjacent to x in B. Since $y, u \in K_w$, it follows that u is adjacent to y in B; consequently $x \neq y$, and so $x \notin K_w$. But now both x and w are in $K_u \setminus K_w$, and Lemma 4.5 holds. □

At this point we fix two vertices $u, w \in V$ as in Lemma 4.5, namely satisfying $|K_u \setminus K_w| = |K_w \setminus K_u| = 2$. Let $K_u \cap K_w = \{v_3, \ldots, v_K\}$, and $K_i = K_{v_i}$ $(i = 3, \ldots, K)$. Let $K_u \setminus K_w = \{x_1, x_2\}$ and $K_w \setminus K_u = \{y_1, y_2\}$. Then $v_i \in K_u \setminus K_i$, and so by Lemma 4.3, there exists $p_i \in K_i \setminus K_u$ such that $v_i p_i$ is an edge of $R \setminus B$. Consequently, $p_i \notin K_u \cup K_w$. Also by Lemma 4.3, the sets $\{x_1, x_2\}$ and $\{y_1, y_2\}$ are matched in $R \setminus B$.

Note next that p_i is not B-complete to $\{x_1, x_2\}$. Otherwise, taking $a_1 = x_1,\ a_2 = x_2,\ a_3 = v_i,\ b_1 = y_1,\ b_2 = y_2,\ b_3 = p_i$ shows that $B|\{x_1, x_2, y_1, y_2, v_i, p_i\} \in \mathcal{F}$. Similarly, p_i is not B-complete to $\{y_1, y_2\}$.

Since p_i is not B-complete to $\{x_1, x_2\}$ nor to $\{y_1, y_2\}$, by Lemma 4.4 p_i is B-complete to $(K_u \cap K_w) \setminus \{v_i\}$. Since $p_i v_i$ is an edge of $R \setminus B$, it follows that the vertices $p_3, \ldots, p_K$ are all distinct.

Our next aim is to show that for all $3 \leq i \leq K$, the vertices p_i make the same choice of which among x_1, x_2 they are connected to in $R \setminus B$, and similarly for y_1, y_2. Fix a specific $i \in \{3, \ldots, K\}$, and without loss of generality assume that $p_i x_2$ and $p_i y_2$ are both edges of $R \setminus B$. Then $x_2, y_2 \notin K_i$, and so, by Lemma 4.4, $K_u \setminus K_i = \{x_2, v_i\}$ and $K_w \setminus K_i = \{y_2, v_i\}$. Consequently,

$$K_i = ((K_u \cap K_w) \cup \{p_i, x_1, y_1\}) \setminus \{v_i\}.$$

We next show that the same is true, with the same parameters, for all j between 3 and K. The

Lemma 4.6. *For every $j \in \{3, \ldots, K\}$*

$$K_j = ((K_u \cap K_w) \cup \{p_j, x_1, y_1\}) \setminus \{v_j\}.$$

Proof. By the argument above, applied to j instead of i, we deduce that there exist $k, m \in \{1, 2\}$ such that

$$K_j = ((K_u \cap K_w) \cup \{p_j, x_k, y_m\}) \setminus \{v_j\},$$

It remains to show that $k = m = 1$. Suppose not. Since $x_1, y_1 \in K_i$ it follows that $x_1 y_1$ is an edge of B. On the other hand, Lemma 4.3 implies that $x_1 y_2$ and $x_2 y_1$ are edges of $R \setminus B$. Since K_j is a clique of B, we deduce that $x_k y_m$ is an edge of B, and so $k = m = 2$. But then $K_i \setminus K_j = \{p_i, x_1, y_1\}$, contrary to Lemma 4.4. This proves that $k = m = 1$, as desired. □

We now write $Y = \{x_1, y_1, v_3, \ldots, v_K\}$ and $Z = \{x_2, y_2, p_3 \ldots, p_K\}$.

Lemma 4.7. *The following hold:*

(1) Z is a clique of size K in $R \setminus B$ and Y is a clique of size K in B.
(2) The pairs $x_1 y_2, x_2 y_1$ and $v_j p_j$ for $j \in \{3, \ldots, K\}$ are adjacent in $R \setminus B$, and all other pairs zy with $z \in Z$ and $y \in Y$ are adjacent in B.

Proof. To prove 4.7(1) suppose that Z is not a clique of $R \setminus B$. We showed earlier that $\{x_2, y_2\}$ is $R \setminus B$-complete to $\{p_3, \dots, p_K\}$, and that $p_3, \dots, p_K$ are all distinct. Suppose first that there exist $k, m \in \{3, \dots, K\}$ such that $p_k p_m$ is not an edge of $R \setminus B$. Then

$$X = ((K_u \cap K_w) \cup \{p_k, p_m, x_1, y_1\}) \setminus \{v_k, v_m\}$$

is a clique of size K in B, but $X \setminus K_u = \{p_k, p_m, y_1\}$, contrary to Lemma 4.4. This proves that $\{p_3, \dots, p_K\}$ is a clique of $R \setminus B$. Since $\{p_3, \dots, p_K\}$ is $R \setminus B$-complete to $\{x_2, y_2\}$, but $\{x_2, y_2, p_3, \dots, p_K\}$ is not a clique of $R \setminus B$, it follows that $x_2 y_2$ is not an edge of $R \setminus B$, and therefore x_2 is adjacent to y_2 in B. Consequently, $W = (K_u \cup \{y_2\}) \setminus \{x_1\}$ is a clique of size K in B. But now $K_3 \setminus W = \{x_1, y_1, p_3\}$, contrary to Lemma 4.4. This proves Lemma 4.7(1).

We now prove the second statement of Lemma 4.7. We have already shown that $x_1 y_2, x_2 y_1$ and $v_i p_i$ for $i \in \{3, \dots, K\}$ are adjacent in $R \setminus B$. Next we observe that every other pair (z, y) with $z \in Z$ and $y \in Y$ is contained in at least one of the cliques $K_u, K_v, K_3, \dots, K_K$, and therefore zy is an edge of B. □

Next we use the symmetry between B and R in order to obtain more information about maximum cliques in each of them.

Lemma 4.8. $K = L = \frac{n-1}{2}$.

Proof. Theorem 4.7(1) implies that $L \geq K$. But we assumed that $K \geq L$. Thus $K = L = \frac{n-1}{2}$ (the second equality following by Theorem 4.2), and the lemma follows. □

It now follows from Lemma 4.7(2) and Lemma 4.8 that $V \setminus (Y \cup Z)$ is a set with exactly one vertex. We denote this vertex by v_R. Let us recall what we know about Y, Z and v_R:

- $V \setminus \{v_R\} = Z \cup Y$, and
- $Z \cap Y = \emptyset$, and
- Z is a clique of size $\frac{n-1}{2}$ in $R \setminus B$, and
- Y is a clique of size $\frac{n-1}{2}$ in B, and
- the vertices of Z can be numbered $z_1, \dots, z_K$, and the vertices of Y can be numbered $y_1, \dots, y_K$, such that for $i, j \in \{1, \dots, K\}$, the pair $z_i y_j \in B$ if and only if $i \neq j$.

Exchanging the roles of R and B, we deduce also that there exists a vertex $v_B \in V$ and sets Y' and Z' such that

- $V \setminus \{v_B\} = Z' \cup Y'$, and
- $Z' \cap Y' = \emptyset$, and
- Y' is a clique of size $\frac{n-1}{2}$ in $B \setminus R$, and
- Z' is a clique of size $\frac{n-1}{2}$ in R, and
- the vertices of Z' can be numbered $z'_1, \dots, z'_K$, and the vertices of Y' can be numbered $y'_1, \dots, y'_K$, such that for $i, j \in \{1, \dots, K\}$, the pair $z'_i y'_j \in R$ if and only if $i \neq j$.

We now analyze the way v_R attaches to Y and Z.

Lemma 4.9. *Let $i, j \in \{1, \dots, K\}$. If v_R is B-complete to $\{z_i, y_j\}$, then $z_i y_j$ is an edge of $R \setminus B$.*

Proof. Suppose that v_R is B-complete to $\{z_i, y_j\}$ and $z_i y_j$ is an edge of B. Then $i \neq j$. Since $(Y \cup \{v_R, z_i\}) \setminus \{y_i\}$ is not an clique of size $K+1$ in B, it follows that there exists $t \in \{1, \dots, K\} \setminus \{i\}$ such that $v_R y_t$ is an edge of $R \setminus B$. Then $t \neq j$. But now $B|\{v_R, z_i, y_j, y_t, y_i, z_j\}$ is a member of $\mathcal{F}$, a contradiction. This proves Lemma 4.9. □

We are finally ready to establish the existence of certain induced subgraphs in B and R.

Lemma 4.10. *At least one of the following holds:*

(1) B contains P_0^c, or

(2) B contains P_1 or P_2, and v_R is $R \setminus B$-complete to Y, or
(3) B contains P_0, and there exists $z \in Z$ such that v_R is $R \setminus B$-complete to $(Y \cup Z) \setminus \{z\}$.

Proof. Since $Z \cup \{v_R\}$ is not a clique of size $K+1$ in R, it follows that v_R has a neighbor in Z in $B \setminus R$. We may assume that $v_R z_1$ is an edge of $B \setminus R$. Since z_1 is B-complete to $Y \setminus \{y_1\}$, Lemma 4.9 implies that v_R is $R \setminus B$-complete to $Y \setminus \{y_1\}$.

Suppose v_R has a neighbor in $Z \setminus \{z_1\}$ in B, say $v_R z_2$ is an edge of B. Then by Lemma 4.9 v_R is adjacent in $R \setminus B$ to y_1, and so v_R is $R \setminus B$-complete to Y. Also, $B|\{y_1, y_2, y_3, z_1, z_2, z_3, v_R\}$ is isomorphic to P_1 if $v_R z_3$ is an edge of $R \setminus B$, and to P_2 if $v_R z_3$ is an edge of B, and the second conclusion of the theorem holds.

So we may assume that v_R is $R \setminus B$-complete to $Z \setminus \{z_1\}$. Now if $v_R y_1$ is an edge of B, then $B|\{y_1, y_2, y_3, z_1, z_2, z_3, v_R\}$ is isomorphic to P_0^c, and the first conclusion of the theorem holds; and if $v_R y_1$ is an edge of $R \setminus B$, then $B|\{y_1, y_2, y_3, z_1, z_2, z_3, v_R\}$ is isomorphic to P_0, v_R is $R \setminus B$-complete to $(Y \cup Z) \setminus \{z_1\}$, and the third conclusion of the theorem holds. This proves Lemma 4.10 □

Applying 4.10 with the roles of R and B reversed, we deduce that either

(1) R contains P_0^c, or
(2) R contains P_1 or P_2, and v_B is $B \setminus R$-complete to Z', or
(3) R contains P_0, and there exists $y' \in Y'$, such that v_B is $B \setminus R$-complete to $(Y' \cup Z') \setminus \{y'\}$.

To complete the proof of Theorem 1.10, we now analyze the possible conclusions of 4.10. Observe first that by Lemma 4.10, each of B, R either contains P_0^c, or contains a member of $\mathcal{P}$. Thus, if the first conclusion of Lemma 4.10 holds for at least one of B, R (in other words, one of B, R contains P_0^c), we get a contradiction to the third, fourth or fifth assumption at the start of this section.

So we may assume that either the second or the third conclusion of Lemma 4.10 holds for B, and the same for R. Therefore v_R is $R \setminus B$-complete to Y. We claim that every vertex of V has at least two neighbors in $R \setminus B$. Since by Lemmas 4.1 and 4.8 $|Y|, |Z| \geq 3$, it follows that v_R has at least two neighbors in Y in $R \setminus B$, and that every vertex of Z has at least two neighbors in Z in $R \setminus B$. Since v_R is $R \setminus B$-complete to Y, and every vertex of Y has a neighbor in Z in $R \setminus B$, the claim follows. Similarly, every vertex of V has at least two neighbors in $B \setminus R$.

Next we observe that if the third conclusion of Lemma 4.10 holds for B, then v_R has at most one neighbor in B, and if the third conclusion of Lemma 4.10 holds for R, then v_B has at most one neighbor in R. This implies that the third conclusion of Lemma 4.10 does not hold for either B or R, and thus the second conclusion of Lemma 4.10 holds for both B and R; consequently each of B and R contains P_1 or P_2. But both P_1 and P_2 contain C_5, contrary to the second assumption at the start of this section. This completes the proof of Theorem 1.10. ∎

5. Further problems

As we have already mentioned, Tardos' theorem was extended in [1] to pairs of chordal graphs on the same vertex set. With some trepidation we venture to conjecture that the theorem is valid also for a pair of graphs as in Theorem 1.10:

Conjecture 5.1. *Let B be a $\{C_4, C_5\}$-free graph and let R be a C_4-free graph with $V(B) = V(R) = V$. Let k be the maximal size of a stable set in $G(B, R)$. Then there exist k cliques $C_1, \ldots, C_k$ in B and k cliques $C_{k+1}, \ldots, C_{2k}$ in R such that $V = \bigcup_{i=1}^{2k} C_i$.*

It is tempting to ask also about the chromatic number of the union of two graphs. For a graph G, we denote its chromatic number by $\chi(G)$. Let us restrict ourselves in this case to chordal graphs:

Conjecture 5.2. *If B and R are chordal graphs on the same vertex set then $\chi(G(B, R)) \leq \chi(B) + \chi(R)$.*

Using the simplicial decomposition of chordal graphs, it is easy to show that in a chordal graph G the average degree of the vertices does not exceed $2(\omega(G)-1) = 2(\chi(G)-1)$. From this follows "half" of the conjecture, namely: if B and R are chordal then $\chi(G(B,R)) \leq 2(\chi(B)+\chi(R))$.

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