Coloring Quasi-line Graphs

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Abstract

A graph G is a quasi-line graph if for every vertex v, the set of neighbors of v can be expressed as the union of two cliques. The class of quasi-line graphs is a proper superset of the class of line graphs. A theorem of Shannon's implies that if G is a line graph then it can be properly colored using no more than $\frac{3}{2}\omega(G)$ colors, where $\omega(G)$ is the size of the largest clique in G. In this paper we extend this result to all quasi-line graphs. We also show that this bound is tight.

1 Introduction

Let G be a finite graph. We denote the set of its vertices by $V(G)$ and the set of its edges by $E(G)$. For an integer k, a function $c : V(G) \rightarrow \{1,\ldots,k\}$ is a *k-coloring* of G. A *proper coloring* is a coloring such that no two adjacent vertices have the same color. The chromatic number of G, denoted $\chi(G)$, is the smallest k for which there exists a proper k-coloring of G. Similarly, a function $c : E(G) \rightarrow \{1,\ldots,k\}$ is a *k-edge coloring* of G. A *proper edge coloring* is an edge coloring such that no two distinct edges that share a vertex have the same color. A *clique* in G is a set of vertices that are all pairwise adjacent. The maximum size of a clique in G is denoted by $\omega(G)$. For $v \in V(G)$, we denote the set of neighbors of v in G by $N_G(v)$.

The *line graph* of a graph G, denoted by $L(G)$, is a graph whose vertices are the edges of G, and if $u, v \in E(G)$ then $uv \in E(L(G))$ if u and v share a vertex in G. In 1948, Shannon [4] proved that the edges of any graph can be properly colored using no more than $\frac{3}{2}\Delta(G)$ colors, where $\Delta(G)$ is the maximum degree of a vertex in G. This of course immediately implies that if G is a line graph then $\chi(G) \leq \frac{3}{2}\omega(G)$.

A question that arises is whether Shannon's Theorem can be extended to larger classes of graphs that include line graphs as a proper subset. One natural class of graphs to consider is the family of "claw-free" graphs. We say that $X \subseteq V(G)$ is a claw if the subgraph of G induced on X, denoted by $G|X$, is isomorphic to the complete bipartite graph $K_{1,3}$. A graph

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G is then *claw-free* if no subset of $V(G)$ is a claw. Clearly, all line graphs are claw-free, but as it turns out, there are claw-free graphs whose structure is very different from that of line graphs [1]. In a recent work, the first author and Seymour [1] showed that a result similar to Shannon's can be proved for all claw-free graphs, namely $\chi(G) \leq 2\omega(G)$, and the constant 2 is best possible.

In this paper, we consider a class of graphs that is a proper subset of all claw-free graphs and a proper superset of line graphs, the set referred to as *quasi-line graphs.* A graph G is a quasi-line graph if for every vertex v, the set of neighbors of v can be expressed as the union of two cliques. Note that this is a partition of the vertex set of the neighborhood of G. The main result of this paper is the following:

Theorem 1.1. *Let G be a quasi-line graph. Then $\chi(G) \leq \frac{3}{2}\omega(G)$.*

In fact, we show that the bound in the theorem is tight. Our proof of Theorem 1.1 uses a structure theorem for quasi-line graphs that appears in [1]. The structure theorem asserts that every quasi-line graph belongs to one of two classes: the first is the class of the so-called "fuzzy circular interval graphs", and the second is "compositions of fuzzy linear interval strips", which is a generalization of line graphs. The word "fuzzy" in both cases refers to the presence of a certain structure in a graph, that is called a "non-trivial homogeneous pair" (we give precise definitions in the next section). The remainder of this paper is organized as follows. In Section 2 we state the structure theorem for quasi-line graphs and all of the necessary definitions. In Section 3 and Section 4 we prove Theorem 1.1 for circular interval graphs and compositions of linear interval strips, respectively (those are precisely the quasi-line graphs that have no non-trivial homogeneous pairs). In Section 5 we use the results of the two previous sections and deal with non-trivial homogeneous pairs, to complete the proof of Theorem 1.1.

2 Structure theorem for quasi-line graphs

We start this section by introducing some definitions from [2] and [1] and then state the structure theorem of [1].

Let Σ be a circle and let $F_1, \ldots, F_k$ be subsets of Σ, each homeomorphic to the closed interval $[0, 1]$. Let V be a finite subset of Σ, and let G be the graph with vertex set V in which $v_1, v_2 \in V$ are adjacent if and only if $v_1, v_2 \in F_i$ for some i. Such a graph is called a *circular interval graph.* A *linear interval graph* is constructed in the same manner except we take Σ to be a line instead of a circle. It is easy to see that all linear interval graphs are also circular interval graphs.

The structure theorem that we use states that there are two types of quasi-line graphs. The first subclass is a generalization of the class of circular interval graphs and we proceed to describe it below. Once again, we start with a few definitions.

Let X, Y be two subsets of $V(G)$ with $X \cap Y = \emptyset$. We say that X and Y are *complete* to each other if every vertex of X is adjacent to every vertex of Y, and we say that they are *anticomplete* if no vertex of X is adjacent to a member of Y. Similarly, if $A \subseteq V(G)$ and $v \in V(G) \setminus A$, then v is A-*complete* if it is adjacent to every vertex in A, and A-*anticomplete* if it has no neighbor in A. A pair A, B of disjoint subsets of $V(G)$ is called a *homogeneous pair* in G if for every vertex $v \in V(G) \setminus (A \cup B)$, v is either A-complete or A-anticomplete and either B-complete or B-anticomplete.

Let G be a circular interval graph with $V(G) = \{v_1, \ldots, v_n\}$ in order clockwise. An edge joining v_j to v_k is called a *maximal edge* if v_j and v_k are ends of every interval including both of them. In this case v_j and v_k can be replaced by two cliques A and B, such that every member of A has the same neighbors as v_j and every member of B has the same neighbors as v_k in $V(G) \setminus \{v_j, v_k\}$, and the edges between A and B are arbitrary. The pair (A, B) is then a homogeneous pair of cliques. Let H be a graph obtained from a circular interval graph by choosing a matching of maximal edges and replacing each of them by a homogeneous pair of cliques as described above. Then H is called a *fuzzy circular interval graph.*

Let (A, B) be a homogeneous pair of cliques in a circular interval graph. We say that (A, B) is *non-trivial* if there exists an induced 4-cycle in G with exactly two vertices in A and exactly two vertices in B. It is easy to see that a fuzzy circular interval graph is not a circular interval graph if and only if it has a non-trivial homogeneous pair.

We proceed with the construction of graphs that belong to the second subclass of quasi-line graphs. A vertex $v \in V(G)$ is *simplicial* if the set of neighbors of v is a clique. A claw-free graph S together with two distinguished simplicial vertices a, b is called a *strip* (S, a, b), with *ends* a and b. If S is a linear interval graph with $V(S) = \{v_1, \ldots, v_n\}$ in order and with $n > 1$, then v_1, v_n are simplicial, and so (S, v_1, v_n) is a strip, called a *linear interval strip.* Since linear interval graphs are also circular interval graphs we can define *fuzzy linear interval strips* by introducing homogeneous pairs of cliques in the same manner as before.

Let (S, a, b) and (S', a', b') be two strips . Then they can be composed as follows. Let A, B be the set of neighbors of a, b in S respectively, and define A', B' analogously. Consider the disjoint union of $S \setminus \{a, b\}$ and $S' \setminus \{a', b'\}$, and make A complete to A' and B complete to B'.

This method of composing two strips described above can be used as follows. Let S_0 be a graph which is the disjoint union of complete graphs with $|V(S)| = 2n$. We arrange the vertices into pairs $(a_1, b_1), \ldots, (a_n, b_n)$. For $i = 1, \ldots, n$, let (S'_i, a'_i, b'_i) be a strip and let S_i be the graph obtained by composing (S_{i-1}, a_i, b_i) and (S'_i, a'_i, b'_i). The resulting graph S_n is then called a *composition* of the strips (S'_i, a'_i, b'_i).

We are finally ready to state the structure theorem for quasi-line graphs [1] that we will use to prove our main results.

Theorem 2.1. *Let G be a connected, quasi-line graph. Then G is either a fuzzy circular interval graph or a composition of fuzzy linear interval strips.*

3 Circular interval graphs

We begin the proof of Theorem 1.1 by proving the result for circular interval graphs.

Theorem 3.1. *If G is a circular interval graph then $\chi(G) \leq \frac{3}{2}\omega(G)$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ in order clockwise, and let $n \equiv k \pmod{\omega}$, where $0 \leq k < \omega$ and $\omega = \omega(G)$. The proof breaks into three cases depending on the values of k and n. Note that in order for a coloring to be proper it suffices to show that every consecutive ω vertices on the circle have distinct colors. For an integer i, let $r(i) = i \pmod{\omega}$ if i is not divisible by ω, and let $r(i) = w$ otherwise.

(1) If $k \leq \frac{\omega}{2}$, then the theorem holds.

Let the function $c : V(G) \to \mathbb{N}$ be defined as

$$c(v_i) = \begin{cases} r(i) & \text{for} \quad i \leq n-k \\ \omega + i - n + k & \text{for} \quad n-k < i \leq n \end{cases}$$

It can be easily verified that any ω consecutive vertices on the circle all have distinct colors. So c describes a proper coloring of G and since $k \leq \frac{\omega}{2}$ we have used no more than $\frac{3\omega}{2}$ colors. This proves (1).

(2) If $k > \frac{\omega}{2}$ and $n \geq 2\omega$, then the theorem holds.

We define the function $c : V(G) \to \mathbb{N}$ as

$$c(v_i) = \begin{cases} i & \text{for} \quad i \leq k + \lfloor\frac{\omega}{2}\rfloor \\ r(i - k + \lfloor\frac{\omega}{2}\rfloor) & \text{for} \quad k + \lfloor\frac{\omega}{2}\rfloor < i \leq n - \lfloor\frac{\omega}{2}\rfloor \\ i + \lfloor\frac{3\omega}{2}\rfloor - n & \text{for} \quad i > n - \lfloor\frac{\omega}{2}\rfloor \end{cases}$$

Since $n \geq 2\omega$, it follows that

$$(n - \lfloor\frac{\omega}{2}\rfloor) - (k + \lfloor\frac{\omega}{2}\rfloor) \geq n - k - \omega \geq \omega.$$

Therefore, every ω consecutive vertices on the circle all have distinct colors. Once again, we have used no more than $\frac{3\omega}{2}$ colors so this proves (2).

It remains to prove that if $n < 2\omega$ then the theorem holds. Let $v \in V(G)$ and suppose that $|N_G(v)| < \frac{3}{2}\omega$. Inductively, $G \setminus \{v\}$ can be properly colored with $\frac{3}{2}\omega(G \setminus \{v\})$ colors. Since $\omega(G \setminus \{v\}) \leq \omega$, this coloring of $G \setminus \{v\}$ can be extended to a proper coloring of G using no more than $\frac{3}{2}\omega$ colors.

Therefore, we may assume that $|N_G(v)| \geq \frac{3}{2}\omega$. Since the set of vertices $S = N_G(v) \cup \{v\}$ can be expressed as the union of two cliques, the complement of the graph $G|S$ is bipartite. Therefore, $G|S$ is perfect as shown in [3] and so can be properly colored using at most ω colors. Furthermore, since $n < 2\omega$ it follows that the set of non-neighbors of v has no more than $\frac{\omega}{2}$ elements and hence can be properly colored using no more than $\frac{\omega}{2}$ colors. Therefore

G can be properly colored using no more than $\frac{3}{2}\omega$ colors. This completes the proof of the theorem.

□

4 Compositions of linear interval strips

In this section we prove Theorem 1.1 for compositions of linear interval strips (meaning that every strip is a linear interval graph rather than a fuzzy linear interval graph). We call a strip that is also a line graph a *line graph strip*. We begin with the following two lemmas.

Lemma 4.1. *The composition of two line graph strips is a line graph.*

Proof. Let (S_1, a_1, b_1) and (S_2, a_2, b_2) be two line graph strips, let R_1, R_2 be graphs such that $L(R_1) = S_1$ and $L(R_2) = S_2$. Suppose that $a_1 = u_1v_1 \in E(R_1)$. Let A be the set of edges of R_1 incident with both u_1,v_1, let B be the set of edges of R_1 incident with u_1 but not with v_1, and let C be the set of edges of R_1 incident with v_1 but not with u_1.

Since the set of neighbors of a_1 is a clique, every two edges in $A \cup B \cup C$ share a vertex. Suppose that both B and C are non-empty, then there exists a vertex x such that every edge in $B \cup C$ is incident with x. Let R_1' be the graph obtained from R_1 by deleting all the edges of B and adding $|B|$ new edges between v_1 and x. Then $S_1 = L(R_1')$. We may therefore assume that B is empty.

We proceed to compose the graphs R_1 and R_2 as follows. Let $a_i = u_iv_i \in E(R_i)$, and let $b_i = s_it_i \in E(R_i)$. The argument of the previous paragraph shows that we may assume that every edge of R_i incident with u_i is also incident with v_i, and every edge of R_i incident with s_i is also incident with t_i. Let $\tilde{R}_i = R_i \setminus \{a_i, b_i\}$ and let R be the graph obtained from the disjoint union of $\tilde{R}_1$ and $\tilde{R}_2$ by identifying v_1 with v_2 and t_1 with t_2. Then the composition of the strips (S_1, a_1, b_1) and (S_2, a_2, b_2) is the line graph of R, and this completes the proof. □

Before we proceed with the proof of the second lemma, we need to introduce one more concept. Distinct vertices u, v of a graph G are called *twins* if they are adjacent and have exactly the same neighbors in $V(G) \setminus \{u, v\}$. A *set of twins* in G is a set of vertices all of which are pairwise twins. If S is a set of twins in G, by *reducing* S we mean deleting all but one of the elements of S. Notice that if $G = L(H)$ we may assume that every pair of twins in G is a pair of parallel edges in H. Now reducing all twins in G is equivalent to deleting all parallel edges in H. Hence, after such a reduction, G still remains a line graph.

Lemma 4.2. *Let (S, a, b) be a linear interval strip. Let $A = N_G(a)$ and $B = N_G(b)$. Then there exists a graph R that with the following properties:*

1. *(R, x, y) is a line graph strip, $|N_R(x)| = |A|$, and $|N_R(y)| = |B|$*
2. *$\omega(R) \leq \omega(S)$*

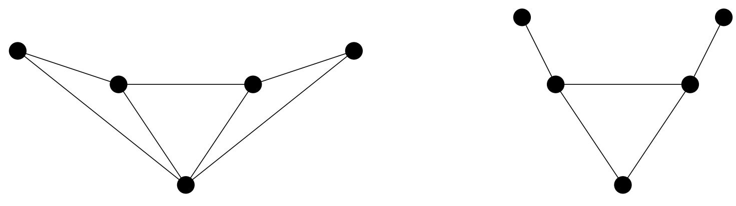

Figure 1:

3. For every proper coloring c_R of R there exists a proper coloring c_S of S with at most $\omega(S)$ colors such that $c_R(N_R(x)) = c_S(A)$ and $c_R(N_R(y)) = c_S(B)$

Proof. First, note that S can be properly colored using at most $\omega(S)$ colors since S is a linear interval graph, and therefore has at least one simplicial vertex. Let c be a proper coloring of S with at most $\omega(S)$ colors. By permuting the colors of c, we may assume that for some $1 \leq m \leq \omega(S)$, each of $1, \ldots, m$ appears in both A and B, and each of $m+1, \ldots, \omega(S)$ appears in at most one of A, B. Let A', B' be the subsets of A, B, respectively, such that the elements of B' and the elements of A' have colors $1, \ldots, m$. Let A'' denote the set $A \setminus A'$ and B'' denote the set $B \setminus B'$.

Let R be a graph with $V(R) = \{x, y\} \cup K \cup L \cup M$ such that $|K| = m$, $|L| = |A''|$, and $|M| = |B''|$, $K \cup L \cup M$ is a clique, x is $K \cup L$-complete, y is $K \cup M$-complete, and there are no other edges in the graph. We claim that R satisfies (1), (2), and (3).

To prove (1), we show that (R, x, y) is a line graph strip. Since $N_R(x) = K \cup L$ and $N_R(y) = K \cup M$ are both cliques, it follows that R is a strip. To show that R is a line graph we notice that by reducing all maximal sets of twins in R, we obtain a graph that is an induced subgraph of the graph E in Figure 1. Since E is the line graph of the graph F shown in the same figure, (1) follows.

To show that (2) holds we notice that the maximal cliques of R are $Q_1 = K \cup L \cup \{x\}$, $Q_2 = K \cup M \cup \{y\}$ and $Q_3 = K \cup L \cup M$. But $|Q_1| \leq |Q_3|$ and $|Q_2| \leq |Q_3|$; and $|Q_3| = |A| + |B| - m \leq \omega(S)$. Hence, $\omega(R) \leq \omega(S)$.

To prove (3), let c_R be a proper coloring of R. Since $\{x\} \cup K \cup L$, $\{y\} \cup K \cup M$, $K \cup L \cup M$ are cliques it follows that $|c_R(\{x\} \cup K \cup L)| = m + |A''| + 1$, $|c_R(\{y\} \cup K \cup M)| = m + |B''| + 1$, and that $c_R(K), c_R(L), c_R(M)$ are pairwise disjoint. We can therefore permute the colors of c to obtain a coloring c_S of S satisfying (3). This completes the proof of Lemma 4.2.

□

For a strip (S, a, b) we call a line graph strip (R, x, y) as in Lemma 4.2 the *line graph image* of (S, a, b). We are now ready to prove the main result of this section.

Theorem 4.3. *Let G be a connected, quasi-line graph that is a composition of linear interval strips. Then $\chi(G) \leq \frac{3}{2}\omega(G)$.*

Proof. Let $n > 0$ be an integer and let $\mathbb{S}$ be a family of strips (S_i, a_i, b_i) with $1 \leq i \leq n$ such that G is a composition of the members of $\mathbb{S}$. Let k be the number of members of $\mathbb{S}$ which are not line graph strips. The proof is by induction on k.

If $k = 0$ the result follows from Lemma 4.1 and Shannon's theorem. So we may assume $k > 0$ and (S_1, a_1, b_1) is not a line graph strip. Let (R, x, y) be the line graph image of (S_1, a_1, b_1). Let H be the graph obtained from G by deleting $V(S_1)$ and adding two new vertices a, b, such that a is complete to $N_G(a_1)$, b is complete to $N_G(b_1)$, and there are no other edges in H incident with a or b. Then (H, a, b) is a strip. Let F be the composition of (H, a, b) with (R, x, y). Since by Lemma 4.2 $|N_R(x)| = |N_{S_1}(a_1)|$, $|N_R(y)| = |N_{S_1}(b_1)|$, and $\omega(R) \leq \omega(S_1)$, it follows that $\omega(F) \leq \omega(G)$. Since F is a composition of linear interval strips at most $k - 1$ of which are not line graph strips, it follows inductively that $\chi(F) \leq \frac{3}{2}\omega(F) \leq \frac{3}{2}\omega(G)$. Let c_H be a proper coloring of H with at most $\frac{3}{2}\omega(H)$ colors. By Lemma 4.2, there exists a proper coloring c_{S_1} of S_1 with $\leq \omega(S_1)$ colors such that $c_H(N_R(x)) = c_{S_1}(N_{S_1}(a_1))$ and $c_H(N_R(y)) = c_{S_1}(N_{S_1}(b_1))$. Let c_G be a coloring of G defined as follows: for $v \in V(G) \cap V(F)$ let $c_G(v) = c_H(v)$ and for $v \in V(G) \cap V(S_1)$ let $c_G(v) = c_{S_1}(v)$. Then c_G is a proper coloring of G with at most $\frac{3}{2}\omega(G)$ colors. This completes the proof of Theorem 4.3. □

5 Quasi-line graphs with non-trivial homogeneous pairs

Theorems 3.1 and 4.3 establish the result of 1.1 for circular interval graphs and graphs that are compositions of liner interval strips. In order to complete the proof of Theorem 1.1, we need to handle non-trivial homogeneous pairs in quasi-line graphs. To do that, we use induction on the number of non-trivial homogeneous pairs in a graph, and theorems 3.1 and 4.3 serve as the base case of the induction.

Lemma 5.1. *Let G be a quasi-line graph and let (A, B) be a non-trivial homogeneous pair of cliques. Then there exists a graph H with the following properties:*

1. *H is a quasi-line graph with one fewer non-trivial homogeneous pair than G.*
2. *$\omega(H) \leq \omega(G)$.*
3. *For every proper coloring of H there is a proper coloring of G with the same number of colors.*

Proof. By Theorem 2.1 G is either a fuzzy circular interval graph or a composition of fuzzy linear interval strips; and in the latter case, A and B are included in the vertex set of some strip, which is a fuzzy linear interval graph. Let C be the set of vertices of G that are A-complete and B-complete, D be the set of vertices of G that are A-complete and B-anticomplete, E the set of vertices of G that are A-anticomplete and B-complete, and F the set of vertices of G that are A-anticomplete and B-anticomplete.

Since the elements of C are complete to both A and B, it follows from the structure of fuzzy circular interval graphs and fuzzy linear interval strips that C is a clique. Moreover,

$A \cup C$ is a clique and so $X = A \cup B \cup C$ is the union of two cliques. This means that the complement of $G|X$ is bipartite, and so $G|X$ is perfect as shown in [3]. Let c be a proper coloring of $G|X$ with $\omega(G|X)$ colors. This implies that for some $1 \leq m \leq \omega(G|X)$, we may assume by permuting the colors, that each of the colors $1, \dots, m$ appear in both A, B and each of the colors $m+1, \dots, \omega(G|X)$ appear in at most one of A, B. Let A', B' be the subsets of A, B, respectively, so that the elements of A' and the elements of B' have colors $1, \dots, m$.

We construct the graph H as follows. Let $V(H) = V(G)$ and $E(H) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$ where

$$\begin{aligned}
E_1 &= E(G|(C \cup D \cup E \cup F)) \\
E_2 &= \{e = uv \mid u \neq v \text{ and } u \in A, v \in A \text{ or } u \in B, v \in B\} \\
E_3 &= \{e = uv \mid u \in A \setminus A', v \in B \text{ or } u \in B \setminus B', v \in A\} \\
E_4 &= \{e = uv \mid u \in A, v \in C \cup D\} \\
E_5 &= \{e = uv \mid u \in B, v \in C \cup E\}
\end{aligned}$$

We claim that for every $v \in V(H)$, $N_H(v)$ is the union of two cliques in H. If $v \in C \cup E \cup F$, then $G|N_G(v) = H|N_H(v)$, and therefore the $N_H(v)$ is the union of two cliques in H, and the claim holds. If $v \in A$, then $N_H(v) \subseteq A \cup C \cup D \cup B$, and since both $A \cup D$ and $B \cup C$ are cliques, the claim follows. Similarly, the claim holds for $v \in B$, and so we may assume that $v \in C$. Since (A, B) is a non-trivial homogeneous pair in G, there exist vertices $a \in A$ and $b \in B$ that are non-adjacent in G. Since a, b, f are not three pairwise non-adjacent vertices in $N_G(v)$ for any $f \in F$, it follows that v is anticomplete to F in G, and therefore in H. Let $D' = N_G(v) \cap D$ and $E' = N_G(v) \cap E$. Then, since C is a clique,

$$N_G(v) = N_H(v) = A \cup B \cup (C \setminus \{v\}) \cup D' \cup E'.$$

If some vertex $c \in C$ has both a non-neighbor $d \in D'$ and a non-neighbor $e \in E'$, then c, d, e are three pairwise non-adjacent vertices in $N_G(v)$, contrary to the fact that G is a quasi-line graph. Therefore, every vertex of C is either complete to D' or to E'. Let C_1 be the vertices of C that are complete to D', let $C_2 = C \setminus (C_1 \cup \{v\})$; and let $K_1 = A \cup D' \cup C_1$ and $K_2 = B \cup E' \cup C_2$. Then both K_1 and K_2 are cliques in both H and G, and $N_H(v) = K_1 \cup K_2$. This completes the proof of the claim.

The claim in the previous paragraph implies that H is a quasi-line graph. However, (A, B) is no longer a non-trivial homogeneous pair. Hence, H has the same non-trivial homogeneous pairs as G except (A, B), and (1) holds.

To verify (2) we just need to check that the edges of H that were not present in G do not increase the size of the maximum clique. The only such edges are those between $A \setminus A'$ and B and between $B \setminus B'$ and A, which create the new cliques $A \cup C \cup (B \setminus B')$ and $B \cup C \cup (A \setminus A')$. But both of these cliques have at most $\omega(G)$ members since they were colored using no more than $\omega(G|X) \leq \omega(G)$ colors. Hence $\omega(H) \leq \omega(G)$.

Finally we need to verify that every proper coloring of H can be used to obtain a proper coloring of G using the same number of colors. Let c_H be a coloring of H. The only edges that are present in G and not present in H are those between the members of A' and the members of B'. Recall that these two sets were chosen so that $G|(A' \cup B')$ can be properly colored with $m = |A'| = |B'|$ colors. Since A' and B' are both cliques, it follows that every proper coloring of $H|(A' \cup B')$, and in particular c_H, must use at least m colors. Also, since permuting colors between the members of A' or B' does not affect the coloring of the rest of the graph, we can permute the colors of c_H that appear on $A' \cup B'$ so that it becomes a proper coloring of $G|(A' \cup B')$. Keeping the colors of the vertices of $V(G) \setminus (A' \cup B')$ unchanged, we obtain a proper coloring of G, with the same number of colors as used by c_H. Thus (3) holds and this completes the proof of the lemma. □

Now we are ready to prove the main theorem.

Proof of Theorem 1.1. Let G be a quasi-line graph. The proof is by induction on the number of non-trivial homogeneous pairs in G. We may assume that G is connected. If G has no non-trivial homogeneous pairs then, by theorem 2.1, G is either a circular interval graphs, or a composition of liner interval strips, and the result follows from Theorem 3.1 and Theorem 4.3. Otherwise, let (A, B) be a non-trivial homogeneous pair in G. Let H be a graph as in Lemma 5.1. Inductively, since H has one fewer non-trivial homogeneous pair than G, H can be properly colored with at most $\frac{3}{2}\omega(H)$ colors. It now follows from Lemma 5.1 that $\omega(H) \leq \omega(G)$ and that G can be properly colored with at most $\frac{3}{2}\omega(H)$ colors. Therefore, $\chi(G) \leq \frac{3}{2}\omega(G)$. This completes the proof. □

Finally, we provide an infinite family of quasi-line graphs for which the bound in Theorem 1.1 is tight. Let $k \in \mathbb{N}$ be even and let G be a circular interval graph with $|V(G)| = 3k-1$, such that for every k consecutive vertices on the circle there is an interval containing all of these vertices and no others. Then it is easy to verify that $\omega(G) = k$. Furthermore, let c be a proper coloring of G using m colors. Notice that for every three vertices of G, at least two of them are less than $\omega(G) - 1$ vertices apart on the circle, and therefore lie in some common interval and are adjacent. Hence, no three vertices can have the same color. Therefore, $m \geq \lceil \frac{|V(G)|}{2} \rceil = \lceil \frac{3}{2}\omega(G) - \frac{1}{2} \rceil$ and so $\chi(G) \geq \frac{3}{2}\omega(G)$. Then by Theorem 1.1 $\chi(G) = \frac{3}{2}\omega(G)$.

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