# Three steps towards Gyárfás' conjecture

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**Abstract**

Gyárfás conjectured in 1985 that for all $k, l$, every graph with no clique of size more than $k$ and no odd hole of length more than $l$ has chromatic number bounded by a function of $k, l$. We prove three weaker statements:

- Every triangle-free graph with sufficiently large chromatic number has an odd hole of length different from five;
- For all $l$, every triangle-free graph with sufficiently large chromatic number contains either a 5-hole or an odd hole of length more than $l$;
- For all $k, l$, every graph with no clique of size more than $k$ and sufficiently large chromatic number contains either a 5-hole or a hole of length more than $l$.

# 1 Introduction

All graphs in this paper are finite, and without loops or parallel edges. A *hole* in a graph $G$ is an induced subgraph which is a cycle of length at least four, and an *odd hole* means a hole of odd length. (The *length* of a path or cycle is the number of edges in it, and we sometimes call a hole of length $n$ an *$n$-hole*.) A famous conjecture of A. Gyárfás [1] from 1985 asserts:

**1.1 Conjecture:** *For all integers $k, l$ there exists $n(k, l)$ such that every graph $G$ with no clique of cardinality more than $k$ and no odd hole of length more than $l$ has chromatic number at most $n(k, l)$.*

We might as well assume that $k \geq 2$, and $l \geq 3$ and is odd; and in a recent paper [3], two of us proved that this is true for all pairs $(k, l)$ when $l = 3$. No other cases have been settled, and the cases when $k = 2$ are presumably the simplest to attack next. Here we settle the first open case, when $k = 2$ and $l = 5$. That asserts that all pentagonal graphs have bounded chromatic number, where we say a graph is *pentagonal* if every induced odd cycle in it has length five (and in particular, it has no triangles). Pentagonal graphs might all be 4-colourable as far as we know (the 11-vertex Grötzsch graph is pentagonal and not 3-colourable), but at least they do indeed all have bounded chromatic number. The following is our main result:

**1.2** *Every pentagonal graph is* 82200*-colourable.*

The proof of 1.2 occupies almost the whole paper. (Much of the proof needs just that $G$ is triangle-free and has no odd hole of length more than $l$, for any fixed $l$, and so we have written it in this generality wherever we could.) We prove:

- if $G$ has no triangle and no odd hole of length more than $l$, and for every vertex $v$ the set of vertices with distance at most three from $v$ has chromatic number at most some $k$, then $\chi(G)$ is bounded by a function of $k$ and $l$;
- if $G$ is pentagonal, and $\chi(G)$ is large, then there is an induced subgraph with large chromatic number in which for every vertex $v$ the set of vertices with distance at most three from $v$ has chromatic number at most 20.

Together these imply that every pentagonal graph has bounded chromatic number. Both of these are consequences of a lemma, a variant of a theorem of [3], asserting roughly that for all $l$, if $G$ is triangle-free and has no odd hole of length more than $l$, and $\chi(G)$ is large, then there is an induced subgraph $H$ such that for some vertex $v_0$ of $H$, if we partition $V(H)$ by distance in $H$ from $v_0$, then all these "level sets" are stable except for one with large $\chi$. We prove this lemma first, and then apply it to prove the two bulleted statements in later sections.

Gyárfás' conjecture 1.1 has a number of other interesting special cases that still remain open; for instance

- **Conjecture:** For all $l$ every triangle-free graph $G$ with sufficiently large chromatic number has an odd hole of length more than $l$;
- **Conjecture:** For all $k, l$ every graph with no clique of size more than $k$ and sufficiently large chromatic number has a hole of length more than $l$.

At the end of this paper, we prove that both these statements are true if in addition we assume that $G$ contains no 5-hole. More precisely, we prove the next two results, where $\omega(G)$ denotes the size of the largest clique of $G$:

**1.3** *Let $l \geq 2$ be an integer, and let $G$ be a triangle-free graph with no 5-hole and no odd hole of length more than $2l+1$. Then $\chi(G) \leq (l+1)4^{l-1}$.*

**1.4** *Let $l \geq 3$ be an integer, and let $G$ be a graph with no 5-hole and no hole of length more than $l$. Then*

$$\chi(G) \leq (2l-2)^{2^{\omega(G)}}.$$

The last was proved (but not published) by the second author some time ago, and improves on [2].

## 2 Lollipops

In [1], Gyárfás gave a neat proof that for any fixed path $P$, all graphs with no induced subgraph isomorphic to $P$ and with bounded clique number also have bounded chromatic number, and in this section we use basically the same proof for a lemma that we need later. If $X \subseteq V(G)$, the subgraph of $G$ induced on $X$ is denoted by $G[X]$, and we sometimes write $\chi(X)$ for $\chi(G[X])$ when there is no danger of ambiguity. If $x \in V(G)$ and $Y \subseteq V(G)$, the *distance* in $G$ of $x$ from $Y$ (or of $Y$ from $x$) is the length of the shortest path containing $x$ and a vertex in $Y$. Let us say a *lollipop* in a graph $G$ is a pair $(C, T)$ where $C \subseteq V(G)$ and $T$ is an induced path with vertices $t_1\text{-}\cdots\text{-}t_k$ in order, say, with $k \geq 2$, satisfying:

- $V(T) \cap C = \emptyset$;
- $G[C]$ is connected;
- $t_k$ has a neighbour in $C$; and
- $t_1, \ldots, t_{k-1}$ have no neighbours in $C$.

The *cleanliness* of a lollipop $(C, T)$ in $G$ is the maximum $l$ such that $t_1, \ldots, t_l$ all have distance (in $G$) at least three from $C$ (or 0 if $t_1$ has distance two from $C$). We call $t_1$ the *end* of the lollipop. If $(C, T)$ and $(C', T')$ are lollipops in $G$, we say the second is a *licking* of the first if $C' \subseteq C$, and they have the same end, and $T$ is a subpath of $T'$, and $V(T') \subseteq V(T) \cup C$ (and consequently the cleanliness of $(C', T')$ is at least that of $(C, T)$). We observe first:

**2.1** *Let $(C, T)$ be a lollipop in $G$, and let $C' \subseteq C$ be non-null, such that $G[C']$ is connected. Then there is a path $T'$ of $G$ such that $(C', T')$ is a licking of $(C, T)$.*

**Proof.** Let $T$ be $t_1\text{-}\cdots\text{-}t_k$, where $(C, T)$ has end $t_1$. Since $t_k$ has a neighbour in $C$, there is a path $P$ of $G$ with one end $t_k$, and with $V(P) \subseteq C \cup \{t_k\}$, such that the other end of $P$ has a neighbour in $C'$. Choose a minimal such path $P$. Then $V(P) \cap C' = \emptyset$, and $P' = T \cup P$ is an induced path. No vertex of $P'$ has a neighbour in $C'$ except its last, and so $(C', T \cup P)$ is a licking of $(C, T)$ as required. This proves 2.1. ∎

For a vertex $v$ of $G$, we denote the set of neighbours of $v$ in $G$ by $N(v)$ or $N_G(v)$, and for $r \ge 1$, we denote the set of vertices at distance exactly $r$ from $v$ by $N^r(v)$ or $N^r_G(v)$. We need the following:

**2.2** *Let $h, \kappa \ge 0$ be integers. Let $G$ be a graph such that $\chi(N^2(v)) \le \kappa$ for every vertex $v$; and let $(C,T)$ be a lollipop in $G$, with $\chi(C) > h\kappa$. Then there is a licking $(C',T')$ of $(C,T)$, with cleanliness at least $h$ more than the cleanliness of $(C,T)$, and such that $\chi(C') \ge \chi(C) - h\kappa$.*

**Proof.** We proceed by induction on $h$. If $h = 0$ we may take $(C',T') = (C,T)$; so we assume that $h > 0$, and that the result holds for $h-1$. Let $(C,T)$ have cleanliness $c$ say (where possibly $c = 0$), and let $T$ have vertices $t_1\text{-}\cdots\text{-}t_k$ in order, where $t_1$ is the end. Thus $t_i$ has distance at least three from $C$ for $1 \le i \le c$, and so $k \ge c+2$. Since $\chi(N^2(t_{c+1})) \le \kappa$, it follows that

$$\chi(C \setminus N^2(t_{c+1})) \ge \chi(C) - \kappa > (h-1)\kappa \ge 0,$$

and so there is a component $C''$ of $C \setminus N^2(t_{c+1})$ with $\chi(C'') > (h-1)\kappa$. By 2.1, there is a licking $(C'',T'')$ of $(C,T)$. Since $t_{c+1}$ has distance at least three from $C''$, it follows that $(C'',T'')$ has cleanliness at least $c+1$. From the inductive hypothesis, there is a licking $(C',T')$ of $(C'',T'')$ and hence of $(C,T)$ that satisfies the theorem. This proves 2.2. ∎

## 3 Stable levelling

Let $G$ be a graph. A *levelling* $\mathcal{L}$ in $G$ is a sequence $L_0, L_1, \ldots, L_k$ of disjoint subsets of $V(G)$, with the following properties:

- $|L_0| = 1$;
- for each $i$ with $1 \le i \le k$, every vertex in $L_i$ has a neighbour in $L_{i-1}$; and
- for $0 \le i, j \le k$ with $|j-i| > 1$, there are no edges between $L_i$ and $L_j$.

The levelling $\mathcal{L}$ is called *stable* if each of the sets $L_0, \ldots, L_{k-1}$ is stable (we do not require $L_k$ to be stable). For $1 \le i \le k$, a *parent* of $v \in L_i$ is a neighbour $u$ of $v$ in $L_{i-1}$ (and we also say $v$ is a *child* of $u$).

The next result is a variant of a theorem proved in [3]; we could use that theorem directly, but the modification here works better numerically. Let the *odd hole number* of $G$ be the length of the longest induced odd cycle in $G$ (or 1, if $G$ is bipartite). If $L_0, \ldots, L_k$ is a stable levelling, we call $L_k$ its *base*.

**3.1** *Let $G$ be a triangle-free graph with odd hole number at most $2l+1$, such that $\chi(N^2(v)) \le \kappa$ for every vertex $v$. Let $L_0, L_1, \ldots, L_k$ be a levelling in $G$. Then there is a stable levelling in $G$ with base of chromatic number at least $(\chi(L_k) - (2l-1)\kappa)/2$.*

**Proof.** We may assume $l \ge 1$, since otherwise $G$ is bipartite and the result is trivial. Also we may assume that $\chi(L_k) > (2l-1)\kappa$, because otherwise the stable levelling $L_0, L_1$ satisfies the theorem. We proceed by induction on $|V(G)|$, and so we may assume:

- $V(G) = L_0 \cup L_1 \cup \cdots \cup L_k$;

- $G[L_k]$ is connected; and
- for $0 \le i < k$ and every vertex $u \in L_i$, there exists $v \in L_{i+1}$ such that $u$ is its only parent.

Let $L_0 = \{s_0\}$, and inductively for $1 \le i \le k$, choose $s_i \in L_i$ such that $s_{i-1}$ is its only parent. Then $s_0$-$s_1$-$\cdots$-$s_k$ is an induced path $S$ say.

Now $s_{k-2}$ has no neighbour in $L_k$, so $(L_k, s_{k-2}\text{-}s_{k-1})$ is a lollipop. By 2.2, there is a licking of this lollipop, say $(C', T')$, with cleanliness at least $2l-1$ and with $\chi(C') \ge \chi(L_k) - (2l-1)\kappa$. Let the first $2l-1$ vertices of $T'$ be $s_{k-2}$-$s_{k-1}$-$t_1$-$\cdots$-$t_{2l-3}$.

Let $N(S)$ be the set of vertices of $G$ not in $S$ but with a neighbour in $S$. If $v \in L_i \cap N(S)$, then $v$ is adjacent to exactly one of $s_i, s_{i-1}$ and has no other neighbour in $S$; because every neighbour of $v$ belongs to one of $L_{i-1}, L_i, L_{i+1}$, and $G$ is triangle-free, and $v$ is not adjacent to $s_{i+1}$ since $s_i$ is the only parent of $s_{i+1}$. So every vertex in $L_i \cap N(S)$ has one of two possible types. We say the *type* of a vertex $v \in L_i \cap N(S)$ is $\alpha$ where $\alpha = 1$ or 2 depending whether $v$ is adjacent to $s_{i-1}$ and not to $s_i$, or adjacent to $s_i$ and not to $s_{i-1}$.

Let us fix a type $\alpha$. Let $V(\alpha)$ be the minimal subset of $V(G) \setminus V(S)$ such that

- every vertex in $N(S)$ of type $\alpha$ belongs to $V(\alpha)$; and
- for every vertex $v \in V(G) \setminus (V(S) \cup N(S))$, if some parent of $v$ belongs to $V(\alpha)$ then $v \in V(\alpha)$.

Consequently, for every vertex $v \in V(\alpha)$, there is a path starting at $v$ and ending at some vertex in $N(S)$ of type $\alpha$, such that each vertex of the path (except $v$) is the parent of the previous vertex, and no vertex of the path belongs to $N(S)$ except the last.

There are only two types $\alpha$, and so there is a type $\alpha$ such that $\chi(V(\alpha) \cap C') \ge \chi(C')/2 > 0$. Let $C$ be the vertex set of a component of $G[V(\alpha) \cap C']$ with maximum chromatic number, so

$$\chi(C) \ge \chi(C')/2 \ge (\chi(L_k) - (2l-1)\kappa)/2.$$

By 2.1, there is a path $T$ such that $(C, T)$ is a licking of $(C', T')$.

Let $J_k = C$, and for $i = k-1, k-2, \ldots, 1$ choose $J_i \subseteq V(\alpha) \cap L_i$ minimal such that every vertex in $J_{i+1} \setminus N(S)$ has a neighbour in $J_i$. It follows from the cleanliness of $(C', T')$ that $J_{k-1} \cap N(S) = \emptyset$, and no vertex in $J_{k-1}$ is adjacent to any of $s_{k-2}, s_{k-1}, t_1, \ldots, t_{2l-3}$.

(1) *For $1 \le i \le k-1$, if $v \in J_i$ and $v$ is nonadjacent to $s_i$, then there is an induced path $P_v$ between $v$ and $s_i$ of length at least $2l-3+2(k-i)$ with interior in $L_{i+1} \cup \cdots \cup L_k$, such that*

- *if $i \le k-2$, no vertex in $J_i$ different from $v$ has a neighbour in the interior of $P_v$*
- *if $i = k-1$, and $u \in J_i \setminus \{v\}$ has a neighbour in the interior of $P_v$, then the induced path between $u, s_{k-1}$ with interior in $V(P_v)$ has length at least $2l-1$.*

Since $v \in J_i$, $v$ has a neighbour in $J_{i+1} \setminus N(S)$ with no other parent in $J_i$; and so there is a path $v = p_i$-$p_{i+1}$-$\cdots$-$p_k$ such that

- $p_j \in J_j$ for $i \le j \le k$
- $p_j \notin N(S)$ for $i < j \le k$

- $p_{j-1}$ is the only parent of $p_j$ in $J_{j-1}$ for $i < j \leq k$.

Since $p_{k-1} \in J_{k-1}$, and no vertex in $J_{k-1}$ is adjacent to any of $s_{k-2}, s_{k-1}, t_1, \dots, t_{2l-3}$, it follows that there is an induced path from $p_{k-1}$ to $s_{k-1}$ with interior in $L_k$ containing all of $t_1, \dots, t_{2l-3}$ and at least one more vertex of $L_k$, and therefore with length at least $2l - 1$. Its union with the path $p_i$-$\cdots$-$p_{k-1}$ and the path $s_{k-1}$-$s_i$ is an induced path between $v$ and $s_i$, of length at least $2l - 3 + 2(k - i)$. If $u \in J_i \setminus \{v\}$ and has a neighbour in the interior of $P_v$, then since $u$ is nonadjacent to all of $s_{i+1}, \dots, s_{k-1}, p_{i+1}, \dots, p_{k-1}$ (because $u$ has no neighbour in $L_{i+2} \cup \cdots \cup L_k$, and $s_{i+1}$ has a unique parent $s_i$, and $p_{i+1}$ has no parent in $J_i$ except $p_i$), it follows that $i = k - 1$; and since no vertex in $J_{k-1}$ is adjacent to any of $s_{k-2}, s_{k-1}, t_1, \dots, t_{2l-3}$, this proves (1).

For $1 \leq i \leq k$ and for every vertex $v \in J_i$, either $v \in N(S)$ or it has a parent in $J_{i-1}$; and so there is a path $r_i$-$r_{i-1}, \dots, r_h$ for some $h \leq i$, such that $r_j \in J_j$ for $h \leq j \leq i$, and $r_h \in N(S)$, and $r_j \notin N(S)$ for $h + 1 \leq j \leq i$. Since $r_h$ has a neighbour in $S$, one of

$$r_i\text{-}r_{i-1}\text{-}\cdots\text{-}r_h\text{-}s_{h-1}\text{-}s_h\text{-}s_{h+1}\text{-}\cdots\text{-}s_i,$$

$$r_i, r_{i-1}\text{-}\cdots\text{-}r_h\text{-}s_h\text{-}s_{h+1}\text{-}\cdots\text{-}s_i$$

is an induced path (the first if $\alpha = 1$ and the second if $\alpha = 2$). We choose some such path and call it $R_v$. Note that for all $v \in J_1 \cup \cdots \cup J_k$, the path $R_v$ has even length if $\alpha = 1$, and odd length otherwise.

(2) *For $0 \leq i \leq k - 1$, $J_i$ is stable.*

Suppose that $u, v \in J_i$ are adjacent. Since $G$ is triangle-free, not both $u, v \in N(S)$. Suppose that $u \in N(S)$, and hence $v \notin N(S)$. Since $N(S) \cap J_{k-1} = \emptyset$ it follows that $i \leq k - 2$. Consequently $u$ has no neighbour in the interior of $P_v$, where $P_v$ is as in (1), and so $P_v \cup R_v$, $s_i$-$P_v$-$v$-$u$-$R_u$-$s_i$ are both holes of length at least $2l + 2$, of different parity, which is impossible. So $u, v \notin N(S)$. We claim that there is a path $P$ of length at least $2l - 1$, from one of $u, v$ to $s_i$, with interior in $L_{i+1} \cup \cdots \cup L_k$, such that the other (of $u, v$) has no neighbour in its interior. For if $u$ has no neighbour in the interior of $P_v$ then we may take $P = P_v$, where $P_v$ is as in (1); and if $u$ has such a neighbour, let $P$ be the induced path between $u$ and $s_i$ with interior a subset of the interior of $P_v$. Note that in the second case, $v$ has no neighbour in the interior of $P$, since $G$ is triangle-free. This proves that the desired path $P$ exists; say from $v$ to $s_i$. Now the union of $P$ and $R_v$ is a hole of length at least $2l + 2$, and so $P, R_v$ have the same parity. But the union of $P$ and the path $v$-$u$-$R_u$-$s_i$ is also a hole, of length at least $2l + 3$, and since $R_u, R_v$ have the same parity this is impossible. This proves (2).

If $\alpha = 1$ let $M_i = \{s_i\} \cup J_i$ for $0 \leq i \leq k$, and if $\alpha = 2$ let $M_0 = \{s_1\}, M_i = \{s_{i+1}\} \cup J_i$ for $1 \leq i < k$, and $M_k = J_k$. In each case $M_0, \dots, M_k$ is a levelling satisfying the theorem. This proves 3.1. ∎

We deduce:

**3.2** *Let $G$ be pentagonal, and let $n \geq 1$ be an integer. If $\chi(G) \geq 10n - 9$, there is a stable levelling in $G$ with base of chromatic number at least $n$.*

**Proof.** Let $G'$ be a component of $G$ with $\chi(G') = \chi(G)$. Choose $v_0 \in V(G')$, and for $i \geq 0$ let $L_i$ be the set of vertices in $G'$ with distance $i$ from $v_0$. There exists $k$ such that $\chi(L_k) \geq \chi(G)/2$ and hence $\chi(L_k) \geq 5n-4$. Now $L_0, \ldots, L_k$ is a levelling in $G$. By 3.1, taking $l = 2$ and $\kappa = n-1$, either there is a levelling $M_0, M_1, M_2$ with $\chi(M_2) \geq n$, necessarily stable, or there is a stable levelling $M_0, \ldots, M_k$ in $G$ with $\chi(M_k) \geq (\chi(L_k) - 3(n-1))/2 \geq n - 1/2$. In either case the theorem holds. ∎

We also include, for convenient reference:

**3.3** *Let $G$ be pentagonal such that $\chi(N^2(v)) \leq 5$ for every vertex $v$, and let $n \geq 1$ be an integer. If $\chi(G) \geq 4n + 27$, there is a stable levelling in $G$ with base of chromatic number at least $n$.*

**Proof.** As before, choose a levelling $L_0, \ldots, L_k$ with $\chi(L_k) \geq \chi(G)/2$ and hence $\chi(L_k) \geq 2n + 14$. By 3.1 with $\kappa = 5$, there is a stable levelling $M_0, \ldots, M_k$ in $G$ with $\chi(M_k) \geq (\chi(L_k) - 15)/2 \geq n - 1/2$ and the result follows. ∎

# 4 Reducing to bounded radius

Let $L_0, \ldots, L_k$ be a levelling. If $0 \leq i \leq j \leq k$ and $u \in L_i$ and $v \in L_j$, and there is a path between $u, v$ of length $j - i$ with one vertex in each of $L_i, L_{i+1}, \ldots, L_j$, we say that $u$ is an *ancestor* of $v$ and $v$ is a *descendant* of $u$.

**4.1** *Let $G$ be a triangle-free graph with odd hole number at most $2l+1$. For $r = 2, 3$, let $\chi(N^r(v)) \leq \kappa_r$ for every vertex $v$. Then $\chi(G) \leq (12l - 2)\kappa_2 + 4\kappa_3 + 8$.*

**Proof.** Suppose that $\chi(G) > (12l - 2)\kappa_2 + 4\kappa_3 + 8$. There is a levelling in $G$ with base of chromatic number at least $\chi(G)/2$, and so by 3.1, there is a stable levelling $L_0, \ldots, L_k$ in $G$ with

$$\chi(L_k) \geq \chi(G)/4 - (l - 1/2)\kappa_2 > 2l\kappa_2 + \kappa_3 + 2.$$

We may choose it in addition such that $G[L_k]$ is connected, and for $0 \leq i < k$ every vertex in $L_i$ has a descendant in $L_k$. Since $\chi(L_k) > 1$ it follows that $k > 1$. Choose $a_{k-2} \in L_{k-2}$. Let $X_1$ be the set of descendants of $a_{k-2}$ in $L_k$; thus $\chi(X_1) \leq \kappa_2$, and since $\chi(L_k) > \kappa_2$, there is a component $C_1$ of $G[L_k \setminus X_1]$ with

$$\chi(C_1) \geq \chi(L_k) - \kappa_2 > (2l - 1)\kappa_2 + \kappa_3 + 2.$$

Since $G[L_k]$ is connected and $X_1 \neq \emptyset$, there exists $a_k \in X_1$ with a neighbour in $C_1$. Let $a_{k-1}$ be a parent of $a_k$ and child of $a_{k-2}$.

Let $X_2$ be the set of neighbours of $a_k$ in $C_1$; then $X_2$ is stable and nonempty, and since $\chi(C_1) > 1$, there is a component $C_2$ of $C_1 \setminus X_2$ with

$$\chi(C_2) \geq \chi(C_1) - 1 > (2l - 1)\kappa_2 + \kappa_3 + 1,$$

and a neighbour $b_k$ of $a_k$ with a neighbour in $C_2$. Let $b_{k-1}$ be a parent of $b_k$. Thus $b_{k-1}, a_{k-2}$ are nonadjacent since $X_1 \cap C_1 = \emptyset$. Also $b_{k-1}, a_{k-1}$ are nonadjacent since $L_{k-1}$ is stable, and $b_{k-1}, a_k$ are nonadjacent since $G$ is triangle-free. Consequently $a_{k-2}$-$a_{k-1}$-$a_k$-$b_k$-$b_{k-1}$ is an induced path of $G$.

Let $X_3$ be the set of all children of $b_{k-1}$; then since $X_3$ is stable, and $\chi(C_2) > 1$, it follows that there is a component $C_3$ of $C_2 \setminus X_3$ with

$$\chi(C_3) \geq \chi(C_2) - 1 > (2l-1)\kappa_2 + \kappa_3,$$

and a child $c_k$ of $b_{k-1}$ with a neighbour in $C_3$, taking $c_k = b_k$ if $b_k$ has a neighbour in $C_3$. Thus $(C_3, b_{k-1}\text{-}c_k)$ is a lollipop. By 2.2, since $\chi(C_3) > (2l-1)\kappa_2$, there is a licking $(C_4, T)$ of $(C_3, b_{k-1}\text{-}c_k)$, with cleanliness at least $2l-1$, such that

$$\chi(C_4) \geq \chi(C_3) - (2l-1)\kappa_2 > \kappa_3.$$

Let $T$ have vertices $t_1\text{-}t_2\text{-}t_3\text{-}\cdots\text{-}t_m$ say, where $m \geq 2l$ and $t_1 = b_{k-1}$ and $t_2 = c_k$. Note that if $b_k \neq c_k$ then $b_k$ has no neighbour in $C_3$ and in particular $b_k$ has no neighbour in $T$ except $t_1$.

Let $X_4$ be the set of all vertices of $C_4$ with distance three from $b_{k-1}$. Since $\chi(X_4) \leq \kappa_3$, and $\chi(C_4) - \kappa_3 > 0$, there is a component $C_5$ of $C_4 \setminus X_4$. By 2.1, there is a licking $(C_5, S)$ say of $(C_4, T)$. Let $S$ have vertices $t_1\text{-}\cdots\text{-}t_n$ say where $n \geq m$. Let $t_{n+1} \in V(C_5)$ be adjacent to $t_n$, and let $d_{k-1}$ be a parent of $t_{n+1}$. Choose $i$ with $1 \leq i \leq n+1$ minimum such that $d_{k-1}$ is adjacent to $t_i$. Note that $d_{k-1}$ is nonadjacent to all of $t_1, \ldots, t_h$ since $(C_4, T)$ has cleanliness at least $2l-1$ and hence so does $(C_5, S)$; and so $i > 2l-1$. Let $P$ be the path $t_2\text{-}t_3\text{-}t_i\text{-}d_{k-1}$. This path $P$ is induced and has length $i - 1 \geq 2l - 1$.

Let $d_{k-2}$ be a parent of $d_{k-1}$. Now $a_{k-2} \neq d_{k-2}$, since $a_{k-2}$ has no descendant in $C_1$, and $d_{k-2}$ has a descendant $t_{n+1}$ in $C_5$ and hence in $C_1$. For the same reason $a_{k-1} \neq d_{k-1}$, and $b_{k-1} \neq d_{k-1}$ since $b_{k-1}$ has no children in $C_3$. Also, $b_{k-1}, d_{k-2}$ are nonadjacent, since $t_{n+1}$ is in $C_5$ and so there is no three-edge path between $b_{k-1}$ and $t_{n+1}$.

Since $L_0, \ldots, L_{k-1}$ are stable, there is an induced path between $b_{k-1}, d_{k-1}$ of even length with interior in $L_0 \cup \cdots \cup L_{k-2}$, and its union with the path $b_{k-1}\text{-}t_2\text{-}P\text{-}d_{k-1}$ is a hole of length at least $2l+2$, which consequently has even length; and so $P$ has odd length. Now there is an even induced path $Q$ between $a_{k-1}, d_{k-1}$ with interior in $L_0 \cup \cdots \cup L_{k-2}$, not containing any neighbour of $b_{k-1}$; for if $a_{k-1}, d_{k-2}$ are adjacent then the path $a_{k-1}\text{-}d_{k-2}\text{-}d_{k-1}$ satisfies our requirements, and otherwise any even induced path between $a_{k-2}, d_{k-2}$ with interior in $L_0 \cup \cdots \cup L_{k-3}$ (extended by the edges $a_{k-1}a_{k-2}$ and $d_{k-1}d_{k-2}$) provides the desired path. If $b_k \neq c_k$ then

$$a_{k-1}\text{-}a_k\text{-}b_k\text{-}b_{k-1}\text{-}c_k\text{-}P\text{-}d_{k-1}\text{-}Q\text{-}a_{k-1}$$

is an odd hole of length at least $2l+5$, while if $b_k = c_k$ then

$$a_{k-1}\text{-}a_k\text{-}b_k\text{-}P\text{-}d_{k-1}\text{-}Q\text{-}a_{k-1}$$

is an odd hole of length at least $2l+3$, in either case a contradiction. This proves 4.1. ∎

## 5 The Grötzsch graph

Let $G$ be a graph, and $H$ an induced subgraph of $G$. We say a levelling $L_0, \ldots, L_k$ in $G$ is *over $H$* if $V(H) \subseteq L_k$. An *$n$-covering (in $G$, over $H$)* is a sequence of graphs $H = G_0, G_1, \ldots, G_n = G$, such that for $1 \leq i \leq n$ there is a stable levelling in $G_i$ over $G_{i-1}$. For $n \geq 1$, let us say a graph $H$ is

$n$*-coverable* if there is an $n$-covering over $H$ in some pentagonal graph $G$ (and in particular, $H$ itself is pentagonal).

The *Grötzsch graph* has vertex set $\{a_1, \ldots, a_5, b_1, \ldots, b_5, c\}$, where $a_1$-$a_2$-$\cdots$-$a_5$-$a_1$ is a cycle, $a_i, b_i$ are both adjacent to $a_{i-1}$ and $a_{i+1}$ for $1 \leq i \leq 5$ (reading subscripts modulo 5), and $c$ is adjacent to $b_1, \ldots, b_5$. We call the 5-hole $a_1$-$a_2$-$\cdots$-$a_5$-$a_1$ its *rim* and $c$ its *apex*.

**5.1** *The Grötzsch graph is not* 1*-coverable.*

**Proof.** Suppose it is, and let $G$ be pentagonal, with a stable levelling $L_0, \ldots, L_k$, such that $G[L_k]$ has an induced subgraph $H$ isomorphic to the Grötzsch graph. Let $V(H)$ be labelled as above. We may assume that $L_k = V(H)$, and $L_{k-1}$ is minimal such that every vertex in $V(H)$ has a neighbour in $L_{k-1}$. For each $v \in L_{k-1}$, let $H(v)$ denote the set of neighbours of $v$ in $V(H)$. Consequently:

(1) *For each $v \in L_{k-1}$, there exists $u \in H(v)$ with no neighbour in $L_{k-1}$ except $v$.*

We call such a vertex $u$ a *dependent* of $v$. If $u, v \in L_{k-1}$, by a $u$-$v$ *gap* we mean an induced path $P$ of $G$, with one end in $H(u)$ and the other in $H(v)$, and with no other vertex in $H(u) \cup H(v)$ (a vertex in $H(u) \cap H(v)$ forms a 1-vertex gap.) Thus a $u$-$v$ gap is the interior of an induced path between $u$ and $v$. A $u$-$v$ gap is *odd* if it has odd length, and *even* similarly.

(2) *For all $u, v \in L_{k-1}$, every odd $u$-$v$ gap has length one.*

For suppose some $u$-$v$ gap has odd length at least three; then there is an induced path between $u, v$ of odd length at least five, with interior in $L_k$. But $u, v$ have neighbours in $L_{k-2}$, and so are joined by an induced path of even length with interior in the top of the levelling; and the union of these two paths is an odd hole of length at least seven, which is impossible.

(3) *For every four-vertex induced path $u_1$-$u_2$-$u_3$-$u_4$ of $H$, if $v, v' \in L_{k-1}$ and $u_1 \in H(v)$ and $u_4 \in H(v')$, then either one of $u_1, u_2 \in H(v')$, or one of $u_3, u_4 \in H(v)$.*

Because $H(v), H(v')$ are stable sets since $G$ is triangle-free; and from (2) this path is not a $u$-$v$ gap; and the claim follows.

(4) $|H(v_0)| \geq 2$ *for all $v_0 \in L_{k-1}$ with $c \in H(v_0)$.*

For suppose that $H(v_0) = \{c\}$. Then by (1), $c$ has no other neighbour in $L_{k-1}$. So for every four-vertex path of $H$ ending at $c$, say $u_1$-$u_2$-$u_3$-$c$, and for all $v \in L_{k-1}$ with $u_1 \in H(v)$, (3) implies that $u_3 \in H(v)$ (because $u_1, u_2 \notin H(v_0)$ since $|H(v_0)| = 1$, and $c \notin H(v')$ since $c$ is a dependent of $v_0$). Choose $v_1 \in L_{k-1}$ with $a_1 \in H(v_1)$. From $a_1$-$a_5$-$b_1$-$c$ it follows that $b_1 \in H(v_1)$, and similarly $b_3, b_4 \in H(v_1)$. Since $H(v_1)$ is stable, and the set $\{a_1, b_1, b_3, b_4\}$ is a maximal stable set of $H$, it follows that $H(v_1) = \{a_1, b_1, b_3, b_4\}$. Choose $v_3 \in L_{k-1}$ with $a_3 \in H(v_3)$; then (from the symmetry of $H$ taking $a_3$ to $a_1$) it follows that $H(v_3) = \{a_3, b_3, b_1, b_5\}$. But $b_4$-$a_5$-$a_4$-$b_5$ is an odd $v_1$-$v_3$ gap, contrary to (2). This proves (1).

(5) $|H(v_0)| = 3$ *for all $v_0 \in L_{k-1}$ with $c \in H(v_0)$.*

No stable set of $H$ containing $c$ has cardinality more than three, so we just need to show that $|H(v_0)| \neq 2$. Suppose not; then from the symmetry of $H$, we may assume that $H(v_0) = \{c, a_1\}$. One of $c, a_1$ is a dependent of $v_0$.

Suppose first that $c$ is a dependent of $v_0$. Choose $v_5 \in L_{k-1}$ with $a_5 \in H(v_5)$. From $a_5$-$a_4$-$b_3$-$c$ and (3) it follows that $b_3 \in H(v_5)$, and from $a_5$-$a_4$-$b_5$-$c$ that $b_5 \in H(v_5)$. Since $a_2, b_3$ are adjacent it follows that $a_2 \notin H(v_5)$; choose $v_2 \in L_{k-1}$ with $a_2 \in H(v_2)$. Then from the symmetry of $H$ exchanging $a_2, a_5$ and fixing $a_1$, it follows $b_2, b_4 \in H(v_2)$. From $a_5$-$b_1$-$c$-$b_2$ and (3) it follows that $b_2 \in H(v_5)$ (since $a_5, b_1 \notin H(v_2)$ because they both have neighbours in $H(v_2)$, and $c \notin H(b_5)$ because it is a dependent of $v_1$). From the same symmetry, $b_5 \in H(v_2)$; and so $a_3 \notin H(v_2)$ and $a_4 \notin H(v_5)$. But then $a_5$-$a_4$-$a_3$-$a_2$ is a $v_5$-$v_2$ gap, contrary to (2).

This shows that $c$ is not a dependent of $v_0$, and so $a_1$ is its dependent. Choose $v_3 \in L_{k-1}$ with $a_3 \in H(v_3)$; then $b_5 \in H(v_3)$ from $a_3$-$a_4$-$b_5$-$a_1$, and $a_5 \in H(v_3)$ from $a_3$-$b_4$-$a_5$-$a_1$. Now $a_4 \notin H(v_3)$; choose $v_4 \in L_{k-1}$ with $a_4 \in H(v_4)$, and then similarly $a_2, b_2 \in H(v_4)$. But then $a_5$-$b_1$-$c$-$b_2$ is a $v_3$-$v_2$ gap, a contradiction. This proves (5).

In view of (5) and the symmetry we may assume henceforth that $H(v_0) = \{a_5, a_2, c\}$. One of $a_5, a_2, c$ is a dependent of $v_0$. Suppose first that $c$ is a dependent of $v_0$. Choose $v_3 \in L_{k-1}$ with $a_3 \in H(v_3)$; then $b_5 \in H(v_3)$ from $a_2$-$a_4$-$b_5$-$c$, and $b_2 \in H(v_3)$ from $a_2$-$a_4$-$b_2$-$c$. Similarly, let $a_4 \in H(v_4)$; then $b_2, b_4 \in H(v_4)$. From $b_4$-$a_5$-$a_1$-$b_5$ it follows that $a_5 \in H(v_3)$, and similarly $a_2 \in H(v_4)$. But then $a_5$-$b_1$-$c$-$b_2$ is a $v_3$-$v_4$ gap, a contradiction.

From the symmetry between $a_2, a_5$, we may therefore assume that $a_2$ is a dependent of $v_0$. Let $b_2 \in H(v_2)$; then $a_4 \in H(v_2)$ from $b_2$-$a_3$-$a_4$-$a_5$, and $b_4 \in H(v_2)$ from $b_2$-$a_3$-$b_4$-$a_5$. Also, $a_2 \in H(v_2)$ from $a_4$-$b_5$-$a_1$-$a_2$. Let $a_3 \in H(v_3)$; then $a_1 \in H(v_3)$ from $a_3$-$b_2$-$a_1$-$a_5$, and $c \in H(v_3)$ from $a_1$-$b_5$-$c$-$b_4$. But then $a_4$-$a_5$-$b_1$-$c$ is a $v_2$-$v_3$ gap, a contradiction. This proves 5.1. ∎

# 6 Bounded radius

In this section we prove a bound on $\chi(N^2(v))$ for 2-coverable graphs, and on $\chi(N^3(v))$ for 3-coverable graphs, to allow us to apply 4.1. We begin with:

**6.1** *Let $L_0, \dots, L_k$ be a stable levelling in a pentagonal graph $G$, and let $P$ be a 5-hole of $G[L_k]$. Choose $S \subseteq L_{k-1}$ minimal such that every vertex in $P$ has a neighbour in $X$. Then*

- $|S| = 3$*;*
- *we can label the vertices of $P$ as $p_1$-$\cdots$-$p_5$-$p_1$ in order, and label the elements of $S$ as $a, b, c$, such that the edges of $G$ between $S$ and $V(P)$ are $ap_1, ap_3, bp_2, bp_4, cp_5$ and possibly $cp_3$;*
- *there exists $z \in L_{k-2}$ adjacent to every vertex in $S$.*

**Proof.** We begin by proving the first two assertions. Each vertex in $S$ has at most two neighbours in $P$, because its neighbours form a stable set. Suppose that every vertex in $S$ has exactly two neighbours in $P$. We may assume that $a \in S$ is adjacent to $p_1, p_3$; then choose $b \in S$ adjacent to $p_2$. It follows that $b$ is adjacent to one of $p_4, p_5$, say $p_4$. Choose $c \in S$ adjacent to $p_5$; then $c$ might also be adjacent to one of $p_2, p_3$, and from the symmetry we may assume it is not adjacent to $p_2$; and so $S = \{a, b, c\}$, and the first two assertions of the theorem hold. We may therefore assume that some vertex in $S$, say $c$, has only one neighbour in $P$, say $p_5$. From the minimality of $S$, no other vertex in $S$ is adjacent to $p_5$. Choose $a \in S$ adjacent to $p_3$. If $a$ has no more neighbours in $P$, then the path $a$-$p_3$-$p_2$-$p_1$-$p_5$-$c$ can be completed via an even path joining $a, c$ with interior in $L_0 \cup \cdots \cup L_{k-2}$ to an odd hole of length at least seven, which is impossible. So $a$ has another neighbour in $P$, and since $a$ is not adjacent to $p_5$ it is adjacent to $p_1$. Similarly, choose $b \in S$ adjacent to $p_2$; then $b$ is also adjacent to $p_4$. From the minimality of $S$, $S = \{a, b, c\}$ and again the first two assertions hold.

For the third assertion, choose $Z \subseteq L_{k-2}$ minimal containing a neighbour of each member of $S$. Suppose that there are distinct $z_1, z_2 \in Z$. From the minimality of $Z$, there exist $s_1, s_2 \in S$ such that for $1 \le i, j \le 2$, $z_i$ is adjacent to $s_j$ if and only if $i = j$. But from the second assertion of the theorem, there is a three-edge path joining $s_1, s_2$ with interior in $V(P)$, say $s_1$-$p_1$-$p_2$-$s_2$, where $p_1p_2$ is an edge of $P$. Then $z_1$-$s_1$-$p_1$-$p_2$-$s_2$-$z_2$ is an induced path, and can be completed to an odd hole of length at least seven via an even induced path joining $z_1, z_2$ with interior in $L_0 \cup \cdots \cup L_{k-3}$, which is impossible. Thus $|Z| = 1$, and so the third assertion holds. This proves 6.1. ∎

We also need the following lemma.

**6.2** *Let $G$ be pentagonal, and let $L_0, \ldots, L_k$ be a stable covering in $G$ of a graph $H$. Let $z \in V(H)$, and $A$ be the set of all vertices $v \in N^2_H(z_0)$ such that every neighbour of $v$ in $L_{k-1}$ is adjacent to $z$. Then $\chi(A) \le 2$.*

**Proof.** Suppose that $\chi(A) > 2$; then there is a 5-hole $P$ of $G[A]$. Choose a minimal subset $S$ of $N^1_H(z)$ such that every vertex in $p$ has a neighbour in $S_1$; then by 6.1 we may assume that $S = \{a, b, c\}$, where the edges between $S$ and $V(P)$ are $ap_1, ap_3, bp_2, bp_4, cp_5$ and possibly $cp_3$. Choose $v \in L_{k-1}$ adjacent to $p_5$; by hypothesis, $v$ is adjacent to $z$. Choose $a', b' \in L_{k-1}$ adjacent to $a, b$ respectively. Consequently $a', b'$ are not adjacent to $z$, and so have no neighbours in $V(P)$; and in particular, $a', b'$ are different from $v$. There is an even induced path between $v, a'$ with interior in $L_0 \cup \cdots \cup L_{k-2}$, and so the odd path $v$-$p_5$-$p_4$-$p_3$-$a$-$a'$ is not induced, since its union with the previous path would form an odd hole of length at least seven. But $a'$ has no neighbour in $P$ (because $V(P) \subseteq A$), and $v$ is not adjacent to $a$ (because $G$ is triangle-free) and $v$ is not adjacent to $a'$ (because $L_{k-1}$ is stable), and it follows that $v$ is adjacent to $p_3$. The same arguments applied to the path $v$-$p_5$-$p_1$-$p_2$-$b$-$b'$ show that $v$ is adjacent to $p_2$; yet not both of these are true since $G$ is triangle-free, a contradiction. This proves 6.2. ∎

We deduce:

**6.3** *If $H$ is a 2-coverable graph and $z \in V(H)$ then $\chi(N^2_H(z)) \le 5$.*

**Proof.** Since $H$ is 2-coverable, there is a 1-coverable graph $G$ and a stable levelling $L_0, \ldots, L_k$ in $G$ over $H$. Let $A$ be the set of all vertices $v$ in $N^2_H(z)$ such that every neighbour of $v$ in $L_{k-1}$ is adjacent to $z$, and let $B = N^2_H(z) \setminus A$. By 6.2, $\chi(A) \le 2$, so we may assume (for a contradiction) that $\chi(B) > 3$.

Choose $z_0 \in L_{k-1}$ adjacent to $z$. Since $N_G(z_0)$ is stable, it follows that $\chi(B \setminus N_G(z_0)) \ge 2$; and so there is a 5-hole $P$ with $V(P) \subseteq B$, such that $z_0$ has no neighbours in $P$. Let $S_1 \subseteq N_H(z)$ be minimal such that every vertex in $P$ has a neighbour in $S_1$. Each vertex in $P$ has a neighbour in $L_{k-1}$ nonadjacent to $z$, and so there exists a minimal subset $S_2$ of $L_{k-1} \setminus N_G(z)$ such that every vertex in $P$ has a neighbour in $S_2$. By 6.1, $|S_1| = |S_2| = 3$.

(1) *If $a_1 \in S_1$ and $a_2 \in S_2$ are joined by a three-edge path with interior in $V(P)$, then $a_1, a_2$ are adjacent. In particular, if $a_1 \in S_1$ and $a_2 \in S_2$ both have two neighbours in $V(P)$ and have a common neighbour in $V(P)$ then they have the same neighbours in $V(P)$.*

Let $a_1, a_2$ be adjacent to $p_1, p_2$ respectively, where $p_1p_2$ is an edge of $P$. If $a_1, a_2$ are not adjacent, then the path $z_0$-$z$-$a_1$-$p_1$-$p_2$-$a_2$ is induced, and can be completed to an odd hole of length at least seven via an even induced path between $z_0, a_2$ with interior in $L_0 \cup \cdots \cup L_{k-2}$, which is impossible. This proves the first claim of (1). For the second, suppose that $a_1, a_2$ have a common neighbour in $V(P)$; then they are nonadjacent, and so cannot be joined by a three-edge path with interior in $V(P)$, by the first claim. This proves (1).

Let $S_i = \{a_i, b_i, c_i\}$ for $i = 1, 2$. By 6.1, for $i = 1, 2$ we may assume that $a_i, b_i$ each have two neighbours in $V(P)$, and have no common neighbour in $V(P)$. So one of $a_2, b_2$, say $a_2$, is adjacent to a neighbour of $a_1$ in $V(P)$, and hence $a_1, a_2$ have the same neighbours in $V(P)$, by the second claim of (1). Therefore $b_2$ and $b_1$ have a common neighbour in $V(P)$, and so by the same argument, $b_1, b_2$ have the same neighbours in $V(P)$. If $c_1$ has two neighbours in $V(P)$, then it has a common neighbour in $V(P)$ with one of $a_2, b_2$, and so by the second claim of (1) it has the same neighbours in $V(P)$ as one of $a_2, b_2$, and hence the same as one of $a_1, b_1$, which is impossible by the minimality of $S_1$. Thus $c_1$ has exactly one neighbour in $P$, and similarly $c_2$ has exactly one neighbour in $P$, and the same neighbour as $c_1$.

We may therefore assume that for $i = 1, 2$, $a_i$ is adjacent to $p_2, p_4$ and $b_i$ to $p_3, p_5$, and $c_i$ to $p_1$. By the first claim of (1), it follows that $a_1$ is adjacent to $b_2, c_2$, and $b_1$ to $a_2, c_2$, and $c_1$ to $a_2, b_2$. But then the subgraph induced on

$$\{p_1, p_2, p_4, p_5, a_1, b_1, c_1, a_2, b_2, c_2, z\}$$

is isomorphic to the Grötzsch graph (with rim $a_2$-$c_1$-$p_1$-$p_5$-$b_1$-$a_2$ and apex $a_1$), contradicting 5.1 since $G$ is 1-coverable. This proves 6.3. ∎

**6.4** *If $H$ is a 3-coverable graph and $z \in V(H)$ then $\chi(N^3_H(z)) \leq 20$.*

**Proof.** Since $H$ is 3-coverable, there is a 2-coverable graph $G$ and a stable levelling $L_0, \ldots, L_k$ in $G$ over $H$. Choose $z_0 \in L_{k-1}$ adjacent to $z$. Let

$$A = N^3_H(z) \setminus (N^2_G(z) \cup N^2_G(z_0)).$$

By 6.3, we can partition $N^2_H(z)$ into five stable sets $D_1, \ldots, D_5$. For $1 \leq i \leq 5$, let $A_i$ be the set of vertices in $A$ with a neighbour in $D_i$. Thus $\{z\}, N_H(z), D_i, A_i$ is a stable levelling in $G$, and $A = A_1 \cup \cdots \cup A_5$.

(1) $\chi(A_i) \leq 2$ *for* $1 \leq i \leq 5$.

For suppose this is false for some $i$, and let $P$ be a 5-hole with $V(P) \subseteq A_i$. Choose $S_1 \subseteq D_i$ minimal such that every vertex in $P$ has a neighbour in $S_1$. By 6.1, there exists $n \in N_H(z)$ adjacent to every vertex in $S_1$. By 6.2, some vertex $x \in L_{k-1}$ has a neighbour in $V(P)$ and is nonadjacent to $n$. Let $x$ be adjacent to $p_1 \in V(P)$ say. Choose $y \in S_1$ adjacent to $p_1$; then the path $x$-$p_1$-$y$-$n$-$z$-$z_0$ is not induced, since $x, z_0$ are joined by an even induced path with interior in $L_0 \cup \cdots \cup L_{k-2}$. But $z_0$ is not adjacent to any of $x, p_1, y$, since $p_1 \in A$ and therefore has distance at least three from $z_0$ in $G$; and $x$ is not adjacent to $z$, because $p_1$ has distance three from $z$ in $G$; and $x$ is not adjacent to $n$, a contradiction. This proves (1).

From (1) we deduce that $\chi(A) \leq 10$. But every vertex of $N^3_H(z)$ belongs to one of $A, N^2_G(z_0), N^2_G(z)$, and by 6.3 the latter two sets both have chromatic number at most five. This proves 6.4. ∎

Now we complete the proof of 1.2, which we restate:

**6.5** *Every pentagonal graph is* 82200*-colourable.*

**Proof.** Define $n_1 = 199$, $n_2 = 4n_1 + 27$, $n_3 = 10n_2 - 9$, and $n_4 = 10n_3 - 9$. Suppose that there is a pentagonal graph $G_4$ with $\chi(G_4) \geq n_4$. By 3.2, there is a stable levelling in $G_4$ over some graph $G_3$ with $\chi(G_3) \geq n_3$. Similarly there is a stable levelling in $G_3$ over some $G_2$ with $\chi(G_2) \geq n_2$. By 6.3, $\chi(N^2_{G_2}(v)) \leq 5$ for every vertex $v$ of $G_2$. By 3.3 there is a stable levelling in $G_2$ over some graph $G_1$ with $\chi(G_1) \geq n_1$; and $\chi(N^3_{G_1}(v)) \leq 20$ for every vertex $v$ of $G_1$, by 6.4 applied to the 2-cover $G_3, G_2, G_1$. By 4.1, setting $l = 2$, $\kappa_2 = 5$ and $\kappa_3 = 20$, it follows that

$$\chi(G_1) \leq (12l - 2)\kappa_2 + 4\kappa_3 + 8 = 198,$$

a contradiction. Thus there is no such $G_4$, and hence every pentagonal graph has chromatic number at most $n_4 - 1 = 82200$. This proves 6.5. ∎

# 7 Long holes

In this section we prove 1.3 and 1.4. The first is implied by the next result with $m = 2$:

**7.1** *Let $l \geq m \geq 2$ be integers, and let $G$ be a triangle-free graph with no odd hole of length at most $2m + 1$ and no odd hole of length more than $2l + 1$. Then $\chi(G) < (3 + 4l)4^{l-m} - 4l$.*

**Proof.** We proceed by induction on $l - m$. If $m = l$ then $G$ is bipartite and the result is true, so we assume that $m < l$. Suppose that $\chi(G) \geq (3 + 4l)4^{l-m} - 4l$. Then we may choose a levelling in $G$ with base of chromatic number at least $\chi(G)/2 \geq (6 + 8l)4^{l-m-1} - 2l$. Since $G$ has no odd cycle of length at most five, it follows that $N_2(v)$ is stable for every vertex $v$; and so by 3.1 with $\kappa = 1$, there is a stable levelling $L_0, L_1, \ldots, L_k$ in $G$ with $\chi(L_k) \geq (3 + 4l)4^{l-m-1} - 2l$, and we may choose it such that $G[L_k]$ is connected. It follows that $k \geq 3$. For $0 \leq i \leq k$ choose $s_i \in L_i$ such that $s_0$-$s_1$-$\cdots$-$s_k$ is a path. Since $\chi(L_k) > 2l$ and $(L_k, s_{k-2}$-$s_{k-1})$ is a lollipop, 2.2 with $\kappa = 1$ implies that there is a licking $(C, T_1)$ of this lollipop with

$$\chi(C) \geq \chi(L_k) - 2l = (3 + 4l)4^{l-m-1} - 4l$$

and cleanliness at least $2l$. From the inductive hypothesis, there is a $(2m+3)$-hole $P$ in $C$, with vertices $p_1$-$\cdots$-$p_{2m+3}$-$p_1$ say. By 2.1 there is a licking $(P,T)$ of $(C,T_1)$. Let $T$ have vertices

$$s_{k-2}\text{-}s_{k-1}\text{-}t_1\text{-}\cdots\text{-}t_r$$

say; thus $t_r$ has a neighbour in $P$, and since the lollipop $(P,T)$ has cleanliness at least $2l$, it follows that $r \geq 2l$ and each of $s_{k-2}, s_{k-1}, t_1, \ldots, t_{2l-2}$ has distance at least three from $V(P)$.

Now since $G$ has no odd cycle of length less than $2m+3$, it follows that every vertex of $G$ not in $P$ either has at most one neighbour in $P$, or has exactly two neighbours in $P$ with distance two in $P$. We may therefore assume that $t_r$ is adjacent to $p_1$ and to no other vertex of $P$ except possibly $p_{2m+2}$. For $i = 3,4$, choose $a_i \in L_{k-1}$ adjacent to $p_i$. It follows that $a_3, a_4$ are nonadjacent to $s_{k-2}, s_{k-1}, t_1, \ldots, t_{2l-2}$. Since $L_0, \ldots, L_{k-1}$ are stable, for $i = 3,4$ there is an even induced path $R_i$ between $a_i$ and $s_{k-1}$ with interior in $L_0 \cup \cdots \cup L_{k-2}$.

(1) *$a_4$ has a neighbour in $V(T)$.*

Because suppose not. Then $R_4 \cup T$ is an induced path from $a_4$ to $t_r$, of length at least $r+2 \geq 2l+2$. But there is an odd induced path and an even induced path between $a_4$ and $t_r$ with interior in $V(P)$ (since $a_4$ has no neighbours in $P$ except $p_4$ and possibly $p_2, p_6$, and $t_r$ has no neighbours in $P$ except $p_1$ and possibly $p_{2m+2}$; one of $a_4$-$p_4$-$p_3$-$p_2$-$p_1$-$t_4$, $a_4$-$p_2$-$p_1$-$t_r$ is the desired odd path, and the even path goes the other way around $P$.) But then the union of one of these paths with $R_4 \cup Q$ is an odd hole of length at least $2l+4$, which is impossible. This proves (1).

Choose $i \leq r$ minimum such that $t_i$ is adjacent to one of $a_4, a_3$. By (1), such a choice is possible. Since $a_3, a_4$ are nonadjacent to $s_{k-2}, s_{k-1}, t_1, \ldots, t_{2l-2}$, it follows that $i > 2l-2$. Since $G$ has no odd cycle of length at most five, $t_i$ is not adjacent to both $a_3, a_4$; let $t_i$ be adjacent to $a_h$ and not to $a_j$, where $\{h,j\} = \{3,4\}$. Let $Q$ be a minimal path between $a_h, s_{k-1}$ with interior in $V(T)$. It follows that $Q$ has length at least $2l$. Consequently $Q \cup R_h$ is a hole of length at least $2l+2$, and so it is even; and hence $Q$ is even. Now $a_h$ has no neighbour in $R_j$, since $a_h$ is not adjacent to the parent of $a_j$ (because $G$ has no 5-holes) and $a_h$ is nonadjacent to $s_{k-2}$ (because $(P,T)$ is a lollipop of cleanliness at least one). Thus

$$a_h\text{-}p_h\text{-}p_j\text{-}a_j\text{-}R_j\text{-}s_{k-1}\text{-}Q\text{-}a_h$$

is an odd hole of length at least $2l+5$, which is impossible. This proves 7.1. ∎

Finally we turn to the proof of 1.4. It follows from the next result.

**7.2** *Let $l \geq 3$ and $\kappa \geq 1$ be integers, and let $G$ be a graph with no hole of length more than $l$, such that $\chi(N(v)), \chi(N^2(v)) \leq \kappa$ for every vertex $v$. Then $\chi(G) \leq (2l-2)\kappa$.*

**Proof.** Suppose not; then there is a levelling $L_0, \ldots, L_k$ in $G$ with $\chi(L_k) > (l-1)\kappa$. Let $C'$ be the vertex set of a component $C'$ of $G[L_k]$ with $\chi(C') > (l-1)\kappa$. Since $l-1 > 1$, it follows that $k \geq 2$. For $i = k-2, k-1$ choose $s_i \in L_i$, such that $s_{k-2}, s_{k-1}$ are adjacent and $s_{k-1}$ has a neighbour in $C'$. Since $\chi(C') > (l-1)\kappa$ and $(V(C'), s_{k-2}\text{-}s_{k-1})$ is a lollipop, by 2.2 there is a licking $(C,T)$ of it with cleanliness at least $l-1$ and with $\chi(C) \geq \chi(C') - (l-1)\kappa > 0$. Choose $a \in L_{k-1}$ with a neighbour in $C$. Now $a$ might have neighbours in $T$, but since $(C, T)$ has cleanliness at least $l-1$, $a$ is nonadjacent to the first $l-1$ vertices of $T$. Let $P$ be an induced path between $s_{k-1}$ and $a$ with interior in $V(T) \cup C$; thus $P$ has length at least $l-1$. But $a, s_{k-1}$ are joined by an induced path with interior in $L_0 \cup \cdots \cup L_{k-2}$, and the union of this path with $P$ is a hole of length at least $l+1$, a contradiction. This proves 7.2. ∎

We deduce 1.4, which we restate, slightly strengthened.

**7.3** *Let $l \geq 3$ be an integer, and let $G$ be a graph with no 5-hole and no hole of length more than $l$. Then*

$$\chi(G) \leq (2l-2)^{2^{\omega(G)-1}-1}.$$

**Proof.** We proceed by induction on $\omega(G)$. If $\omega(G) = 1$ the result is true, so we assume $\omega(G) > 1$. Let

$$n = (2l-2)^{2^{\omega(G)-2}-1}.$$

From the inductive hypothesis, every induced subgraph $H$ of $G$ with $\omega(H) < \omega(G)$ is $n$-colourable.

(1) *For every vertex $v$ of $G$, $\chi(N(v)) \leq n$, and $\chi(N^2(v)) \leq n^2$.*

The graph $G[N(v)]$ contains no clique of size $\omega(G)$, and so is $n$-colourable. Let $A_1, \ldots, A_n$ be a partition of $N(v)$ into $n$ stable sets, and for $1 \leq i \leq n$ let $B_i$ be the set of vertices in $N^2(v)$ with a neighbour in $A_i$. Suppose that there is a clique $C$ of cardinality $\omega(G)$ with $C \subseteq B_i$ for some $i$. Choose $a \in A_i$ with as many neighbours in $C$ as possible; then there exists $c' \in C$ nonadjacent to $a$, since $G$ has no $(\omega(G)+1)$-clique. Choose $a' \in A_i$ adjacent to $c'$; then from the choice of $a$, there exists $c \in C$ adjacent to $a$ and not to $a'$. But then the subgraph induced on $\{v, a, a', c, c'\}$ is a 5-hole, which is impossible. Thus there is no such clique $C$, and so $\chi(A_i) \leq n$. Since this holds for all $i$, it follows that $\chi(N^2(v)) \leq n^2$. This proves (1).

From (1) and 7.2, it follows that

$$\chi(G) \leq (2l-2)n^2 = (2l-2)(2l-2)^{2^{\omega(G)-1}-2} = (2l-2)^{2^{\omega(G)-1}-1}.$$

This proves 7.3. ∎

# References


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